

DEFINABILITY VIA KALIMULLIN PAIRS IN THE STRUCTURE OF THE ENUMERATION DEGREES

HRISTO A. GANCHEV[†] AND MARIYA I. SOSKOVA^{†‡}

ABSTRACT. We give an alternative definition of the enumeration jump operator. We prove that the class of total enumeration degrees and the class of low enumeration degrees are first order definable in the local structure of the enumeration degrees.

1. INTRODUCTION

The main focus in degree theory, established as one of the core areas in Computability Theory, is to understand a mathematical structure, which arises as a formal way of classifying the computational strength of an object. The most studied examples of such structures are that of the Turing degrees, \mathcal{D}_T , based on the notion of Turing reducibility, as well as its local substructures, of the Turing degrees reducible to the first jump of the least degree, $\mathcal{D}_T(\leq \mathbf{0}_T')$, and of the computably enumerable degrees, \mathcal{R} . In investigating such a mathematical structure among the main question that we ask is: how complex is this structure. The complexity of a structure can be inspected from many different aspects: how rich is it algebraically; how complicated is its theory; what sets are definable in it; does it have nontrivial automorphisms. The question about definability, in particular, is interrelated with all of the other questions, and can be seen as a key to understanding the natural concepts that are approximated by the corresponding mathematical formalism. Research of the Turing degrees has been successful in providing a variety of results on definability. For the global theory of the Turing degrees, among the most notable results is the definability of the jump operator by Slaman and Shore [20]. The method used in the proof of this result, as well as many other definability results in \mathcal{D}_T , leads Slaman and Woodin to conjecture that every definable set in second order arithmetic is definable in \mathcal{D}_T . This is a consequence of their Biinterpretability conjecture, which is shown to be equivalent to the rigidity of \mathcal{D}_T [21].

In the local theory Nies, Shore and Slaman [15] show a weakening of the biinterpretability conjecture for the computably enumerable degrees and obtain from it the first order definition of the jump classes H_n and L_{n+1} in \mathcal{R} , for every natural number n . Recently Shore [18] has shown that the same weakening of the biinterpretability conjecture holds in the Δ_2^0 Turing degrees, and so the classes H_n

1991 *Mathematics Subject Classification.* 03D30.

Key words and phrases. Enumeration reducibility, first order definability, enumeration jump, total enumeration degrees, low enumeration degrees.

This research was supported by a BNSF grant No. DMU 03/07/12.12.2011 and by a Sofia University SF grant No. 131/09.05.2012. The second author was supported by a Marie Curie international outgoing fellowship STRIDE (298471) within the 7th European Community Framework Programme and the Isaac Newton Institute for Mathematical Sciences in the programme ‘Semantics and Syntax’.

and L_{n+1} for every natural number n are definable in $\mathcal{D}_T(\leq \mathbf{0}_T')$ as well. One class of degrees which has managed to elude every attempt at definability in both local structures is that of the low_1 degrees, L_1 , the degrees whose jump is the least possible degree, $\mathbf{0}_T'$.

Another approach for understanding a structure, often used in mathematics, is to place this structure in a richer context, a context which would reveal new hidden relationships. The most promising candidate for such a larger context is the structure of the enumeration degrees, introduced by Friedberg and Rogers [5]. This structure is induced by a weaker form of relative computability: a set A is enumeration reducible to a set B if every enumeration of the set B can be effectively transformed into an enumeration of the set A . The induced structure of the enumeration degrees, \mathcal{D}_e , is an upper-semilattice with jump operation and least element. The Turing degrees can be embedded in the enumeration degrees via the standard embedding ι which maps the Turing degree of a set A to the enumeration degree of $A \oplus \bar{A}$. This embedding preserves the order, the least upper bound and the jump operation. The range of ι is therefore a substructure of \mathcal{D}_e , which is isomorphic to \mathcal{D}_T . This structure will be denoted by \mathcal{TOT} and its elements will be called the total enumeration degrees. An important question, which immediately arises in this context, first set by Rogers [16], is whether \mathcal{TOT} is first order definable in \mathcal{D}_e . Rozinas [17] proves that every enumeration degree is the greatest lower bound of two total enumeration degrees, thus the total enumeration degrees are an automorphism base for \mathcal{D}_e . This gives further motivation for studying the issue of the definability of \mathcal{TOT} in \mathcal{D}_e , as it would provide a strong link between the automorphism problem for the structures of the Turing degrees and the enumeration degrees. If \mathcal{TOT} is definable in \mathcal{D}_e then a nontrivial automorphism of \mathcal{D}_e would yield a nontrivial automorphism of \mathcal{D}_T .

Definability in the enumeration degrees has had its successes as well. Kalimullin [12] has shown that the enumeration jump is definable in \mathcal{D}_e . The method used in his proof is significantly less complex than that used to prove the corresponding result in the Turing degrees. The definition of the enumeration jump is closer to the much sought natural definition, see Shore [19], and is based on the first order definability of a relativized version of the notion of a Kalimullin pair, \mathcal{K} -pair. Here we will give an alternative proof of the definability of the enumeration jump, which does not use relativization and we see as more natural in a sense that will be made precise.

The jump operation gives rise to a local structure in the enumeration degrees, \mathcal{G}_e , consisting of all enumeration degrees that are reducible to the first jump, $\mathbf{0}_e'$, of the least degree, $\mathbf{0}_e$. As ι preserves the jump operation, it follows that $\mathcal{TOT} \cap \mathcal{G}_e$ is a structure, which is isomorphic to $\mathcal{D}_T(\leq \mathbf{0}_T')$, the local structure of the Turing degrees. In [6] we have shown that \mathcal{K} -pairs are first order definable in \mathcal{G}_e , providing the first step in the investigation of the definability theme for the local structure of the enumeration degrees. The local definition of \mathcal{K} -pairs unlocked numerous further results in the study of \mathcal{G}_e . For example in [6] we show that the classes of the upwards properly Σ_2^0 enumeration degrees and the downwards properly Σ_2^0 enumeration degrees are first order definable in \mathcal{G}_e and in [8] we show that the first order theory of true arithmetic can be interpreted in \mathcal{G}_e , using coding methods based on \mathcal{K} -pairs.

In this article we give two more examples of sets of degrees with natural first order definitions in \mathcal{G}_e . The first one gives a positive answer to the local version of Rogers' question. We show that the set of total Σ_2^0 enumeration degrees is first order definable in \mathcal{G}_e .

The second example supplies further evidence that studying the structure of the Turing degrees within the larger context of the enumeration degrees can provide us with more insight. We show that the set of low enumeration degrees is first order definable in \mathcal{G}_e . Combined with the local definability of the total enumeration degrees this gives the first instance of a local first order definition of an isomorphic copy of the low Turing degrees.

2. PRELIMINARIES

We refer to Cooper [3] for a survey of basic results on the structure of the enumeration degrees and to Sorbi [22] for a survey of basic results on the local structure \mathcal{G}_e . We outline here basic definitions and properties of the enumeration degrees used in this article.

Definition 1. *A set A is enumeration reducible (\leq_e) to a set B if there is a c.e. set Φ such that:*

$$A = \Phi(B) = \{n \mid \exists u(\langle n, u \rangle \in \Phi \ \& \ D_u \subseteq B)\},$$

where D_u denotes the finite set with code u under the standard coding of finite sets. We will refer to the c.e. set Φ as an enumeration operator.

A set A is *enumeration equivalent* (\equiv_e) to a set B if $A \leq_e B$ and $B \leq_e A$. The equivalence class of A under the relation \equiv_e is the enumeration degree $\mathbf{d}_e(A)$ of A . The structure of the enumeration degrees $\langle \mathcal{D}_e, \leq \rangle$ is the class of all enumeration degrees with relation \leq defined by $\mathbf{d}_e(A) \leq \mathbf{d}_e(B)$ if and only if $A \leq_e B$. It has a least element $\mathbf{0}_e = \mathbf{d}_e(\emptyset)$, the set of all c.e. sets. We can define a least upper bound operation, by setting $\mathbf{d}_e(A) \vee \mathbf{d}_e(B) = \mathbf{d}_e(A \oplus B)$.

The enumeration jump of a set A is defined by Cooper [2].

Definition 2. *The enumeration jump of a set A is denoted by $J_e(A)$ and is defined as $K_A \oplus \overline{K_A}$, where $K_A = \{\langle e, x \rangle \mid x \in \Phi_e(A)\}$. The enumeration jump of the enumeration degree of a set A is $\mathbf{d}_e(A)' = \mathbf{d}_e(J_e(A))$.*

By iterating the jump operation, we define inductively the n -th jump of a degree \mathbf{a} for every n : $\mathbf{a}^0 = \mathbf{a}$ and $\mathbf{a}^{n+1} = (\mathbf{a}^n)'$.

Definition 3. *A set A is called total if $A \equiv_e A \oplus \overline{A}$. An enumeration degree is called total if it contains a total set. The collection of all total degrees is denoted by \mathcal{TOT} .*

As noted above, the structure \mathcal{TOT} is an isomorphic copy of the Turing degrees. The map ι , defined by

$$\iota(\mathbf{d}_T(A)) = \mathbf{d}_e(A \oplus \overline{A})$$

is an embedding of \mathcal{D}_T in \mathcal{D}_e , which preserves the order, the least upper bound and the jump operation.

The local structure of the enumeration degrees, denoted by \mathcal{G}_e , is the substructure with domain, consisting of all enumeration degrees, which are reducible to $\mathbf{0}_e'$. As noted above, the elements of \mathcal{G}_e are the enumeration degrees which contain Σ_2^0 sets, or equivalently, which consist entirely of Σ_2^0 sets.

Definition 4. A set A will be called low if $J_e(A) \equiv_e J_e(\emptyset)$. An enumeration degree $\mathbf{a} \in \mathcal{G}_e$ is low, if $\mathbf{a}' = \mathbf{0}_e'$.

3. SEMI-RECURSIVE SETS

In this section we will examine the properties of semi-recursive sets in the context of enumeration reducibility. This analysis extends the one that can be found in Arslanov, Cooper and Kalimullin [1].

Definition 5. We say that a set of natural numbers A is semi-recursive if there is a total computable selector function $s_A : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, such that for any $x, y \in \mathbb{N}$, $s_A(x, y) \in \{x, y\}$ and whenever $\{x, y\} \cap A \neq \emptyset$, $s_A(x, y) \in A$.

Jockusch [11] showed that every nonzero Turing degree contains a semi-recursive set A , such that both A and \bar{A} are not c.e. In the context of enumeration reducibility this property can be translated as follows. A nonzero enumeration degree \mathbf{t} is total if and only if there is a semi recursive set A , which is not c.e. or co-c.e. such that

$$\mathbf{t} = \mathbf{d}_e(A) \vee \mathbf{d}_e(\bar{A}).$$

Thus if we could show that the pairs of enumeration degrees of a semi-recursive sets and its complement are first order definable, then we would be able to define the class of total enumeration degrees. This fact motivates the study of the structural properties of semi recursive sets.

Our first discovery is that we can limit our investigations of semi-recursive sets to the study of semi-recursive sets of a particular kind. We describe these below.

