INTERPRETING TRUE ARITHMETIC IN THE LOCAL STRUCTURE OF THE ENUMERATION DEGREES.

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1. Introduction

Degree theory studies mathematical structures, which arise from a formal notion of reducibility between sets of natural numbers based on their computational strength. In the past years many reducibilities, together with their induced degree structures have been investigated. One of the aspects in these investigations is always the characterization of the theory of the studied structure. One distinguishes between global structures, ones containing all possible degrees, and local structures, containing all degrees bounded by a fixed element, usually the degree which contains the halting set. It has become apparent that modifying the underlying reducibility does not influence the strength of the first order theory of either the induced global or the induced local structure. The global first order theories of the Turing degrees [15], of the many-one degrees [11], of the 1-degrees [11] are computably isomorphic to second order arithmetic. The local first order theories of the computably enumerable degrees [1], of the many-one degrees [12], of the Δ^0_2 Turing degrees [14] are computably isomorphic to the theory of first order arithmetic.

In this article we consider enumeration reducibility, and the induced structure of the enumeration degrees. Enumeration reducibility introduced by Friedberg and Rogers [4] arises as a way to compare the computational strength of the positive information contained in sets of naturals numbers. A set A is enumeration reducible to a set B if given any enumeration of the set B, one can effectively compute an enumeration of the set A. The induced structure of the enumeration degrees $\mathcal{D}_e$ is an upper semilattice with least element and jump operation. This structure raises particular interest as it can be viewed as an extension of the structure of the Turing degrees. There is an isomorphic copy of the the Turing degrees in $\mathcal{D}_e$. The elements of this copy are called the total enumeration degrees.

The enumeration jump operation gives rise to a local substructure, $\mathcal{G}_e$,consisting of all degrees in the interval enclosed by the least degree and its first jump. Cooper [1] shows that the elements of $\mathcal{G}_e$ are precisely the enumeration degrees which contain Σ^0_2 sets, or equivalently are made up entirely of Σ^0_2 sets, which we call Σ^0_2 degrees. This structure can in turn be viewed as an extension of the structure of the Δ^0_2 Turing degrees, which is isomorphic to the Σ^0_2 total degrees.

Slaman and Woodin [16] prove that the theory of the global structure of the enumeration degrees, $\mathcal{D}_e$, is computably isomorphic to the theory of second order arithmetic and show that the local theory is undecidable. In the same article

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[1]This result is due to Harrington and Slaman, see [13] for a published proof.

under question 3.5 they ask whether or not the theory of the local structure, $\mathcal{G}_e$, is computably isomorphic to the theory of first order arithmetic. The same question appears first in [2] in 1990, and then again in other articles, see [17], [3] and [18].

Recently Kent [10] has characterized the first order theory of a substructure of $\mathcal{G}_e$. He proves that the first order theory of the Δ^0_2 enumeration degrees is computably isomorphic to the theory of true arithmetic. However as there are properly Σ^0_2 enumeration degrees, degrees which contain no Δ^0_2 set, this does not settle the problem for the theory of $\mathcal{G}_e$.

The goal of this article is to give an answer to this longstanding question. Note that since the degrees in $\mathcal{G}_e$ are the degrees of the Σ^0_2 sets we can associate a natural number a to every $\mathbf{a} \in \mathcal{G}_e$ in such a way that the relation $\preceq$ on natural numbers, defined by $a \preceq b \iff \mathbf{a} \leq \mathbf{b}$, is arithmetic. Thus there is a computable translation ϕ of every sentence of $\mathcal{G}_e$ into a sentence of arithmetic such that

$$\mathcal{G}_e \models \theta \iff \mathbb{N} \models \phi(\theta).$$

For the converse direction it is enough to show the existence of a class of definable with parameters structures $\mathcal{N}(\overline{\mathbf{q}})$ in $\mathcal{G}_e$ and a formula SMA, such that the following two statements are true.

$$\text{If } \mathcal{G}_e \models SMA(\overline{\mathbf{q}}), \text{ then } \mathcal{N}(\overline{\mathbf{q}}) \text{ is a standard model of arithmetic.} \tag{1.1}$$

$$\text{There are parameters } \overline{\mathbf{q}}, \text{ such that } \mathcal{G}_e \models SMA(\overline{\mathbf{q}}). \tag{1.2}$$

The methods used to prove this result rely on the notion of a $\mathcal{K}$-pair, introduced by Kalimullin [9] and used to show the definability of the enumeration jump operation. In [5] we show that $\mathcal{K}$-pairs are first order definable in $\mathcal{G}_e$. In Section 3 we use this result to improve the local version of Slaman and Woodin's coding lemma [16], showing that a larger class of relations can be coded with parameters in $\mathcal{G}_e$. Using this result we prove the existence of a formula SMA satisfying (1.1) and (1.2).

Finally we show the existence of a parameterless interpretation of true arithmetic.

2. Preliminaries

In this section we shall introduce all the notions and results that will be needed throughout the paper. We start with the notion of enumeration reducibility. As noted above, intuitively a set A is enumeration reducible to a set B, denoted by $A \leq_e B$ if and only if there is an algorithm transforming every enumeration of B into an enumeration of A. Formally

$$A \leq_e B \iff \exists i[A = W_i(B)],$$

where W_i is the c.e. set with Gödel index i and $W_i(B)$ stands for the set

$$W_i(B) = \{x \mid \exists \langle x, u\rangle \in W_i \ \&\ D_u \subseteq B\},$$

where D_u is the finite set with canonical index u.

