COMPLEXITY PROFILES AND GENERIC MUCHNIK REDUCIBILITY

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ABSTRACT. We study the relative computational complexity of expansions of Cantor and Baire space in terms of generic Muchnik reducibility. We show that no expansion of Cantor space by countably many unary or closed relations can give a generic Muchnik degree strictly between the degree of Cantor space and the degree of Baire space. Similarly, assuming $\boldsymbol{\Delta}^1_2$-Wadge determinacy we show that no expansion of Baire space by countably many unary or closed relations can give a generic Muchnik degree strictly between the degree of Baire space and the Borel complete degree. On the other hand, we provide a construction of a degree strictly between $\mathcal{C}$ and $\mathcal{B}$ and also between $\mathcal{B}$ and $\mathcal{BC}$.

1. Introduction

In computable structure theory, we analyze and compare the computational complexity of mathematical structures. One method of saying a structure $\mathcal{N}$ is computationally simpler than a structure $\mathcal{M}$ is if every copy of $\mathcal{M}$ computes a copy of $\mathcal{N}$. This notion, called Muchnik reducibility and written $\mathcal{N} \leq_w \mathcal{M}$, works well for countable structures, the standard domain of classical computation. Several methods have been proposed to extend computable model theory to structures of higher cardinalities. See, for example, [GK13, Sac90] for computability on admissible ordinals, [PER89, Wei93] for computability on separable structures modeled by computing a dense countable substructure, [Mil13] for a view of uncountable structures as being built out of countable structures, and [HMSW08] for an approach based on infinite time Turing machines. These methods give a notion of computable structures beyond the countable, but they intrinsically change the notion of computation. For example, in computability on admissible ordinals, every countable structure becomes computable.

In his thesis, the third author introduced the notion of generic Muchnik reducibility which allows us to maintain our familiar notion of computation and compare the computational content of uncountable structures. The idea is to change our model of set theory.

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We benefited from our interaction with Greenberg, Igusa, Turetsky, and Westrick, who were also trying to build a generic Muchnik degree strictly between Cantor space and Baire space. In particular, the main idea used in the proof of Theorem 4.2 belongs to them. While he was working on an undergraduate thesis with Miller, Kirill Gura helped clarify some of the themes in this paper, including the use of complexity profiles. There is no doubt that his influence is present in our work. Finally, we are grateful to Arnie Miller for pointing out that we had reproved (in a weaker form) a result of Hurewicz (Lemma 6.2).

Definition 1.1. A structure $\mathcal{N}$ is *generic Muchnik reducible* to a structure $\mathcal{M}$, written $\mathcal{N} \leq_w^* \mathcal{M}$, if in some forcing extension of the universe in which $\mathcal{M}$ and $\mathcal{N}$ are countable, every copy of $\mathcal{M}$ computes a copy of $\mathcal{N}$.

It follows from Shoenfield's absoluteness theorem [Sho61] that generic Muchnik reducibility is set-theoretically robust:

Lemma 1.2 (Schweber [KMS16])**.** *If $\mathcal{N} \leq_w^* \mathcal{M}$, then $\mathcal{N} \leq_w \mathcal{M}$ in every forcing extension that makes $\mathcal{M}$ and $\mathcal{N}$ countable. In particular, if $\mathcal{M}$ and $\mathcal{N}$ are countable, then $\mathcal{N} \leq_w^* \mathcal{M}$ if and only if $\mathcal{N} \leq_w \mathcal{M}$.*

An underlying theme of logic is understanding the definable sets in a structure and the complexity of their definitions. Much of model theory focuses on analyzing definable sets with a special focus on finding a collection of sets that allow quantifier-elimination. In computability, the complexity of a defining formula for a subset of $\mathbb{N}$ corresponds to its arithmetic complexity. In descriptive set theory, this leads to the topological hierarchies. In computable model theory, this often corresponds to the notion of a relatively intrisically Σ^0_k set—that is, a set that is Σ^0_k in every presentation of a model. Equivalently, these are the sets definable in $\mathcal{L}_{\omega_1,\omega}$ by a computable Σ_k formula, a Σ_k formula where conjunctions and disjunctions are over c.e. sets. The main tool introduced in this paper is the complexity profile, which brings this analysis to the setting of the generic Muchnik degrees.

What we find is that the natural analog of the relatively intrinsically Σ^0_k set, that of a subset of $\mathcal{M}$ being Σ^0_k in every presentation in the forcing extension, once again corresponds to it being definable by a computable Σ_k formula in $\mathcal{L}_{\omega_1,\omega}$. We define the complexity profile for a structure to capture this notion, and extend the notion to allow us to compare structures with different domains. We compute these complexity profiles for the familiar generic Muchnik degrees and see that they correspond naturally to levels of the topological hierarchy. We show how they can be used to determine the generic Muchnik degree of structures. The surprising effectiveness of this approach is borne out by the sharp dichotomies we prove, which all hinge on determining a level of the complexity profile.

Several structures have arisen as particularly interesting uncountable structures to consider. These are Cantor space, Baire space, and the field of real numbers, along with various expansions of each. We write $\mathcal{C}$ for the structure of Cantor space: the domain of this structure is 2^ω and the language is $(U_i)_{i\in\omega}$, where $U_i(x)$ holds if and only if $x(i) = 1$. Similarly, $\mathcal{B}$ is the structure of Baire space with domain is ω^ω and language consisting of $(U_{i,j})_{i,j\in\omega}$ and $U_{i,j}(x)$ holds if and only if $x(i) = j$. The following theorem summarizes what is known regarding their generic Muchnik degrees:

Theorem 1.3 (Knight–Montalban–Schweber [KMS16], Igusa–Knight [IK16], Downey—Greenberg–Miller [DGM16], Igusa–Knight–Schweber [IKS17], Andrews–Knight–Kuyper–Miller–Soskova [AKK+])**.**

$$\begin{aligned} \mathcal{C} <_w^* (\mathcal{C}, \oplus) \equiv_w^* \mathcal{B} \equiv_w^* (\mathbb{R}, +, \cdot) \equiv_w^* (\mathbb{R}, +, <) \\ \equiv_w^* (\mathbb{R}, +, \cdot, \{f_i\}_{i\in\omega}) <_w^* (\mathcal{C}, \oplus, ') \equiv_w^* (\mathcal{B}, \oplus, '). \end{aligned}$$

In $(\mathbb{R}, +, \cdot, \{f_i\}_{i\in\omega})$, the sequence of f_i is any countable sequence of continuous functions on a Cartesian power of $\mathbb{R}$.

We denote by $\mathcal{BC}$ the structure $(\mathcal{B}, \oplus, ')$. Andrews et al. [AKK+] prove that this structure represents the largest generic Muchnik degree of a Borel structure, i.e., it is *Borel complete*.

In this paper, we examine expansions of $\mathcal{C}$ and $\mathcal{B}$ and try to understand their generic Muchnik degrees. We show that an expansion of $\mathcal{C}$ by countably many unary relations cannot be strictly between $\mathcal{C}$ and $\mathcal{B}$. Further, any expansion of $\mathcal{C}$ by countably many Borel unary relations is either $\equiv^*_w \mathcal{C}$ or is $\geq^*_w \mathcal{B}$. We also show that any expansion of $\mathcal{C}$ by countably many closed (not necessarily unary) relations is either $\equiv^*_w \mathcal{C}$ or $\geq^*_w \mathcal{B}$.

We give an analogous analysis of expansions of Baire space, assuming $\mathbf{\Delta^1_2}$-Wadge determinacy. In particular, any expansion of $\mathcal{B}$ by countably many unary relations cannot be strictly between $\mathcal{B}$ and $\mathcal{BC}$. Further, any expansion of $\mathcal{B}$ by countably many unary $\mathbf{\Delta^1_2}$-relations is either $\equiv^*_w \mathcal{B}$ or $\geq^*_w \mathcal{BC}$. Also, any expansion of $\mathcal{B}$ by countably many closed (not necessarily unary) relations is either $\equiv^*_w \mathcal{B}$ or $\equiv^*_w \mathcal{BC}$.

Finally, we answer a question from [Sch16] by giving an example of a generic Muchnik degree strictly between $\mathcal{C}$ and $\mathcal{B}$. The same construction also allows us to produce a generic Muchnik degree strictly between $\mathcal{B}$ and $\mathcal{BC}$. The construction is built around the intuition that a linear order codes no sets in its jump, but sets can be coded in its double-jump. We append a linear order $\mathcal{L}$ to $\mathcal{C}$ in such a way as to make $\mathcal{C} \leq^*_w \mathcal{C} \sqcup \mathcal{L} \leq^*_w \mathcal{B}$. We show that for any linear order, it is always true that $\mathcal{C} \sqcup \mathcal{L} \not\geq^*_w \mathcal{B}$, because $\mathcal{B}$ codes more sets in its jump than $\mathcal{C} \sqcup \mathcal{L}$ can. To get $\mathcal{C} \sqcup \mathcal{L}$ to be strictly between $\mathcal{C}$ and $\mathcal{B}$, we construct $\mathcal{L} \leq^*_w \mathcal{B}$ so that the third jump of $\mathcal{L}$ codes a strictly $\mathbf{\Pi^1_2}$ set. In contrast, the sets coded in three jumps over $\mathcal{C}$ are the $\mathbf{\Delta^1_2}$ sets.

2. Preliminaries

Throughout, we will use the notion of relatively intrinsically Σ_n and relatively intrinsically Δ_n sets. Recall that a presentation of a countable structure is a an isomorphic structure with universe ω. We introduce the following notation.

Definition 2.1. Let $\mathcal{M}$ be a countable structure. A set $X \subseteq \mathcal{M}^k$ is *relatively intrinsically Σ^0_n in $\mathcal{M}$*, written $\Sigma^*_n(\mathcal{M})$, if for every presentation $\mathcal{A}$ of $\mathcal{M}$, X is Σ^0_n relative to the atomic diagram of $\mathcal{A}$.

The following is the classic theorem regarding definability of relatively intrinsically Σ^0_n sets. We note that it holds in our expanded framework.

Theorem 2.2 (Ash–Knight–Manasse–Slaman–Chisholm)**.** *Let $V[G]$ be a generic extension and $X \in V[G]$ be a set that is $\Sigma^*_n(\mathcal{M})$ in the sense of $V[G]$. Then there is a Σ^c_n formula in the sense of V that defines X in V. In particular, $X \in V$.*

Proof. The classic Ash–Knight–Manasse–Slaman–Chisholm theorem [AKMS89, Chi90] shows that there is a Σ^c_n formula $\varphi(x, \vec{a})$ in $V[G]$ that defines X (with finitely many parameters) in $V[G]$. But computability is absolute, so φ is Σ^c_n and in V. Similarly, satisfaction of φ on $\mathcal{M}$ is absolute, so the fact that X is defined by φ is absolute. ☐

We can now expand the definition of $\Sigma^*_n(\mathcal{M})$ to arbitrary structures.

Definition 2.3. Let $\mathcal{M}$ be any structure. A set $X \subseteq \mathcal{M}^k$ is *relatively intrinsically Σ^0_k* if there is some generic G so that $\mathcal{M}$ is countable in $V[G]$ and $X \in \Sigma^*_k(\mathcal{M})$ in the sense of $V[G]$. In this case, we write $X \in \Sigma^*_k(\mathcal{M})$.

A set $X \subseteq \mathcal{M}^k$ is $\Delta^*_k(\mathcal{M})$ if both it and its complement are $\Sigma^*_k(\mathcal{M})$.

By Theorem 2.2, the collection of X in V that are $\Sigma^*_n(\mathcal{M})$ or $\Delta^*_n(\mathcal{M})$ does not depend on the generic G.

The following definition captures a very useful property of $\mathcal{C}$ and $\mathcal{B}$.

Definition 2.4. A structure $\mathcal{M}$ is said to be Δ^0_2-*relatively-generically-categorical* if whenever A and B are two copies of $\mathcal{M}$ in $V[G]$, where $V[G]$ makes $\mathcal{M}$ countable, then there is a $\Delta^0_2(A \oplus B)$-isomorphism in $V[G]$ between them.

The fact that $\mathcal{C}$ and $\mathcal{B}$ are Δ^0_2-relatively-generically-categorical will allow us to define complexity profiles on their domains beginning at the Σ^0_2 level.

Theorem 2.5. *Both $\mathcal{C}$ and $\mathcal{B}$ are Δ^0_2-relatively-generically-categorical.*

Proof. It is computable in $A \oplus B$ to see that an element $a \in A$ has the same nth value as an element $b \in B$. Thus to check whether a has all the same values as b is $\Delta^0_2(A \oplus B)$. This gives the isomorphism. ☐

When we are presenting $\mathcal{C}$ or $\mathcal{B}$ along with unary predicates, it will be convenient to not have to worry about repetitions of elements. The following theorem shows that we can always remove such repetitions from a structures without increasing their complexity.

Theorem 2.6. *Let $\mathcal{M}$ be a structure (such as $\mathcal{C}$ or $\mathcal{B}$) in which any two elements have different quantifier-free types. Further suppose that the language of $\mathcal{M}$ contains only unary relations. Let $\mathcal{N}$ be any structure that agrees with $\mathcal{M}$ but with possible repetitions of elements; then $\mathcal{N} \geq^*_w \mathcal{M}$.*

Proof. Fix a presentation $\mathcal{A}$ of $\mathcal{N}$, enumerated as $(a_i)_{i\in\omega}$. Then consider the c.e. set of elements a_i in $\mathcal{A}$ so that $(\forall j < i)(\exists R \in \mathcal{L})\ [R(a_j) \not\leftrightarrow R(a_i)]$. This c.e. subset is isomorphic to $\mathcal{M}$. Thus we have a computable presentation of $\mathcal{M}$ from any presentation of $\mathcal{N}$. ☐

2.1. **Topological background.** We will apply the following definitions in this paper exclusively to the spaces $\mathcal{C}$ and $\mathcal{B}$, though we state them in a more general form below.

Definition 2.7. For a Polish space X:

- The *Borel sets* of X are the sets in the σ-algebra generated by the open sets of X.
- $\mathbf{\Sigma^0_1}$ is the collection of open subsets of X.