Let A be a set of natural numbers. We view the characteristic function of A as an infinite binary string χ_A , the $(n+1)$ -th element of which, $\chi_A(n)$, is determined by the membership of n in A . Namely, $\chi_A(n) = 0$ if $n \notin A$ and $\chi_A(n) = 1$ if $n \in A$. We will denote by $\chi \upharpoonright n$ the initial segment of χ_A of length n .

Finite binary strings are naturally ordered by the lexicographical ordering. Let σ and ρ be finite binary strings. We say that σ is to the left of ρ , denoted by $\sigma <_L \rho$, if there is a finite binary string τ , such that $\tau * 0 \subseteq \sigma$ and $\tau * 1 \subseteq \rho$. We can extend this relation to make it reflexive and linear as follows: $\sigma \leq_L \rho$ if $\sigma \subseteq \rho$ or $\sigma <_L \rho$.

Denote the length of a finite binary string σ by $|\sigma|$. If σ is a finite binary string and χ is an infinite binary string then $\sigma \leq_L \chi$ and $\sigma <_L \chi$ will be shorthand for $\sigma \leq_L \chi \upharpoonright |\sigma|$ and $\sigma <_L \chi \upharpoonright |\sigma|$ respectively. The set of all finite binary strings to the left of or along χ_A will be denoted by L_A :

$$L_A = \{\sigma \in 2^{<\omega} \mid \sigma \leq_L \chi_A\}.$$

The complement of L_A , will be denoted by R_A . Note that R_A can be described as follows.

$$R_A = \overline{L_A} = \{\sigma \in 2^{<\omega} \mid \sigma \not\leq_L \chi_A \upharpoonright |\sigma|\} = \{\sigma \in 2^{<\omega} \mid \chi_A \upharpoonright |\sigma| <_L \sigma\}.$$

Both sets L_A and R_A are easily shown to be semi-recursive, as the relation \leq_L is a computable linear ordering on finite binary strings. The selector function s_{L_A} simply outputs the smaller, with respect to \leq_L , of its two arguments. In the context of enumeration reducibility, the following properties make the pair $\{L_A, R_A\}$ particularly useful.

Proposition 1. For every set of natural numbers A the following holds.

- (1) $L_A \leq_e A$;

- (2) $R_A \leq_e \bar{A}$;
- (3) $L_A \oplus R_A \equiv_e A \oplus \bar{A}$;
- (4) A is semi-recursive if and only if $A \leq_1 L_A$, i.e. if there is a total computable injective function g , such that $n \in A$ if and only if $g(n) \in L_A$.

Proof. Let A be a set of natural numbers. For every finite binary string σ let $D_\sigma = \{n < |\sigma| \mid \sigma(n) = 1\}$. We note here that if $D_\sigma \subseteq A$ then $\sigma \leq_L \chi_A$. If we assume otherwise: $\sigma \not\leq_L \chi_A$ then there is a finite string τ of length n , which is a prefix of both σ and χ_A , and such that $\sigma(n) = 1$ and $\chi(n) = 0$. But then $n \in D_\sigma \setminus A$. Note that the reverse is not necessarily true.

- (1) A finite string σ is in L_A if and only if there is a τ , of the same length as σ , such that $\sigma \leq_L \tau$ and $D_\tau \subseteq A$. Indeed, if $\sigma \in L_A$ and σ is of length n , then $\tau = \chi_A \upharpoonright (n+1)$ is such a string. If on the other hand $\sigma \leq_L \tau$ and $D_\tau \subseteq A$ then $\tau \leq_L \chi_A$ and hence by transitivity $\sigma \leq_L \chi_A$.
- (2) Here it is worth noting that $R_A \cup \{\sigma \mid \sigma \subseteq A\}$ is 1-equivalent to $L_{\bar{A}}$ via the computable permutation which maps a finite string σ to its mirror image $\bar{\sigma}$, inverting every bit. Thus $\sigma \in R_A$ if and only if there exists τ such that $\bar{\sigma} <_L \tau$ and $D_\tau \subseteq \bar{A}$.
- (3) From (1) and (2) it follows that $L_A \oplus R_A \leq_e A \oplus \bar{A}$. On the other hand for every natural number n , $n \in \bar{A}$ if and only if there is a string σ of length n such that $\sigma * 0 \in L_A$ and $\sigma * 1 \in R_A$. Finally $n \in A$ if and only if there is a string σ of length $n+1$ ending in 1 in L_A , such that for every string τ of length $n+1$, if $\sigma <_L \tau$ then $\tau \in R_A$.
- (4) Suppose A is semi-recursive with a selector function s_A . Then for every n we build a string σ_n of length $n+1$, ending in 1, as follows. For every $m < n$:

$$\sigma_n(m) = \begin{cases} 0, & \text{if } s_A(n, m) = n \\ 1, & \text{if } s_A(n, m) = m. \end{cases}$$

We claim that $n \in A$ if and only if $\sigma_n \in L_A$. Suppose that $n \in A$ then for every $m \leq n$, $\sigma_n(m) = 1$ implies $m \in A$ by the properties of the selector function. So if $n \in A$ then $D_{\sigma_n} \subseteq A$ and hence $\sigma_n \leq_L \chi_A$.

If $n \notin A$ then $\sigma_n(m) = 0$ implies $m \in \bar{A}$ by the properties of the selector function. Thus $D_{\bar{\sigma}_n} \subseteq \bar{A}$ and hence $\bar{\sigma}_n \in L_{\bar{A}}$. Here $\bar{\sigma}_n$ is, as in (2), the mirror image of σ_n . Thus $\sigma_n \in R_A \cup \{\sigma \mid \sigma \subseteq \chi_A\}$. As $\sigma_n(n) = 1$ and $n \notin A$, it follows that $\sigma_n \notin \chi_A$, so $\sigma_n \in R_A = \overline{L_A}$. This shows that $A \leq_1 L_A$ via the function g , defined by $g(n) = \sigma_n$.

On the other hand if $A \leq_1 L_A$ via a computable injective function g then we can construct the selector function for A , using g and the selector function for L_A :

$$s_A(x, y) = g^{-1}(s_{L_A}(g(x), g(y))).$$

□

We note here that the first three properties are in fact a proof of Jockusch's theorem for all sets A whose Turing degree is not computably enumerable. The fourth property in Proposition 1 shows that up to enumeration equivalence all semi-recursive sets can be regarded of the form L_A for some A .

In terms of structure, the enumeration degrees of a semi-recursive A and its complement behave as a minimal pair in a very strong sense. Arslanov, Cooper and Kalimullin [1] showed that for every set of natural numbers X :

$$(\mathbf{d}_e(A) \vee \mathbf{d}_e(X)) \wedge (\mathbf{d}_e(\bar{A}) \vee \mathbf{d}_e(X)) = \mathbf{d}_e(X).$$

Unfortunately this statement cannot be reversed. In fact the class of enumeration degrees for which the statement can be reversed brings us to the next topic: \mathcal{K} -pairs.

4. \mathcal{K} -PAIRS AND THE DEFINABILITY OF THE ENUMERATION JUMP

In this section we will define the notion of a \mathcal{K} -pair, give examples of this notion, discuss basic properties and give an alternative first order definition of the enumeration jump.

Definition 6. *Let A and B be sets of natural numbers. The pair $\{A, B\}$ is a Kalimullin pair¹ (\mathcal{K} -pair) if there is a c.e. set W , such that:*

$$A \times B \subseteq W \ \& \ \bar{A} \times \bar{B} \subseteq \bar{W}.$$

As a first example of a \mathcal{K} -pair consider a c.e. set U and an arbitrary set of natural numbers A . Then U and A form a \mathcal{K} -pair via the c.e. set $U \times \mathbb{N}$. \mathcal{K} -pairs of this sort are considered trivial and we will not be interested in them. A \mathcal{K} -pair $\{A, B\}$ is nontrivial if A and B are not c.e.

Non-trivial \mathcal{K} -pairs exist. As anticipated, if A is semi-recursive, then $\{A, \bar{A}\}$ is a \mathcal{K} -pair. Indeed let s_A be the selector function for A and let

$$\bar{s}_A(n, m) = \begin{cases} n, & \text{if } s_A(n, m) = m \\ m, & \text{if } s_A(n, m) = n. \end{cases}$$

Now consider the c.e. set $W = \{(s_A(n, m), \bar{s}_A(n, m)) \mid n, m \in \mathbb{N}\}$ and notice that $A \times \bar{A} \subseteq W$ and $\bar{A} \times \bar{A} = \bar{A} \times A \subseteq \bar{W}$.

Kalimullin [12] has shown that the enumeration degrees of \mathcal{K} -pairs are precisely the degrees which satisfy the strong minimal-pair property described in the previous section. He shows that the property of being a \mathcal{K} -pair is degree theoretic and first order definable in the global structure. A pair of sets $\{A, B\}$ is a \mathcal{K} -pair if and only if

$$\forall \mathbf{x} \in \mathcal{D}_e[\mathbf{x} = (\mathbf{x} \vee \mathbf{d}_e(A)) \wedge (\mathbf{x} \vee \mathbf{d}_e(B))].$$

Thus we can lift the notion of a \mathcal{K} -pair to the enumeration degrees. A pair of enumeration degrees \mathbf{a} and \mathbf{b} shall be called a \mathcal{K} -pair if every member of \mathbf{a} forms a \mathcal{K} -pair with every member of \mathbf{b} . By $\mathcal{K}(\mathbf{a}, \mathbf{b})$ we will denote the first order formula, which is true of \mathbf{a} and \mathbf{b} if and only if they form a \mathcal{K} -pair.

Some additional properties of \mathcal{K} -pairs, that will become important later, are listed below. A proof of these properties can be found in [12].

Proposition 2. *Let A and B be a nontrivial \mathcal{K} -pair.*

- (1) $A \leq_e \bar{B}$ and $\bar{A} \leq_e B \oplus J_e(\emptyset)$. Similarly $B \leq_e \bar{A}$ and $\bar{B} \leq_e A \oplus J_e(\emptyset)$;
- (2) The enumeration degrees $\mathbf{d}_e(A)$ and $\mathbf{d}_e(B)$ are incomparable and quasiminimal, i.e. the only total degree bounded by either of them is $\mathbf{0}_e$.
- (3) The class of enumeration degrees of sets that form a \mathcal{K} -pair with A is an ideal.

¹Kalimullin's original term for this notion is *e-ideal*.