The relation $\leq_e$ is reflexive and transitive (but not antisymmetric) and so it gives rise to a nontrivial equivalence relation $\equiv_e$ defined by

$$A \equiv_e B \iff A \leq_e B \ \&\ B \leq_e A.$$

We denote $\mathbf{d}_e(A) = \{B \mid A \equiv_e B\}$. The equivalence classes under $\equiv_e$ are called enumeration degrees. We shall denote by $\mathbf{D}_e$ the collection of all enumeration degrees. The preorder $\leq_e$ on sets induces a partial order $\leq$ on degrees, defined by

$$\mathbf{a} \leq \mathbf{b} \iff \exists A \in \mathbf{a} \exists B \in \mathbf{b}[A \leq_e B].$$

The degree $\mathbf{0}_e$ consisting of all c.e. sets is the least degree in $\mathbf{D}_e$. Furthermore the degree $\mathbf{d}_e(A \oplus B)$, where $A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$, is the least degree which is greater then or equal to the degrees $\mathbf{d}_e(A)$ and $\mathbf{d}_e(B)$. Thus $\mathcal{D}_e = (\mathbf{D}_e, \leq)$ is an upper semilattice with least element.

Besides the join of two set, we shall need the notion of a uniform join of a system of sets. Let $I \subseteq \mathbb{N}$ and let $\{X_i \mid i \in I\}$ be a system of sets of natural numbers. We define the uniform join of the system to be

$$\bigoplus_{i \in I} X_i = \{\langle x, i\rangle \mid x \in X_i, i \in I\}.$$

The uniform join of a system $\{X_i \mid i \in I\}$ is the least uniform upper bound for it, in the sense that $X_i \leq_e A$ uniformly in i and $I \leq_e A$ if and only if $\bigoplus_{i \in I} X_i \leq_e A$. Note that for a finite system $\{X_i \mid i \leq n\}$ we have $\bigoplus_{i \leq n} X_i \equiv_e X_0 \oplus X_1 \oplus \cdots \oplus X_n$.

We introduce the enumeration jump on sets by setting $A' = K_A \oplus (\mathbb{N} \setminus K_A)$, where $K_A = \bigoplus_{i<\omega} W_i(A)$. The jump operation on sets has the property $A \leq_e B \Rightarrow A' \leq_e B'$ and hence we can define a jump operation on degrees by setting

$$\mathbf{d}_e(A)' = \mathbf{d}_e(A').$$

Furthermore $A \leq_e A'$ and $A' \not\leq_e A$, so that the jump operation on degrees is strictly monotone.

This paper is dedicated to the degrees that lie between the least degree $\mathbf{0}_e$ and its first jump $\mathbf{0}_e'$, i.e. the degrees in the interval $[\mathbf{0}_e, \mathbf{0}_e']$. We shall denote by $\mathcal{G}_e$ the substructure $\mathcal{G}_e = ([\mathbf{0}_e, \mathbf{0}_e'], \leq)$ of $\mathcal{D}_e$. The theory of $\mathcal{G}_e$ is referred to as the local theory of the enumeration degrees. As noted above, Cooper [1] has shown that the degrees in $\mathcal{G}_e$ are exactly the degrees of the Σ^0_2 sets, so that the local theory of the enumeration degrees is actually the theory of the Σ^0_2 enumeration degrees.

A special role in this paper shall be played by the so called low degrees. A degree $\mathbf{a}$ is low if and only if $\mathbf{a}' = \mathbf{0}_e'$. The sets contained in a low degree are called low sets. Thus A is a low set if and only if $A' \equiv_e \emptyset'$. The low sets have the following characterization:

$$A \text{ is low} \iff W_i(A) \text{ is a } \Delta^0_2 \text{ set for every } i.$$

In particular every low set is a Δ^0_2 set.

An instance of low degrees (and sets) are the so called Kalimullin pairs, or briefly $\mathcal{K}$-pairs, of degrees in $\mathcal{G}_e$ (or Σ^0_2 sets). We say that the pair of sets $\{A, B\}$ is a $\mathcal{K}$-pair if both A and B are not c.e.[2] and $A \times B \subseteq W_i$, $\overline{A} \times \overline{B} \subseteq \overline{W_i}$ for some i, where $A \times B = \{\langle a, b\rangle \mid a \in A, b \in B\}$. We say that the pair of degrees $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair, if there are $A \in \mathbf{a}$ and $B \in \mathbf{b}$, such that $\{A, B\}$ is a $\mathcal{K}$-pair of sets. The following properties proved by Kalimullin [9] shall be important for us:

(1) Let $\{\mathbf{a}, \mathbf{b}\}$ be a $\mathcal{K}$-pair and $\mathbf{a}, \mathbf{b} \leq \mathbf{0}_e'$. Then both $\mathbf{a}$ and $\mathbf{b}$ are low.

(2) Let $\{\mathbf{a}, \mathbf{b}\}$ be a $\mathcal{K}$-pair. Then for every $\mathbf{x}$

$$\mathbf{x} = (\mathbf{a} \vee \mathbf{x}) \wedge (\mathbf{b} \vee \mathbf{x}).$$

In particular every $\mathcal{K}$ pair is a minimal pair. Furthermore if $\mathbf{a}_1 \leq \mathbf{a}$, $\mathbf{b}_1 \leq \mathbf{b}$ and $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair, so is the pair $\{\mathbf{a}_1, \mathbf{b}_1\}$.

[2] In the original definition this is not required, but it is useful for our goals.

In [5] we have seen that there is a formula $\mathcal{L}K(\boldsymbol{a},\boldsymbol{b})$ that locally defines the $\mathcal{K}$-pairs, i.e. for arbitrary $\mathbf{a},\mathbf{b}\in\mathcal{G}_e$

$$\{\mathbf{a},\mathbf{b}\} \text{ is a } \mathcal{K}\text{-pair} \iff \mathcal{G}_e \models \mathcal{L}K(\mathbf{a},\mathbf{b}).$$

Thus we have a first order definable class of low degrees in $\mathcal{G}_e$, namely the class of the degrees in $\mathcal{G}_e$ that are part of $\mathcal{K}$-pairs. Since Jockusch [7] has proved that each total degree is the least upper bound of a $\mathcal{K}$-pair, this class is not empty.

