Cupping Δ_2^0 enumeration degrees to $\mathbf{0}_e'$

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Abstract. In this paper we prove that every nonzero Δ_2^0 e-degree is cuppable to $\mathbf{0}_e'$ by a 1-generic Δ_2^0 e-degree (so low and nontotal) and that every nonzero ω-c.e. e-degree is cuppable to $\mathbf{0}_e'$ by an incomplete 3-c.e. e-degree.

1 Introduction

Intuitively, we say that a set A is *enumeration reducible* to a set B, denoted as $A \leq_e B$, if there is an effective procedure to enumerate A, given any enumeration of B. More formally, $A \leq_e B$ if there is a computably enumerable set W such that

$$A = \{x : (\exists u)[\langle x, u\rangle \in W \ \& \ D_u \subseteq B]\}$$

where D_u is the finite set with canonical index u.

Let $\equiv_e$ denote the equivalence relation generated by $\leq_e$ and let $[A]_e$ be the equivalence class of A — the *enumeration degree* (*e-degree*) of A. The degree structure $\langle \mathcal{D}_e, \leq\rangle$ is defined by setting $\mathcal{D}_e = \{[A]_e : A \subseteq \omega\}$ and setting $[A]_e \leq [B]_e$ if and only if $A \leq_e B$. The operation of least upper bound is given by $[A]_e \vee [B]_e = [A \oplus B]_e$ where $A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\}$. The structure $\mathcal{D}_e$ is an upper semilattice with least element $\mathbf{0}_e$, the collection of computably enumerable sets. Gutteridge [9] proved that $\mathcal{D}_e$ does not have minimal degrees (see Cooper [1]).

An important substructure of $\mathcal{D}_e$ is given by the Σ_2^0 e-degrees i.e. the e-degrees of Σ_2^0 sets. Cooper [2] proved that Σ_2^0 e-degrees are the e-degrees below $\mathbf{0}_e'$, the e-degree of $\overline{K}$. An e-degree is Δ_2^0 if it contains a Δ_2^0 set, a set A with a computable approximation f such that for every element x, $f(x, 0) = 0$ and $\lim_s f(x, s)$ exists and equals to $A(x)$. Cooper and Copestake [5] proved that

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below $\mathbf{0}'_e$ there are e-degrees that are not Δ^0_2. These e-degrees are called *properly* Σ^0_2 *e*-degrees.

In this paper we are mainly concerned with the cupping property of Δ^0_2 e-degrees. An e-degree $\mathbf{a}$ is cuppable if there is an incomplete e-degree $\mathbf{c}$ such that $\mathbf{a} \cup \mathbf{c} = \mathbf{0}'_e$. In [6], Cooper, Sorbi and Yi proved that all nonzero Δ^0_2 e-degrees are cuppable and that there are noncuppable Σ^0_2 e-degrees.

Theorem 1. *(Cooper, Sorbi and Yi* [6]*) Given a nonzero* Δ^0_2 *e-degree* $\mathbf{a}$*, there is a total* Δ^0_2 *e-degree* $\mathbf{c}$ *such that* $\mathbf{a} \cup \mathbf{c} = \mathbf{0}'_e$*, where an e-degree is total if it contains the graph of a total function. Meanwhile, noncuppable e-degrees exist.*

In this paper we first prove that each nonzero Δ^0_2 e-degree $\mathbf{a}$ is cuppable to $\mathbf{0}'_e$ by a non-total Δ^0_2 e-degree.

Theorem 2. *Given a nonzero* Δ^0_2 *e-degree* $\mathbf{a}$*, there is a 1-generic* Δ^0_2 *e-degree* $\mathbf{b}$ *such that* $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'_e$*. Since 1-generic e-degrees are quasi-minimal and 1-generic* Δ^0_2 *e-degrees are low,* $\mathbf{b}$ *is nontotal and low.*

Here a set A is *1-generic* if for every computably enumerable set S of $\{0,1\}$-valued strings there is some initial segment σ of A such that either S contains σ or S contains no extension of σ. An enumeration degree is 1-generic if it contains a 1-generic set. Obviously, no nonzero e-degree below a 1-generic e-degree contains a total function and hence 1-generic e-degrees are quasi-minimal. Copestake proved that a 1-generic e-degree is low if and only if it is Δ^0_2 (see [7]).

Our second result is concerned with cupping ω-c.e. e-degrees to $\mathbf{0}'_e$. A set A is n-c.e. if there is an effective function f such that for each x, $f(x,0) = 0$, $|\{s+1 \mid f(x,s) \neq f(x,s+1)\}| \leq n$ and $A(x) = \lim_s f(x,s)$. A is is ω-c.e. if there are two computable functions $f(x,s), g(x)$ such that for all x, $f(x,0) = 0$, $|\{s+1 \mid f(x,s) \neq f(x,s+1)\}| \leq g(x)$ and $\lim_s f(x,s) \downarrow = A(x)$.