For each $k > 0$,

- $\mathbf{\Pi^0_k}$ is the collection of complements of $\mathbf{\Sigma^0_k}$ sets.
- $\mathbf{\Sigma^0_{k+1}}$ is the collection of countable unions of $\mathbf{\Pi^0_k}$ sets.
- $\mathbf{\Delta^0_k}$ is $\mathbf{\Sigma^0_k} \cap \mathbf{\Pi^0_k}$.

In particular, $\mathbf{\Sigma^0_2}$ is the collection of F_σ subsets of X and $\mathbf{\Pi^0_2}$ is the collection of G_δ subsets of X.

Definition 2.8. For a Polish space X:

- A set $A \subseteq X$ is $\mathbf{\Sigma^1_1}$ if for some Polish Y and Borel $B \subseteq X \times Y$, A is the projection of B on X.

For each $k > 0$,

- $\mathbf{\Pi^1_k}$ is the collection of complements of $\mathbf{\Sigma^1_k}$ sets.
- A set $A \subseteq X$ is $\mathbf{\Sigma^1_{k+1}}$ if for some Polish Y and $\mathbf{\Pi^1_k}$ set $B \subseteq X \times Y$, A is the projection of B on X.
- $\mathbf{\Delta^1_k}$ is $\mathbf{\Sigma^1_k} \cap \mathbf{\Pi^1_k}$.
- A set is *projective* if it is in $\bigcup_{k\in\omega} \mathbf{\Sigma^1_k}$.

The following series of facts about the notions defined above can be found in any standard text on descriptive set theory, such as for example [Kec95].

Fact 1. *A set is $\mathbf{\Delta^1_1}$ if and only if it is Borel.*

Fact 2. *In the definitions above, the Polish space Y can always be taken to be $\mathcal{C}$ or $\mathcal{B}$ or any other uncountable Polish space.*

Using this fact, when we are considering subsets of $\mathcal{C}$, we will generally simply consider $\mathbf{\Sigma^1_{k+1}}$ sets to be projections in $\mathcal{C}$ of $\mathbf{\Pi^1_k}$ subsets of $\mathcal{C}^2$, and similarly in $\mathcal{B}$. The following fact is Exercise 14.3 in Kechris [Kec95].

Fact 3. *A subset of $\mathcal{B}$ is $\mathbf{\Sigma^1_1}$ if and only if it is the projection of a $\mathbf{\Pi^0_1}$ subset of $\mathcal{B} \times \mathcal{B}$.*
A subset of $\mathcal{C}$ is $\mathbf{\Sigma^1_1}$ if and only if it is the projection of a $\mathbf{\Pi^0_2}$ subset of $\mathcal{C} \times \mathcal{C}$.

Our last fact is Exercise 37.2 in Kechris [Kec95] and is closely related to Fact 2:

Fact 4. *If $X \subseteq \mathcal{C} \subseteq \mathcal{B}$ then X is a $\mathbf{\Sigma^1_k}$ subset of $\mathcal{C}$ if and only if it is a $\mathbf{\Sigma^1_k}$ subset of $\mathcal{B}$.*

2.2. **Wadge reduction and determinacy.** For showing dichotomy theorems between $\mathcal{B}$ and $\mathcal{BC}$, we will assume some amount of Wadge determinacy. This section introduces the necessary terminology.

Definition 2.9. Let $X, Y \subseteq \mathcal{B}$. We say that X is Wadge reducible (or continuously reducible) to Y (written $X \leq_W Y$) if there is a continuous function $f\colon \mathcal{B} \to \mathcal{B}$ so that $x \in X$ if and only if $f(x) \in Y$ for every $x \in \mathcal{B}$.

For any two sets $X, Y \in \mathcal{B}$, the Wadge Game $WG(X, Y)$ is played by having player 1 and player 2 alternate playing elements of ω. Thus player 1 is playing a sequence of numbers giving an element $x \in \mathcal{B}$ and player 2 is playing a sequence of numbers giving an element $y \in \mathcal{B}$. Player 2 wins if and only if $x \in X \leftrightarrow y \in Y$ holds.

For a topological class Γ, Γ-Wadge determinacy says that the game $WG(X, Y)$ is determined for any $X, Y \in \Gamma$.

In one application, we will use projective Wadge determinacy. Note that Wadge determinacy only declares that Wadge games on projective sets are determined, which are a particular type of game, so this assumption is a priori weaker than the more familiar axiom of Projective Determincay (PD). In our other applications, we will use the weaker assumption of $\mathbf{\Delta^1_2}$-Wadge determinacy

Lemma 2.10 (Wadge's Lemma, [Wad83, Proposition II.A.1] or [Kec95, Theorem 21.14])**.** *For any $X, Y \in \mathcal{B}$, if $WG(X, Y)$ is determined, then either $X \leq_W Y$ or $Y \leq_W \mathcal{B} \smallsetminus X$.*

Proof. A winning strategy for player 2 witnesses that $X \leq_W Y$, as it gives a continuous function from $\mathcal{B}$ to itself and $x \in X \leftrightarrow f(x) \in Y$ because player 2 is winning. A winning strategy for player 1 gives a continuous function from $\mathcal{B}$ to $\mathcal{B}$ satisfying $y \in Y \not\leftrightarrow f(y) \in X$. In other words, $y \in Y \leftrightarrow f(y) \in \mathcal{B} \smallsetminus X$. □

Lemma 2.11. *Suppose that $\boldsymbol{\Sigma}^1_1 \subseteq \Gamma \subseteq \mathcal{P}(\mathcal{B})$. Then Γ-Wadge determinacy implies that any $X \in \Gamma \smallsetminus \boldsymbol{\Delta}^1_1$ is either $\boldsymbol{\Sigma}^1_1$-hard or $\boldsymbol{\Pi}^1_1$-hard under continuous reduction.*

Proof. Fix $X \in \Gamma \smallsetminus \boldsymbol{\Delta}^1_1$ and suppose that X is not $\boldsymbol{\Sigma}^1_1$. By Wadge's Lemma, for any $\boldsymbol{\Sigma}^1_1$ set Y, either X continuously reduces to Y or Y continuously reduces to the complement of X. The former is impossible since X is not $\boldsymbol{\Sigma}^1_1$, so it follows that the complement of X is $\boldsymbol{\Sigma}^1_1$-hard for continuous reduction, i.e., X is $\boldsymbol{\Pi}^1_1$-hard for continuous reduction. Similarly, if X is not $\boldsymbol{\Pi}^1_1$, then it is $\boldsymbol{\Sigma}^1_1$-hard for continuous reduction. □

3. Complexity Profiles

It will be important for us to know which subsets of each of the structures we analyze are Σ^*_n for various n. To be able to compare structures that have different domains, we consider the subsets of a particular Δ^0_2-relatively-generically-categorical domain to ensure that the collection of sets that are Σ^*_n on that domain is well-defined. The two domains that we will use throughout this paper are $\mathcal{C}$ and $\mathcal{B}$.

Definition 3.1. Fix a Δ^0_2-relatively-generically-categorical structure $\mathcal{A}$ in signature $\mathcal{L}$. Let $\mathcal{M}$ be an $\mathcal{L}'$-structure where $\mathcal{L}' \supseteq \mathcal{L} \cup \{U\}$ where U is a unary predicate symbol not in $\mathcal{L}$ and so that the relations in $\mathcal{L}$ restricted to the set defined by U give a copy of $\mathcal{A}$. The U $\mathcal{A}$*-complexity profile of* $\mathcal{M}$ is the sequence $(\mathcal{A}\Sigma^{\mathcal{M}}_i)_{i>1}$, where $\mathcal{A}\Sigma^{\mathcal{M}}_i$ is the set of sets $X \subseteq \mathcal{A}^k$ for some k (identifying $\mathcal{A}$ with $U^{\mathcal{M}}$) such that X is $\Sigma^*_i(\mathcal{M})$.

If both X and its complement are in $\mathcal{A}\Sigma^{\mathcal{M}}_i$, then we say X is in $\mathcal{A}\Delta^{\mathcal{M}}_i$.

Observation 3.2. *If $\mathcal{M}$ is an expansion of $\mathcal{A}$, i.e., U defines all of $\mathcal{M}$, then $\mathcal{A}\Sigma^{\mathcal{M}}_n = \Sigma^*_n(\mathcal{M})$.*

The point is that we will use this definition to allow us to talk about the complexity trace of $\mathcal{M}$ on $\mathcal{A}$ when $\mathcal{M}$ is not necessarily an expansion of $\mathcal{A}$.

The following proposition shows that the $\mathcal{A}$-complexity profile captures information inherent in the $\equiv^*_w$-degree of the structure. In particular, if $\mathcal{M}$ has two copies of $\mathcal{A}$ (i.e., two different unary predicates U^0, U^1), then the U^0 $\mathcal{A}$-complexity profile of $\mathcal{M}$ is the same as the U^1 $\mathcal{A}$-complexity profile of $\mathcal{M}$.

Proposition 3.3. *For $i \geq 2$, $\mathcal{A}\Sigma^{\mathcal{M}}_i$ is $\equiv^*_w$-invariant: Suppose that $\mathcal{M}^i$ is an $\mathcal{L}^i$-structure with $\mathcal{L}^i \supseteq \mathcal{L}$, and $U^i \in \mathcal{L}^i$ defines a copy of $\mathcal{A}$ in $\mathcal{M}^i$ for $i \in \{0,1\}$. Suppose further that $\mathcal{M}^0 \geq^*_w \mathcal{M}^1$. Then $\mathcal{A}\Sigma^{\mathcal{M}_0}_i \supseteq \mathcal{A}\Sigma^{\mathcal{M}_1}_i$ for each $i > 1$.*

Proof. This is a consequence of the assumption that $\mathcal{A}$ is Δ^0_2-relatively-generically-categorical: Let X be a degree that presents a copy of $\mathcal{M}_0$. Then X also presents a copy of $\mathcal{M}_1$. So X' computes an isomorphism f between $(U^0, \mathcal{L})$ and $(U^1, \mathcal{L})$. Thus, for any $n \geq 1$, for any subset Y of U^1 that is $X^{(n)}$-enumerable, $f[Y]$ is $X^{(n)}$-enumerable. Hence $\mathcal{A}\Sigma^{\mathcal{M}_0}_i \supseteq \mathcal{A}\Sigma^{\mathcal{M}_1}_i$. □

We now extend the definition to structures that do not have a copy of $\mathcal{A}$ directly definable, but are only $\geq^*_w \mathcal{A}$.

Definition 3.4. Let $\mathcal{M}$ be an $\mathcal{L}'$-structure that is $\geq^*_w$ the $\mathcal{L}$-structure $\mathcal{A}$ (we are not assuming that $\mathcal{L} \subseteq \mathcal{L}'$). The $\mathcal{A}$*-complexity profile of* $\mathcal{M}$ is the U $\mathcal{A}$-complexity profile of the $\mathcal{L} \cup \mathcal{L}' \cup \{U\}$-structure $\mathcal{M} \sqcup \mathcal{A}$, where U defines the set $\mathcal{A}$ and predicate symbols in $\mathcal{L}$ are interpreted to hold on tuples from $\mathcal{A}$ as in the structure $\mathcal{A}$.

Proposition 3.5. *If $\mathcal{X} \equiv_w^* \mathcal{Y} \geq_w^* \mathcal{A}$, then the $\mathcal{A}$-complexity profile of $\mathcal{X}$ equals the $\mathcal{A}$-complexity profile of $\mathcal{Y}$.*

Proof. Since $\mathcal{X} \sqcup \mathcal{A} \equiv_w^* \mathcal{X} \equiv_w^* \mathcal{Y} \equiv_w^* \mathcal{Y} \sqcup \mathcal{A}$, Proposition 3.3 shows that the U $\mathcal{A}$-complexity profiles of $\mathcal{X} \sqcup \mathcal{A}$ and of $\mathcal{Y} \sqcup \mathcal{A}$ are equal, and these are precisely the $\mathcal{A}$-complexity profiles of $\mathcal{X}$ and $\mathcal{Y}$. ☐

Definition 3.6. By Proposition 3.5, the $\mathcal{A}$-complexity profile of $\mathcal{M}$ only depends on the degree of $\mathcal{M}$. So, we define the *$\mathcal{A}$-complexity profile of a generic Muchnik degree* as the $\mathcal{A}$-complexity profile of any structure in that degree.

This invariance tells us that if $\mathcal{M}_1$, $\mathcal{M}_2$ have different $\mathcal{A}$-complexity profiles, then that yields they have different generic Muchnik degrees. Surprisingly, in many cases, it turns out that the complexity profiles are a sensitive enough measure to provide important dichotomies.

The following two lemmas will be useful in understanding the complexity profiles of expansions of familiar structures.

Lemma 3.7. *Let $n, m \geq 1$. Let $\mathcal{M}$ be any structure and $X \subseteq \mathcal{M}$ be $\Delta_n^*(\mathcal{M})$. Then for any $\mathcal{A} \leq_w^* \mathcal{M}$, $\mathcal{A}\Sigma_m^{(\mathcal{M},X)} \subseteq \mathcal{A}\Sigma_{n+m-1}^{\mathcal{M}}$.*

Proof. Let D be a set that computes a copy of $\mathcal{M}$ (in some generic extension that makes $\mathcal{M}$ and $\mathcal{A}$ countable.). Then $D^{(n-1)}$ presents $(\mathcal{M}, X)$, and hence $(D^{(n-1)})^{(m-1)}$ enumerates any set in $\mathcal{A}\Sigma_m^{(\mathcal{M},X)}$. So any such set is enumerable from $D^{(n+m-2)}$, i.e., it is in $\mathcal{A}\Sigma_{m+n-1}^{\mathcal{M}}$. ☐

The next lemma is essentially the same but utilizes the free uniformity over $\mathcal{C}$:

Lemma 3.8. *Let $\mathcal{M} \geq_w^* \mathcal{C}$. If $(X_i)_{i\in\omega}$ is a countable collection of $\Delta_n^*(\mathcal{M})$ sets. Then for any $\mathcal{A} \leq_w^* \mathcal{M}$, $\mathcal{A}\Sigma_m^{(\mathcal{M},(X_i)_{i\in\omega})} \subseteq \mathcal{A}\Sigma_{n+m-1}^{\mathcal{M}}$.*

Proof. Let D be a set that computes a copy of $\mathcal{M}$ (in some generic extension that makes $\mathcal{M}$, $\mathcal{A}$, and $\mathcal{C}$ countable. Then $D^{(n-1)}$ presents $(\mathcal{M}, (X_i)_{i\in\omega})$. To see this, we use a parameter from $\mathcal{C}$ to make the sets X_i uniformly $\Delta_n^*(\mathcal{M})$. Thus $(D^{(n-1)})^{(m-1)}$ enumerates any set in $\mathcal{A}\Delta_m^{(\mathcal{M},(X_i)_{i\in\omega})}$. So any such set is enumerable from $D^{(n+m-2)}$, i.e., is in $\mathcal{A}\Sigma_{n+m-1}^{\mathcal{M}}$. ☐

We now calculate the $\mathcal{C}$-complexity profiles and $\mathcal{B}$-complexity profiles of the familiar structures. Note that $\mathcal{C}$, $\mathcal{B}$, and $\mathcal{BC}$ have different $\mathcal{C}$-complexity profiles at each level.