Low degrees allow us to encode a special kind of antichains, i.e. sets of pairwise incomparable degrees. Consider the formula $SW(\boldsymbol{z},\boldsymbol{a},\boldsymbol{p}_1,\boldsymbol{p}_2)$ defined by

$$\boldsymbol{z}\leq\boldsymbol{a}\;\&\;\boldsymbol{z}\neq(\boldsymbol{z}\vee\boldsymbol{p}_1)\wedge(\boldsymbol{z}\vee\boldsymbol{p}_2)\;\&\;\forall\boldsymbol{y}<\boldsymbol{z}[\boldsymbol{y}=(\boldsymbol{y}\vee\boldsymbol{p}_1)\wedge(\boldsymbol{y}\vee\boldsymbol{p}_2)],$$

i.e. $\boldsymbol{z}$ is less or equal to $\boldsymbol{a}$ and it is a minimal solution to $\boldsymbol{x}\neq(\boldsymbol{x}\vee\boldsymbol{p}_1)\wedge(\boldsymbol{x}\vee\boldsymbol{p}_2)$. Obviously the set $Z(\mathbf{a},\mathbf{p}_1,\mathbf{p}_2)=\{\mathbf{z}\in\mathcal{G}_e\mid\mathcal{G}_e\models SW(\mathbf{z},\mathbf{a},\mathbf{p}_1,\mathbf{p}_2)\}$ is an antichain for every choice of $\mathbf{a}$, $\mathbf{p}_1$ and $\mathbf{p}_2$ in $\mathcal{G}_e$. On the other hand it is not known whether every antichain in $\mathcal{G}_e$ can be encoded by the formula SW. However, the following result of Slaman and Woodin [16] will be enough for our purposes:

Theorem 1 (Slaman, Woodin [16]). *Let $k\leq\omega$. Let A be a low set and let $\{Z_i\mid i<k\}$ be a system of incomparable reals (i.e. for each $i\neq j$, $Z_i\not\leq_e Z_j$ and $Z_j\not\leq_e Z_i$) uniformly e-reducible to A. Then there are degrees $\mathbf{p}_1,\mathbf{p}_2\in\mathcal{G}_e$, such that for all $\mathbf{z}$*

$$\mathcal{G}_e\models SW(\mathbf{z},\mathbf{d}_e(A),\mathbf{p}_1,\mathbf{p}_2)\iff\mathbf{z}=\mathbf{d}_e(Z_i)\text{ for some } i<\omega.$$

The notion of a $\mathcal{K}$-system introduced in [6] is a natural extension of the notion of a $\mathcal{K}$-pair. We say that the finite system of sets $\{A_1,\dots,A_n\}$ (degrees $\{\mathbf{a}_1,\dots,\mathbf{a}_n\}$), $n\geq 2$, is a $\mathcal{K}$-system if any pair $\{A_i,A_j\}$ ($\{\mathbf{a}_i,\mathbf{a}_j\}$) for $i\neq j$ is a $\mathcal{K}$-pair. Finite $\mathcal{K}$-systems have the following property easily derived from the definition:

Proposition 1 (GS[6]). *Let $\{A_1,\dots,A_n\}$ be a finite $\mathcal{K}$-system. For arbitrary disjoint subsets R_1 and R_2 of $\{1,\dots,n\}$, the pair $\{\bigoplus_{i\in R_1}A_i,\bigoplus_{i\in R_2}A_i\}$ is a $\mathcal{K}$-pair. Furthermore for every $R_1,R_2\subseteq\{1,\dots,n\}$ we have*

$$\bigoplus_{i\in R_1}A_i\leq_e\bigoplus_{i\in R_2}A_i\iff R_1\subseteq R_2$$

In order to be able to prove a property analogous to the above proposition, in the infinite case we introduce a further uniformity condition. We say that a system of sets $\{A_i|i<\omega\}$ is a uniform $\mathcal{K}$-system, if $\forall i[A_i\not\leq_e\emptyset]$ and there is a computable function r, such that for each $i\neq j$

$$A_i\times A_j\subseteq W_{r(i,j)},\overline{A_i}\times\overline{A_j}\subseteq\overline{W_{r(i,j)}},$$

i.e. $\{A_i,A_j\}$ is a $\mathcal{K}$-pair via the c.e. set $W_{r(i,j)}$. The following property holds for uniform $\mathcal{K}$-systems:

Proposition 2 (GS[6]). *Let $\{A_i\mid i<\omega\}$ be a uniform $\mathcal{K}$-system. Then for arbitrary disjoint computable sets R_1 and R_2, $\{\bigoplus_{i\in R_1}A_i,\bigoplus_{i\in R_2}A_i\}$ is a $\mathcal{K}$-pair. Furthermore for every computable R_1 and R_2*

$$\bigoplus_{i\in R_1}A_i\leq_e\bigoplus_{i\in R_2}A_i\iff R_1\subseteq R_2$$

Proposition 1 and Proposition 2 show that every $\mathcal{K}$-system of sets, either finite or uniform, is a system of incomparable reals. Thus whenever a $\mathcal{K}$-system of degrees is bounded by a low degree it can be encoded by three parameters via the formula SW.

Existence of finite $\mathcal{K}$-systems consisting of three sets is proven by Kalimullin [9]. The existence of uniform $\mathcal{K}$-system (and thus in particular of finite $\mathcal{K}$-systems of arbitrary cardinality) is proven in [6].

Proposition 3. *Let B be a non c.e. Δ^0_2 set. Then there is a uniform $\mathcal{K}$-system $\{A_i \mid i < \omega\}$, such that $\bigoplus_{i<\omega} A_i \leq_e B$.*

3. Coding sets and relations in $\mathcal{G}_e$

In this section we follow the lines of Slaman and Woodin's [16] coding of countable sets and relations in $\mathcal{D}_e$. The coding of arbitrary countable relations in $\mathcal{D}_e$ relies on the following two assertions:

(i) There is a formula coding every countable antichain via parameters.
(ii) For every set A, there is a set $\mathcal{C} = \{C_i \mid i < \omega\}$ of incomparable reals, such that for every $X, Y \leq_e A$ and every $i, j < \omega$

$$C_i \oplus X \leq_e C_j \oplus Y \iff i = j \ \& \ X \leq_e Y. \tag{3.1}$$

In order to prove a coding lemma for the local theory we shall need properties analogous to (i) and (ii). The analogue of (i) is provided by the local version of the Slaman-Woodin antichains coding theorem (Theorem 1). However, this theorem is not as powerful as the global one. Indeed, it guarantees that a set of Σ^0_2 incomparable reals is definable by parameters in $\mathcal{G}_e$ only in the case when it is uniformly reducible to a low set. Thus we need to prove a stronger version of (ii), namely we need to require that the set $\mathcal{C}$ is uniformly reducible to a low set.