To show that the jump is definable, Kalimullin then introduces a relativized version of a \mathcal{K} -pair. If U is a set of natural numbers then A and B form a \mathcal{K} -pair over U if there is a set $W \leq_e U$ such that $A \times B \subseteq W$ and $\overline{A} \times \overline{B} \subseteq \overline{W}$. His proof of the definability of \mathcal{K} -pairs also relativizes: a pair of sets $\{A, B\}$ is \mathcal{K} -pair over a set U if and only if

$$\forall \mathbf{x} \in \mathcal{D}_e[\mathbf{x} \vee \mathbf{d}_e(U) = (\mathbf{x} \vee \mathbf{d}_e(U) \vee \mathbf{d}_e(A)) \wedge (\mathbf{x} \vee \mathbf{d}_e(U) \vee \mathbf{d}_e(B))].$$

A triple of degrees \mathbf{a} , \mathbf{b} and \mathbf{c} , such that each pair $\{\mathbf{a}, \mathbf{b}\}$, $\{\mathbf{a}, \mathbf{c}\}$ and $\{\mathbf{b}, \mathbf{c}\}$ is a nontrivial \mathcal{K} -pair (over \mathbf{u}) will be called a \mathcal{K} -triple (over \mathbf{u}). The first order definition of the enumeration jump given by Kalimullin is equivalent to the following. For every enumeration degree $\mathbf{u} \in \mathcal{D}_e$, \mathbf{u}' is the greatest enumeration degree, which can be represented as the join of a \mathcal{K} -triple over \mathbf{u} .

Here we give an alternative definition of the enumeration jump, which does not use the relativized version of a \mathcal{K} -pair and is in that sense simpler. The proof of this result is for the most part an application of the properties of \mathcal{K} -pairs of the form $\{L_A, R_A\}$ for some A , discussed above.

Theorem 1. *For every nonzero enumeration degree $\mathbf{u} \in \mathcal{D}_e$, \mathbf{u}' is the largest among all least upper bounds $\mathbf{a} \vee \mathbf{b}$ of nontrivial \mathcal{K} -pairs $\{\mathbf{a}, \mathbf{b}\}$, such that $\mathbf{a} \leq \mathbf{u}$.*

Proof. First we observe that if the sets A and B form a nontrivial \mathcal{K} -pair then $A \oplus B \leq_e J_e(A)$. This follows from the fact that $K_A \equiv_e A$ and so by the third property in Proposition 2, K_A and B form a nontrivial \mathcal{K} -pair. By the first property of Proposition 2, $B \leq_e \overline{K_A}$, so $A \oplus B \leq_e K_A \oplus \overline{K_A} = J_e(A)$.

Now fix a nonzero enumeration degree \mathbf{u} and a nontrivial \mathcal{K} -pair $\{\mathbf{a}, \mathbf{b}\}$ such that $\mathbf{a} \leq \mathbf{u}$. Then $\mathbf{a} \vee \mathbf{b} \leq \mathbf{a}' \leq \mathbf{u}'$ by the monotonicity of the jump operation. This establishes the first direction of the proof.

The second direction must be split in two cases. Suppose that \mathbf{u} is not low. Fix $U \in \mathbf{u}$. Then $L_{K_U} \oplus R_{K_U} \equiv_e K_U \oplus \overline{K_U} \equiv_e J_e(U)$ by the third property in Proposition 1 and $L_{K_U} \leq_e U$ by the first property in Proposition 1. As U is not low and $R_{K_U} = \overline{L_{K_U}}$, neither of the two sets L_{K_U} or R_{K_U} is c.e. Let $\mathbf{l} = \mathbf{d}_e(L_{K_U})$ and $\mathbf{r} = \mathbf{d}_e(R_{K_U})$. Then $\{\mathbf{l}, \mathbf{r}\}$ is a nontrivial \mathcal{K} -pair, such that $\mathbf{l} \leq \mathbf{u}$ and $\mathbf{l} \vee \mathbf{r} = \mathbf{u}'$.

Now suppose \mathbf{u} is low. Then we cannot guarantee that L_{K_U} and R_{K_U} form a nontrivial \mathcal{K} -pair. However here we can use Theorem 4 from [7] which proves that for every nonzero Δ_2^0 enumeration degree \mathbf{a} there exists a nontrivial Δ_2^0 \mathcal{K} -pair $\{\mathbf{b}, \mathbf{c}\}$, such that $\mathbf{a} \vee \mathbf{b} = \mathbf{b} \vee \mathbf{c} = \mathbf{0}_{e'}$. Now as every low enumeration degree is Δ_2^0 , we can apply this theorem to \mathbf{u} and obtain a nontrivial \mathcal{K} -pair $\{\mathbf{b}, \mathbf{c}\}$, such that $\mathbf{u} \vee \mathbf{b} = \mathbf{b} \vee \mathbf{c} = \mathbf{0}_{e'}$. Finally we apply the property $\mathcal{K}(\mathbf{b}, \mathbf{c})$ to \mathbf{u} and get:

$$\mathbf{u} = (\mathbf{b} \vee \mathbf{u}) \wedge (\mathbf{c} \vee \mathbf{u}) = \mathbf{0}_{e'} \wedge (\mathbf{c} \vee \mathbf{u}) = (\mathbf{c} \vee \mathbf{u}),$$

hence $\mathbf{c} \leq \mathbf{u}$. □

We note here that this definition has the disadvantage, that it cannot be applied to define $\mathbf{0}_{e'}$ in a simple way. A definition of $\mathbf{0}_{e'}$ can still be obtained, from the facts that $\mathbf{0}_{e'}$ is the least possible enumeration jump and that there are nonzero low degrees.

An immediate corollary of the proof of the alternative definition of the enumeration jump operation is McEvoy's jump inversion theorem [14].

Corollary 1 (McEvoy). *For every total degree $\mathbf{a} \geq \mathbf{0}_{e'}$ there is a quasi-minimal enumeration degree \mathbf{b} , such that $\mathbf{b}' = \mathbf{a}'$.*

5. MAXIMAL \mathcal{K} -PAIRS AND THE LOCAL DEFINABILITY OF THE TOTAL DEGREES

Let us consider again the special case of a \mathcal{K} -pair given by a semi-recursive non c.e. set and its non-c.e. complement, say $\{A, \bar{A}\}$. This \mathcal{K} -pair can be considered as a maximal \mathcal{K} -pair, in the sense that there is no \mathcal{K} -pair $\{C, D\}$, such that $A \leq_e C$ and $\bar{A} \leq_e D$ and one of these reductions is strict. Indeed suppose there were such a pair $\{C, D\}$, and suppose for definiteness that $\bar{A} <_e D$. By the third property of Proposition 2, as $A \leq_e C$ and $\{C, D\}$ is a \mathcal{K} -pair, A would also form a \mathcal{K} -pair with D . By the first property of Proposition 2, $D \leq_e \bar{A}$, contradicting the strong inequality $\bar{A} <_e D$.

Definition 7. *We say that $\{A, B\}$ is a maximal \mathcal{K} -pair if for every \mathcal{K} -pair $\{C, D\}$, such that $A \leq_e C$ and $B \leq_e D$, we have $A \equiv_e C$ and $B \equiv_e D$.*

Using the second property in Proposition 2, we can restate Jockusch's theorem about the existence of semi-recursive sets once again as follows:

Corollary 2. *Every nonzero total set is enumeration equivalent to the join of the components of a maximal \mathcal{K} -pair.*

The first order definability of the total enumeration degrees would then follow, if it were true that maximality is the additional structural property needed to capture \mathcal{K} -pairs of the form $\{A, \bar{A}\}$. If this were true than we can further argue that the definition of the enumeration jump given by Theorem 1 is natural as follows:

Consider the relation *c.e. in* between Turing degrees defined by: \mathbf{x} is c.e. in \mathbf{u} if there are sets $X \in \mathbf{x}$ and $U \in \mathbf{u}$, such that X is c.e. in U .

Proposition 3. *Let \mathbf{x} and \mathbf{u} be Turing degrees such that \mathbf{u} is nonzero. Then \mathbf{x} is c.e. in \mathbf{u} if and only if there is a \mathcal{K} -pair $\{A, \bar{A}\}$ such that $\mathbf{d}_e(A) \leq_e \iota(\mathbf{u})$ and $\iota(\mathbf{x}) = \mathbf{d}_e(A) \vee \mathbf{d}_e(\bar{A})$.*

Proof. Suppose that \mathbf{x} is c.e. in \mathbf{u} . Let $X \in \mathbf{x}$ and $U \in \mathbf{u}$ be sets, such that X is c.e. in U . X is c.e. in U if and only if $X \leq_e U \oplus \bar{U}$. Consider the \mathcal{K} -pair $\{L_X, R_X\}$. By Proposition 1, $L_X \leq_e X \leq_e U \oplus \bar{U}$ and $L_X \oplus R_X \equiv_e X \oplus \bar{X}$. Thus $\mathbf{d}_e(L_X) \leq \mathbf{d}_e(U \oplus \bar{U}) = \iota(\mathbf{u})$ and $\mathbf{d}_e(L_X) \oplus \mathbf{d}_e(R_X) = \mathbf{d}_e(X \oplus \bar{X}) = \iota(\mathbf{x})$.

Suppose $\iota(\mathbf{x}) = \mathbf{d}_e(A) \vee \mathbf{d}_e(\bar{A})$ for some \mathcal{K} -pair $\{A, \bar{A}\}$ such that $\mathbf{d}_e(A) \leq_e \iota(\mathbf{u})$. Again let $X \in \mathbf{x}$ and $U \in \mathbf{u}$. Then $A \leq_e U \oplus \bar{U}$ and hence A is c.e. in U . On the other hand $A \oplus \bar{A} \equiv_e X \oplus \bar{X}$, hence $A \equiv_T X$. Thus \mathbf{x} is c.e. in \mathbf{u} . \square

Thus if every maximal \mathcal{K} -pair is of the form $\{A, \bar{A}\}$ for some A then the total degrees would be definable and the relation *c.e. in* between nonzero total degrees would be definable. The definition of the enumeration given in Theorem 1 restricted to the total degrees can then be read as \mathbf{u}' is the largest total enumeration degree which is c.e. in \mathbf{u} .

In the local structure of the enumeration degrees, we can implement this plan in full. Let us consider maximal \mathcal{K} -pairs of Σ_2^0 enumeration degrees. The universal quantifier in the first order definition of \mathcal{K} -pairs makes it nontrivial to show that their definability is preserved when restricted to the local structure \mathcal{G}_e . In [6] we show that this is nevertheless true.

Theorem 2 ([6]). *There is a first order formula $\mathcal{L}K$, such that for any Σ_2^0 sets A and B , $\{A, B\}$ is a non-trivial \mathcal{K} -pair if and only if $\mathcal{G}_e \models \mathcal{L}K(\mathbf{d}_e(A), \mathbf{d}_e(B))$.*

Thus to prove that the class of total degrees is first order definable in \mathcal{G}_e , it suffices to show that the join of every maximal \mathcal{K} -pair of Σ_2^0 sets is enumeration

equivalent to a total set. We prove something stronger. We prove that every nontrivial \mathcal{K} -pair $\{A, B\}$ can be extended to a maximal \mathcal{K} -pair of the form $\{C, \overline{C}\}$.