Theorem 3.9.

1. *The $\mathcal{C}$-complexity profile of $\mathcal{C}$ is given by $\mathcal{C}\Sigma_2^{\mathcal{C}} = \mathbf{\Sigma_2^0}$, and $\mathcal{C}\Sigma_i^{\mathcal{C}} = \mathbf{\Sigma_{i-2}^1}$ for $i \geq 3$.*
2. *The $\mathcal{B}$-complexity profile of $\mathcal{B}$ is given by $\mathcal{B}\Sigma_i^{\mathcal{B}} = \mathbf{\Sigma_{i-1}^1}$.*
3. *The $\mathcal{C}$-complexity profile of $\mathcal{B}$ is given by $\mathcal{C}\Sigma_i^{\mathcal{B}} = \mathbf{\Sigma_{i-1}^1}$.*
4. *The $\mathcal{B}$-complexity profile of $\mathcal{BC}$ is given by $\mathcal{B}\Sigma_i^{\mathcal{BC}} = \mathbf{\Sigma_i^1}$.*
5. *The $\mathcal{C}$-complexity profile of $\mathcal{BC}$ is given by $\mathcal{C}\Sigma_i^{\mathcal{BC}} = \mathbf{\Sigma_i^1}$.*

Proof. (1) We first show that $\mathbf{\Sigma_2^0} \subseteq \mathcal{C}\Sigma_2^{\mathcal{C}}$: If X is $\mathbf{\Sigma_2^0}$, X is a countable union of closed sets. Every closed set is $\Pi_1^*(\mathcal{C})$ because it can be defined by a Π_1^c formula using as parameter an element of $\mathcal{C}$ that gives us the tree that defines the closed set. Furthermore, we can fix a parameter in $\mathcal{C}$ such that the countably many closed sets are uniformly $\Pi^*_1(\mathcal{C})$, so their union is $\Sigma^*_2(\mathcal{C})$.

Next we show that $\mathcal{C}\Sigma^{\mathcal{C}}_2 \subseteq \mathbf{\Sigma^0_2}$. Let X be in $\mathcal{C}\Sigma^{\mathcal{C}}_2$. By Theorem 2.2 and the fact that quantifier-free formulas in $\mathcal{C}$ only mention finitely many bits so tuples can be coded as joins by re-indexing which bits are mentioned, X is described by an $\mathcal{L}^c_{\omega_1,\omega}$-formula of the form: $\bigvee_{i\in\omega} \exists f \in \mathcal{C} \bigwedge_{j\in\omega} \forall g \in \mathcal{C}\ R_{i,j}(\bar{x}, f, g, h)$, where $\{R_{i,j}\}_{i,j<\omega}$ is a computable list of quantifier free formulas and h is a fixed parameter in $\mathcal{C}$. Since each $R_{i,j}$ defines a clopen subset of $\mathcal{C}$, each $\bigwedge_{j\in\omega} \forall g \in \mathcal{C}\ R_{i,j}(f, g)$ defines a closed, thus compact, subset of $\mathcal{C}$. This means that each $\exists f \in \mathcal{C}$ can be replaced by $\exists \sigma \in 2^{<\omega}$. Therefore, we have X defined by a countable union of closed sets, i.e., $X \in \mathbf{\Sigma^0_2}$.

We now show that $\mathcal{C}\Sigma^{\mathcal{C}}_i \subseteq \mathbf{\Sigma^1_{i-2}}$, for $i \geq 3$, by induction on i. Let $X \in \mathcal{C}\Sigma^{\mathcal{C}}_i$. By Theorem 2.2, X can be defined by a formula of the form $\bigvee_{j\in\omega} \exists f \in \mathcal{C}\ \Psi_j(f)$, where Ψ_j is a Π^c_{i-1}-formula. Thus each Ψ_j defines a $\Pi^*_{i-1}(\mathcal{C})$ set. By the inductive hypothesis, Ψ_j defines a $\mathbf{\Pi^0_2}$ set if $i = 3$ and a $\mathbf{\Pi^1_{i-3}}$ set otherwise. Thus $X \in \mathbf{\Sigma^1_{i-2}}$.

Finally, we show that $\mathbf{\Sigma^1_{i-2}} \subseteq \mathcal{C}\Sigma^{\mathcal{C}}_i$, for $i \geq 3$. Let X be $\mathbf{\Sigma^1_{i-2}}$. Then for some R that is either $\mathbf{\Pi^0_2}$ or $\mathbf{\Sigma^0_2}$, X is the set of elements $x \in \mathcal{C}$ so that $(\exists s_1 \in \mathcal{C})(\forall s_2 \in \mathcal{C}) \cdots (Q s_{i-2} \in \mathcal{C})\ R(\bar{x}, s_1, \ldots, s_{i-2})$. Since we know that $\mathbf{\Sigma^0_2}$ sets are all $\Sigma^*_2(\mathcal{C})$, and quantifiers over $\mathcal{C}$ are countable quantifiers in the large model of set theory, counting quantifiers shows that this is $\Sigma^*_{i-2}(\Delta^*_3)(\mathcal{C}) = \Sigma^*_i(\mathcal{C})$.

(2) First, we show that $\mathbf{\Sigma^1_j} \subseteq \mathcal{B}\Sigma^{\mathcal{B}}_{j+1}$, for all $j \geq 1$. Let $X \subseteq \mathcal{B}$ be $\mathbf{\Sigma^1_j}$. Then X is the set of $\bar{x} \in \mathcal{B}^k$ so that $(\exists s_1 \in \mathcal{B}) \cdots (Q s_j \in \mathcal{B})\ R(\bar{x}, s_1, \ldots, s_j)$, where R is either a closed or open subset of $\mathcal{B}$. Every closed or open subset of $\mathcal{B}$ is $\Delta^*_2(\mathcal{B})$—it can be defined by a Π^c_1 or Σ^c_1 formula that uses as a parameter the set in $\omega^{<\omega}$ that defines the open or closed set. Thus $X \in \Sigma^*_{j+1}(\mathcal{B})$.

Now we want to show that $\mathcal{B}\Sigma^{\mathcal{B}}_{j+1} \subseteq \mathbf{\Sigma^1_j}$, for $j \geq 1$. First, let $j = 1$ and let X be in $\mathcal{B}\Sigma^{\mathcal{B}}_2$. By Theorem 2.2, we know that X is defined by a formula of the form $\bigvee_{i\in\omega} \exists f \in \mathcal{B} \bigwedge_{j\in\omega} \forall g \in \mathcal{B}\ R_{i,j}(\bar{x}, f, g)$, where each $R_{i,j}$ is first-order and quantifier free. But this means that each $R_{i,j}$ defines a clopen set, so $\bigwedge_{j\in\omega} \forall g \in \mathcal{B}\ R_{i,j}(\bar{x}, f, g)$ is closed and X is $\mathbf{\Sigma^1_1}$. Using this as the base case, a simple induction using Theorem 2.2 shows that $\mathcal{B}\Sigma^{\mathcal{B}}_{j+1} \subseteq \mathbf{\Sigma^1_j}$ for all $j > 1$.

(3) Next we show that a set X is in $\mathcal{C}\Sigma^{\mathcal{B}}_i$ if and only if X, when considered as a subset of $\mathcal{B}^k$ via the inclusions $X \subseteq \mathcal{C}^k \subseteq \mathcal{B}^k$ is in $\mathcal{B}\Sigma^{\mathcal{B}}_i$. Suppose X is in $\mathcal{C}\Sigma^{\mathcal{B}}_i$. Then every copy of the structure $\mathcal{B} \sqcup \mathcal{C}$, makes X a Σ^0_i subset of $\mathcal{C}^k$ (the second part of the structure). If A presents the structure then A' can identify $\mathcal{C}$ as a subset of $\mathcal{B}$. Further, $\mathcal{A}'$ can compute the embedding ι of the second component of $\mathcal{B} \sqcup \mathcal{C}$ into $\mathcal{B}$. Thus $\iota(X)$ is also Σ_i in A, so $\iota(X)$ is in $\mathcal{B}\Sigma^{\mathcal{B}}_i$. For the reverse implication, use the same argument noting that ι^{-1} is computable in A'. By Fact 4 and the characterization of $\mathcal{B}\Sigma^{\mathcal{B}}_i$, a subset of $\mathcal{C}$ is in $\mathcal{B}\Sigma^{\mathcal{B}}_i$ if and only if it is $\mathbf{\Sigma^1_{i-1}}$ as a subset of $\mathcal{C}$.

(4) We now show that $\mathbf{\Sigma^1_i} \subseteq \mathcal{B}\Sigma^{\mathcal{BC}}_i$. Let $X \subseteq \mathcal{B}^k$ be $\mathbf{\Sigma^1_i}$. Then X is the set of $\bar{x} \in \mathcal{B}^k$ so that $(\exists s_1 \in \mathcal{B}) \cdots (Q s_i \in \mathcal{B})\ R(\bar{x}, s_1, \ldots, s_i)$, where R is either open or closed. Since $\mathcal{BC}$ is Borel complete, we have that $\mathcal{B}\Sigma^{\mathcal{BC}}_i \supseteq \mathcal{B}\Sigma^{(\mathcal{B},R)}_i$ because $\mathcal{BC} \geq^*_w (\mathcal{B}, R)$. In this structure, R is quantifier-free, so X is clearly defined by a Σ^c_i-formula, meaning that $X \in \mathcal{B}\Sigma^{(\mathcal{B},R)}_i \subseteq \mathcal{B}\Sigma^{\mathcal{BC}}_i$.

Now we want to prove that $\mathcal{B}\Sigma_i^{\mathcal{BC}} \subseteq \mathbf{\Sigma^1_i}$. Since $\mathcal{BC} = (\mathcal{B}, \oplus, ')$ and $\oplus$ and $'$ are both $\Delta_2^*(\mathcal{B})$, Lemma 3.7 shows that $\mathcal{B}\Sigma_k^{\mathcal{BC}} \subseteq \Sigma_{k+1}^{\mathcal{B}} = \mathbf{\Sigma^1_k}$.

(5) Finally, we get that $\mathcal{C}\Sigma_k^{\mathcal{BC}} = \mathbf{\Sigma^1_k}$ from the fact that $\mathcal{B}\Sigma^{\mathcal{BC}} = \mathbf{\Sigma^1_k}$ just as we did when computing $\mathcal{C}\Sigma_k^{\mathcal{B}}$. □

Note that these calculations prove that $\mathcal{C} <_w^* \mathcal{B}$, or more precisely that $\mathcal{B} \not\leq_w^* \mathcal{C}$. This separation has already been shown by Igusa and Knight [IK16] and by Downey, Greenberg, and Miller [DGM16], but we consider our new proof to be the most elucidating.[1] Similarly, the previous theorem shows that $\mathcal{BC} \not\leq_w^* \mathcal{B}$, a result proved by Andrews, Knight, Kuyper, Miller, and Soskova [AKK+] using a quite different method.

The following is a sample use of complexity profiles.

Lemma 3.10. *Let $\mathcal{M} \geq_w^* \mathcal{C}$ and suppose that the set of finite elements in $\mathcal{C}$ is in $\mathcal{C}\Delta_2^{\mathcal{M}}$. Then $\mathcal{M} \geq_w^* \mathcal{B}$.*

Proof. We will give an enumeration of $\mathcal{B}$ with repetitions from any presentation of $\mathcal{M}$. Fix a presentation $\mathcal{A}$ of $\mathcal{M}$. For any non-finite element x of Cantor space, we will enumerate the distance function of x. That is, $f_x(n)$ is the number of 0s between the nth 1 and the $(n+1)$st 1 in x.

To construct our enumeration of $\mathcal{B}$ with repetitions, we use that we have a presentation of $\mathcal{C}$ computable from $\mathcal{A}$ along with a Δ_2^0-approximation from $\mathcal{A}$ of the set of finite elements in our enumeration of $\mathcal{C}$. At each stage that declares (via the Δ_2^0-approximation) that x is non-finite, we begin enumerating f_x into our enumeration of $\mathcal{B}$. If the Δ_2^0-approximation changes at a later stage to say that x is finite, we will extend whatever we have enumerated by all 0's. Since every $f \in \mathcal{B}$ is the distance function of a non-finite element in $\mathcal{C}$, we enumerate every member of $\mathcal{B}$ (once the Δ_2^0-approximation settles for this x). By Theorem 2.6, this suffices to show that we can compute a presentation of $\mathcal{B}$ from $\mathcal{A}$. □

The final result of this section is a particularly useful application of Lemma 3.8 that we isolate and highlight here.

Lemma 3.11. *If $(X_i)_{i\in\omega}$ is a countable collection of $\mathbf{\Delta^0_2}$ subsets of $\mathcal{C}$, then $\mathcal{C}\Delta_2^{(\mathcal{C},(X_i)_{i\in\omega})} \subseteq \mathbf{\Delta^1_1}$.*

Proof. By Theorem 3.9, each X_i is $\Delta_2^*(\mathcal{C})$. By Lemma 3.8, $\Delta_2^*(\mathcal{C}, (X_i)_{i\in\omega}) \subseteq \Delta_3^*(\mathcal{C})$. Now using Theorem 3.9 again, $\Delta_3^*(\mathcal{C}) = \mathbf{\Delta^1_1}$. □

4. Expansions of $\mathcal{C}$ or $\mathcal{B}$ by unary relations that do not increase the Generic Muchnik degree

Theorem 4.1. *Let $\mathcal{A}$ be either $\mathcal{C}$ or $\mathcal{B}$. Let X be a relation of any arity and suppose that $(\mathcal{A}, X) \leq_w^* \mathcal{A}$. Then $X \in \Delta_2^*(\mathcal{A})$.*

Proof. Since $(\mathcal{A}, X) \equiv_w^* \mathcal{A}$, it must have the same $\mathcal{A}$-complexity profile as $\mathcal{A}$ for $n \geq 2$. In particular, $X \in \mathcal{A}\Delta_2^{(\mathcal{A},X)} = \mathcal{A}\Delta_2^{\mathcal{A}} = \Delta_2^*(\mathcal{A})$. □

[1] However, it should be noted that the ideas of [DGM16] are extended in our forthcoming paper [AMSS] to produce a structure $\mathcal{M}$ such that $\mathcal{C} <_w^* \mathcal{M}$ ($<_w^* \mathcal{B}$), but $\mathcal{C}$ and $\mathcal{M}$ have the same complexity profile.