An enumeration degree is n-c.e. (ω-c.e.) if it contains an n-c.e. (ω-c.e.) set. It's easy to see that the 2-c.e. e-degrees are all total and coincide with the Π_1 e-degrees, see [3]. Cooper also proved the existence of a 3-c.e. nontotal e-degree. As the construction presented in [6] actually proves that any nonzero n-c.e. e-degree can be cupped to $\mathbf{0}'_e$ by an $(n+1)$-c.e. e-degree, we will prove that any nonzero ω-c.e. e-degree is cuppable to $\mathbf{0}'_e$ by a 3-c.e. e-degree.

Theorem 3. *Given a nonzero* ω*-c.e. e-degree* $\mathbf{a}$*, there is a 3-c.e. e-degree* $\mathbf{b}$ *such that* $\mathbf{a} \cup \mathbf{b} = \mathbf{0}'_e$*.*

This is the strongest possible result. We explain it as follows. Consider the standard embedding ι of $\mathcal{D}_T$ to $\mathcal{D}_e$ given by: $\iota(deg_T(A)) = deg_e(\chi_A)$ where χ_A denotes the graph of the characteristic function of A. It is well-known that ι is an order-preserving mapping and that the Π_1 enumeration degrees are exactly the images of the Turing c.e degrees under ι. Consider a noncuppable c.e. degree $\mathbf{a}$. $\iota(\mathbf{a})$ is Π_1, hence ω-c.e., and $\iota(\mathbf{a})$ is not cuppable by any Π_1 e-degree, as ι preserves the least upper bounds. Therefore, no 2-c.e. e-degree cups $\iota(\mathbf{a})$ to $\mathbf{0}'_e$.

We use standard notation, see [4] and [10].

2 Basic ideas of Cooper-Sorbi-Yi's cupping

In this section we describe the basic ideas of Cooper-Sorbi-Yi's construction given in [6]. Let $\{A_s\}_{s<\omega}$ be a Δ^0_2 approximation of the given Δ^0_2 set A which is assumed to be not computably enumerable. We will construct two Δ^0_2 sets B and E (auxiliary) and an enumeration operator Γ such that the following requirements are satisfied:

$$S : \Gamma^{A,B} = \overline{K}$$
$$N_\Phi : E \neq \Phi^B$$

The first requirement is the global cupping requirement and it guarantees that the least upper bound of the degrees of A and B is $\mathbf{0}'_e$. Here $\Gamma^{A,B}$ denotes an enumeration operation relative to the enumerations of A and B.

The second group of requirements N_Φ, where Φ ranges over all enumeration operators, guarantees that the degree of B is not complete. Indeed, we have a witness — the degree of E is not below that of B.

To satisfy the global requirement S we will construct by stages an enumeration operator Γ such that $\overline{K} = \Gamma^{A,B}$. That is, at stage s we find all $x < s$ such that $x \in \overline{K}_s$ but $x \notin \Gamma^{A,B}[s]$, the approximation of $\Gamma^{A,B}$ at stage s, we define two markers a_x (bound of the A-part) and b_x (bound of the B-part and $b_x \in B$) and enumerate x into $\Gamma^{A,B}$ via the axiom $\langle x, A_s \upharpoonright a_x + 1, B_s \upharpoonright b_x + 1\rangle$. If x leaves $\overline{K}$ later, we can make this axiom invalid by extracting b_x from B or by a change (from 1 to 0) of A on $A_s \upharpoonright a_x + 1$. Intuitively we must use A-changes in the definition of Γ since otherwise B would be complete, contradicting the N-requirements. Since A is not in our control, if A does not provide such changes then we have to extract b_x out of B. We call this process the rectification of Γ at x.

Note that after stage s, at stage $t > s$ say, if $x \in \overline{K}_t$ but $A_t \upharpoonright a_x + 1 \not\subseteq A_t$ or $B_s \upharpoonright b_x + 1 \not\subseteq B_t$ then we need to put x into $\Gamma^{A,B}$ by enumerating a new axiom into Γ. If this happens infinitely often then x is not in $\Gamma^{A,B}$ and we cannot ensure that $\Gamma^{A,B}(x) = \overline{K}(x)$. To avoid this at stage t, when we re-enumerate x into $\Gamma^{A,B}$, we keep a_x the same as before, but let b_x be a bigger number. We put $b_x[t]$ into B and extract $b_x[s]$ from B (we want only one valid axiom enumerating x into $\Gamma^{A,B}$). Assuming that the G-strategies also do not change a_x after a certain stage, as A is Δ^0_2 there can be only finitely many changes in $A \upharpoonright a_x$ and hence we will eventually stop enumerating axioms for x in Γ.