Theorem 3. *For every nontrivial Σ_2^0 \mathcal{K} -pair $\{A, B\}$ there is a Σ_2^0 \mathcal{K} -pair $\{C, \overline{C}\}$, such that $A \leq_e C$ and $B \leq_e \overline{C}$.*

The proof of Theorem 3 is presented below. We can now combine Corollary 2, Theorem 2 and Theorem 3 to establish:

Theorem 4. *The set of total Σ_2^0 enumeration degrees is first order definable in \mathcal{G}_e .*

Proof. Consider the formula:

$$\begin{aligned} \mathcal{TOT}(\mathbf{x}) \iff & \mathbf{x} = \mathbf{0}_e \vee \exists \mathbf{a} \exists \mathbf{b} [\mathcal{LK}(\mathbf{a}, \mathbf{b}) \ \& \ \mathbf{x} = (\mathbf{a} \vee \mathbf{b}) \ \& \\ & \forall \mathbf{c} \forall \mathbf{d} (\mathcal{LK}(\mathbf{c}, \mathbf{d}) \ \& \ \mathbf{c} \geq \mathbf{a} \ \& \ \mathbf{d} \geq \mathbf{b} \rightarrow \mathbf{c} = \mathbf{a} \ \& \ \mathbf{d} = \mathbf{b})] \end{aligned}$$

A Σ_2^0 enumeration degree \mathbf{x} is total if and only if $\mathcal{G}_e \models \mathcal{TOT}(\mathbf{x})$. □

5.1. Extending to maximal \mathcal{K} -pairs. To prove Theorem 3 we will use the following dynamic characterization of \mathcal{K} -pairs. It is stated in [12], however it is not formally proved there, so we supply our own proof for it.

Lemma 1 (Kalimullin). *A pair of non-c.e. Σ_2^0 sets $\{A, B\}$ is a \mathcal{K} -pair if and only if there are Δ_2^0 approximations $\{A_i\}_{i < \omega}$ to A and $\{B_i\}_{i < \omega}$ to B , such that:*

$$\forall i (A_i \subseteq A \vee B_i \subseteq B).$$

Proof. Let A and B be Σ_2^0 sets. Suppose that A and B form a nontrivial \mathcal{K} -pair in \mathcal{G}_e . The first property in Proposition 2 implies that A and B are Δ_2^0 sets, as $\overline{A}, \overline{B} \leq_e J_e(\emptyset)$ and hence are also Σ_2^0 .

Fix Δ_2^0 approximations $\{\hat{A}_i\}_{i < \omega}$ and $\{\hat{B}_i\}_{i < \omega}$ to A and B respectively. Without loss of generality we may assume that there are infinitely many stages i , such that $\hat{A}_i \subseteq A$ and that there are infinitely many stages i , such that $\hat{B}_i \subseteq B$. Indeed every Δ_2^0 approximation can be transformed effectively into a Δ_2^0 approximation with the requested property. This can be seen for example in Lachlan and Shore [13]. Stages i at which $\hat{A}_i \subseteq A$ will be called good stages for A . Similarly Stages i at which $\hat{B}_i \subseteq B$ will be called good stages for B . We assume furthermore that $\hat{A}_0 = \hat{B}_0 = \emptyset$

Let W be a c.e. set such that

$$A \times B \subseteq W \ \& \ \overline{A} \times \overline{B} \subseteq \overline{W}.$$

Fix a Σ_1^0 approximation $\{W_i\}_{i < \omega}$ to W . From the given approximations to A and B we construct the required approximations $\{A_i\}_{i < \omega}$ and $\{B_i\}_{i < \omega}$ as follows:

Set $A_0 = \hat{A}_0$ and $B_0 = \hat{B}_0$. Suppose that we have defined $A_s = \hat{A}_{i_s}$ and $B_s = \hat{B}_{j_s}$. At stage $s + 1$ find the greatest pair $\langle i, j \rangle < s$, such that $i_s \leq i$, $j_s \leq j$ and $\hat{A}_i \times \hat{B}_j \subseteq W_s$. Set $A_{s+1} = A_i$ and $B_{s+1} = \hat{B}_j$.

To see that $\{A_s\}_{s < \omega}$ and $\{B_s\}_{s < \omega}$ are the required approximations we argue as follows. For every s , $A_s \times B_s \subseteq W_s \subseteq W$. As $\overline{A} \times \overline{B} \subseteq \overline{W}$, it follows that $A_s \subseteq A$ or $B_s \subseteq B$.

By our choice of approximations to A and B with infinitely many good stages and the fact that $A \times B \subseteq W$, it follows that the sequences $\{i_s\}_{s < \omega}$ and $\{j_s\}_{s < \omega}$ are unbounded. Indeed fix i_s and j_s . Fix good stages for A and B , respectively,

such that $i > i_s$ and $j > j_s$. Then $A_i \times B_j \subseteq A \times B \subseteq W$, hence there is a stage s^* such that $A_i \times B_i \subseteq W_{s^*}$. Then by construction $i_{s^*} \geq i > i_s$ and $j_{s^*} \geq j > j_s$. A straightforward consequence of this is that $\{A_i\}_{i < \omega}$ and $\{B_i\}_{i < \omega}$ are Δ_2^0 approximations to A and B .

For the opposite direction fix Δ_2^0 approximations $\{A_i\}_{i < \omega}$ to A and $\{B_i\}_{i < \omega}$ to B , such that:

$$\forall i (A_i \subseteq A \vee B_i \subseteq B).$$

Consider the c.e. set W , defined by:

$$W = \bigcup_{i < \omega} A_i \times B_i.$$

We show that W is the c.e. set witnessing that A and B form a \mathcal{K} -pair. Fix $\langle a, b \rangle \in A \times B$. Let i_a be a stage, such that $a \in A_i$ for all $i > i_a$ and let j_b be a stage, such that $b \in B_j$ for all $j > j_b$. Then let $s = \max(i_a, j_b)$. Then $\langle a, b \rangle \in A_s \times B_s \subseteq W$. On the other hand if $\langle \bar{a}, \bar{b} \rangle \in \bar{A} \times \bar{B}$ then for every i , we have that $\bar{a} \notin A_i$ or $\bar{b} \notin B_i$. It follows that for every i , $\langle \bar{a}, \bar{b} \rangle \notin A_i \times B_i$, hence $\langle \bar{a}, \bar{b} \rangle \in \bar{W}$. \square

Approximations to sets, which form a nontrivial \mathcal{K} -pair, with the property above will be called \mathcal{K} -approximations.

Now we can start the proof of Theorem 3.

Proof. Fix a nontrivial Σ_2^0 \mathcal{K} -pair $\{A, B\}$ and let $\{A_i\}_{i < \omega}$ and $\{B_i\}_{i < \omega}$ be their respective Δ_2^0 \mathcal{K} -approximations. We build two Σ_2^0 sets C and D which will satisfy the following requirements:

- (R1) $A = \{x \mid \exists j[2\langle x, j \rangle \in C]\}$, $B = \{x \mid \exists j[2\langle x, j \rangle + 1 \in D]\}$;
- (R2) C and D are Δ_2^0 ;
- (R3) $\bar{C} = D$;
- (R4) $\{C, D\}$ is a \mathcal{K} -pair.

To ensure that these requirements are met, we construct respective Σ_2^0 approximations $\{C_i\}_{i < \omega}$ and $\{D_i\}_{i < \omega}$, which will have the following properties:

- (P1) $A_i = \{x \mid \exists j[2\langle x, j \rangle \in C_i]\}$ and $B_i = \{x \mid \exists j[2\langle x, j \rangle + 1 \in D_i]\}$. Assuming that the constructed approximations are Δ_2^0 this will ensure that $A \supseteq \{x \mid \exists j[2\langle x, j \rangle \in C]\}$ and $B \supseteq \{x \mid \exists j[2\langle x, j \rangle + 1 \in D]\}$.
- (P2) $\forall i[A_i \not\subseteq A \Rightarrow D_i \subseteq D]$ and $\forall i[B_i \not\subseteq B \Rightarrow C_i \subseteq C]$. This property will ensure that the constructed approximations are Δ_2^0 , i.e. (R2) holds. Indeed, if we assume that for some $d \notin D$ the set $I(d) = \{i \mid d \in D_i\}$ is infinite, then $\{A_i\}_{i \in I(d)}$ is a c.e. approximation to A contradicting that A is not c.e. Furthermore together with (P1) it ensures the inclusions $A \subseteq \{x \mid \exists j[2\langle x, j \rangle \in C]\}$ and $B \subseteq \{x \mid \exists j[2\langle x, j \rangle + 1 \in D]\}$. The argument is similar: there are infinitely many stages i , such that $B_i \not\subseteq B$, otherwise B_i would turn out c.e. So for every $x \in A$ we can find a stage i such that $x \in A_i$ and $B_i \not\subseteq B$. By (P1) there is a number j , such that $2\langle x, j \rangle \in C_i$. By (P2) $C_i \subseteq C$. The second inclusion is proved in a similar way.
- (P3) $\forall i[C_i \cap D_i = \emptyset]$ and every natural number is eventually enumerated in one of the sets. This will ensure (R3);
- (P4) $\forall i[C_i \subseteq C \vee D_i \subseteq D]$. This will ensure (R4).

Note that the property (P2) is a consequence of properties (P1) and (P4), so let us consider in more detail what the property (P4) is expressing. Suppose that $x \notin C$, but for some i , $x \in C_i$. Then i is a bad stage for C , i.e. $C_i \not\subseteq C$, and we must ensure that all the elements in D_i are ultimately enumerated in D . Thus from this point on the element x is connected to all elements in D_i , in the sense that we should not enumerate x in D_k at a further stage $k > i$ unless we also ensure that $D_i \subseteq D_k$. This suggests the following relations for every stage j :

$$r_j(x, y) \iff \exists i \leq j [x \in C_i \ \& \ y \in D_i].$$

The main property (MP) of the construction is as follows: for every stage j and every x and y , if $r_j(x, y)$, then

$$x \in D_j \implies y \in D_j \quad \text{and} \quad y \in C_j \implies x \in C_j.$$

Note that (MP) automatically ensures that the two constructed approximations have the \mathcal{K} -pair property. The construction must therefore ensure that (P1), (P3) and (MP) are true. The other properties are implied by these.

5.1.1. Construction. We introduce the following piece of notation: with \mathbf{c}_i^a we shall denote the natural number $2\langle a, i \rangle$, and by \mathbf{d}_i^b the natural number $2\langle b, i \rangle + 1$. If $a \in A_i$ we shall say that \mathbf{c}_i^a is a follower of a , and similarly if $b \in B_i$ we shall say that \mathbf{d}_i^b is a follower of b . Note that by the properties of the construction we will have that $a \in A$ if and only if at least one of its followers is in C and $b \in B$ if and only if at least one of its followers is in D . During the construction each follower will have one of the following two states: *free* or *not free*. Intuitively a follower is free if it is not currently enumerated in either of the sets C or D . By *Free* we denote the set of all followers that are currently free.

The construction will be carried out in stages. Every stage consists of two parts - *Extracting* and *Adding*. We shall describe the construction formally and supply a brief description of the intuition for every action. The main intuition of the construction is that followers \mathbf{c}_j^a *want* to end up in the set D and followers \mathbf{d}_j^b *want* to end up in the set C . A follower \mathbf{c}_j^a remains in the set C (\mathbf{d}_j^b remains in the set D) only if it is forced to do so by other followers to which it is connected.