For $\mathcal{C}$, the result below is due to Greenberg, Igusa, Turetsky, and Westrick (personal communication). They had the idea of coding countably many unary relations into the graph of a function and looking at the complexity of that graph. Although they did not state it in terms of $\Delta^*_2(\mathcal{C})$, the condition they used is easily seen to be equivalent to being a $\boldsymbol{\Delta^0_2}$ subset of $\mathcal{C}$.

Theorem 4.2. *Let $\mathcal{A}$ be either $\mathcal{C}$ or $\mathcal{B}$. Let $(S_i)_{i\in\omega}$ be a countable sequence of unary relations. Let $F\colon \mathcal{A}\to\mathcal{A}$ be given by $F(x)=y$ where $y(n)=1$ if $x\in S_n$ and $y(n)=0$ if $x\notin S_n$ (so F actually has range in $\mathcal{C}\subseteq\mathcal{A}$). Then $(\mathcal{A},(S_i)_{i\in\omega})\equiv^*_w\mathcal{A}$ if and only if the graph of F is $\Delta^*_2(\mathcal{A})$.*

Proof. Suppose that $(\mathcal{A},(S_i)_{i\in\omega})\equiv^*_w\mathcal{A}$. The graph of F is $\Delta^*_2(\mathcal{A},(S_i)_{i\in\omega})$ and $\Delta^*_2(\mathcal{A},(S_i)_{i\in\omega})=\Delta^*_2(\mathcal{A})$ by Proposition 3.3, so the graph of F is $\Delta^*_2(\mathcal{A})$.

Let $\mathcal{M}=(\mathcal{A},(S_i)_{i\in\omega})$. Next suppose that the graph of F is $\Delta^*_2(\mathcal{A})$. We build a presentation of $\mathcal{M}$ given any presentation of $\mathcal{A}$. Let A be a presentation of $\mathcal{A}$. We will build a presentation of $\mathcal{M}$ allowing repetitions; this suffices by Theorem 2.6.

We first fix (nonuniformly) in our copy of $\mathcal{A}$ a parameter that encodes the information we will need for the construction. This parameter specifies, for each basic open set $[\sigma]$, and each finite boolean combination R of the predicates S_i, whether there exists a member of $R\cap[\sigma]$. Furthermore, for each basic open set $[\sigma]$ such that $R\cap[\sigma]\neq\emptyset$, the parameter specifies a particular member of $R\cap[\sigma]$ (specifically, it gives us the digits of a member, not the index in any particular enumeration of $\mathcal{A}$). Finally, for each of the members $r\in\mathcal{A}$ that are specified as above, the parameter tells us $\{n\mid r\in S_n\}$. Such a parameter can be fixed in the ground model and can thus be found in any enumeration of $\mathcal{A}$.

Using this parameter, we can speedup the $\Delta^0_2(A)$-approximation of the graph of F to ensure that at every stage s and every $a\in\mathcal{A}$, if the approximations to the graph of F says that $a\in R$ where R is a boolean combination of $\{S_i\mid i\le s\}$, then $R\cap[a\upharpoonright s]\neq\emptyset$. Assume the approximation of the graph of F has this property.

We now give a presentation $\mathcal{N}$ of a copy of $\mathcal{M}$ with repetitions as follows: At every stage t, we will have determined the first t digits of some elements, committed to all the digits of some other elements, and determined whether or not each S_i for $i\le t$ holds on each element. At each stage, we will have two types of elements in $\mathcal{N}$: copy elements and trash elements. If x is a copy element, it has a parameter z_x, which is the index of an element in $\mathcal{A}$ that it is copying.

At stage $t+1$, we check for each existing copy element x in $\mathcal{N}$ whether the $\Delta^0_2(A)$-approximation to $F(z_x)$ changed at stage t. If so, then we make x a trash element. We use our fixed parameter from $\mathcal{A}$ to give us (the digits of) a particular element y of $\mathcal{A}$ so that $x\upharpoonright t=y\upharpoonright t$ and $y\in R$, where R is the boolean combination of S_i for $i\le t$ (i.e., the commitments we have already made for x). By the setup above, this necessarily exists. At this point we are completely committed to making the digits of x in $\mathcal{N}$ equal to those in y. Furthermore, we know $\{n\mid x\in S_n\}$ for any trash element x (from the parameter). At every future stage s, we determine $S_s(x)$ accordingly.

For each copy element where the $\Delta^0_2(A)$-approximation to $F(z_x)$ has not changed, we let the $(t+1)$st digit of x agree with the $(t+1)$st digit of z_x, and we let $S_{t+1}(x)$ hold if and only if $F(z_x)(t+1)=1$. Lastly, we create $t+1$ new elements in $\mathcal{N}$, call them copy elements, assign their parameters z_x to be the first $t+1$ elements in $\mathcal{A}$, enumerate their first $t+1$ digits, and let each $S_i(x)$ hold if and only if the stage $t+1$ approximation of $F(z_x)(n) = 1$.

For every $a \in \mathcal{A}$, let t be a stage large enough that the $\Delta^0_2(A)$-approximations to $F(a)$ has settled before stage t and also a is among the first t elements in $\mathcal{A}$. Then at stage t we create a copy element with $a = z_x$, and since the $\Delta^0_2(A)$-approximation for $F(a)$ never changes after stage t, this element copies a. It follows that we have presented $\mathcal{M}$ with repetitions. □

Corollary 4.3. *Let $\mathcal{A}$ be either $\mathcal{C}$ or $\mathcal{B}$ and let $S_1, \ldots, S_n$ be subsets of $\mathcal{A}$. Then we have $(\mathcal{A}, S_1, \ldots, S_n) \equiv^*_w \mathcal{A}$ if and only if each of the sets $S_1, \ldots, S_n$ are $\Delta^*_2(\mathcal{A})$.*

Proof. If $(\mathcal{A}, S_1, \ldots, S_n) \equiv^*_w \mathcal{A}$ then each S_i is $\Delta^*_2(A)$ by 4.1.

If each S_i is $\Delta^*_2(\mathcal{A})$, then consider the structure $\mathcal{M} = (\mathcal{A}, S_1, \ldots, S_n, S_1, S_1, S_1, \ldots)$. Since each S_i is $\Delta^*_2(\mathcal{A})$, the graph of F from Theorem 4.2 is $\Delta^*_2(\mathcal{A})$. Thus $\mathcal{A} \leq^*_w (\mathcal{A}, S_1, \ldots, S_n) \leq^*_w \mathcal{M} \leq^*_w \mathcal{A}$, and they are all equivalent. □

5. Taming expansions of $\mathcal{C}$ or $\mathcal{B}$ by closed relations

When it comes to expanding $\mathcal{C}$ or $\mathcal{B}$ by relations of arity > 1, since different elements interact with each other the analysis for unary relations breaks down immediately (including Theorem 2.6). We are, however, able to analyze expansions by closed relations of arbitrary arity. In this section, we figure out which degrees are above a given expansion of $\mathcal{C}$ or $\mathcal{B}$ by closed predicates.

In the following definition, we collect the crucial information needed to build a presentation $\mathcal{M}$ of an expansion of $\mathcal{C}$ or $\mathcal{B}$ by closed predicates. In particular, as we build $\mathcal{M}$ we will only ever commit to positive information regarding the atomic formulas. That is, we give an enumeration of the tuples in the closed predicates. Since the complements of the predicates are open, these complements are automatically enumerable, so this suffices to give a computable presentation of the structure. To enumerate positive information about the atomic formulas, we need to know the positive existential type of whatever tuple we have already built. This lets us determine which configurations we can safely move to. We call the predicate that tells us this the *Safe Move Analysis*.

Definition 5.1. A formula is *positive existential, also written $\exists^+_1$,* if it is in the closure of atomic formulas by the operations $\wedge, \vee, \exists v$.

Let $\mathcal{A} \in \{\mathcal{C}, \mathcal{B}\}$ and let $\mathcal{M}$ be an expansion of $\mathcal{A}$. We define $\mathrm{SMA}_k(\mathcal{M}) \subseteq \mathcal{A}^{k+1}$ to be the relation defined by: $\mathrm{SMA}_k(\bar{x}, y)$ holds if $|\bar{x}| = k$ and the positive existential type of $\bar{x}$ in $\mathcal{M}$ equals y. That is, y defines the characteristic function of the positive existential type of $\bar{x}$, identifying formulas with numbers via their Gödel codes.

We define $\mathrm{SMA}(\mathcal{M})$ to be a unary relation on $\mathcal{A}$ given by $\mathrm{SMA}(z)$ holds if $z = n \frown (x_1 \oplus \cdots \oplus x_n \oplus y)$ and $\mathrm{SMA}_n(\bar{x}, y)$.

Lemma 5.2. *Let $\mathcal{M}$ be an expansion of $\mathcal{C}$ or $\mathcal{B}$ by countably many closed relations. Then $SMA(\mathcal{M}) \in \Delta^*_2(\mathcal{M})$.*

Proof. We first observe that since $\oplus$ is $\Delta^*_2(\mathcal{M})$, it suffices to show that the relations SMA_k are uniformly $\Delta^*_2(\mathcal{M})$.

Now let us check that SMA_k is $\Sigma^*_2(\mathcal{M})$. For this, we claim that $\mathrm{SMA}_k(\bar{x}, y)$ holds if and only if there exists a function $f(\varphi, n)$ (coded by an element of $\mathcal{M}$) such that for each formula $\varphi(\bar{v}) := \exists \bar{z}\, \theta(\bar{v}, \bar{z})$, where $|\bar{v}| = k$ and θ is a positive boolean combination of atomics, $f(\varphi, -)$ is (the join of the components of) a witness $\bar{z}$ to $\varphi(\bar{v})$, if there is such a witness. In other words, there is a function f such that for all φ of the above form,

- $y(\ulcorner \varphi(\bar{x}) \urcorner) = 1$ and $(\forall s)\, \theta \cap [\,(\bar{x} \upharpoonright s, f(\varphi, -) \upharpoonright s)\,] \neq \emptyset$, or
- $y(\ulcorner \varphi(\bar{x}) \urcorner) = 1$ and $\mathcal{M} \models \neg\varphi(\bar{x})$.

Since the atomic relations are all closed and θ is formed by conjunctions and disjunctions of these, it follows that θ defines a closed relation. Thus the first line suffices to say that $f(\varphi, -)$ gives a function providing the digits of a witness of $\varphi(\bar{x})$. The first line is $\Pi^*_1(\mathcal{M})$ using a parameter that specifies for each positive quantifier free formula the collection of σ so that $\theta \cap [\sigma] = \emptyset$, and the second line is also $\Pi^*_1(\mathcal{M})$, since φ is existential. Thus SMA is $\Sigma^*_2(\mathcal{M})$. Since SMA is a function, it follows that SMA is $\Delta^*_2(\mathcal{M})$ ☐

The following theorem is our characterization of the degrees that bound a closed expansion of $\mathcal{C}$ or $\mathcal{B}$.

Theorem 5.3. *Let $\mathcal{A}$ be $\mathcal{C}$ or $\mathcal{B}$ and let $\mathcal{H} \geq^*_w \mathcal{A}$. Let $\mathcal{M}$ be an expansion of $\mathcal{A}$ by countably many closed relations. Then $\mathcal{M} \leq^*_w \mathcal{H}$ if and only if SMA$(\mathcal{M}) \in \mathcal{A}\Delta^{\mathcal{H}}_2$.*

Proof. Suppose that $\mathcal{M} \leq^*_w \mathcal{H}$. Then $\Delta^*_2(\mathcal{M}) \subseteq \mathcal{A}\Delta^{\mathcal{H}}_2$ by Observation 3.2 and Proposition 3.3. The previous lemma shows that $\mathrm{SMA}(\mathcal{M}) \in \Delta^*_2(\mathcal{M})$.

Now we suppose that $\mathrm{SMA}(\mathcal{M}) \in \mathcal{A}\Delta^{\mathcal{H}}_2$ and prove that $\mathcal{M} \leq^*_w \mathcal{H}$. We first prove a lemma about computably extending types. The purpose of this lemma is to help us handle the fall-out from our construction while we have the wrong guess at the element y such that $\mathrm{SMA}(\bar{x}, y)$.

Definition 5.4. In what follows we will consider positive existential (written $\exists^+_1$) formulas. These are formulas of the form $\exists \bar{z}\, \theta(\bar{x}, \bar{z})$ where θ is a positive boolean combination of atomic formulas. That is, θ is built from atomic formulas using only $\wedge$ and $\vee$.

For any tuple $\bar{a}$ in a model, $\mathrm{tp}_{\exists^+_1}(\bar{a})$ is the set of positive existential formulas true of $\bar{a}$. A positive existential type is a set $\mathrm{tp}_{\exists^+_1}(\bar{a})$ for some tuple $\bar{a}$ in some structure. A positive existential type $p(\bar{x})$ is realized in a structure M if there is a tuple $\bar{a} \in M$ so that $p = \mathrm{tp}_{\exists^+_1}(\bar{a})$.

We will also refer to a partial positive existential type $r(\bar{x})$ which is simply a set of $\exists^+_1$ formulas. We say that r is realized in a structure M if there is a tuple $\bar{b}$ so that for every formula $\varphi \in r(\bar{x})$, $M \models \varphi(\bar{b})$. Note that realizing a partial positive existential type is a far weaker notion than realizing a positive existential type.