In the global theory property (3.1) is satisfied by every countable set $\mathcal{C}$ of reals that are mutually Cohen generic with respect to meeting all dense sets that are arithmetic in A. Due to the genericity of its elements, $\mathcal{C}$ is not bounded by $A^{(n)}$ for any n and hence it is not usable in the local theory. If we relax the condition of genericity only to the dense sets that are necessary to meet the property (3.1), we would obtain (by means of the usual forcing argument) an antichain $\mathcal{C}$, for which the best estimated upper bound is the first jump of A which is obviously not low. However it turns out that in $\mathcal{G}_e$ we can use uniform $\mathcal{K}$-systems instead of generic reals.

Lemma 1. *Let $\{A, B\}$ be a nontrivial $\mathcal{K}$-pair and let $\mathcal{C} = \{C_i \mid i < \omega\}$ be a uniform $\mathcal{K}$-system bounded by B. Then for every $X, Y \leq_e A$ and every $i, j < \omega$*

$$C_i \oplus X \leq_e C_j \oplus Y \iff i = j \ \& \ X \leq_e Y.$$

Proof. Suppose that $X, Y \leq_e A$ and that $C_i \oplus X \leq_e C_j \oplus Y$. The second inequality implies $C_i \leq_e C_j \oplus Y$. On the other hand $C_i \leq_e C_i \oplus Y$ and hence if it was the case $i \neq j$, $\{C_i, C_j\}$ would be a $\mathcal{K}$-pair and we would have $C_i \leq_e Y \leq_e A$. But $C_i \not\leq_e A$, so that $i = j$.

Thus $C_i \oplus X \leq_e C_i \oplus Y$. In particular $X \leq_e C_i \oplus Y$. On the other hand $X \leq_e X \oplus Y$. But $X \leq_e A$ and $C_i \leq_e B$ so that $\{X, C_i\}$ is a $\mathcal{K}$-pair. Hence $X \leq_e Y$.

□

Lemma 1 shows that $\mathcal{K}$-systems are antichains satisfying (3.1) and thus appropriate for coding in $\mathcal{G}_e$ sets and relations bounded by a half of a nontrivial $\mathcal{K}$-pair. We will prove that there is a sufficiently large class of sets and relations definable via parameters in $\mathcal{G}_e$.

Definition 1. *Let* $\mathbf{a} \in \mathcal{G}_e$ *and let* R *be a* k*-ary relation in the interval* $[\mathbf{0}_e, \mathbf{a}]$*. We shall say that* $\mathbf{R}$ *is e-presentable beneath* $\mathbf{a}$ *if there is an* $A \in \mathbf{a}$ *and a c.e. set* W*, such that*

$$\mathbf{R} = \{(\mathbf{d}_e(W_{i_1}(A)), \mathbf{d}_e(W_{i_2}(A)), \dots, \mathbf{d}_e(W_{i_k}(A))) \mid \langle i_1, i_2, \dots, i_k\rangle \in W\}.$$

In particular we shall say that $\mathbf{R}$ *is e-presentable beneath* $\mathbf{a}$ *via* A *and* W

Let $\mathbf{R}$ be an n-ary relation on degrees. For $1 \leq k \leq n$ let us denote by $\mathbf{R}(k)$ the k-th projection of $\mathbf{R}$, i.e.

$$\mathbf{R}(k) = \{\mathbf{r} \mid \exists \mathbf{r}_1, \dots, \mathbf{r}_{k-1}, \mathbf{r}_{k+1}, \dots, \mathbf{r}_k[(\mathbf{r}_1, \dots, \mathbf{r}_{k-1}, \mathbf{r}, \mathbf{r}_{k+1}, \dots, \mathbf{r}_k) \in \mathbf{R}]\}.$$

Note that if $\mathbf{R}$ is bounded by $\mathbf{a}$, i.e. the domain of $\mathbf{R}$ is bounded by $\mathbf{a}$, then $\mathbf{R}(k)$ is also bounded by $\mathbf{a}$. Furthermore if $\mathbf{R}$ is e-presentable beneath $\mathbf{a}$ via A and W, then $\mathbf{R}(k)$ is e-presentable beneath $\mathbf{a}$ via A and $W(k)$, where

$$W(k) = \{i \mid \exists i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_k[\langle i_1, \dots, i_{k-1}, i, i_{k+1}, \dots, i_k\rangle \in W]\}.$$

Theorem 2. *For every* $n \geq 1$ *there is a formula* $\varphi_n(\boldsymbol{x}_1, \dots, \mathbf{x}_n, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{p}_1, \dots, \boldsymbol{p}_{4n+2})$*, such that for every half of a nontrivial* $\mathcal{K}$*-pair* $\mathbf{a}$*, and e-presentable beneath* $\mathbf{a}$ n*-ary relation* $\mathbf{R}$*, there are parameters* $\mathbf{b}$*,* $\mathbf{p}_1$*,…,* $\mathbf{p}_{4n+2}$*, such that*

$$(\mathbf{x}_1, \dots, \mathbf{x}_n) \in \mathbf{R} \iff \mathcal{G}_e \models \varphi_n(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{a}, \mathbf{b}, \mathbf{p}_1, \dots, \mathbf{p}_{4n+2})$$

Proof. Let $\mathbf{a}$ be a half of $\mathcal{K}$-pair and let $\mathbf{a}_1$ be such that $\{\mathbf{a}, \mathbf{a}_1\}$ is a $\mathcal{K}$-pair. Take a $\mathcal{K}$-pair $\{\mathbf{b}, \mathbf{b}_1\}$ beneath $\mathbf{a}_1$. Then $\{\mathbf{a}, \mathbf{b}, \mathbf{b}_1\}$ is a $\mathcal{K}$-system and in particular $\{\mathbf{a}, \mathbf{b}\}$ is a $\mathcal{K}$-pair. Note that since $\{\mathbf{a} \vee \mathbf{b}, \mathbf{b}_1\}$ is also a $\mathcal{K}$-pair, $\mathbf{a} \vee \mathbf{b}$ is low.