Now we consider how to satisfy a N_Φ-requirement. We use variant of the Friedberg-Muchnik strategy. Namely, we select x as a witness, enumerate it into E and wait for $x \in \Phi^B$. If x never enters Φ^B then N_Φ is satisfied. Otherwise we will extract x from E, preserving $B \upharpoonright \phi(x)$ where $\phi(x)$ denotes the use function of the computation $\Phi^B(x) = 1$.

The need to preserve $B \upharpoonright \phi(x)$ conflicts with the need to rectify Γ. To avoid this before choosing x the N_Φ-strategy will first choose a (big) number k as its threshold and try to achieve $b_n > \phi(x)$ for all $n \geq k$. For elements $n < k$, S will be allowed to rectify Γ at its will. Whenever $\overline{K}$ changes below $k + 1$ we reset this N_Φ-strategy by cancelling all associated parameters except for this k. Since

k is fixed such a *resetting* process can happen at most $k+1$ many times, so we can assume that after a stage large enough this N_Φ-strategy will never be reset anymore.

If k enters K, the threshold is moved automatically to the next number in $\overline{K}$. Since $\overline{K}$ is infinite, eventually, the threshold will stop changing its value. This threshold will be the real threshold of the corresponding N_Φ-strategy.

In order to be able to preserve some initial segment of B for the diagonalization, N_Φ will first try to move all markers b_n for elements $n \geq k$ above the restraint. A useful A-change will facilitate this. In the event that no such useful change appears we will be able to argue that A is c.e. contrary to hypothesis. To do this we will have an extra parameter U, aimed to construct a c.e. set approximating A.

The N_Φ-strategy works as follows at stage s:

Setup: Define a threshold k to be a big number. Choose a witness $x > k$ and enumerate it in E.

K**-Check:** If a marker b_n for an element $n \leq k$ has been extracted from B during Γ-rectification then restart the attack.

Attack:

1. If $x \in \Phi^B$ go to step 2. Otherwise return to step 1 at the next stage.
2. Approximate A by $A_s \upharpoonright a_k$ at stage s. Extract $b_k[s]$ from B. Cancel all markers a_n and b_n for $n \geq k$. Define a_k new, bigger than any element seen so far in the construction. Go to step 3.
3. Initialize all strategies of lower priority. If a previous approximation of A defined at stage $t < s$ is not true then enumerate $b_k[t]$ back in B, extract x from E and go to step 4, otherwise go back to step 1.
4. While the observed change in A is still apparent, do nothing. Otherwise enumerate x back in E and extract $b_k[t]$ from B, go back to step 3.

If after a large enough stage the strategy waits at 1 or 4 forever then the N_Φ-requirement is obviously satisfied. In the latter case $\Phi^B(x) = 1 \neq 0 = E(x)$ and the construction of Γ will never change the enumeration of $\Phi^B(x) = 1$ since all γ-markers are lifted to bigger values by the changes of A below $a_k[s]+1$. This strategy will not go from 1 or 4 back to 3 infinitely often and hence the N_Φ-requirement is satisfied. Otherwise as A is Δ^0_2 it would pass through 2 infinitely often. Let $t_1 < t_2 < \cdots < t_n < \cdots$ be the stages at which this strategy passes through 2. Then for each i, $A_{t_i} \upharpoonright a_k[t_i]+1 \subset A$. By this property we argue that A is computably enumerable as follows: for each x, x is in A if and only if x is in A_{t_i} for some i, or

$$x \in A \quad \Leftrightarrow \quad \exists i (x \in A_{t_i}).$$

This contradicts our assumption on A.

3 Cupping by 1-generic degrees

In this section we give a proof of Theorem 2. That is, given an non-c.e. Δ^0_2 set A, we will construct a Δ^0_2 1-generic B satisfying the following requirements:

$S : \Gamma^{A,B} = \overline{K}$;
$G_i : (\exists \lambda \subset B)[\lambda \in W_i \vee (\forall \mu \supseteq \lambda)[\mu \notin W_i]]$.

If all requirements G_i together with the global requirement S are satisfied then B will have the intended properties. It is well known that the degree of a 1-generic set can not be complete.

Definition 1. *The tree of outcomes will be a perfect binary tree T. Each node $\alpha \in T$ of length i will be labelled by the requirement G_i. We will say that α is a G_i-strategy.*

At stage 0 $B = \emptyset$, $\Gamma = \emptyset$, $U_\alpha = \emptyset$ for all α and all thresholds and witnesses will be undefined.