Lemma 5.5. *For every $\bar{a} \in \mathcal{M}$ with $p(\bar{a}) = \mathrm{tp}_{\exists^+_1}(\bar{a})$ and $\psi(\bar{a}, b)$ a $\exists^+_1$-formula so that $\exists y\, \psi(\bar{a}, y) \in p$, there exists an $\exists^+_1$-type $q(\bar{a}, b)$ containing $p \cup \{\psi(\bar{a}, b)\}$ so that q is realized in $\mathcal{M}$. Further, q can be computed uniformly from p and ψ.*

Proof. We construct an auxiliary partial $\exists^+_1$ ω-type $r(\bar{a}, b, (c_i)_{i \in \omega})$. The type q will be the restriction of r to the variables $\bar{a}, b$.

We fix an enumeration of all $\exists^+_1$-formulas $\rho_i(\bar{a}, b)$. We write $\rho_i(\bar{a}, b) \equiv \exists \bar{z}\, \theta_i(\bar{a}, b, \bar{z})$, where θ_i is a positive boolean combination of atomic formulas. At stage s, let $\Gamma_s(\bar{a}, b, \bar{y})$ be the set of formulas that we have decided by stage s are in r. This will be finitely many $\exists_1^+$-formulas in addition to p. At every stage, we maintain the inductive hypothesis that $\exists b \exists \bar{y}\ \Gamma_s(\bar{a}, b, \bar{y})$ is in p, where $\Gamma_s(\bar{a}, b, \bar{y})$ is shorthand for the conjunction of formulas it comprises. We call this consistency with p.

We construct r in stages, beginning with $\Gamma_0 = \{\psi(\bar{a}, b)\}$, alternately performing E-steps and D-steps:

The kth E-step: We want to add ρ_k to r if possible. We check whether $\exists b \exists \bar{y}\ (\rho_k(\bar{a}, b) \wedge \Gamma_s(a, b, \bar{y}))$ is in p. If so, we add ρ_k to r. Further, we choose symbols $\bar{c}$ from the c_i's that have not yet been used, and we add $\theta_k(\bar{a}, b, \bar{c})$ to r. Then $\Gamma_{s+1} = \Gamma_s \cup \{\rho_k, \theta_k(\bar{a}, b, \bar{c})\}$.

The D-steps: For every $x \in \{b\} \cup \{c_i \mid i \in \omega\}$, and each $n \in \omega$, we want to pick an m and add to r the instance of the unary predicate in $\mathcal{A}$ that expresses that the nth digit of x is m (notation: $x(n) = m$). One such requirement handles a single x and a single n. We search for an m such that $\exists b \exists \bar{y}\ (\Gamma_s(a, b, \bar{y}) \wedge x(n) = m)$ is in p. Claim 1 below will show that some m will be found. Once such an m is found, we let Γ_{s+1} be $\Gamma_s \cup \{x(n) = m\}$.

Claim 1. *Every D-step will find an m.*

Proof. Since Γ_s is a finite set of formulas, $p = \mathrm{tp}_{\exists_1^+}(\bar{a})$, and $\exists b \exists \bar{y}\ \Gamma_s(\bar{a}, b, \bar{y})$ is in p, there must be a tuple $b'\bar{c}'$ realizing Γ_s. So, for any $x \in \{b, \bar{c}\}$, let $m = x'(n)$. The tuple $b', \bar{c}'$ shows that $\Gamma_s \cup \{x(n) = m\}$ is consistent with p. □

This describes the construction of the partial $\exists_1^+$ ω-type $r(\bar{a}, b, (c_i)_{i \in \omega})$. Let $q(\bar{a}, b)$ be the restriction of r to the formulas using only the variables $\bar{a}, b$.

Claim 2. *r is computable from p. Thus q is computable from p.*

Proof. For any formula of the form $x(n) = m$ for x among the variables of r, whether $x(n) = m$ is in r is determined at some finite D-stage.

For any $\exists_1^+$-formula $\rho_k(\bar{a}, b)$, either ρ_k is put into r at the kth E-step or we see at the kth E-step that it would be inconsistent with p to do so. Since we maintain at every stage consistency with p, we know that at no later stage will we put ρ_k into r. Thus, we can determine at this finite stage that $\rho_k \notin r$.

The only other formulas that we put in r are positive quantifier-free in the variables $\bar{a}b$ and a tuple $\bar{c}$ of variable that were unused at that point. So, to determine if $\theta(\bar{a}, b, \bar{c})$ is in r, we run the construction until we see that an element of $\bar{c}$ is not new anymore. Then just check if this θ is in Γ_s at that stage. (Note that if θ does not use any of the c_i's, then it is ρ_k for some k, so it is handled.) □

Claim 3. *q is existentially isolated over p. That is, for any $\exists_1^+$-formula $\rho(\bar{a}, b)$, either $\rho(\bar{a}, b) \in q$ or there is some other $\exists_1^+$-formula $\xi(\bar{a}, b) \in q$ so that $\exists y\ (\rho(\bar{a}, y) \wedge \xi(\bar{a}, y))$ is not in p.*

Proof. Let $\rho = \rho_k$ and consider the kth E-stage. We either add ρ to q in which case $\rho(\bar{a}, b) \in q$ or $\exists b \exists \bar{y}\ (\rho(\bar{a}, b) \wedge \Gamma_s(\bar{a}, b, \bar{y}))$ is not in p. So, $\xi = \exists \bar{y}\ \Gamma_s(\bar{a}, b, \bar{y})$ is as needed. □

Claim 4. *r is realized in $\mathcal{M}$. Hence q is realized in $\mathcal{M}$.*

Proof. For each $x \in b \cup \{c_i \mid i \in \omega\}$, let x' be the element in $\mathcal{A}$ with the same digits as x. This must exist because the sequence of digits of x are determined in r (via D-steps) and are thus computable from p; so it forms a set in the ground model, and thus is an element in $\mathcal{A}$. We claim that this tuple realizes all of r. For each formula of the form $x(n) = m$, this is by definition. We need to verify the $\exists_1^+$-formulas $\rho_k(\bar{a}, b)$ that are in q and the positive quantifier-free formulas holding on a tuple $\bar{a}b\bar{c}$. Since we place these quantifier-free formulas to ensure the realization of the positive existential formulas, we need only verify the latter.

Since θ is positive quantifier-free, θ defines a closed condition. Since every finite fragment of r is consistent with p, we have that for every n, p contains the sentence

$$\exists b \exists \bar{y} \ \left(b \restriction n = b' \restriction n \wedge \bar{y} \restriction n = \bar{c}' \restriction n \wedge \theta(\bar{a}, b, \bar{y})\right).$$

By closedness of θ, $\mathcal{A} \models \theta(\bar{a}, b', \bar{c}')$. Thus $(\bar{a}, b', (c_i')_{i\in\omega})$ satisfies r. □

□Lemma 5.5

We now produce a copy $\mathcal{N}$ of $\mathcal{M}$ computably from a given a copy $\mathfrak{H}$ of $\mathcal{H}$. Note that we automatically have access to a copy $\mathfrak{A}$ of $\mathcal{A}$; we will also build a $\Delta_2^0(\mathfrak{H})$ bijection $f \colon \mathcal{N} \to \mathfrak{A}$ by approximations. To build $\mathcal{N}$, we give an enumeration of the elements of $\mathcal{A}$ (digitwise) and we have to determine where each closed relation R holds. To do this, we will enumerate the tuples on which we have R hold. This will suffice since the negation of R is open, so we can enumerate where $\neg R$ holds by having determined enough digits of the elements and using a parameter encoding the open set that is $\neg R$. Fix nonuniformly an element $a_0 \in \mathfrak{A}$ that codes the $\exists_1^+$-type of the emptyset. That is, $\mathrm{SMA}_0(\emptyset, a_0)$. At every stage of the construction, we will have a finite partial function f_s sending a portion $\bar{b}$ of $\mathcal{N}$ to a tuple $f_s(\bar{b}) \in \mathfrak{A}$. We may have committed to information about the quantifier-free type of a larger tuple $\bar{b}\bar{c}$. At every stage s, and every tuple $\bar{a} \in \mathfrak{A}$, we have via our $\Delta_2^0(\mathfrak{H})$-approximation to SMA, a current guess at the element c so that $\mathrm{SMA}(\bar{a}, c)$. This will be called $\mathrm{SMA}_s(\bar{a})$. We define $\mathrm{SMA}_s(\emptyset) = a_0$ for all s.

There are two requirements we must satisfy to ensure that $\lim_s f_s$ is a function from $\mathcal{N}$ to $\mathfrak{A}$ and is a surjection (injectivity will be built into the construction):

R_a : For $a \in \mathfrak{A}$, there is a b so that $f_s(b) = a$ for all sufficiently large s.
S_b : For $b \in \mathcal{N}$, there is an $a \in \mathfrak{A}$ so that $f_s(b) = a$ for all sufficiently large s.

At every stage, we maintain the consistency requirement: For every $\bar{b}_0 \subseteq \bar{b}$, the positive-quantifier-free type of $\bar{b}_0, (\bar{b} \smallsetminus \bar{b}_0)\bar{c}$ is consistent with $\mathrm{SMA}_s(f(\bar{b}_0))$. When the approximations SMA_s change, we immediately undefine f_s on some tuple to maintain the consistency requirement. Note that since we have $\mathrm{SMA}_s(\emptyset) = a_0$ is correct, we will always at least be maintaining consistency with the positive existential theory of $\mathcal{M}$.

R_a-Requirements are handled by choosing the first element c of $\bar{c}$ for which adding $f(c) = a$ maintains the consistency requirement. If no such c exists, we create a new element d and define $f(d) = a$. The only addition to the consistency requirement for the function f defined on the tuple $\bar{b}d$ over the consistency requirement of the function f on the tuple $\bar{b}$ is for the full tuple. Since d is new, all this requirement says about d is that it is not equal to any element of $\bar{c}$. This is necessarily consistent otherwise we could have extended f by making some $f(c) = a$.

Requirements of the second form will critically use Lemma 5.5. In order to find an image for b, we take the least element $a \in \mathfrak{A}$ and $t > s$ so that $\mathrm{SMA}_t(f(\bar{b})a) \restriction t = q \restriction t$ where q is as constructed in Lemma 5.5 from $p = \mathrm{tp}_{\exists_1^+}(f(\bar{b}))$ and ψ the formula asserting the existence of the tuple $\bar{c}$ with all of our previous commitments (note that since we only commit to positive occurrences of our closed relations, this is a positive existential formula). Further, at every later stage, we add our own consistency requirement: Namely, that whatever we commit to is consistent with this q. If for any $r > t$, $\mathrm{SMA}_r(f(\bar{b})a) \upharpoonright r \neq q \upharpoonright r$, we will undefine f all the way back to make $\mathrm{dom}(f) = \bar{b}$ and search anew for the least element having SMA agree with our q.

Lastly, we must assign when a closed relation R holds on tuples from $\bar{b}$ and determine digits for elements in $\bar{b}$. We determine R holds on a tuple $\bar{e}$ if $\bar{e} \subseteq \bar{b}$ and it maintains the consistency requirements to add $R(\bar{e})$. For each element x of $\bar{b}$, we determine the nth digit as follows: Search for some m so that it maintains the consistency requirements to add $x(n) = m$. In the meantime, search for a $t > s$ so that $\mathrm{SMA}_t(\bar{b}_0) \neq \mathrm{SMA}_s(\bar{b}_0)$ for each subtuple $\bar{b}_0$ of $\bar{b}$. If this is found, then undefine f as necessary to maintain the consistency requirement. Repeat the search (again simultaneously for a change in SMA and for a bit so that adding $x(n) = m$ maintains the consistency requirement. By Lemma 5.5, if $\mathrm{SMA}_s(\bar{b}_0)$ is correct for every tuple $\bar{b}_0 \subseteq \bar{b}$, then our $\exists_1^+$-type q exists and is realized consistent with our current commitments, so there is some way to maintain the consistency requirement and determine the nth digit of x.

Lemma 5.6. *Each requirement is satisfied.*

Proof. Suppose otherwise. Let Q be the highest priority requirement not satisfied. In this case, there is a tuple $\bar{b}$ and an s by which higher-priority requirements have determined $f(\bar{b})$, i.e. $f_t(\bar{b}) = f(\bar{b})$ for all $t \geq s$.

Suppose $Q = R_a$. Let $t > s$ be a stage so that $\mathrm{SMA}_r(\bar{d})$ is constant for $r > t$ for every $\bar{d} \subseteq f(\bar{b})a$. Let c be chosen by R_a at a stage $r > t$ to make $f(c) = a$. Since SMA does not change on any subtuple of $f(\bar{b})a$, there is never a reason to undefine f on c and thus the R_a-requirement is satisfied after all.

Suppose $Q = S_b$. Let q be the type constructed by the S_b-requirement. Let $a \in \mathfrak{A}$ be least so that $\mathrm{SMA}(f(\bar{b})a, r)$ for an element r coding q (this exists since q is realized). Further, let t be a stage large enough that SMA has settled for every subtuple of the initial segment of ω containing $f(\bar{b})$ and a. Let $r > t$ be a stage where S_b next acts. It will send b to a and f can never be undefined again. Thus S_b is satisfied after all. □

Lemma 5.7. *Each bit of every tuple is determined.*

Proof. Since every S_b-requirement is satisfied, every element is eventually in $\bar{b}$, so we will determine its bits successively at stages. □

Lemma 5.8. $\mathcal{N} \cong \mathcal{M}$.

Proof. We aim to show that f is an isomorphism between $\mathcal{N}$ and $\mathcal{M}$. The satisfaction of the requirements shows that f is a bijection from $\mathcal{N}$ to $\mathfrak{A}$. The consistency requirements ensures that the bits of $x \in \mathcal{N}$ equals the bits of $f(x) \in \mathfrak{A}$. It remains to show that we defined the closed relations correctly in $\mathcal{N}$. Let R be a closed predicate in the language of $\mathcal{M}$. We first show that if $f(\bar{b})$ is in R, then at some stage we enumerate R onto $\bar{b}$: Let t be a stage late enough that $f(\bar{b})$ has settled and $\mathrm{SMA}(f(\bar{b}))$ has settled. At this stage, since $\mathrm{SMA}(f(\bar{b}))$ says that $R(f(\bar{b}))$ holds, we will enumerate R onto the tuple $\bar{b}$. Thus $f(\bar{b}) \in R$ implies that we will enumerate R onto $\bar{b}$. Now, suppose that we enumerate R onto $\bar{b}$. Then take a stage t late enough that $f(\bar{b})$ has settled and $\mathrm{SMA}(f(\bar{b}))$ has settled. Then the consistency requirement at stage t implies that $f(\bar{b}) \in R$.