Fix an integer $n \geq 1$. For an arbitrary $i \in \mathbb{N}$ denote by $Div(i)$ the divisor and by $Rem(i)$ the remainder (or residue) resulting from the division of i by n.

Let $\mathbf{R}$ be an n-ary e-presentable relation beneath $\mathbf{a}$ via A and W. Fix $B \in \mathbf{b}$ and a uniform $\mathcal{K}$-system $\mathcal{C} = \{C_i \mid i < \omega\}$ beneath B. For $1 \leq k \leq n$ denote by $\mathcal{C}(k)$ the uniform $\mathcal{K}$-system $\mathcal{C}(k) = \{C_i \mid Rem(i) = k\}$. Note that $\mathcal{C}(k)$ is uniformly beneath B and hence the set $\mathbf{C}(k) = \{\mathbf{d}_e(C) \mid C \in \mathcal{C}(k)\}$ is definable with parameters. From this definition we obtain the parameters $\mathbf{p}_1, \dots, \mathbf{p}_{2n}$. We shall use $\mathbf{C}(k)$ to code the projections $\mathbf{R}(k)$.

Let $\mathcal{C}(k) + \mathcal{R}(k) = \{C_i \oplus W_j(A) \mid Div(i) = j \in W(k) \;\&\; Rem(i) = k\}$ for $1 \leq k \leq n$. Clearly $\mathcal{C}(k) + \mathcal{R}(k)$ is uniformly reducible to $A \oplus B$. Furthermore, Lemma 1 yields that $\mathcal{C}(k) + \mathcal{R}(k)$ is a set of incomparable reals. Thus $\mathbf{C}(k) + \mathbf{R}(k) = \{\mathbf{d}_e(Y) \mid Y \in \mathcal{C}(k) + \mathcal{R}(k)\}$ is uniformly definable via parameters in $\mathcal{G}_e$. This gives us parameters $\mathbf{p}_{2n+1} \dots \mathbf{p}_{4n}$. Besides Lemma 1 yields

$$\forall \mathbf{x} \leq \mathbf{a}\, [\, \mathbf{x} \in \mathbf{R}(k) \iff \exists \mathbf{c} \in \mathbf{C}(k)(\mathbf{c} \vee \mathbf{x} \in \mathbf{C}(k) + \mathbf{R}(k))\,],$$

and hence each of the projections $\mathbf{R}(k)$ is uniformly definable via parameters.

In order to code the relation $\mathbf{R}$ we shall need one more antichain. Consider the set

$$\mathcal{C}_W = \{C_{i_1} \oplus \dots \oplus C_{i_n} \mid Rem(i_k) = k, \text{ for } 1 \leq k \leq n \text{ and } \langle Div(i_1), \dots, Div(i_n)\rangle \in W\}.$$

We claim that $\mathcal{C}_W$ is a system of incomparable reals uniformly beneath B. Indeed, suppose that $C_{i_1} \oplus \dots \oplus C_{i_n} \leq_e C_{j_1} \oplus \dots \oplus C_{j_n}$ for some $C_{i_1} \oplus \dots \oplus C_{i_n} \in \mathcal{C}_W$ and $C_{j_1} \oplus \dots \oplus C_{j_n} \in W$. Hence, according to Proposition 2, $\{i_1, \dots, i_n\} \subseteq \{j_1, \dots, j_n\}$. On the other hand $Rem(i_1) = Rem(j_1) = 1$,…, $Rem(i_n) = Rem(j_n) = n$, so that $i_1 = j_1$,…, $i_n = j_n$. Thus the set $\mathbf{C}_W = \{\mathbf{d}_e(C) \mid C \in \mathcal{C}_W\}$ is an antichain uniform in $\mathbf{b}$ and so is definable with parameters $\mathbf{b}, \mathbf{p}_{4n+1}$ and $\mathbf{p}_{4n+2}$.

Finally let $X_1, \dots, X_n \leq_e A$ be such that there is a $C_{i_1} \oplus \cdots \oplus C_{i_n} \in \mathcal{C}_W$, such that for each $1 \leq k \leq n$, $C_{i_k} \oplus X_k$ is enumeration equivalent to some $C_{j_k} \oplus W_{Div(j_k)}(A) \in \mathcal{C}(k) \oplus \mathcal{R}(k)$. Then Lemma 1 yields $i_k = j_k$ and $X_k \equiv_e W_{Div(i_k)}(A)$ for $1 \leq k \leq n$. But $\langle Div(i_1), \dots, Div(i_n) \rangle \in W$ and hence $(\mathbf{d}_e(X_1), \dots, \mathbf{d}_e(X_n)) \in \mathbf{R}$.

Thus for arbitrary $\mathbf{x}_1, \dots, \mathbf{x}_n \leq \mathbf{a}$, the n-tuple $(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is an element of the relation $\mathbf{R}$ if and only if

$$\exists \mathbf{c}_1 \in \mathbf{C}(1) \dots \exists \mathbf{c}_n \in \mathbf{C}(n)[\forall 1 \leq k \leq n(\mathbf{c}_k \vee \mathbf{x}_k \in \mathbf{C}(k) + \mathbf{R}(k))] \ \&\ \mathbf{c}_1 \vee \cdots \vee \mathbf{c}_n \in \mathbf{C}_W].$$

□

4. Interpreting true arithmetic in $\mathcal{G}_e$

Let us fix a finite axiomatization PA^- of arithmetic in the language $\{+, \times, <\}$, such that every model of PA^- has a standard part. Let $\mathcal{B}_{PA^-}$ be the class of all models of PA^-. For arbitrary $\mathfrak{N} \in \mathcal{B}_{PA^-}$ and $\mathbf{x} \in \mathfrak{N}$ let us denote by $L_{\mathfrak{N}}(\mathbf{x})$ the set of all elements of $\mathfrak{N}$ less or equal to $\mathbf{x}$ in $\mathfrak{N}$, i.e.

$$L_{\mathfrak{N}}(\mathbf{x}) = \{\mathbf{z} \in \mathfrak{N} \mid \mathfrak{N} \models \mathbf{z} \leq \mathbf{x}\} \tag{4.1}$$

We have the following characterisation of the standard models of PA^- in $\mathcal{B}_{PA^-}$.