At stage s we start by rectifying Γ and then construct a path through the tree δ_s of length s visiting all nodes $\alpha \subset \delta_s$ and performing actions as stated in the construction.

The Γ-rectification module for satisfying the global S requirement is as follows:

***Γ*-rectification module.** Scan all elements $n < s$ and perform the following actions for the elements n such that $\Gamma^{A,B}(n) \neq \overline{K}(n)$:

- $n \in \overline{K}$.
  1. If $a_n \uparrow$, define $a_n = a_{n-1} + 1$(if n= 0, define $a_n = 1$). Note that this is the only case when the Γ-module changes the value of a_n. Once defined a_n can only be redefined due to a G-strategy. The idea is that eventually G-strategies will stop cancelling a_n, so that we can approximate $A \upharpoonright a_n$ correctly and obtain a true axiom for n.
  2. If $b_n \downarrow$ then extract it from B and cancel all markers $b_{n'}$ for $n' > n$.
  3. Define b_n to be big, i.e a number greater than any number mentioned in the construction so far, and enumerate it in B.
  4. Enumerate in Γ the axiom $\langle n, A \upharpoonright a_n + 1, \{b_m | m \leq n\}\rangle$.
- $n \notin \overline{K}$

  Then find all valid axioms in Γ for n – $\langle n, A \upharpoonright a + 1, M_n\rangle$ and extract the greatest element of M_n from B.

Construction of δ_s. We will define $\delta_s(n)$ for all $n < s$ by induction on n. Suppose we have already defined $\delta_s \upharpoonright i = \alpha$ working on requirement G_W. We will perform the actions assigned to α and choose its outcome $o \in \{0, 1\}$. Then $\delta_s(i) = o$.

α will be equipped with a threshold k and a witness λ, a finite binary string. When α is visited for the first time after initialization it starts from *Setup*. At further stages it always performs *Check* first. If the *Check* does not empty U_α then it continues with the *Attack* module from where it was directed to at the previous α-true stage. Otherwise it continues with the *Setup* to define λ again and then proceeds to step 1 of *Attack*.

Setup: If a threshold has not been defined or is cancelled then define k to be big – bigger than any element appeared so far in the construction. If a witness has not yet been defined choose a binary string λ of length $b_k + 1$ so that $\lambda = B \restriction b_k + 1$.

Check: If a marker b_n for an element $n \leq k$ has been extracted from B during Γ-rectification at a stage t such that $s- < t \leq s$ where $s-$ is the previous α-true stage then initialize the subtree below α, empty U.

If $k \notin \overline{K}$ then define k to be the least $k' > k$ such that $k' \in \overline{K}$. I nitialize the subtree below α, empty U.

If b_k has changed since the last α-true stage and $\lambda \not\subseteq B$ then define λ to be $B \restriction b_k$. Do not empty U.

Attack:

1. Check if there is a finite binary string $\mu \supseteq \lambda$ in W . If not then the outcome is $o = 1$. Return to step 1 at the next stage. If there is such a μ then remember the least one and go to step 2.
2. Enumerate in the guess list U a new entry $\langle A_s \restriction a_k, \mu, b_k \rangle$. Extract b_k from B. Let $\hat{\mu}$ be the string μ but with position $b_k = 0$. For all elements $n > |\lambda|$ such that $\hat{\mu}(n)$ is defined let $B(n) = \hat{\mu}(n)$. Cancel all markers a_n and b_n for $n \geq k$. Define a_k to be bigger. Note that $\hat{\mu} \subset B$ and at the next stage Check will define a new value of λ to be $B \restriction b_k + 1$ so that $\lambda \supseteq \hat{\mu}$. Go to step 3.
3. Initialize all strategies below α. Scan the guess list U for errors. The entries in the guess list will be of the following form $\langle U_t, \mu_t, b_t \rangle$ where U_t is a guess of A and b_t is the marker that was extracted from B when this guess was made at stage t. Note that to make $\mu_t \subset B$ we only need to enumerate b_t in B. If there is an error in the guess list, i.e. some $U_t \not\subseteq A_s$, then enumerate b_t in B and go to step 4 with current guess $G = \langle U_t, \mu_t, b_t \rangle$ where t is the least index of an error in U. If all elements are scanned and no errors are found go back to step 1.
4. If the current guess $G = \langle U_t, \mu, b_t \rangle$ has the property $U_t \not\subseteq A_s$ then let the outcome be $o = 0$. Come back to step 4 at the next stage. Otherwise extract b_t from B. If the Γ-rectification module has extracted a marker m for an axiom that includes b_t in its $B-$part since the last stage on which this strategy was visited then enumerate m back in B. Go back to step 3.