Start of construction.

We set $C_0 = \{\mathbf{c}_0^a \mid a \in A_0\}$ and $D_0 = \{\mathbf{d}_0^b \mid b \in B_0\}$. At stage $i > 0$ we construct C_i and D_i by modifying C_{i-1} and D_{i-1} respectively as follows:

Initially we set $C_i = C_{i-1}$ and $D_i = D_{i-1}$.

Part 1: Extracting

- (E1) For all $\mathbf{c}_j^a \in C_i$ such that $a \notin A_i$ we extract \mathbf{c}_j^a from C_i .
 For all $\mathbf{d}_j^b \in D_i$ such that $b \notin B_i$ we extract \mathbf{d}_j^b from D_i .

Intuition: This action ensures that $\{a \mid \exists j [\mathbf{c}_j^a \in C_i]\} \subseteq A_i$ and that $\{b \mid \exists j [\mathbf{d}_j^b \in D_i]\} \subseteq B_i$.

- (E2) For all $\mathbf{d}_j^b \in C_i$ such that $\{\mathbf{c}_k^a \mid r_{i-1}(\mathbf{c}_k^a, \mathbf{d}_j^b)\} \not\subseteq C_i$ we extract \mathbf{d}_j^b from C_i .
 For all $\mathbf{c}_j^a \in D_i$ such that $\{\mathbf{d}_k^b \mid r_{i-1}(\mathbf{c}_j^a, \mathbf{d}_k^b)\} \not\subseteq D_i$ we extract \mathbf{c}_j^a from D_i .

Intuition: The follower \mathbf{d}_j^b is only allowed to remain in C_i if all of the elements to which it has been connected at a previous stage, i.e. $\{\mathbf{c}_k^a \mid r_{i-1}(\mathbf{c}_k^a, \mathbf{d}_j^b)\}$ are still in C_i . These elements we can consider as *requested* in

C_i by \mathbf{d}_j^b . If one of these requests cannot be fulfilled (due to the properties of A_i for example and rule (E1)), \mathbf{d}_j^b must also be extracted from C_i . Similar reasoning is applied to followers \mathbf{c}_j^a and their membership to D_i . These actions ensure that the main property of the construction is true.

- (E3) For all $\mathbf{c}_j^a \in C_i$ such that $\{\mathbf{d}_k^b \mid r_{i-1}(\mathbf{c}_j^a, \mathbf{d}_k^b)\} \cap C_i = \emptyset$ we extract \mathbf{c}_j^a from C_i .
 For all $\mathbf{d}_j^b \in D_i$ such that $\{\mathbf{c}_k^a \mid r_{i-1}(\mathbf{c}_k^a, \mathbf{d}_j^b)\} \cap D_i = \emptyset$ we extract \mathbf{d}_j^b from D_i .

Intuition: A follower \mathbf{c}_j^a was forced into C_{i-1} because of a request by some \mathbf{d}_k^b , to which it is connected. However at this stage the follower that made this request is not any longer in C_i , (it was extracted under (E2) as one of its other requests was not fulfilled). In other words \mathbf{c}_j^a is not requested any longer in C_i , so it is free to leave and attempt entering D_i .

All extracted elements become free.

Part 2 (Adding)

- (A1) For all free \mathbf{d}_j^b such that $\{\mathbf{c}_k^a \mid r_{i-1}(\mathbf{c}_k^a, \mathbf{d}_j^b)\} \subseteq C_i \cup \text{Free}$ and $\{a \mid r_{i-1}(\mathbf{c}_k^a, \mathbf{d}_j^b)\} \subseteq A_i$ we enumerate \mathbf{d}_j^b and $\{\mathbf{c}_k^a \mid r_{i-1}(\mathbf{c}_k^a, \mathbf{d}_j^b)\}$ in C_i .
 All enumerated elements become *not free*.
 For all free \mathbf{c}_j^a such that $\{\mathbf{d}_k^b \mid r_{i-1}(\mathbf{c}_j^a, \mathbf{d}_k^b)\} \subseteq D_i \cup \text{Free}$ and $\{b \mid r_{i-1}(\mathbf{c}_j^a, \mathbf{d}_k^b)\} \subseteq B_i$ we enumerate \mathbf{c}_j^a and $\{\mathbf{d}_k^b \mid r_{i-1}(\mathbf{c}_j^a, \mathbf{d}_k^b)\}$ in D_i .
 All enumerated elements become *not free*.

Intuition: This is the action that allows followers \mathbf{d}_j^b to enter C_i and respectively \mathbf{c}_j^a to enter D_i . This can be done only if all of their requests can be fulfilled at the same time. These requests must also not injure the actions of rule (E1).

- (A2) For all $a \in A_i$ we enumerate \mathbf{c}_i^a in C_i .
 For all $b \in B_i$ we enumerate \mathbf{d}_i^b in D_i .
 All enumerated elements become *not free*.

Intuition: This action ensures that $\{a \mid \exists j[\mathbf{c}_j^a \in C_i]\} \supseteq A_i$ and that $\{b \mid \exists j[\mathbf{d}_j^b \in D_i]\} \supseteq B_i$ and together with (E1), property (P1).

- (A3) For all $a, j \leq i$ such that $a \notin A_j$ we enumerate $\mathbf{c}_j^a \in D_i$.
 For all $b, j \leq i$ such that $b \notin B_j$ we enumerate $\mathbf{d}_j^b \in C_i$.

Intuition: This action handles elements that are not followers. As our aim is to construct D as \overline{C} , these elements also need to be enumerated in one of the two constructed sets. Note that even elements are enumerated in D_i and odd elements are enumerated in C_i . At the following stage an even number \mathbf{c}_j^a , which was enumerated in D_i under this action, cannot be extracted under rules (E1) and (E3). Furthermore as \mathbf{c}_j^a has never been enumerated into an approximating set to C , the set $\{\mathbf{d}_k^b \mid r_i(\mathbf{c}_j^a, \mathbf{d}_k^b)\}$ is empty, so it cannot be extracted under rule (E2). Thus this element remains in D_k at all further stages $k > i$. Similar reasoning is applied to odd numbers, enumerated in D_i under this action.

End of construction.

5.1.2. *Verification of the construction.* We prove that the described construction produces sets C and D , which have the properties listed as (P1)-(P4) and (MP). We start with the easiest property: (P1).

Proposition 4. *For every i , $A_i = \{a \mid \exists j[\mathbf{c}_j^a \in C_i]\}$ and $B_i = \{b \mid \exists j[\mathbf{d}_j^b \in D_i]\}$.*

Proof. The claims of the proposition follow directly from rules (E1), (A1) and (A2). Indeed (A2) guarantees the inclusion \subseteq , as $A_i = \{a \mid \mathbf{c}_i^a \in C_i\}$ and $B_i = \{b \mid \mathbf{d}_i^b \in D_i\}$. On the other hand (E1) and (A1) enforce that $A_i \supseteq \{a \mid \exists j[\mathbf{c}_j^a \in C_i]\}$ and $B_i \supseteq \{b \mid \exists j[\mathbf{d}_j^b \in D_i]\}$. □

The following proposition is a direct consequence of the construction. We state it nevertheless for completeness.

Proposition 5. *For all i , $C_i \cap D_i = \emptyset$.*

Next we turn to the main property of the construction (MP). One particular case of it will be used frequently in the rest of the proof and we will state and prove it here separately.

Proposition 6. *If \mathbf{c}_j^a is a follower and $\mathbf{c}_j^a \in D_i$ then $\{\mathbf{d}_k^b \mid r_i(\mathbf{c}_j^a, \mathbf{d}_k^b)\} \subseteq D_i$;*

If \mathbf{d}_j^b is a follower and $\mathbf{d}_j^b \in C_i$ then $\{\mathbf{c}_k^a \mid r_i(\mathbf{c}_k^a, \mathbf{d}_j^b)\} \subseteq C_i$.

Proof. We prove the first statement. The second statement is proved similarly. Let \mathbf{c}_j^a be a follower, (i.e. $a \in A_j$), such that $\mathbf{c}_j^a \in D_i$. If \mathbf{c}_j^a is enumerated in D_i at stage i under rule (A1) then by construction the set $\{\mathbf{d}_k^b \mid r_i(\mathbf{c}_j^a, \mathbf{d}_k^b)\}$ is also enumerated in D_i . As no more elements are extracted from D_i after the execution of step (A1), it follows that $\{\mathbf{d}_k^b \mid r_i(\mathbf{c}_j^a, \mathbf{d}_k^b)\} \subseteq D_i$.

The other possibility is that $\mathbf{c}_j^a \in D_{i-1}$ and during stage i , \mathbf{c}_j^a is not extracted from D_i . But then the prerequisites of rule (E2) are not valid for \mathbf{c}_j^a at stage i and hence before starting the execution of (E3) it is true that $\{\mathbf{d}_k^b \mid r_i(\mathbf{c}_j^a, \mathbf{d}_k^b)\} \subseteq D_i$. During the execution of (E3) it is the case that for every $\mathbf{d}_k^b \in \{\mathbf{d}_k^b \mid r_i(\mathbf{c}_j^a, \mathbf{d}_k^b)\}$, $\mathbf{c}_j^a \in \{\mathbf{c}_l^a \mid r_{i-1}(\mathbf{c}_l^a, \mathbf{d}_k^b)\} \cap D_i$. By (E3) this intersection must be empty in order to extract \mathbf{d}_k^b from D_i , so none of the elements in $\{\mathbf{d}_k^b \mid r_i(\mathbf{c}_j^a, \mathbf{d}_k^b)\}$ are extracted from D_i during the execution of (E3). Thus finally $\{\mathbf{d}_k^b \mid r_i(\mathbf{c}_j^a, \mathbf{d}_k^b)\} \subseteq D_i$. □

We are now ready to prove the main property (MP).

Lemma 2 (Main Lemma). *Let x and y be natural numbers, such that $r_i(x, y)$, for some natural number i . Then the following two conditions are true.*

(C1) $x \in D_i \implies y \in D_i$.

(C2) $y \in C_i \implies x \in C_i$.

Proof. The claim of the lemma is trivial when either x or y are not followers, as every such element is only enumerated once under (A3) in its corresponding set and is never extracted. For followers x and y we shall consider three different cases.

Case 1. $x = \mathbf{c}_j^a$ and $y = \mathbf{d}_k^b$. This is a direct consequence of Proposition 6.

Case 2. $x = \mathbf{c}_l^a$ and $y = \mathbf{c}_l^a$ (or $x = \mathbf{d}_k^b$ and $y = \mathbf{d}_l^b$). Let s be the least natural number for which $r_s(x, y)$. We shall prove simultaneously claims (C1), (C2) and that

$$(1) \quad \{\mathbf{d}_k^b \mid r_i(y, \mathbf{d}_k^b)\} \subseteq \{\mathbf{d}_k^b \mid r_i(x, \mathbf{d}_k^b)\}$$

by induction on $i \geq s$. For $i = s$ claims (C1) and (C2) are trivially true, as by the definition of the relation r_s and the choice of s we have $x \in C_s$ and $y \in D_s$. For claim (1) suppose that \mathbf{d}_k^b is such that $r_s(y, \mathbf{d}_k^b)$. Since $y \in D_s$, Proposition 6 implies that $\mathbf{d}_k^b \in D_s$ and hence from $x \in C_s$ we obtain $r_s(x, \mathbf{d}_k^b)$.