Lemma 2. *Let $\mathcal{A} \subseteq \mathcal{B}_{PA^-}$ be a class of models of PA^-, containing a standard model of arithmetic. Then for arbitrary $\mathfrak{N}_1 \in \mathcal{B}_{PA^-}$, $\mathfrak{N}_1$ is a standard model of arithmetic, if and only if for every $\mathbf{x}_1 \in \mathfrak{N}_1$ and every $\mathfrak{N}_2 \in \mathcal{A}$, there is an $\mathbf{x}_2 \in \mathfrak{N}_2$ such that the sets $L_{\mathfrak{N}_1}(\mathbf{x}_1)$ and $L_{\mathfrak{N}_2}(\mathbf{x}_2)$ have the same cardinality.*

Proof. Since every model of PA^- has a standard part, $\mathfrak{N}_1$ is a standard model of arithmetic if and only if the set $L_{\mathfrak{N}_1}(\mathbf{x})$ is finite for every $\mathbf{x}_1 \in \mathfrak{N}_1$. Suppose that $\mathfrak{N}_1 \in \mathcal{B}_{PA^-}$ is a standard model of arithmetic and let $\mathbf{x}_1 \in \mathfrak{N}_1$. Then the set $L_{\mathfrak{N}_1}(\mathbf{x}_1)$ has finite cardinality, say n. Take an arbitrary $\mathfrak{N}_2 \in \mathcal{A}$. Since $\mathfrak{N}_2$ has a standard part, then there is an $\mathbf{x}_2 \in \mathfrak{N}_2$, such that $L_{\mathfrak{N}_2}(\mathbf{x}_2)$ has cardinality n.

For the converse direction suppose that $\mathfrak{N}_1 \in \mathcal{B}_{PA^-}$ is such that for every $\mathbf{x}_1 \in \mathfrak{N}_1$ and every $\mathfrak{N}_2 \in \mathcal{A}$, there is $\mathbf{x}_2 \in \mathfrak{N}_2$, such that the sets $L_{\mathfrak{N}_1}(\mathbf{x}_1)$ and $L_{\mathfrak{N}_2}(\mathbf{x}_2)$ have the same cardinality. Since there is a standard model of arithmetic in $\mathcal{A}$, then the set $L_{\mathfrak{N}_1}(\mathbf{x}_1)$ is finite for arbitrary $\mathbf{x}_1 \in \mathfrak{N}_1$ and hence $\mathfrak{N}_1$ is also standard.

□

Fix a formula θ_{PA^-} expressing the following facts for an arbitrary degree $\mathbf{a}$ and parameters $\mathbf{b}_N$, $\overline{\mathbf{p}}_N$, $\mathbf{b}_+$, $\overline{\mathbf{p}}_+$, $\mathbf{b}_\times$, $\overline{\mathbf{p}}_\times$, $\mathbf{b}_<$ and $\overline{\mathbf{p}}_<$ (we shall denote such a list of parameters by $\overline{\mathbf{q}}$):

- The relations $\mathbf{R}_+ = \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) \mid \mathcal{G}_e \models \varphi_3(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{a}, \mathbf{b}_+, \overline{\mathbf{p}}_+)\}$ and $\mathbf{R}_\times = \{(\mathbf{x}, \mathbf{y}, \mathbf{z}) \mid \mathcal{G}_e \models \varphi_3(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{a}, \mathbf{b}_\times, \overline{\mathbf{p}}_\times)\}$ are (the graphs of two) binary operations on $\mathbf{R}_N = \{\mathbf{x} \mid \mathcal{G}_e \models \varphi_1(\mathbf{x}, \mathbf{a}, \mathbf{b}_N, \overline{\mathbf{p}}_N)\}$.
- The relation $\mathbf{R}_< = \{(\mathbf{x}, \mathbf{y}) \mid \mathcal{G}_e \models \varphi_2(\mathbf{x}, \mathbf{y}, \mathbf{a}, \mathbf{b}_<, \overline{\mathbf{p}}_<)\}$ is a binary relation on $\mathbf{R}_N$.
- $(\mathbf{R}_N; \mathbf{R}_+, \mathbf{R}_\times, \mathbf{R}_<)$ is a model of PA^-. We shall denote this model by $\mathfrak{N}(\mathbf{a}, \overline{\mathbf{q}})$.

Thus for every $\mathbf{a} \in \mathcal{G}_e$ the formula θ_{PA^-} defines a class $\mathcal{N}(\mathbf{a})$ of models of PA^- bounded by $\mathbf{a}$.

Now suppose that $\mathbf{a}$ is a half of a nontrivial $\mathcal{K}$-pair. We claim that the class $\mathcal{N}(\mathbf{a})$ is nonempty and contains a standard model of arithmetic. Indeed, fix an independent system of reals $\mathcal{X} = \{X_i \mid i < \omega\}$ uniform in $A \in \mathbf{a}$ and denote by

$\mathbf{x}_i$ the degree of X_i. Let $\mathbf{R}_N^{\mathcal{X}} = \{\mathbf{x}_i \mid i < \omega\}$, $\mathbf{R}_+^{\mathcal{X}} = \{(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_{i+j}) \mid i, j < \omega\}$, $\mathbf{R}_\times^{\mathcal{X}} = \{(\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_{i\times j}) \mid i, j < \omega\}$ and $\mathbf{R}_<^{\mathcal{X}} = \{(\mathbf{x}_i, \mathbf{x}_j) \mid i < j < \omega\}$. It is clear that these relations are e-presentable beneath $\mathbf{a}$ and hence they are definable via parameters. Furthermore $(\mathbf{R}_N^{\mathcal{X}}; \mathbf{R}_+^{\mathcal{X}}, \mathbf{R}_\times^{\mathcal{X}}, \mathbf{R}_<^{\mathcal{X}})$ is a standard model of arithmetic.

Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be two models in $\mathcal{N}(\mathbf{a})$ coded by $\overline{\mathbf{q}}_1$, and $\overline{\mathbf{q}}_2$ respectively. For any $\mathbf{x}_1 \in \mathfrak{N}_1$ and $\mathbf{x}_2 \in \mathfrak{N}_2$ we shall say that $(\mathbf{x}_1, \overline{\mathbf{q}}_1) \approx (\mathbf{x}_2, \overline{\mathbf{q}}_2)$ if and only if there are parameters $\mathbf{b}_\approx$ and $\overline{\mathbf{p}}_\approx$ for the formula φ_2, such that the relation $\mathbf{R}_\approx = \{(\mathbf{z}_1, \mathbf{z}_2) \mid \mathcal{G}_e \models (\mathbf{z}_1, \mathbf{z}_2, \mathbf{a}, \mathbf{b}_\approx, \overline{\mathbf{p}}_\approx)$ is the graph of a bijection from $L_{\mathfrak{N}_1}(\mathbf{x}_1)$ into $L_{\mathfrak{N}_2}(\mathbf{x}_2)$ (in particular the sets $L_{\mathfrak{N}_1}(\mathbf{x}_1)$ and $L_{\mathfrak{N}_2}(\mathbf{x}_2)$ have the same cardinality).

Note that if $\mathbf{x}_1$ and $\mathbf{x}_2$ represent the same standard natural number then $(\mathbf{x}_1, \mathbf{q}_1) \approx (\mathbf{x}_2, \mathbf{q}_2)$. Indeed, in this case the sets $L_{\mathfrak{N}_1}(\mathbf{x}_1)$ and $L_{\mathfrak{N}_2}(\mathbf{x}_2)$ have the same finite cardinality, so that the set $L_{\mathfrak{N}_1}(\mathbf{x}_1) \cup L_{\mathfrak{N}_2}(\mathbf{x}_2)$ is finite and hence e-presentable beneath $\mathbf{a}$. Therefore every bijection from $L_{\mathfrak{N}_1}(\mathbf{x}_1)$ onto $L_{\mathfrak{N}_2}(\mathbf{x}_2)$ is e-presentable beneath $\mathbf{a}$ and hence definable by the formula φ_2. Thus $(\mathbf{x}_1, \mathbf{q}_1) \approx (\mathbf{x}_2, \mathbf{q}_2)$.

Now for every half of a nontrivial $\mathcal{K}$-pair $\mathbf{a}$ we can select the standard models of arithmetic in $\mathcal{N}(\mathbf{a})$ in the following way: A model $\mathfrak{N}_1 \in \mathcal{N}(\mathbf{a})$ coded via the parameters $\mathbf{q}_1$ is a standard model of arithmetic if and only if for every $\mathbf{x} \in \mathfrak{N}_1$ and every $\mathfrak{N}_2 \in \mathcal{N}(\mathbf{a})$ (coded by, say, $\mathbf{q}_2$), there is a $\mathbf{y} \in \mathfrak{N}_2$, such that $(\mathbf{x}, \mathbf{q}_1) \approx (\mathbf{y}, \mathbf{q}_2)$. Thus we have proven the following theorem.

Theorem 3. *There is a formula SMA such that for every half $\mathbf{a}$ of a nontrivial $\mathcal{K}$-pair the following assertions hold:*

(i) *For every choice of parameters $\overline{\mathbf{q}}$, if $\mathcal{G}_e \models SMA(\mathbf{a}, \overline{\mathbf{q}})$, then $\mathfrak{N}(\mathbf{a}, \overline{\mathbf{q}})$ is a standard model of arithmetic.*
(ii) *There are parameters $\overline{\mathbf{q}}$, such that $\mathcal{G}_e \models SMA(\mathbf{a}, \overline{\mathbf{q}})$.*

Thus we have defined a class of standard models of arithmetic in $\mathcal{G}_e$, namely

$$\mathcal{N}_{\mathcal{G}_e} = \{\mathfrak{N}(\mathbf{a}, \overline{\mathbf{q}}) \mid \mathbf{a} \text{ is a half of a } \mathcal{K}\text{-pair and } \mathcal{G}_e \models SMA(\mathbf{a}, \overline{\mathbf{q}})\}.$$

Hence for every arithmetical sentence θ, $\mathbb{N} \models \theta$ if and only if for every $\mathcal{K}$-pair $\{\mathbf{a}, \mathbf{b}\}$ there are parameters $\overline{\mathbf{q}}$, such that $\mathcal{G}_e \models SMA(\mathbf{a}, \overline{\mathbf{q}})$ and $\mathcal{N}(\mathbf{a}, \overline{\mathbf{q}}) \models \theta$. From here we obtain the computable translation of the arithmetical sentences into sentences of $\mathcal{G}_e$.

The results so far can be extended to a definition of a parameterless standard model of arithmetic in $\mathcal{G}_e$. In order to do this, it is enough to show that the equivalence relation on tuples of the form $(\mathbf{x}, \mathbf{a}, \overline{\mathbf{q}})$, where $\mathbf{a}$ is a half of a $\mathcal{K}$-pair, $\mathcal{G}_e \models SMA(\mathbf{a}, \overline{\mathbf{q}})$ and $\mathbf{x} \in \mathfrak{N}(\mathbf{a}, \overline{\mathbf{q}})$, defined by

$$(\mathbf{x}_1, \mathbf{a}_1, \overline{\mathbf{q}}_1) \simeq (\mathbf{x}_2, \mathbf{a}_2, \overline{\mathbf{q}}_2) \iff \mathbf{x}_1 \text{ and } \mathbf{x}_2 \text{ represent the same natural number},$$

is definable without parameters in $\mathcal{G}_e$.