The Proof. Define the true path $f \subset T$ to be the leftmost path through the tree that is visited infinitely many times, i.e. $\forall n \exists^\infty t (f \restriction n \subseteq \delta_t)$ and $\forall n \exists t_n \forall t > t_n (\delta_t \not<_L f \restriction n)$.

Lemma 1. *For each strategy $f \restriction n$ the following is true:*

1. *There is a stage $t_1(n) > t_n$ such that at all $f \restriction n$-true stages $t > t_1(n)$ Check does not empty U.*
2. *There is a stage $t_2(n) > t_1(n)$ such that at all $f \restriction n$-true stages $t > t_2(n)$ the Attack module never passes through step 3 and hence the strategies below $f \restriction n$ are not initialized anymore, B is not modified by $f \restriction n$, and the markers a_n for any elements n are not moved by $f \restriction n$*

Proof. Suppose the two conditions are true for $m < n$. Let $f \restriction n = \alpha$. Let t_0 be an α-true stage bigger than $t_2(m)$ for all $m < n$ and t_n.

Then after stage t_0 α will not be initialized anymore.

After stage t_0 all elements $n < k$ have permanent markers a_n. Indeed none of the strategies above α modify them anymore according to the induction hypothesis, strategies to the left are not accessible anymore and strategies to the right are initialized on stage t_0, hence the next time they are accessed they will have new thresholds greater than k.

The threshold k will stop shifting its value as $\overline{K}$ is infinite and we will eventually find the true threshold $k \in \overline{K}$.

As A is Δ^0_2, eventually all $A \restriction a_n$ for element $n < k$ will have their final value and so will $\overline{K} \restriction k$. Hence there is a stage $t_1(n) > t_0$ after which no markers b_n for elements $n \leq k$ will be extracted from B by the Γ-rectification and the *Check* module at α will never empty U again.

To prove the second clause suppose that the module passes through step 3 infinitely many times and consider the set $V = \bigcup L(U)$ where $L(U)$ denotes the left part of entries in the guess list U, that is the actual guesses at the approximation of A. By assumption A is not c.e. hence $A \neq V$.

If $V \nsubseteq A$ then there is a least stage t' and element p such that $p \in U_{t'} \backslash A$ and all U_t for $t < t'$ are subsets of A. Let $t_p > t_2$ be a stage such that the Δ^0_2 approximation of A settles down on p, i.e. for all $t > t_p$, $A_t(p) = A(p) = 0$. Then when we pass through step 3 after stage t_p we will spot this error, go to step 4 and never again return to step 3.

If $V \subset A$, let p be the least element such that $p \in A \backslash V$. Every guess in U is eventually correct and returns to step 1. To access step 3 again we pass through step 2, i.e. we pass through step 2 infinitely often. As a result a_k grows unboundedly and will eventually reach a value greater than p. As on all but finitely many stages t, $p \in A_t$, p will enter V. □

Corollary 1. *Every G_i-requirement is satisfied.*

Proof. Consider the G_i-strategy $\alpha = f \restriction i$. Choose a stage $t_3 > t_2(i)$ from Lemma 1, after which the Attack module is stuck at step 1 or step 4, we have a permanent value for a_k and $A \restriction a_k$ remains unchanged. Then so will the marker b_k and we will never modify λ again and $\lambda \subseteq B_t$ at all $t > t_3$.

If the module is stuck at step 1 we have found a string λ such that $\lambda \subset B$ and no string $\mu \supset \lambda$ is in the set W_i.

If the module is stuck at step 4 we have found a string μ from the guess $G = \langle U_t, \mu, b_t \rangle$ which is in W_i. It follows from the construction that $\mu \subset B$. The current markers b_n, for $n \geq k$ at stage t were cancelled and $b_k[t] = b_t$ was extracted from B. Any axiom defined after stage t has b-marker greater than $|\mu|$. Hence the Γ-rectifying procedure will not extract any element below the restraint $B \restriction |\mu|$ from B. It does not extract markers of elements $n < k$. If $n \geq k$ and $n \in \overline{K}$ then its current marker is greater than $|\mu|$. If $n > k$ and $n \notin \overline{K}$ then any axiom defined before stage t is invalid, because its b-marker is already extracted from B at a previous stage $t_0 < t$ or else it has an A-component that contains as a subset $U_t \nsubseteq A$. □

Lemma 2. *The S-requirement is satisfied.*

Proof. At each stage s we make sure that Γ is rectified. For elements $n < s$, we have $\Gamma^{A,B}(n)[s] = \overline{K}(n)[s]$. This is enough to prove that $n \notin \overline{K} \Rightarrow n \notin \Gamma^{A,B}$. Indeed if we assume that $n \in \Gamma^{A,B}$ then there is an axiom $\langle n, A_n, M_n \rangle \in \Gamma$ and $A_n \subseteq A$, $M_n \subset B$. Hence this axiom is valid on all but finitely many stages. But according to our construction we will ensure $M_n \nsubseteq B$ on infinitely many stages, a contradiction.