Now let $i > s$. In order to prove (C1) suppose that $x \in D_i$. Then according to Proposition 6, $\{\mathbf{d}_k^b \mid r_i(x, \mathbf{d}_k^b)\} \subseteq D_i$. Now using the induction hypothesis for (1) and that $\{\mathbf{d}_k^b \mid r_{i-1}(x, \mathbf{d}_k^b)\} \subseteq \{\mathbf{d}_k^b \mid r_i(x, \mathbf{d}_k^b)\}$ we obtain $\{\mathbf{d}_k^b \mid r_{i-1}(y, \mathbf{d}_k^b)\} \subseteq D_i$. As by Proposition 5 we have that $D_i \cap C_i = \emptyset$, it follows that at stage i when we reach step (E3), $\{\mathbf{d}_k^b \mid r_i(y, \mathbf{d}_k^b)\} \cap C_i = \emptyset$, which implies that $y \notin C_i$. This means that if y is not already in D_i , it is free during the execution of (A1) and we would enumerate it in D_i .

In order to prove (C2) suppose that $y \in C_i$. Then there is a $\mathbf{d}_k^b \in C_i$ such that $r_{i-1}(y, \mathbf{d}_k^b)$, since otherwise y would have been extracted under (E3). From the induction hypothesis for (1) we obtain that $r_{i-1}(x, \mathbf{d}_k^b)$ and hence $x \in C_i$ by Proposition 6.

Finally let us prove (1). We consider two cases. First suppose that $y \notin C_i$. Then $\{\mathbf{d}_k^b \mid r_i(y, \mathbf{d}_k^b)\} = \{\mathbf{d}_k^b \mid r_{i-1}(y, \mathbf{d}_k^b)\}$. On the other hand $\{\mathbf{d}_k^b \mid r_{i-1}(x, \mathbf{d}_k^b)\} \subseteq \{\mathbf{d}_k^b \mid r_i(x, \mathbf{d}_k^b)\}$ and now the claim follows from the induction hypothesis. Secondly let $y \in C_i$. Then

$$\{\mathbf{d}_k^b \mid r_i(y, \mathbf{d}_k^b)\} = \{\mathbf{d}_k^b \mid r_{i-1}(y, \mathbf{d}_k^b)\} \cup \{\mathbf{d}_k^b \mid \mathbf{d}_k^b \in D_i\}.$$

On the other hand by (C2) we have $x \in C_i$ and hence

$$\{\mathbf{d}_k^b \mid r_i(x, \mathbf{d}_k^b)\} = \{\mathbf{d}_k^b \mid r_{i-1}(x, \mathbf{d}_k^b)\} \cup \{\mathbf{d}_k^b \mid \mathbf{d}_k^b \in D_i\}$$

and again the claim follows from the induction hypothesis.

Case 3. $x = \mathbf{d}_k^b$ and $y = \mathbf{c}_j^\alpha$. Let s be again the least stage for which $r_s(x, y)$. In particular $x \in C_s$ and $y \in D_s$. We shall prove simultaneously (C1), (C2) and for all $i \geq s$:

$$(2) \quad \{\mathbf{d}_l^\beta \mid r_i(y, \mathbf{d}_l^\beta)\} \subseteq \{\mathbf{d}_l^\beta \mid r_i(x, \mathbf{d}_l^\beta)\}$$

$$(3) \quad \{\mathbf{c}_l^\alpha \mid r_i(\mathbf{c}_l^\alpha, x)\} \subseteq \{\mathbf{c}_l^\alpha \mid r_i(\mathbf{c}_l^\alpha, y)\}$$

by induction on i . For $i = s$ claims (C1) and (C2) are trivial. In order to prove (2) suppose that \mathbf{d}_l^β is such that $r_s(y, \mathbf{d}_l^\beta)$. Then according to Proposition 6, $\mathbf{d}_l^\beta \in D_s$ and hence $r_s(x, \mathbf{d}_l^\beta)$. The proof of (3) is analogous.

Now let $i > s$. In order to prove (C1) suppose that $x \in D_i$. Then according steps (E3) and (A1) of the construction there is a $\mathbf{c}_l^\alpha \in D_i$, such that $r_{i-1}(\mathbf{c}_l^\alpha, x)$. The induction hypothesis for (3) implies $r_{i-1}(\mathbf{c}_l^\alpha, y)$. Now from claim (C1) of *Case 2* and $\mathbf{c}_l^\alpha \in D_i$ we obtain $y \in D_i$. The proof of (C2) is analogous.

Now let us prove (2). Suppose that for some \mathbf{d}_l^β , $r_i(y, \mathbf{d}_l^\beta)$. We shall consider two cases. First suppose that $y \notin C_i$. Then it should be the case $r_{i-1}(y, \mathbf{d}_l^\beta)$ which together with the induction hypothesis implies $r_{i-1}(x, \mathbf{d}_l^\beta)$ and hence $r_i(x, \mathbf{d}_l^\beta)$. Now let $y \in C_i$. If $r_{i-1}(y, \mathbf{d}_l^\beta)$ we reason in the same way as above, so suppose that $r_{i-1}(y, \mathbf{d}_l^\beta)$ is not true. Then it should be the case $\mathbf{d}_l^\beta \in D_i$. On the other hand (C2) implies $x \in C_i$ and hence $r_i(x, \mathbf{d}_l^\beta)$.

Claim (3) is proved analogously.

□

Next we show that property (P2) is true.

Proposition 7. *For every i the following holds:*

- $A_i \not\subseteq A \implies D_i \subseteq D$;
- $B_i \not\subseteq B \implies C_i \subseteq C$.

Furthermore

- $a \in A_i \setminus A \implies \mathbf{c}_i^a \in D$;
- $b \in B_i \setminus B \implies \mathbf{d}_i^b \in C$.

Proof. Fix an i such that $A_i \not\subseteq A$ and let $a \in A_i \setminus A$. Consider the follower \mathbf{c}_i^a . According to (A2) $\mathbf{c}_i^a \in C_i$, so that for all $y \in D_i$, $r_i(\mathbf{c}_i^a, y)$ and hence $r_j(\mathbf{c}_i^a, y)$ for $j \geq i$. Let $s_1 > i$ be the least stage, such that for all $j \geq s_1$, $a \notin A_j$ (such stage exists since $\{A_i\}_{i < \omega}$ is a Δ_2^0 approximation). Then according to rule (E1) for each $j \geq s_1$, $\mathbf{c}_i^a \notin C_j$ and hence for $j \geq s_1$, $\{\mathbf{d}_k^b \mid r_j(\mathbf{c}_i^a, \mathbf{d}_k^b)\} \cap C_j = \emptyset$. Thus for $j \geq s_1$, $\{\mathbf{d}_k^b \mid r_j(\mathbf{c}_i^a, \mathbf{d}_k^b)\} \subseteq D_j \cup \text{Free}$. Now consider the set $\{b \mid \exists k[r_j(\mathbf{c}_i^a, \mathbf{d}_k^b)]\}$. Note that as for all $j \geq s_1$, $\mathbf{c}_i^a \notin C_j$, it follows that this set is finite and does not change. We claim that

$$(4) \quad \{b \mid \exists k[r_j(\mathbf{c}_i^a, \mathbf{d}_k^b)]\} \subseteq B.$$

Indeed, $r_j(\mathbf{c}_i^a, \mathbf{d}_k^b)$ implies that for some l , $\mathbf{c}_i^a \in C_l$ and $\mathbf{d}_k^b \in D_l$, and in the particular $a \in A_l$ and $b \in B_l$. Thus $A_l \not\subseteq A$, so that by our choice of \mathcal{K} -approximations to A and B , it must be true that $B_l \subseteq B$ and hence $b \in B$.

Fix the least stage $s_2 \geq s_1$, such that for all $j \geq s_2$, $\{b \mid \exists k[r_j(\mathbf{c}_i^a, \mathbf{d}_k^b)]\} \subseteq B_j$ (such a stage exists in virtue of (4)). Then for $j \geq s_2$, $\{\mathbf{d}_k^b \mid \exists k[r_j(\mathbf{c}_i^a, \mathbf{d}_k^b)]\} \subseteq D_j \cup \text{Free}$ and $\{b \mid \exists k[r_j(\mathbf{c}_i^a, \mathbf{d}_k^b)]\} \subseteq B_j$, so that (A1) implies $\mathbf{c}_i^a \in D_j$. Thus $\mathbf{c}_i^a \in D$.

Finally since for all $y \in D_i$ and all $j \geq s_2$, $r_j(\mathbf{c}_i^a, y)$, Lemma 2 implies $D_i \subseteq D_j$ and hence $D_i \subseteq D$. □

Corollary 3. $\{C_i\}_{i < \omega}$ and $\{D_i\}_{i < \omega}$ are Δ_2^0 approximations to C and D respectively.

Proof. Towards a contradiction assume that $\{D_i\}_{i < \omega}$ is not a Δ_2^0 approximation to D . Then there is an element $y \notin D$ such that the set $I(y) = \{i \mid y \in D_i\}$ is infinite. Every $i \in I(y)$ is a bad stage for D and hence according to Proposition 7 it is a good stage for A . Since $I(y)$ is infinite,

$$A = \{a \mid \exists i[i \in I(y) \ \& \ a \in A_i]\}.$$

On the other hand $I(y)$ is computable and hence A is c.e. contrary to what is given.

Similarly one proves that $\{C_i\}_{i < \omega}$ is a Δ_2^0 approximations to C . □

Corollary 4. $A = \{a \mid \exists j[\mathbf{c}_j^a \in C]\}$ and $B = \{b \mid \exists j[\mathbf{d}_j^b \in D]\}$.

Proof. By Proposition 4 for every i , $A_i = \{a \mid \exists j[\mathbf{c}_j^a \in C_i]\}$ and $B_i = \{b \mid \exists j[\mathbf{d}_j^b \in D_i]\}$. Hence if $a \notin A$ there is a stage i_a such that $a \notin A_i$ for all $i > i_a$ and hence for all j and all $i > i_a$, $\mathbf{c}_j^a \notin C_i$. This yields $A \supseteq \{a \mid \exists j[\mathbf{c}_j^a \in C]\}$.

Now let $a \in A$. Let i_a be a stage such that $a \in A_i$ for all $i > i_a$. Let $j > i_a$ be a stage such that $B_j \not\subseteq B$. Such a stage exists, as if we assume otherwise, i.e. that for all $j > i_a$, $B_j \subseteq B$, it would follow that B is c.e. contrary to what is given. At

stage j , as $a \in A_j$, $\mathbf{c}_j^a \in C_j$ by (A2). By Proposition 7, as $B_j \not\subseteq B$, $C_j \subseteq C$. So $\mathbf{c}_j^a \in C$ and $A \subseteq \{a \mid \exists j[\mathbf{c}_j^a \in C]\}$. That $B = \{b \mid \exists j[\mathbf{d}_j^b \in D]\}$ is proved similarly. \square

To complete the verification of the construction in the last two propositions we prove that properties (P3) and (P4) are true.