Thus we enumerate exactly the relation R on $\mathcal{N}$. We can also enumerate $\neg R$ whenever we see enough bits on a tuple which determine that it cannot be in R (using openness of R). Thus we have computably determined exactly the set R. □

$\square_{\text{Theorem 5.3}}$

6. Expansions of $\mathcal{C}$

6.1. **Our main tool to show that an expansion of $\mathcal{C}$ is above $\mathcal{B}$.** Fairly simple expansions of $\mathcal{C}$ might already be above $\mathcal{B}$. For example, the graph R of the left-shift function is $\mathbf{\Pi^0_1}$ since the left-shift function is continuous. But in $(\mathcal{C}, R)$ we can enumerate the finite elements of $\mathcal{C}$ (they are the ones that eventually left-shift to the element 0^ω), and this lets us construct a copy of $\mathcal{B}$ by Lemma 3.10. This phenomenon is captured more generally by the following observation:

Theorem 6.1. *Suppose $\mathcal{M} \geq^*_w \mathcal{C}$ and $A \subseteq \mathcal{C}\Delta_2^{\mathcal{M}}$ is countable, $P \subseteq \mathcal{C}$ is perfect, and $A \cap P$ is dense in P. Then $\mathcal{M} \geq^*_w \mathcal{B}$.*

Proof. Since P is closed, it is the set of paths through a pruned tree $T \in 2^{<\omega}$. There is a code for T in the ground model, and using T we can define a $\Delta^*_2(\mathcal{C})$ homeomorphism $F\colon P \to \mathcal{C}$. Since $A \in \mathcal{C}\Delta_2^{\mathcal{M}}$, we have that $D = F[A] \in \mathcal{C}\Delta_2^{\mathcal{M}}$. Of course, D is countable and dense in $\mathcal{C}$. It is well-known that $\mathcal{C} \smallsetminus D$ is homeomorphic to $\mathcal{B}$; we exploit the existence of a simple homeomorphism below.

Fix a parameter in $\mathcal{C}$ that codes a fixed enumeration $(d_i)_{i\in\omega}$ of the elements of D. Note that from this fixed parameter, we have a listing of the elements of D bitwise, but we do not have indices for the members of D inside any given enumeration of $\mathcal{C}$.

For any $\sigma \in 2^{<\omega}$, we define w_σ to be d_k where k is least so that $d_k \in [\sigma]$. We now describe a way to enumerate a member of $\mathcal{B}$ from an element $c \notin D$. We first define a sequence $\sigma_0 \prec \sigma_1 \prec \sigma_2 \prec \cdots$. Let σ_0 be the empty string. If we have already defined σ_i, let $\sigma_{i+1} \succ \sigma_i$ be the longest common initial segment of c and w_{σ_i}. Finally, define a member of $\mathcal{B}$ by $r(n) = |\sigma_{n+1}| - |\sigma_n| - 1$. Note that this operation is computable from c and our fixed enumeration $(d_i)_{i\in\omega}$. For $c \notin D$, call the output of this algorithm $r(c) \in \mathcal{B}$. If we attempt to apply this algorithm to a member of D, we may get stuck in that w_{σ_i} may equal c, so we will not find a longest common initial segment.

We proceed as follows: For each member x of $\mathcal{C}$, whenever the Δ^0_2-approximation says that $x \notin D$, we begin this enumeration of an element $r \in \mathcal{B}$. If at a later stage, the approximation says $x \in D$, then we stop the enumeration (having already built $r \restriction n$, for some n) and extend this enumeration to give $(r \restriction n) \frown 0^\omega$. The result is that we will enumerate $r(x)$ for each $x \notin D$ once the Δ^0_2-approximation settles down. It is straightforward to see that the map $r\colon \mathcal{C} \smallsetminus D \to \mathcal{B}$ is onto, so we enumerate a copy of $\mathcal{B}$ with repetitions. From Theorem 2.6, this gives us a copy of $\mathcal{B}$. Thus $\mathcal{B} \leq^*_w \mathcal{M}$. □

The following set-theoretic fact will let us transform any set that is Borel but not $\mathbf{\Delta^0_2}$ into the form needed to apply Theorem 6.1.

Lemma 6.2 (Hurewicz, Theorem 21.18 of [Kec95])**.** *If $R \subseteq \mathcal{C}$ is Borel but not $\mathbf{\Delta^0_2}$, then there is a perfect set $P \subseteq \mathcal{C}$ such that either $P \cap R$ or $P \setminus R$ is countable and dense in P.*

Corollary 6.3. *If $\mathcal{M}$ is an expansion of $\mathcal{C}$ such that there is a set $X \in \Delta^*_2(\mathcal{M})$ that is Borel and is not $\mathbf{\Delta^0_2}$, then $\mathcal{M} \geq^*_w \mathcal{B}$.*

Proof. Suppose that $X \subseteq \mathcal{C}^k \cap \Delta^*_2(\mathcal{M})$, X is Borel, and X is not $\mathbf{\Delta^0_2}$. Then consider $X_0 = \{x_0 \oplus \cdots \oplus x_{k-1} \mid \bar{x} \in X\}$. Since $\oplus : \mathcal{C}^k \to \mathcal{C}$ is a homeomorphism, X_0 is still Borel and not $\mathbf{\Delta^0_2}$. Since $\oplus$ is $\Delta^*_2(\mathcal{C})$, X_0 is also in $\Delta^*_2(\mathcal{M})$. Now apply Lemma 6.2, and then apply Theorem 6.1, using either X_0 or its complement for A. □

In particular, this gives us strong information about expansions of $\mathcal{C}$ that are below $\mathcal{B}$.

Corollary 6.4. *If $\mathcal{M} \leq^*_w \mathcal{B}$ is an expansion of $\mathcal{C}$ and $\Delta^*_2(\mathcal{M}) \neq \Delta^*_2(\mathcal{C})$, then $\mathcal{M} \equiv^*_w \mathcal{B}$.*

Proof. Let $X \in \Delta^*_2(\mathcal{M}) \setminus \Delta^*_2(\mathcal{C})$. Since $\mathcal{M} \leq^*_w \mathcal{B}$, X is in $\mathcal{C}\Delta^{\mathcal{B}}_2$ by Proposition 3.3. By Theorem 3.9, X is Borel, but not $\mathbf{\Delta^0_2}$. By Corollary 6.3, we see that $\mathcal{M} \geq^*_w \mathcal{B}$. □

6.2. **Expansions of $\mathcal{C}$ by unary relations.**

Theorem 6.5. *For a finite sequence $U_1, \ldots, U_n \subseteq 2^\omega$, $(\mathcal{C}, U_1, \ldots, U_n) \leq^*_w \mathcal{C}$ if and only if each U_i is $\mathbf{\Delta^0_2}$.*

Proof. This follows immediately from Theorems 4.3 and 3.9. □

What if at least one of the unary relations is Borel but not $\mathbf{\Delta^0_2}$?

Theorem 6.6. *Let $U_1, \ldots, U_n \subseteq \mathcal{C}$ not all be $\mathbf{\Delta^0_2}$ and suppose that each U_i is Borel or that $(\mathcal{C}, U_1, \ldots, U_n) \leq^*_w \mathcal{B}$. Then $(\mathcal{C}, U_1, \ldots, U_n) \geq^*_w \mathcal{B}$ (in fact, it is $\equiv^*_w \mathcal{B}$).*

Proof. Let $U \in \{U_1, \ldots, U_n\}$ be non-$\mathbf{\Delta^0_2}$. Since U is in $\Delta^*_2(\mathcal{C}, U_1, \ldots, U_n)$, Corollary 6.3 and Corollary 6.4 give the result in each of the two cases. For the parenthetical comment, if each U_i is Borel, then by Theorem 7.4 below, we have $(\mathcal{C}, U_1, \ldots, U_n) \leq^*_w (\mathcal{B}, U_1, \ldots, U_n, V) \leq^*_w \mathcal{B}$, where V is a predicate for $\mathcal{C}$ as a subset of $\mathcal{B}$. □

The next result deals with adding countably many Borel unary relations. Note the use of Theorem 4.2, which in the case of $\mathcal{C}$ is due to Greenberg, Igusa, Turetsky, and Westrick. Prior to learning of their result, we had proved the two previous results, so we knew that the dichotomy held for expansions of $\mathcal{C}$ by *finitely* many Borel unary relations.

Theorem 6.7. *Let $(U_i)_{i\in\omega}$ be a countable sequence of unary relations on $\mathcal{C}$ that are each Borel. Then $\mathcal{M} = (\mathcal{C}, (U_i)_{i\in\omega})$ has $\mathcal{M} \leq^*_w \mathcal{C}$ or $\mathcal{M} \geq^*_w \mathcal{B}$.*

Proof. If any one of the U_i are not $\mathbf{\Delta^0_2}$, then Theorem 6.6 shows that $\mathcal{M} \geq^*_w \mathcal{B}$. So, we can assume each U_i is $\mathbf{\Delta^0_2}$. Then the graph of F from Theorem 4.2 is $\Delta^*_2(\mathcal{M})$, which is Borel by Lemma 3.11. If $F \in \Delta^*_2(\mathcal{C})$, then $\mathcal{M} \leq^*_w \mathcal{C}$ by Theorem 4.2. Otherwise, $\mathcal{M} \geq^*_w \mathcal{B}$ by Corollary 6.3. □

We also examine whether it is possible to add countably many unary relations to $\mathcal{C}$ to get a degree strictly between $\mathcal{C}$ and $\mathcal{B}$.

Theorem 6.8. *Let $(U_i)_{i\in\omega}$ be a countable sequence of unary relations on $\mathcal{C}$. Then $\mathcal{M} = (\mathcal{C}, (U_i)_{i\in\omega})$ cannot have generic Muchnik degree strictly between $\mathcal{C}$ and $\mathcal{B}$.*

Proof. Suppose $\mathcal{M} \leq_w^* \mathcal{B}$. Then $\mathcal{C}\Delta_2^{\mathcal{M}} \subseteq \mathcal{C}\Delta_2^{\mathcal{B}} = \mathbf{\Delta_1^1}$. So each U_i is Borel. Thus Theorem 6.7 shows that either $\mathcal{M} \equiv_w^* \mathcal{C}$ or $\mathcal{M} \equiv_w^* \mathcal{B}$. □

6.3. **Expansions of $\mathcal{C}$ by closed relations.** Using the safe move analysis predicate introduced in Section 5 we can extend Theorem 6.8 to expansions of Cantor space by relations of arbitrary arity, as long as they are all closed.

Theorem 6.9. *Suppose that $\mathcal{M}$ is an expansion of $\mathcal{C}$ by countably many closed relations. Then either $\mathcal{M} \equiv_w^* \mathcal{C}$ or $\mathcal{M} \geq_w^* \mathcal{B}$.*

Proof. The dichotomy is determined by whether or not SMA($\mathcal{M}$) is $\Delta_2^*(\mathcal{C})$: If SMA($\mathcal{M}$) $\notin \Delta_2^*(\mathcal{C})$, then it is Borel by Lemma 5.2 and Lemma 3.11 but not $\mathbf{\Delta_2^0}$ by Theorem 3.9. By Corollary 6.3, we get $\mathcal{M} \geq_w^* \mathcal{B}$. On the other hand, if SMA($\mathcal{M}$) $\in \Delta_2^*(\mathcal{C})$, then Theorem 5.3 (applied with $\mathcal{H} = \mathcal{C}$) shows that $\mathcal{M} \equiv_w^* \mathcal{C}$. □

7. Expansions of $\mathcal{B}$

7.1. **Our main tool to show that an expansion of $\mathcal{B}$ is above $\mathcal{BC}$.** In this section, we will need to make some set-theoretic assumptions in order to get clean dichotomies between $\mathcal{B}$ and $\mathcal{BC}$.

First, we start with a purely computability theoretic result (in ZFC) that captures our method of showing that an expansion of $\mathcal{B}$ is above $\mathcal{BC}$.

Lemma 7.1. *Let $\mathcal{M}$ be an expansion of $\mathcal{B}$. Suppose that there is a set $Y \in \Delta_2^*(\mathcal{M})$ that is $\mathbf{\Sigma_1^1}$-hard under continuous reduction. Then $\mathcal{M} \geq_w^* \mathcal{BC}$.*

Proof. Consider the structure $\mathcal{A} = (\mathcal{B}, \oplus, J)$ where J is the ternary relation defined by $J(f, g, h)$ if and only if $f' = g$ and h is a settling time function witnessing that $f' = g$. It is shown in [AKK+, Corollary 4.11] that $\mathcal{A} \equiv_w^* \mathcal{BC}$. Further, $\mathcal{A}$ is formed by adding closed relations to $\mathcal{B}$.

Note that each SMA$_k(\mathcal{A})$ is the intersection of a $\mathbf{\Sigma_1^1}$ set S_1 and a $\mathbf{\Pi_1^1}$ set S_2: SMA$_k(\bar{x}, y)$ holds if

- For each $n \in \omega$, $y(n) = 0$ or $y(n) = 1$. (Borel)
- For each $n \in \omega$, if $y(n) = 1$, then $\mathcal{A} \models \varphi_n(\bar{x})$ where φ_n is the nth positive existential formula. ($\mathbf{\Sigma_1^1}$)
- For each $n \in \omega$, if $y(n) = 0$, then $\mathcal{A} \models \neg\varphi_n(\bar{x})$. ($\mathbf{\Pi_1^1}$)

Thus SMA is $\Delta_2^*(\mathcal{M})$, using continuous reductions to Y to determine both S_1 and S_2 from Y. Since $\mathcal{A}$ is formed by adding closed relations to $\mathcal{B}$ and SMA($\mathcal{A}$) is $\Delta_2^*(\mathcal{M})$, we get that $\mathcal{A} \leq_w^* \mathcal{M}$ by Theorem 5.3. Thus $\mathcal{BC} \leq_w^* \mathcal{M}$. □

The previous result highlights the reason why set-theoretic assumptions are necessary. The goal of these assumptions is to get a sufficiently topologically complex Y to be above every $\mathbf{\Sigma_1^1}$ set under continuous reduction, allowing us to apply the lemma. To this end, we will assume Wadge determinacy for a class of sets containing the $\mathbf{\Sigma_1^1}$ sets and our set Y. As a first application, we assume projective Wadge-determinacy and consider expansions of $\mathcal{B}$ by projective sets.