To prove the other direction, $n \in \overline{K} \Rightarrow n \in \Gamma^{A,B}$, we have to establish that the N-strategies will stop modifying the markers a_n and b_n eventually. Indeed the markers can be modified only by N-strategies with thresholds $k < n$. The way we choose each threshold guarantees that there will be only finitely many nodes on the tree with this property. The nodes to the left of the true path will eventually not be accessible anymore and the nodes to the right will be cancelled and will choose new thresholds, bigger then n. Lemma 1 proves that every node along the true path will eventually stop moving a_n and b_n by property 2.

Suppose the markers are not modified after stage t_1. After stage t_1, a_n has a constant value. As A is Δ^0_2 there will be a stage $t_2 > t_1$ such that for all $t > t_2$ $A \upharpoonright a_n[t] = A \upharpoonright a_n$. At stage $t_2 + 1$ we rectify Γ. If $n \in \Gamma^{A,B}$ then there is an axiom $\langle n, A_n, M_n \rangle$ in Γ such that $A_n \subset A \upharpoonright a_n$ and at all further stages this axiom will remain valid, so the Γ-rectifying procedure will not modify it again. Otherwise it will extract a b-marker for the last time and enumerate an axiom $\langle n, A \upharpoonright a_n, M'_n \rangle$ that will be valid at all further stages. In both cases we have found an axiom for n that is valid on all but finitely many stages, hence $n \in \Gamma^{A,B}$. □

Lemma 3. *B is Δ^0_2.*

Proof. We need to show that for each n, n can be put in and moved out from B at most finitely times. To see this fix n and consider the G_i-strategy along the true path that has a threshold $k_i > n$. As we have already established in Corollary 1 there is a stage $t_3 > t_2(i)$, after which we will never modify λ_i again and $\lambda \subseteq B_t$ on all $t > t_3$. As $n < |\lambda_i|$ then $B_t(n)$ will remain constant on all stages $t > t_3$. This means that $B(n)$ changes at most t_3 many times. □

4 Cupping the ω-c.e. degrees

In this section we give a proof of Theorem 3. Suppose we are given an ω-c.e. set A with bounding function g. We will modify the construction of the set B so that it will turn out to be $3 - c.e.$. The requirements are:

$$S : \Gamma^{A,B} = \overline{K}$$
$$N_\Phi : E \neq \Phi^B$$

The structure of the axioms enumerated in Γ will be more complex. Again we will have an a-marker a_n for each element n, but instead of just one marker b_n we will have a set of b-markers B_n of size g_n+1 where $g_n = \sum_{x<a_n} g(x)$ together

with a counter c_n that will tell us which element we should extract if we need to. Every time $A \upharpoonright a_n$ changes we will extract from B a different element – the c_n-th element $b_n \in B_n$ and then add 1 to c_n to ensure that each element in B will be extracted only once. If we need to restore a computation due to the N-strategies we will enumerate the extracted marker back in B, hence B is $3-c.e.$. Note that if a restored computation has to be destroyed again, we will need to extract a different marker from B. This could destroy further computations. That is why will always try to restore the last computation $\Phi^B(x)$.

***Γ*-rectification module.** Scan all elements $n < s$ and perform the following actions for the elements n such that $\Gamma^{A,B}(n) \neq \overline{K}(n)$:

- $n \in \overline{K}$.
  1. If $a_n \uparrow$ then define $a_n = a_{n-1} + 1$(if n = 0, define $a_n = 1$).
  2. If $B_n \downarrow$. Extract the c_n-th member of B_n. Move the counter c_n to the next position $c_n + 1$. Cancel all $B_{n'}$ for $n' > n$.
  3. If $B_n \uparrow$ then define a set of new markers B_n of size $g_n + 1$ where $g_n = \sum_{x<a_n} g(x)$ and a new counter $c_n = 1$ and enumerate B_n in B.
  4. Enumerate in Γ the axiom $\langle n, A_s \upharpoonright a_n + 1, \bigcup\{B_{n'}(c_{n'}) | n' \leq n\}\rangle$ where $B_{n'}(c_{n'})$ is the set of all elements in $B_{n'}$ with positions greater than or equal to $c_{n'}$.
- $n \notin \overline{K}$
  Then find all valid axioms in Γ for n – $\langle n, A_t \upharpoonright a + 1, M_n\rangle$ where $M_n = \bigcup\{B_{n'} | n' \leq n\}$ and extract the least member of B_n that has not yet been extracted from B. Increment the counter c_n that corresponds to the set of markers B_n.