Proposition 8. $D = \overline{C}$.

Proof. First we claim that $C \cap D = \emptyset$. Indeed, at each stage the rules of the construction guarantee the $C_i \cap D_i = \emptyset$. This together with the fact that $\{C_i\}_{i < \omega}$ and $\{D_i\}_{i < \omega}$ are Δ_2^0 approximations implies $C \cap D = \emptyset$.

Next we prove that $C \cup D = \mathbb{N}$. Fix a natural $x \in \mathbb{N}$. Suppose that x is not a follower. Without loss of generality we may assume that $x = \mathbf{c}_i^a$ for some natural numbers i and a . Then at stage $s = \max\{i, a\}$, x is enumerated in D_s under rule (A3). It is never extracted from D . Indeed it could be extracted at a stage j only under rule (E2), because this is the only rule which extracts an even number from D . However the set $\{\mathbf{d}_k^b \mid r_{j-1}(x, \mathbf{d}_k^b)\} = \emptyset$ so rule (E2) does not apply. Thus $x \in D$.

Now suppose that x is a follower. If $x = \mathbf{c}_i^a$ for some $a \notin A$, or $x = \mathbf{d}_i^b$ for some $b \notin B$ then according to Proposition 7, $x \in D$ or $x \in C$ respectively. So let $x = \mathbf{c}_i^a$ for some $a \in A$ and suppose that $x \notin C$. Then according to Proposition 7 for every j if $x \in C_j$, then $B_j \subseteq B$. Thus if $r_j(x, \mathbf{d}_k^b)$, then $b \in B$. Let s_1 be the least stage, such that for $j \geq s_1$, $x \notin C_j$. Then for $j \geq s_1$ we have $r_j(x, \mathbf{d}_k^b) \iff r_{s_1}(x, \mathbf{d}_k^b)$. Furthermore Proposition 6 implies that for $j \geq s_1$, $\{\mathbf{d}_k^b \mid r_j(x, \mathbf{d}_k^b)\} \cap C_j = \emptyset$ and hence $\{\mathbf{d}_k^b \mid r_j(x, \mathbf{d}_k^b)\} \subseteq D_j \cup \text{Free}$. Let $s_2 \geq s_1$ be the least stage such that for $j \geq s_2$, $\{b \mid \exists k[r_j(x, \mathbf{d}_k^b)]\} \subseteq B_j$. Then at stage s_2 , x is enumerated in D_{s_2} under rule (A1) and is never extracted from D .

Analogously we may prove that if $x = \mathbf{d}_k^b$ for some $b \in B$ and $x \notin D$ then $x \in C$. \square

Proposition 9. For every i , either $C_i \subseteq C$ or $D_i \subseteq D$.

Proof. Suppose that for some i , $C_i \not\subseteq C$ and let $x \in C_i \setminus C$. Fix a stage s such that for each $j \geq s$, $x \in D_j$ (such a stage exists since $D = \overline{C}$ and the approximation to D is Δ_2^0). Take an arbitrary $y \in D_i$. Then for each $j \geq i$, $r_j(x, y)$ and hence according to claim (C1) of Lemma 2 we obtain that for $j \geq s$, $y \in D_j$. Thus $y \in D$ and hence $D_i \subseteq D$. \square

This completes the proof of Theorem 3. \square

5.2. Weakly semi-recursive sets. The constructed maximal \mathcal{K} -pair is of the form $\{C, \overline{C}\}$, just as the ones from our initial examples, coming from a semi-recursive set and its complement. It would be natural to wonder if the sets produced by this construction are semi-recursive as well and in that line of thought, do all \mathcal{K} -pairs of the form $\{C, \overline{C}\}$ consist of semi-recursive sets. In pursuit of this answer we come to a generalization of the notion of a semi-recursive set, weakly semi-recursive sets. Jockusch and Owings [10] have already defined this notion and used it in a completely different context - the theory of bounded queries.

Definition 8. We say that a set of natural numbers, A , is weakly semi-recursive if there is a computable selector function $s_A : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$, such that: for any $x, y \in \mathbb{N}$, if $\{x, y\} \cap A \neq \emptyset$ and $\{x, y\} \cap \overline{A} \neq \emptyset$ then $s_A(x, y)$ is defined and $s_A(x, y) \in \{x, y\} \cap A$.

Every semi-recursive set is weakly semi-recursive. Furthermore every c.e. set is weakly semi-recursive. The selector function for a c.e. set works as follows: on every input (x, y) it starts approximating the c.e. set until one of the arguments appears in the approximation, and outputs this argument. It is not difficult to construct a c.e. set, which is not semi-recursive: for every pair of elements $2e$ and $2e + 1$, one waits until (if ever) the e -th partial computable function φ_e is defined on argument $(2e, 2e + 1)$ and diagonalizes against it by enumerating in the constructed set an element from $\{2e, 2e + 1\} \setminus \{\varphi_e(2e, 2e + 1)\}$.

Proposition 10. *For every set of natural numbers A the following are equivalent:*

- (1) A is weakly semi-recursive.
- (2) $\{A, \bar{A}\}$ form a \mathcal{K} -pair.

Proof. Suppose that A is weakly semi-recursive with selector function s_A . Consider the set $W = \{\langle a, b \rangle \mid s_A(a, b) \downarrow = a\}$. Then $A \times \bar{A} \subseteq W$ and $\bar{A} \times A \subseteq \bar{W}$.

If on the other hand $\{A, \bar{A}\}$ form a \mathcal{K} -pair witnessed by W then the graph of the selector function s_A is $\{\langle \langle a, b \rangle, a \rangle \mid \langle a, b \rangle \in W\}$. \square

Finally we show that in this case as well, up to enumeration equivalence, all weakly semi-recursive sets, just as all semi-recursive sets, can be regarded of the form L_A .

Proposition 11. *If A is weakly semi-recursive then $A \equiv_e L_A$.*

Proof. By Proposition 1 we only need to show that $A \leq_e L_A$. Let s_A be the selector function for A . Consider the enumeration operator Γ , defined as follows:

$$\Gamma = \{\langle n, \sigma \rangle \mid |\sigma| = n + 1 \ \& \ \sigma(n) = 1 \ \& \ \forall m < n (\sigma(m) = 0 \rightarrow s_A(n, m) \downarrow = n)\}$$

Then $A = \Gamma(L_A)$ can be seen as follows. If $n \in A$ then $\langle n, \chi_A \upharpoonright (n + 1) \rangle \in \Gamma$ and $\chi_A \upharpoonright (n + 1) \in L_A$. Suppose that $n \in \Gamma(L_A)$ via the pair $\langle n, \sigma \rangle \in \Gamma$. Towards a contradiction assume that $n \notin A$. We will show that every prefix $\tau \subseteq \sigma$ is an initial segment of χ_A by induction on its length. Suppose that for $\tau = \sigma \upharpoonright m$ we have that $\tau \subseteq \chi_A$. If $\tau * 1 \subseteq \sigma$ then as $\tau * 1 \in L_A$ and $\tau * 1$ is the rightmost extension of the rightmost string in L_A of length $|\tau|$, $\tau * 1 \subseteq \chi_A$. If $\tau * 0 \subseteq \sigma$, i.e. $\sigma(m) = 0$, then $s_A(n, m) \downarrow = n$ and as $n \notin A$ by the properties of a selector function $m \notin A$, hence $\tau * 0 \subseteq \chi_A$. Thus $\sigma \subseteq \chi_A$ and $\sigma(n) = 1$ provides the anticipated contradiction. \square

Thus the statement of Theorem 3 can be further strengthened:

Corollary 5. *For every nontrivial Σ_2^0 \mathcal{K} -pair $\{A, B\}$ there is a semi-recursive set C , such that $A \leq_e C$ and $B \leq_e \bar{C}$.*

The class of nonzero enumeration degrees, which contain Δ_2^0 non-c.e. and non-co c.e. semi-recursive sets is first order definable in \mathcal{G}_e .

Proof. Let $\{A, B\}$ be a nontrivial \mathcal{K} -pair of Σ_2^0 enumeration degrees. By Theorem 3 there is a Σ_2^0 \mathcal{K} -pair $\{C, \bar{C}\}$, such that $A \leq_e C$ and $B \leq_e \bar{C}$. By Proposition 10 the set C is weakly semi-recursive. By Proposition 11 we have that $C \equiv_e L_C$. Thus $A \leq_e L_C$ and hence $B \leq_e \bar{L}_C = R_C$.

Thus \mathbf{a} is the enumeration degree of a Δ_2^0 non c.e., non co-c.e semi-recursive set A if and only if \mathbf{a} is half of a maximal Σ_2^0 \mathcal{K} -pair. \square

5.3. Relativization. Zooming out from the local structure and looking at the whole structure of the enumeration degrees, this is what we have so far.

From the definition of the enumeration jump of a set A , as $K_A \oplus \overline{K}_A$, it follows immediately that the jump of every enumeration degree is total. By Friedberg's Jump Inversion Theorem every total enumeration degree greater than or equal to $\mathbf{0}_e'$ belongs to the range of the enumeration jump operator. This together with the definability of the enumeration jump operator in \mathcal{D}_e yields the first order definition of the total degrees above $\mathbf{0}_e'$. Thus by Theorem 4 the total degrees comparable with $\mathbf{0}_e'$ are first order definable in \mathcal{D}_e .

Furthermore the proof of Theorem 4 can be relativized above any total degree. Consider again the relativized version of a \mathcal{K} -pair: a pair of sets A and B form a \mathcal{K} -pair over a set U if there is a set $W \leq_e U$, such that

$$A \times B \subseteq W \text{ \& } \overline{A} \times \overline{B} \subseteq \overline{W}.$$

Recall that this property is also degree theoretic and first order definable by:

$$\forall \mathbf{x}[\mathbf{x} \vee \mathbf{u} = (\mathbf{x} \vee \mathbf{a} \vee \mathbf{u}) \wedge (\mathbf{x} \vee \mathbf{b} \vee \mathbf{u})].$$

In [7] we relativize the dynamic characterization of \mathcal{K} -pairs as follows:

Lemma 3 ([7]). *Let G be a total set and let B and C be $\Sigma_2^0(G)$ sets. B and C form a \mathcal{K} -pair over G if and only if B and C have $\Sigma_2^0(G)$ approximations $\{B_i\}_{i < \omega}$ and $\{C_i\}_{i < \omega}$ such that for every i either $B_i \subseteq B$ or $C_i \subseteq C$.*

Thus we can relativize the construction in the proof of Theorem 3 and obtain the following theorem:

Theorem 5. *For every total enumeration degree \mathbf{a} the class $\mathcal{TOT} \cap [\mathbf{a}, \mathbf{a}']$ is first order definable in \mathcal{D}_e with parameter \mathbf{a} .*

6. LOCAL DEFINABILITY OF THE LOW ENUMERATION DEGREES

Now we turn to the local definability of the low enumeration degrees. The low enumeration degrees have been characterized in terms of the arithmetical complexity of the degrees that they bound. Cooper and McEvoy [4] show that an enumeration degree \mathbf{a} is low if and only if every $\mathbf{b} \leq \mathbf{a}$ is Δ_2^0 . Giorgi, Sorbi and Yang [9] show that this characterization can be strengthened for the total enumeration degrees.