Lemma 7.2. *If X is projective, and $Y \in \Delta_n^*(\mathcal{B}, X)$ for some $n \in \omega$, then Y is projective.*

Proof. Let k be so $X \in \mathbf{\Delta^1_k}$. Then X is in $\Delta^*_{k+1}(\mathcal{B})$ by Theorem 3.9. Then $Y \in \Delta^*_{k+n}(\mathcal{B})$ by Lemma 3.7. So, $Y \in \mathbf{\Delta^1_{k+n-1}}$ again by Theorem 3.9, and thus is projective. □

Theorem 7.3 (Projective Wadge-determinacy)**.** *Let $\mathcal{M} = (\mathcal{B}, X)$ where X is projective. Suppose $\Delta^*_2(\mathcal{M}) \neq \Delta^*_2(\mathcal{B})$. Then $\mathcal{M} \geq^*_w \mathcal{BC}$.*

Proof. Let Y be in $\Delta^*_2(\mathcal{M}) \setminus \Delta^*_2(\mathcal{B})$. Then Y is projective by Lemma 7.2. By Lemma 3.9, we know that Y is not Borel. By Lemma 2.11, either Y or its complement is $\mathbf{\Sigma^1_1}$-hard under continuous reduction. Finally, Lemma 7.1 shows that $\mathcal{M} \geq^*_w \mathcal{BC}$. □

7.2. **Expansions of $\mathcal{B}$ by unary relations.**

Theorem 7.4. *Let $U_1, \dots, U_n \subseteq \mathcal{B}$. Then $(\mathcal{B}, U_1, \dots, U_n) \leq^*_w \mathcal{B}$ if and only if each U_i is Borel.*

Proof. This follows immediately from Theorems 4.3 and 3.9. □

In what follows, we focus on the $\mathbf{\Delta^1_2}$ sets because they are the sets that might arise in a degree strictly between $\mathcal{B}$ and $\mathcal{BC}$. In the following theorem, if we replace $\mathbf{\Delta^1_2}$ both in the assumption and the result by any topological class that contains $\mathbf{\Delta^1_2}$, then the argument works the same.

Theorem 7.5 ($\mathbf{\Delta^1_2}$-Wadge determinacy)**.** *Let $\mathcal{M}$ be an expansion of $\mathcal{B}$ by countably many $\mathbf{\Delta^1_2}$ unary relations. Then either $\mathcal{M} \equiv^*_w \mathcal{B}$ or $\mathcal{M} \geq^*_w \mathcal{BC}$.*

Proof. Suppose that one of the unary relations X is $\mathbf{\Delta^1_2}$ but not Borel. By $\mathbf{\Delta^1_2}$-Wadge determinacy and Lemma 2.11, either X or its complement is $\mathbf{\Sigma^1_1}$-hard under continuous reduction. Then Lemma 7.1 shows that $\mathcal{M} \geq^*_w \mathcal{BC}$.

Now suppose that all of the unary relations are Borel. Then they are all $\Delta^*_2(\mathcal{B})$, so the graph of the function F from Theorem 4.2 is $\Delta^*_2(\mathcal{M})$, which is contained in $\Delta^*_3(\mathcal{B}) = \mathbf{\Delta^1_2}$ by Lemma 3.8 and Lemma 3.9. If the graph of F is $\Delta^*_2(\mathcal{B})$, i.e. Borel, then Theorem 4.2 shows that $\mathcal{M} \equiv^*_w \mathcal{B}$. Otherwise, by $\mathbf{\Delta^1_2}$-Wadge determinacy, either the graph of F or its complement is $\mathbf{\Sigma^1_1}$-hard under continuous reduction, and Lemma 7.1 shows that $\mathcal{M} \geq^*_w \mathcal{BC}$. □

Corollary 7.6 ($\mathbf{\Delta^1_2}$-Wadge determinacy)**.** *Let $\mathcal{M}$ be an expansion of $\mathcal{B}$ by countably many unary relations. Then $\mathcal{M}$ cannot have degree strictly between $\mathcal{B}$ and $\mathcal{BC}$.*

Proof. Suppose that $\mathcal{M} \leq^*_w \mathcal{BC}$. Then each of the unary relations is $\Delta^*_2(\mathcal{M})$, so in $\Delta^*_2(\mathcal{BC})$. Since $\mathcal{BC} = (\mathcal{B}, \oplus, ')$ and $\oplus$ and $'$ are both $\Delta^*_2(\mathcal{B})$, Lemma 3.8 shows that $\Delta^*_2(\mathcal{BC}) \subseteq \Delta^*_3(\mathcal{B}) = \mathbf{\Delta^1_2}$. Thus by Theorem 7.5, either $\mathcal{M} \equiv^*_w \mathcal{B}$ or $\mathcal{M} \equiv^*_w \mathcal{BC}$. □

7.3. **Expansions of $\mathcal{B}$ by closed relations.** Just like with Cantor space, we can extend our results on expansions of Baire space by countably many unary relation to countably many closed relations of arbitrary arity using the safe move analysis predicate.

Theorem 7.7. *Let $\mathcal{M}$ be an expansion of $\mathcal{B}$ by countably many closed relations. Suppose that $\Delta^*_2(\mathcal{M}) = \Delta^*_2(\mathcal{B})$. Then $\mathcal{M} \equiv^*_w \mathcal{B}$.*

Proof. By Lemma 5.2, $\mathrm{SMA}(\mathcal{M}) \in \Delta^*_2(\mathcal{M}) = \Delta^*_2(\mathcal{B})$. So by Theorem 5.3, we have that $\mathcal{M} \leq^*_w \mathcal{B}$. □

Corollary 7.8 ($\mathbf{\Delta^1_2}$-Wadge determinacy)**.** *Let $\mathcal{M}$ be an expansion of $\mathcal{B}$ by countably many closed relations. Then $\mathcal{M} \equiv^*_w \mathcal{B}$ or $\mathcal{M} \equiv^*_w \mathcal{BC}$.*

Proof. It is immediate that $\mathcal{M} \leq^*_w \mathcal{BC}$.

If $\Delta^*_2(\mathcal{M}) \neq \Delta^*_2(\mathcal{B})$, then by Lemma 3.8, $\Delta^*_2(\mathcal{M}) \subseteq \Delta^*_3(\mathcal{B}) = \mathbf{\Delta^1_2}$. By $\mathbf{\Delta^1_2}$-Wadge determinacy, we have that a member of $\Delta^*_2(\mathcal{M}) \smallsetminus \Delta^*_2(\mathcal{B})$ is $\mathbf{\Sigma^1_1}$-hard under continuous reduction. So Lemma 7.1 shows that $\mathcal{M} \equiv^*_w \mathcal{BC}$.

If $\Delta^*_2(\mathcal{M}) = \Delta^*_2(\mathcal{B})$, then Theorem 7.7 implies that $\mathcal{M} \equiv^*_w \mathcal{B}$. □

8. A structure of intermediate degree

In this section, we produce a structure $\mathcal{M}$ such that $\mathcal{C} <^*_w \mathcal{M} <^*_w \mathcal{B}$, answering Question 9 from [Sch16]. We say that such an $\mathcal{M}$ has *intermediate degree*. By the results of Section 6, such a structure cannot be a simple expansion of Cantor space, and indeed it is still open whether *any* expansion of Cantor space has intermediate degree.

Our intermediate structure will be of the form $\mathcal{C} \sqcup \mathcal{L}$, where $\mathcal{L}$ is an appropriately complicated linear ordering. This is convenient because we will show that adding a linear order does not change the relatively intrinsically Δ^0_2 sets. This is a direct analog of Knight's theorem [Kni86] that if a linear order has a jump degree, then that degree must be $\mathbf{0}'$.

Theorem 8.1. *Consider $\mathcal{M} \geq^*_w \mathcal{C}$ and any linear order $\mathcal{L}$. Then $\mathcal{C}\Delta_2^{\mathcal{M}\sqcup\mathcal{L}} = \mathcal{C}\Delta_2^{\mathcal{M}}$.*

Proof. This proof is exactly as in the proof of Knight [Kni86, Theorem 3.5]. We only need to verify that having a copy of $\mathcal{M}$ present does not change anything in the forcing.

Fix an ω-copy of $\mathcal{M} \sqcup \mathcal{L} \sqcup \mathcal{C}$. We build a generic copy $\mathcal{A}$ of $\mathcal{M} \sqcup \mathcal{L} \sqcup \mathcal{C}$ by forcing with partial permutations of ω, thus building G with to be a generic permutation of ω and letting $\mathcal{A} = G(\mathcal{M} \sqcup \mathcal{L} \sqcup \mathcal{C})$.

We want to show that $\mathcal{C}\Delta_2^{\mathcal{M}\sqcup\mathcal{L}} \subseteq \mathcal{C}\Delta_2^{\mathcal{M}}$. Let X be in $\mathcal{C}\Delta_2^{\mathcal{M}\sqcup\mathcal{L}}$. Then there is an e and a condition p such that that $p \Vdash \varphi_e^{G(\mathcal{M}\sqcup\mathcal{L}\sqcup\mathcal{C})'} = G(X)$. Let $\mathrm{dom}(p)$ consist of $b_0, \ldots, b_{m-1}$ from $\mathcal{M}$ and $a_0 < \cdots < a_{n-1}$ from $\mathcal{L}$. Let I_i be the interval (a_{i-1}, a_i) in $\mathcal{L}$ and let I_0 be the interval $(-\infty, a_0)$ and I_n be the interval (a_{n-1}, ∞) in $\mathcal{L}$. Knight [Kni86, Lemma 3.3] shows that there are computable orderings J_i for $i \leq n$ such that $I_i \equiv_2 J_i$.[2] Further, replacing each I_i by J_i forms a computable linear order $\hat{\mathcal{L}}$ so that $(\mathcal{L}, \mathrm{dom}(p)) \equiv_2 (\hat{\mathcal{L}}, \mathrm{dom}(p))$ by Knight [Kni86, Lemma 3.2]. Since there is no interaction between $\mathcal{M}$, $\mathcal{L}$, and $\mathcal{C}$ in $\mathcal{M} \sqcup \mathcal{L} \sqcup \mathcal{C}$, it follows that $(\mathcal{M} \sqcup \mathcal{L} \sqcup \mathcal{C}, \mathrm{dom}(p)) \equiv_2 (\mathcal{M} \sqcup \hat{\mathcal{L}} \sqcup \mathcal{C}, \mathrm{dom}(p))$. Thus $p \Vdash \varphi_e^{G(\mathcal{M}\sqcup\hat{\mathcal{L}}\sqcup\mathcal{C})'} = G(X)$ since the relevant facts are Σ^c_2. We thus see that in any sufficiently generic copy $\mathcal{A}$ of $\mathcal{M} \sqcup \hat{\mathcal{L}} \sqcup \mathcal{C}$, $X^{\mathcal{A}}$ is $\Delta^0_2(\mathcal{A})$. It follows from the argument in [AKMS89] and [Chi90] that X has both Σ^c_2 and Π^c_2 definitions in $\mathcal{M} \sqcup \hat{\mathcal{L}} \sqcup \mathcal{C}$. In particular, the arguments there only use sufficiently generic copies of the structure, despite the fact that the theorem is usually stated as assuming that X is Δ^0_2 in *every* copy of the structure. Thus X is in $\mathcal{C}\Delta_2^{\mathcal{M}\sqcup\hat{\mathcal{L}}\sqcup\mathcal{C}}$. But since $\mathcal{M} \equiv^*_w \mathcal{M} \sqcup \hat{\mathcal{L}} \sqcup \mathcal{C}$, it follows that $X \in \mathcal{C}\Delta_2^{\mathcal{M}}$. □

Corollary 8.2. *Let $\mathcal{L}$ be any linear order. Then $\mathcal{C} \sqcup \mathcal{L} \not\geq^*_w \mathcal{B}$ and $\mathcal{B} \sqcup \mathcal{L} \not\geq^*_w \mathcal{BC}$.*

[2] $A \equiv_2 B$ means that for all $\bar{a} \in A$, there is a $\bar{b} \in B$ so that $\mathrm{tp}^A_\exists(\bar{a}) = \mathrm{tp}^B_\exists(\bar{b})$ and similarly for all $\bar{b} \in B$ there is a $\bar{a}$ in A so that $\mathrm{tp}^A_\exists(\bar{a}) = \mathrm{tp}^B_\exists(\bar{b})$

Proof. The previous theorem shows that $\mathcal{C}\Delta_2^{\mathcal{A}\sqcup\mathcal{L}} = \mathcal{C}\Delta_2^{\mathcal{A}}$ for any structure $\mathcal{A} \geq_w^* \mathcal{C}$. Since $\mathcal{C}\Delta_2^{\mathcal{BC}}$ is strictly larger than $\mathcal{C}\Delta_2^{\mathcal{B}}$, which is strictly larger than $\mathcal{C}\Delta_2^{\mathcal{C}}$ by Theorem 3.9, it follows that $\mathcal{B} \not\leq_w^* \mathcal{C} \sqcup \mathcal{L}$ and $\mathcal{BC} \not\leq_w^* \mathcal{B} \sqcup \mathcal{L}$. □

We now shift to constructing linear orders such that $\mathcal{C} <_w^* \mathcal{C}\sqcup\mathcal{L} <_w^* \mathcal{B}$ or $\mathcal{B} <_w^* \mathcal{B}\sqcup\mathcal{L} <_w^* \mathcal{BC}$. We begin by giving a definition of a linear order $\mathcal{L}_X$ for any given set $X \subseteq \mathcal{C}$.