Construction of $\boldsymbol{\delta_s}$. Setup: If a threshold has not been defined or is cancelled then define k to be big, bigger than any element appeared so far in the construction. If a witness has not yet been defined choose $x > k$ and enumerate it in E.

Check: If a marker from B_n for an element $n < k$ has been extracted from B during Γ-rectification at a stage t, $s- < t \leq s$ where $s-$ is the previous α-true stage, then initialize the subtree below α, empty U.

If $k \notin \overline{K}$ then shift it to the next possible value and redefine x to be bigger. Again initialize the subtree below α and empty U.

Attack:

1. Check if $x \in \Phi^B$. If not then the outcome is $o = 1$, return to step 1 at the next stage. If $x \in \Phi^B$ go to step 2.

2. Initialize all strategies below α. Scan the guess list U for errors. If there is an error then take the last entry in the guess list, say the one with index t: $\langle U_t, B_t, c_t \rangle \in U$ and $U_t \nsubseteq A_s$. Enumerate the $(c_t - 1)$-th member of B_t back in B. Extract x from E and go to step 4 with current guess $G = \langle U_t, B_t, c_t \rangle$. If all elements are scanned and no errors are found go to step 3.
3. Enumerate in the guess list U a new entry $\langle A_s \upharpoonright a_k, B_k, c_k \rangle$. Extract the c_k-th member of B_k from B and move c_k to the next position $c_k + 1$. Cancel all markers a_n and B_n for $n \geq k$. Define a_k new, bigger than any element seen so far in the construction. Go to back to step 1.
   Note that this ensures that our guesses at the approximation of A are monotone. Hence if there is an error in the approximation, this error will be apparent in the last guess. This allows us to always use the computation corresponding to the last guess. We will always be able to restore it.
4. If the current guess $G = \langle U_t, B_t, c_t \rangle$ has the property $U_t \nsubseteq A_s$ then let the outcome be $o = 0$. Come back to step 4 at the next stage. Otherwise enumerate x back in E and extract the c_t-th member of B_t from B and move the value of the counter to $c_t + 1$. If at this stage during the Γ-rectification procedure a different marker m for an axiom that contains B_t was extracted then enumerate m back in B. Go back to step 1.

The Proof. The construction ensures that for any n, at any stage t, at most one axiom in Γ defines $\Gamma^{A,B}(n)$. Generally, we extract a number from B_n to drive n out of $\Gamma^{A,B}$. When an N-strategy α acts at step 3 of the Attack module, at stage s say, α needs to preserve $\Phi^B(x_\alpha)$. All lower priority strategies are initialized and an element b_1 in B_{k_α} is extracted from B to prevent the S-strategy from changing B on $\phi(x_\alpha)$. Note that all axioms for elements $n \geq k_\alpha$ contain B_{k_α}. So at stage s, when we extract b_1 from B, n is driven out of $\Gamma^{A,B}$. As in the remainder of the construction, at any stage, we will have either that A has changed below a_n or B has changed on B_n, these axioms will never be active again. As the Γ-module acts first, it may still extract a marker m from an axiom for $n > k_\alpha$ if $A \upharpoonright a_n$ has changed back and thereby injure $B \upharpoonright \phi(x_\alpha)$. But when α is visited it will correct this by enumerating m back in B and extracting a further element $b_2 \in B_{k_\alpha}$ from B to keep Γ true. This makes our N-strategies and the S-strategy consistent. We comment here that such a feature is also true in the proof of Theorem 2, but there we do not worry about this as we are constructing a Δ^0_2 set. In the proof of Theorem 3, this becomes quite crucial, as we are constructing B as a 3-c.e. set, and we have less freedom to extract numbers out from B.

The construction ensures that B is a $3 - c.e.$ set. First we prove that the counter c_n never exceeds the size of its corresponding set B_n and therefore we will always have an available marker to extract from B if it is necessary.

Lemma 4. *For every set of markers B_n and corresponding counter c_n at all stages of the construction $c_n < |B_n|$ and the c_n-th member of B_n is in B.*

Proof. For each set of markers B_n only one node along the true path can enumerate its elements back into B. Indeed if B_n enters the guess list U_t at some node α on the tree then at stage t, B_n is the current set of markers for n and n is the threshold for α. When α enumerates B_n in its U_t, it cancels the current markers for the element n. Hence B_n does not belong to any $U^{\beta}_{t'}$ for $t' \leq t$ and any node β or else B_n will not be current and B_n will not enter $U^{\beta}_{t''}$ at any stage $t'' \geq t$ and any node β as it is not current anymore.