Definition 9. *A Σ_2^0 set A is called downwards properly Σ_2^0 if for every non c.e. set B , such that $B \leq_e A$, B is not Δ_2^0 . A degree \mathbf{a} is downwards properly Σ_2^0 if it contains a downwards properly Σ_2^0 set.*

Giorgi, Sorbi and Yang show that every non-low total Σ_2^0 enumeration degree bounds a downwards properly Σ_2^0 enumeration degree.

In [6] we show that the class of downwards properly Σ_2^0 degrees is first order definable in \mathcal{G}_e .

Theorem 6. [6] *A Σ_2^0 degree is downwards properly Σ_2^0 if and only if it does not bound any nontrivial \mathcal{K} -pair.*

Giorgi, Sorbi and Yang's result combined with the local first order definability of the classes of the total enumeration degrees and the downwards properly Σ_2^0 enumeration degrees already gives the first order definition of the low *total* enumeration degrees in \mathcal{G}_e . We could complete the proof of the definability of the low

degrees simply by applying Soskov's Jump Inversion Theorem [23] which implies that a degree is low if and only if it is bounded by a low total degree.

To get a simpler definition we will first generalize the result by Giorgi, Sorbi and Yang.

Theorem 7. *Every non-low Σ_2^0 enumeration degree bounds a downwards properly Σ_2^0 enumeration degree.*

Proof. Let A be a member of a non-low Σ_2^0 enumeration degree. Consider the \mathcal{K} -pair L_{K_A} and R_{K_A} . Then by Proposition 1, $L_{K_A} \oplus R_{K_A} \equiv_e K_A \oplus \overline{K_A} = J_e(A)$. As $L_{K_A} \leq_e K_A \equiv_e A$, L_{K_A} is a Σ_2^0 set. On the other hand, as A is not low, R_{K_A} is not a Σ_2^0 set, and so L_{K_A} cannot be c.e. If we assume that L_{K_A} is not downwards properly Σ_2^0 then by Theorem 6 L_{K_A} bounds a nontrivial \mathcal{K} -pair $\{C, D\}$. By Part 3 of Proposition 2, $\{C, D, R_{K_A}\}$ form a \mathcal{K} -triple with $C \oplus D \oplus R_{K_A} \not\leq_e \emptyset'$. This contradicts Kalimullin's definition of the enumeration jump [12], which proves that $\mathbf{0}_e'$ is the largest enumeration degree, which can be represented as a \mathcal{K} -triple. Thus L_{K_A} is downwards properly Σ_2^0 and bounded by A . \square

Thus a degree is low if and only if it does not bound any downwards properly Σ_2^0 enumeration degree. Incorporating Theorem 6 this translates into a characterization of lowness in terms of the downwards density of \mathcal{K} -pairs: a degree is low if and only if every degree below it bounds a nontrivial \mathcal{K} -pair.

Theorem 8. *The set of low enumeration degrees is first order definable in \mathcal{G}_e .*

Proof. The low enumeration degrees are defined in \mathcal{G}_e by the following formula:

$$\mathcal{LOW}(\mathbf{x}) \iff \forall \mathbf{b} \leq \mathbf{x} \exists \mathbf{c} \leq \mathbf{b} \exists \mathbf{d} \leq \mathbf{b} (\mathcal{LK}(\mathbf{c}, \mathbf{d})).$$

\square

The definition of the total enumeration degrees below $\mathbf{0}_e'$ given in Theorem 4, combined with the statement of Proposition 3 allows us to define the set

$$C = \{\langle \mathbf{x}, \mathbf{u} \rangle \mid \mathbf{x}, \mathbf{u} \in \mathcal{TOT} \ \& \ \mathbf{u} \neq \mathbf{0}_e \ \& \ \iota^{-1}(\mathbf{x}) \text{ c.e. in } \iota^{-1}(\mathbf{u})\}.$$

Combining this with the definability of the low enumeration degrees, we could obtain a first order definition of set of total enumeration degrees which are images of low Turing degrees, c.e. in some nonzero low Turing degree. Unfortunately the next theorem reveals that this set is not very interesting. It also gives a characterization of the Δ_2^0 Turing degrees by an unexpected method.

Theorem 9. *A Turing degree \mathbf{x} is a Δ_2^0 Turing degrees if and only if \mathbf{x} is c.e. in some low Turing degree.*

Proof. One direction is obviously true: if \mathbf{x} is c.e. in a low Turing degree, then \mathbf{x} is Δ_2^0 . So let us concentrate on the opposite direction.

Suppose that \mathbf{x} is a Δ_2^0 Turing degree. If \mathbf{x} is c.e. then it is c.e. in $\mathbf{0}_T$. Suppose that $\mathbf{x} = \mathbf{d}_T(X)$ is not c.e. Then the enumeration degree $\iota(\mathbf{x}) = \mathbf{d}_e(X \oplus \overline{X})$ contains no Π_1^0 set. Consider the sets L_X and R_X . It follows that neither of these sets is c.e., as otherwise $L_X \oplus R_X$ would be enumeration equivalent to a Π_1^0 set, and by Proposition 1 $L_A \oplus R_A \equiv_e X \oplus \overline{X}$. Now we show that L_X is low. To see this we use a familiar trick. As $K_{L_X} \equiv_e L_X$ it follows from Proposition 2 that K_{L_X} forms a \mathcal{K} -pair with R_X , so $\overline{K_{L_X}} \leq_e R_X \oplus J_e(\emptyset)$. Now $J_e(L_X) = K_{L_X} \oplus \overline{K_{L_X}} \leq_e L_X \oplus R_X \oplus J_e(\emptyset) \equiv_e J_e(\emptyset)$.

By Soskov's Jump Inversion Theorem [23] there is a low total degree $\iota(\mathbf{u})$ such that $\mathbf{d}_e(L_X) \leq \iota(\mathbf{u})$. Note that as ι preserves the jump operation the Turing degree \mathbf{u} is also low. So $\iota(\mathbf{x})$ is the least upper bound of a maximal \mathcal{K} -pair $\{\mathbf{d}_e(L_X), \mathbf{d}_e(R_X)\}$ such that $\mathbf{d}_e(L_X) \leq \iota(\mathbf{u})$. By Proposition 3 it follows that \mathbf{x} is c.e. in \mathbf{u} . □

REFERENCES

- [1] M. M. Arslanov, S. B. Cooper, and I. Sh. Kalimullin, *Splitting properties of total enumeration degrees*, Algebra and Logic **42** (2003), 1–13.
- [2] S. B. Cooper, *Partial degrees and the density problem. Part 2: The enumeration degrees of the Σ_2 sets are dense*, J. Symbolic Logic **49** (1984), 503–513.
- [3] ———, *Enumeration reducibility, nondeterministic computations and relative computability of partial functions*, Recursion theory week, Oberwolfach 1989, Lecture notes in mathematics (Heidelberg) (K. Ambos-Spies, G. Muler, and G. E. Sacks, eds.), vol. 1432, Springer-Verlag, 1990, pp. 57–110.
- [4] S. B. Cooper and K. McEvoy, *On minimal pairs of enumeration degrees*, J. Symbolic Logic **50** (1985), no. 4, 983–1001.
- [5] R. M. Friedberg and Jr. H. Rogers, *Reducibility and completeness for sets of integers*, Z. Math. Logik Grundlag. Math. **5** (1959), 117–125.
- [6] H. A. Ganchev and M. I. Soskova, *Cupping and definability in the local structure of the enumeration degrees*, J. Symbolic Logic **77** (2012), no. 1, 133–158.
- [7] ———, *The high/low hierarchy in the local structure of the ω -enumeration degrees*, Ann. Pure Appl. Logic **163** (2012), 547–566.
- [8] H. A. Ganchev and M. I. Soskova, *Interpreting true arithmetic in the local structure of the enumeration degrees*, to appear in J. Symbolic Logic (2012).
- [9] M. Giorgi, A. Sorbi, and Y. Yang, *Properly Σ_2^0 enumeration degrees and the high/low hierarchy*, J. Symbolic Logic **71** (2006), 1125–1144.
- [10] C. Jockusch and J. Owings, *Weakly semirecursive sets*, J. Symbolic Logic **55** (1990), no. 2, 637–644.
- [11] C. G. Jockusch, *Semirecursive sets and positive reducibility*, Trans. Amer. Math. Soc. **131** (1968), 420–436.
- [12] I. Sh. Kalimullin, *Definability of the jump operator in the enumeration degrees*, Journal of Mathematical Logic **3** (2003), 257–267.
- [13] A. H. Lachlan and R. A. Shore, *The n -re a enumeration degrees are dense*, Arch. Math. Logic **31** (1992), 277–285.
- [14] K. McEvoy, *Jumps of quasi-minimal enumeration degrees*, J. Symbolic Logic **50** (1985), 839–848.
- [15] A. Nies, R. A. Shore, and T. A. Slaman, *Interpretability and definability in the recursively enumerable degrees*, Proc. London Math. Soc. **77** (1998), 241–249.
- [16] H. Rogers Jr., *Theory of recursive functions and effective computability*, McGraw-Hill Book Company, New York, 1967.
- [17] M. Rozinas, *The semi-lattice of e -degrees*, Recursive functions (Ivanovo), Ivano. Gos. Univ., 1978, Russian, pp. 71–84.
- [18] R. A. Shore, *Biinterpretability up to double jump in the degrees below $\mathbf{0}'$* , to appear.
- [19] ———, *Degree structures: Local and global investigations*, Bull. Symb. Logic **12** (2006), 369–389.
- [20] R. A. Shore and T. A. Slaman, *Defining the Turing jump*, Math. Res. Lett. **6** (1999), 711–722.
- [21] T. A. Slaman, *Global properties of the turing degrees and the turing jump*, Computational Prospects of Infinity. Part I: Tutorials (C. Chong, Qi Feng, T. A. Slaman, W. H. Woodin, and Y. Yang, eds.), World Scientific, 2005, pp. 83–101.
- [22] A. Sorbi, *The enumeration degrees of the Σ_2^0 sets*, Complexity, Logic and Recursion Theory (New York) (A. Sorbi, ed.), Marcel Dekker, 1975, pp. 303–330.
- [23] I. N. Soskov, *A jump inversion theorem for the enumeration jump*, Arch. Math. Logic **39** (2000), 417–437.

DEFINABILITY VIA KALIMULLIN PAIRS IN THE STRUCTURE OF THE ENUMERATION DEGREES

†FACULTY OF MATHEMATICS, SOFIA UNIVERSITY

‡DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY
E-mail address: ganchev@fmi.uni-sofia.bg, msoskova@fmi.uni-sofia.bg