Definition 8.3. For any element $f \in \mathcal{C}$, we define $\mathcal{J}_f$ to be $\mathbb{Z}+1+\mathbb{Z}$ if $f \notin X$ and to be $\mathbb{Z}$ if $f \in X$. For $f \in \mathcal{C}$, and $n \in \omega$, define $\mathcal{K}_n^f$ to be $2n+3$ if $f(n)=1$ and $2n+4$ if $f(n)=0$. For $f \in \mathcal{C}$, let $\mathcal{I}_f$ be

$$2 + \mathcal{J}_f + (\Sigma_{n\in\omega}\mathcal{K}_n^f + \mathbb{Z}) + 2.$$

Finally, let $\mathcal{L}_X$ be any shuffle sum of the $\mathcal{I}_f$ for every $f \in 2^\omega$. That is, fix any ordering (without endpoints) of $\mathcal{C} \times \omega$ so that for any $f, g, h \in \mathcal{C}$, between any f element and g element, there must be an h element. Then, replace each $\langle f, i\rangle$ by the linear order $\mathcal{I}_f$.

We now give a construction that yields both linear orders simultaneously. Note that the construction is not specific to $\mathcal{C}$ and $\mathcal{B}$, but that the analysis of the complexity profiles allows us to show the necessary reductions and non-reductions.

Theorem 8.4. *Let $\mathcal{C} \leq_w^* \mathcal{M} <_w^* \mathcal{N}$, $\mathcal{C}\Delta_2^{\mathcal{M}} \subsetneq \mathcal{C}\Delta_2^{\mathcal{N}}$, $\mathcal{C}\Delta_4^{\mathcal{M}} \subsetneq \mathcal{C}\Pi_3^{\mathcal{N}}$. Then there is a linear order $\mathcal{L}$ so that $\mathcal{M} <_w^* \mathcal{M}\sqcup\mathcal{L} <_w^* \mathcal{N}$.*

Proof. Let Y be a $\mathcal{C}\Pi_3^{\mathcal{N}} \setminus \mathcal{C}\Delta_4^{\mathcal{M}}$ set and fix $\mathcal{L} = \mathcal{L}_Y$. We will show that $\mathcal{M} <_w^* \mathcal{M}\sqcup\mathcal{L} <_w^* \mathcal{N}$.

Lemma 8.5. *For any X, $X \in \mathcal{C}\Delta_4^{\mathcal{C}\sqcup\mathcal{L}_X}$.*

Proof. First consider the predicate $\mathrm{FinInt}_k(x,y)$ on $\mathcal{L}$ which says that $x<y$ and $[x,y]$ is a maximal discrete interval of size k. FinInt_k is $\Pi_2^*(\mathcal{L})$.

We let $\mathrm{Bounds}(x,y,z,w)$ be the predicate on $\mathcal{L}$ which holds if $x<y<z<w$ and $\mathrm{FinInt}_2(x,y) \wedge \mathrm{FinInt}_2(z,w) \wedge \neg(\exists u,v \in (y,z))\ \mathrm{FinInt}_2(u,v)$. Note that Bounds is $\Pi_3^*(\mathcal{L})$ and that $\mathrm{Bounds}(x,y,z,w)$ holds if and only if x and y are the extreme points in a copy of $\mathcal{I}_f$ for some $f \in 2^\omega$.

Finally, $f \in X$ if and only if there exists l_1, l_2, r_1, r_2 so that

- $\mathrm{Bounds}(l_1, l_2, r_1, r_2)$
- there is z, w so that w is the successor of z and $(l_2, z) \cong \mathbb{Z}$
- For every $n \in \omega$, either $n \in f$ and there is no pair $x, y \in (l_2, r_1)$ so that $\mathrm{FinInt}_{2n+4}(x,y)$ OR $n \notin f$ and there is no pair $x,y \in (l_2, r_1)$ so that $\mathrm{FinInt}_{2n+3}(x,y)$.

It is known that a computable linear order being isomorphic to $\mathbb{Z}$ is a Π_3^0-condition. Using this and counting quantifiers, we see that $f \in X$ is a Σ_4^*-condition in a presentation of $\mathcal{C}\sqcup\mathcal{L}$.

Finally by changing the second condition to say there is z such that z has no successor and $(l_2, z) \cong \mathbb{Z}$, we can also show that $\mathcal{C} \setminus X$ is $\Sigma_4^*(\mathcal{C}\sqcup\mathcal{L})$. □

Corollary 8.6. *$\mathcal{L} \not\leq_w^* \mathcal{M}$.*

Proof. Suppose that $\mathcal{L} \leq_w^* \mathcal{M}$. Then $\mathcal{C}\sqcup\mathcal{L} \leq_w^* \mathcal{M}$, so $Y \in \mathcal{C}\Delta_4^{\mathcal{M}}$. But Y was chosen to be in $\mathcal{C}\Pi_3^{\mathcal{N}} \setminus \mathcal{C}\Delta_4^{\mathcal{M}}$. □

The following lemma is completely standard.

Lemma 8.7. *For any Σ^0_3 set Z, there exists a uniformly computable sequence of computable structures $(M_i)_{i\in\omega}$ so that $M_i \cong \mathbb{Z}+1+\mathbb{Z}$ if $i \in Z$ and $M_i \cong \mathbb{Z}$ if $i \notin Z$.*

Proof. Recall that COF, the set of i such that W_i is co-finite is Σ^0_3 m-complete. Thus it suffices to uniformly provide a structure M_i, for each $i \in \omega$, such that $M_i \cong \mathbb{Z}+1+\mathbb{Z}$ if $i \in$COF and $M_i \cong \mathbb{Z}$ otherwise.

For each $n \in \omega$, say that a stage s is n-expansionary if $n-1 \notin W_{i,s}$ and the largest k so that $[n,k] \subseteq W_{i,s}$ increases at stage s. Uniformly build a sequence of linear orders L_n as follows: Begin with 1 element called c. At every n-expansionary stage, place 2 new elements one directly the left of c and one directly to the right of c. So, if there are infinitely many n-expansionary stages, then $L_n \cong \omega+1+\omega^*$ and otherwise L_n is finite. Now, let $\mathcal{M} = \omega^* + L_0 + L_1 + \cdots$. If $i \in$COF, then exactly one L_n is isomorphic to $\omega+1+\omega^*$ and all the others are finite, so $\mathcal{M} \cong \mathbb{Z}+1+\mathbb{Z}$. Otherwise, every L_n is finite and $\mathcal{M} \cong \mathbb{Z}$. □

Lemma 8.8. *For any structure $\mathcal{A} \geq^*_w \mathcal{C}$, if $X \in \mathcal{C}\Pi^{\mathcal{A}}_3$, then $\mathcal{L}_X \leq^*_w \mathcal{A}$. In particular since $Y \in \mathcal{C}\Pi^{\mathcal{N}}_3$, $\mathcal{L} \leq^*_w \mathcal{N}$.*

Proof. Since $\mathcal{A} \geq^*_w \mathcal{C}$, in any presentation of $\mathcal{A}$, we can build a copy of $\mathcal{L}_X$ with f-labels in place of $\mathcal{J}_f$. That is, instead of putting $\mathcal{J}_f$ in the order, we simply label the interval (note that $\mathcal{J}_f$ has a predecessor and successor in $\mathcal{L}_X$) with the label f. Note that although we did not specify which shuffle-sum we used in constructing $\mathcal{L}_X$, it does not matter, because in $V[G]$, where $\mathcal{C}$ is countable, this is a shuffle sum of only countably many terms, which is uniquely defined. Now, by Lemma 8.7, if $X \in \mathcal{C}\Pi^{\mathcal{A}}_3$, we can fill in this labeled interval uniformly by either $\mathbb{Z}$ or $\mathbb{Z}+1+\mathbb{Z}$, as needed. □

It follows that $\mathcal{N} \geq^*_w \mathcal{M} \sqcup \mathcal{L} >^*_w \mathcal{M}$. The fact that $\mathcal{N} \not\leq^*_w \mathcal{M} \sqcup \mathcal{L}$ follows from Theorem 8.1 applied to $\mathcal{M} \sqcup \mathcal{C}$ and the fact that $\mathcal{C}\Delta^{\mathcal{M}}_2 \subsetneq \mathcal{C}\Delta^{\mathcal{N}}_2$. □

Corollary 8.9. *There is a linear order $\mathcal{L}$ so that $\mathcal{C} \sqcup \mathcal{L}$ has generic Muchnik degree strictly between $\mathcal{C}$ and $\mathcal{B}$.*

Proof. By Theorem 8.4, it suffices to check that $\mathcal{C}\Delta^{\mathcal{C}}_2 \subsetneq \mathcal{C}\Delta^{\mathcal{B}}_2$ and $\mathcal{C}\Delta^{\mathcal{C}}_4 \subsetneq \mathcal{C}\Pi^{\mathcal{B}}_3$. These both follow from Theorem 3.9. □

Corollary 8.10. *There is a linear order $\mathcal{L}$ so that $\mathcal{B} \sqcup \mathcal{L}$ has generic Muchnik degree strictly between $\mathcal{B}$ and $\mathcal{BC}$.*

Proof. By Theorem 8.4, it suffices to check that $\mathcal{C}\Delta^{\mathcal{B}}_2 \subsetneq \mathcal{C}\Delta^{\mathcal{BC}}_2$ and $\mathcal{C}\Delta^{\mathcal{B}}_4 \subsetneq \mathcal{C}\Pi^{\mathcal{BC}}_3$. These both follow from Theorem 3.9. □

The core idea in constructing our structure $\mathcal{C} \sqcup \mathcal{L}$ of intermediate degree was in using the attached structure $\mathcal{L}$ to code a set that alters the $\mathcal{C}$-complexity profile. This suggests that with more work, we can use Marker extensions to construct structures in different intermediate degrees by controlling their complexity profiles—for example, a structure $\mathcal{M}_4$ with complexity profile

$$\mathbf{\Sigma^0_2}, \mathbf{\Sigma^1_1}, \mathbf{\Sigma^1_2}, \mathbf{\Sigma^1_4}, \mathbf{\Sigma^1_5}, \ldots$$

matching $\mathcal{C}$ for the $2 \leq i \leq 4$ and matching $\mathcal{B}$ for $i \geq 5$. This was done by Kirill Gura, shortly following the work of this paper:

Theorem 8.11 (Gura, in preparation)**.** *For each $i \geq 2$, there is a structure $\mathcal{M}_i$ strictly between $\mathcal{C}$ and $\mathcal{B}$ whose $\mathcal{C}$-complexity profile is given by $\mathcal{C}\Sigma_j^{\mathcal{M}} = \mathcal{C}\Sigma_j^{\mathcal{C}}$ for $j \leq i$ and $\mathcal{C}\Sigma_j^{\mathcal{M}} = \mathcal{C}\Sigma_j^{\mathcal{B}}$ for $j > i$. In fact, these are linearly ordered under generic Muchnik reduction as follows:*

$$\mathcal{C} <_w^* \cdots <_w^* \mathcal{M}_4 <_w^* \mathcal{M}_3 <_w^* \mathcal{M}_2 <_w^* \mathcal{B}.$$

9. Open Questions

There are several questions that we have left unanswered. The most obvious are whether expansions can give intermediate degrees in the intervals we have studied:

Question 1. *Is there an expansion of $\mathcal{C}$ that is strictly between $\mathcal{C}$ and $\mathcal{B}$? Similarly, is there an expansion of $\mathcal{B}$ that is strictly between $\mathcal{B}$ and $\mathcal{BC}$?*

To recap, we have presented dichotomy theorems that greatly limit what such expansions can look like. No unary expansion can be intermediate in either case (under an appropriate set-theoretic assumption in the Borel case). Similarly, no closed expansion can be intermediate. Furthermore, the relations in an intermediate expansion cannot be too complicated. In the case of the interval between $\mathcal{C}$ and $\mathcal{B}$, an intermediate expansion—were it to exist—*must* be properly $\boldsymbol{\Delta^0_2}$.

We have seen that any $\boldsymbol{\Delta^0_2}$ unary relation can be added to $\mathcal{C}$ without changing its Muchnik degree. For $\mathcal{B}$, any $\boldsymbol{\Delta^1_1}$ unary relation can be added without changing the degree. In both cases, the Δ_2^* unary relations are tame. The Borel complete structure could behave differently; it comes equipped with $\oplus$, so there is nothing special about unary expansions as opposed to expansions of higher arity. Of course, any expansion of $\mathcal{BC}$ that does not raise its generic Muchnik degree must be $\Delta_2^*(\mathcal{BC}) = \boldsymbol{\Delta^1_2}$.

Question 2. *Characterize those relations $R \subseteq \mathcal{B}$ that can be added to the Borel complete structure without increasing its generic Muchnik degree?*

We have used the complexity profiles of structures to separate them in the generic Muchnik degrees. However, as already mentioned, this is not the only way. In a forthcoming paper [AMSS], we produce a structure $\mathcal{M}$ such that $\mathcal{C} <_w^* \mathcal{M}$, but $\mathcal{C}$ and $\mathcal{M}$ have the same complexity profile. Can this always be done?

Question 3. *Is it true that for every $\mathcal{N}$ there is an $\mathcal{M} >_w^* \mathcal{N}$ such that $\mathcal{N}\Sigma_k^{\mathcal{M}} = \mathcal{N}\Sigma_k^{\mathcal{N}}$ for all $k \geq 2$.*

On the other hand, we can ask if the methods of this paper can be generalized: assuming that $\mathcal{N} <_w^* \mathcal{M}$ and there is enough room between their complexity profiles, can we always build a structure with degree strictly between $\mathcal{N}$ and $\mathcal{M}$ by exploiting this room?

Question 4. *Let $\mathcal{N} <_w^* \mathcal{M}$ be structures so that $\mathcal{N}\Sigma_k^{\mathcal{M}}$ properly contains $\mathcal{N}\Sigma_k^{\mathcal{N}}$ for all k. Must there be a degree strictly between $\mathcal{N}$ and $\mathcal{M}$?*

Igusa and Schweber [Sch16, Ch. 6] present a structure $\mathcal{P}$ such that in the forcing extension $V[G]$, $\mathcal{P}$ is presentable in exactly the degrees that compute every $f \in \mathcal{C} \cap V$. It follows that $\mathcal{P} \leq_w^* \mathcal{C}$. In unpublished work, Kirill Gura proved that $\mathcal{P}$ is strictly below $\mathcal{C}$. He also constructed, using a proof similar to what we do above, a structure between $\mathcal{P}$ and $\mathcal{B}$ that is incomparable with $\mathcal{C}$ in the generic Muchnik degrees. This leaves open:

Question 5. *Is there a degree in the interval between $\mathcal{P}$ and $\mathcal{C}$?*

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