We ensure that n being a threshold is in $\overline{K}$, hence after stage t the Γ-rectification procedure will not modify $B \upharpoonright B_n$. Before stage t while the markers were current the counter c_n was moved only when the Γ-rectification procedure observed a change in $A \upharpoonright a_n$, i.e some element that was in $A \upharpoonright a_n$ at the previous stage is not there anymore. After stage t α will move the marker c_n once at entry in U_t and then only when it observes a change in $A \upharpoonright a_n$, i.e $U_n = A \upharpoonright a_n[t]$ was a subset of A at a previous step but is not currently. Altogether c_n will be moved at most $g_n + 1 < |B_n|$ times.

Otherwise B_n belongs to an axiom which contains the set B_k for a particular threshold k and $n \notin \overline{K}$. Then again its members are enumerated back in B only in reaction to a change in $A \upharpoonright a_n$. □

We will now prove that Lemma 1 is valid for this construction as well. Note that this construction is a bit different, therefore we will need a new proof. The true path f is defined in the same way.

Lemma 5. *For each strategy $f \upharpoonright n$ the following is true:*

1. *There is a stage $t_1(n) > t_n$ such that at all $f \upharpoonright n$-true stages $t > t_1(n)$ Check does not empty U.*
2. *There is a stage $t_2(n) > t_1(n)$ such that at all $f \upharpoonright n$-true stages $t > t_2(n)$ the Attack module never passes through step 2 and hence the strategies below α are not initialized anymore, B is not modified by $f \upharpoonright n$, and the markers a_n for any elements n are not moved by $f \upharpoonright n$*

Proof. Suppose the two conditions are true for $m < n$. Let $f \upharpoonright n = \alpha$. Let t_0 be an α-true stage bigger than $t_2(m)$ for all $m < n$ and t_n.

Then after stage t_0 α will not be initialized anymore. The proof of the the existence of stage $t_1(n)$ satisfying the first property is the same as in Lemma 1.

To prove the second clause suppose that the module passes through step 2 infinitely many times and consider the set $V = \bigcup L(U)$ where $L(U)$ denotes the left part of entries in the guess list U. By assumption A is not c.e. hence $A \neq V$.

If $V \nsubseteq A$ then there is element p such that $p \in V \backslash A$. Let $t_p > t_2$ be a stage such that the approximation of A settles down on p, i.e. for all $t > t_p$, $A_t(p) = A(p) = 0$. Then when we pass through step 2 after stage t_p we will spot this error, go to step 4 and never again return to step 1.

If $V \subset A$, let p be the least element such that $p \in A \backslash V$. Every guess in U is eventually correct and allows us to move to step 3, i.e. we pass through step 3 infinitely often. As a result a_k grows unboundedly and will eventually reach a value greater than p. As on all but finitely many stages t, $p \in A_t$, p will enter V. □

Corollary 2. *Every N_i-requirement is satisfied.*

Proof. Let $\alpha \subset f$ be an N_i-strategy. As a corollary of Lemma 5 there is a stage $t_3 > t_2(i)$ after which the *Attack* module is stuck at step 1, and hence $x \notin \Phi^B$, but $x \in E$. Or else the module is stuck at step 4, in which case $x \in \Phi^B$ and $x \notin E$. Indeed step 4 was accessed with $G = \langle U_t, B_t, c_t \rangle$, belonging to the last entry in the guess list $\langle U_t, B_t, c_t \rangle$. At stage t we had $x \in \Phi^B[t]$. The current markers b_n , for $n \geq k$ were cancelled and $b_k[t]$ was extracted from B. Hence the Γ-rectifying procedure will not extract any element below the restraint $B \upharpoonright \phi(x)$ from B. It does not extract markers of elements $n < k$. If $n \geq k$ and $n \in \overline{K}$ then its current marker is greater than $\phi(x)$. If $n > k$ and $n \notin \overline{K}$ then any axiom defined before stage t is invalid, because one of its b-markers is extracted from B at a previous stage or else it has an A-component $U_t \not\subseteq A$. Any axiom defined after stage t has b-markers greater than $\phi(x)$.

After stage t, if α modifies B it will be in the set of markers B_t, and when step 4 is accessed we have $B_t \subset B$. □

Lemma 2 is now valid for Theorem 3 as well, hence all requirements are satisfied and this concludes the proof of Theorem 3.

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