First of all note that if $\mathbf{a}_1, \mathbf{a}_2 \leq \mathbf{a}$ for some half of $\mathcal{K}$-pair $\mathbf{a}$ we have

$$(\mathbf{x}_1, \mathbf{a}_1, \overline{\mathbf{q}}_1) \simeq (\mathbf{x}_2, \mathbf{a}_2, \overline{\mathbf{q}}_2) \iff (\mathbf{x}_1, \overline{\mathbf{q}}_1) \approx (\mathbf{x}_2, \overline{\mathbf{q}}_2),$$

for arbitrary $\mathbf{q}_1$ and $\mathbf{q}_2$, satisfying $\mathcal{G}_e \models SMA(\mathbf{a}_1, \overline{\mathbf{q}}_1)$ and $\mathcal{G}_e \models SMA(\mathbf{a}_2, \overline{\mathbf{q}}_2)$. Thus we can compare all standard models bounded by a fixed half of $\mathcal{K}$-pair. We extend this result by means of the following lemma.

Lemma 3. *Let A_0 and A_1 be non c.e. Δ^0_2 sets. Then there is a $\mathcal{K}$-pair $\{B_0, B_1\}$ such that $B_0 \leq_e A_0$ and $B_1 \leq_e A_1$.*

Proof. Kallimulin [8] has proved the statement in the case $A_0 = A_1$. Since the proof in the case $A_0 \neq A_1$ is analogous, we give here only a sketch.

Fix Δ^0_2 good approximations $\{A^s_0\}$ and $\{A^s_1\}$ of A_0 and A_1 respectively. Define by induction on s the following computable sequences V^s_0 and V^s_1 of finite sets:

- Set $V^0_0 = V^0_1 = \emptyset$.
- Suppose that V^s_0 and V^s_1 are defined. If $V^s_i(A^s_i) \setminus V^s_i(A^{s+1}_i) = \emptyset$ for $i \leq 1$, set $\hat{V}^s_i = V^s_i$. Otherwise let k be the least natural number for which there is an x and an $i \leq 1$, such that $\langle k, x\rangle \in V^s_i(A^s_i) \setminus V^s_i(A^{s+1}_i) = \emptyset$. Set $\hat{V}^s_i = V^s_i$ and
$$\hat{V}^s_{1-i} = V^s_{1-i} \cup \{\langle\langle r, y\rangle, \emptyset\rangle \mid r \geq k \;\&\; \langle r, y\rangle < s\}.$$
For each e, s and $i \leq 1$, denote by $l(i, e, s)$ the length of agreement between $V^s_i(A^s_i)$ and W^s_e (here $\{W^s_e\}$ is a fixed c.e. approximation of W_e). For each $i \leq 1$ choose the least e_i such that $l(i, e_i, s) > \max\{l(i, e_i, k) \mid k < s\}$. If such an e_i does not exist set $V^{s+1}_i = \hat{V}^s_i$. Otherwise set
$$V^{s+1}_i = \hat{V}^s_i \cup \{\langle\langle e_i, y\rangle, \{y\}\rangle \mid \langle e_i, y\rangle < s\}.$$

It is clear that the sets $V_i = \bigcup V^s_i$ for $i \leq 1$ are c.e. We set $B_i = V_i(A_i)$. Now from the construction of V_0 and V_1 it follows that B_0 and B_1 are not c.e. Furthermore the sequences $\{V^s_0(A^s_0)\}$ and $\{V^s_1(A^s_1)\}$ are Δ^0_2 approximations to B_0 and B_1 respectively, having the following property for arbitrary i and s:
$$\left(V^s_i(A^s_i) \setminus V^{s+1}_i(A^{s+1}_i)\right) \cap \omega^{[k]} \neq \emptyset \Rightarrow \omega^{[\geq k]} \restriction s \subseteq V_{1-i}(A_{1-i}).$$
Kalimullin [8] has proved that the above property is a sufficient condition for $\{V_0(A_0), V_1(A_1)\}$ to be a $\mathcal{K}$-pair.

□

Now let us turn to the proof of the definability in $\mathcal{G}_e$ of the relation $\simeq$. Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be arbitrary halves of $\mathcal{K}$-pairs and let $\overline{\mathbf{q}}_1$ and $\overline{\mathbf{q}}_2$ be such that $\mathcal{G}_e \models SMA(\mathbf{a}_1, \overline{\mathbf{q}}_1)$ and $\mathcal{G}_e \models SMA(\mathbf{a}_2, \overline{\mathbf{q}}_2)$. Let $\mathbf{x}_1 \in \mathfrak{N}(\mathbf{a}_1, \overline{\mathbf{q}}_1)$ and $\mathbf{x}_2 \in \mathfrak{N}(\mathbf{a}_2, \overline{\mathbf{q}}_2)$ represent the same natural number. Fix a $\mathcal{K}$-pair $\{\mathbf{b}_1, \mathbf{b}_2\}$ such that $\mathbf{b}_1 \leq \mathbf{a}_1$, $\mathbf{b}_2 \leq \mathbf{a}_2$ and $\mathbf{b}_1 \vee \mathbf{b}_2$ is half of a $\mathcal{K}$-pair (we can obtain such $\mathbf{b}_1$ and $\mathbf{b}_2$ applying Lemma 3 and the trick used in the proof of Theorem 2). Let $\overline{\mathbf{q}}_{11}$ and $\overline{\mathbf{q}}_{22}$ be such that $\mathcal{G}_e \models SMA(\mathbf{b}_1, \overline{\mathbf{q}}_{11})$ and $\mathcal{G}_e \models SMA(\mathbf{b}_2, \overline{\mathbf{q}}_{22})$ and let $\mathbf{x}_{11} \in \mathfrak{N}(\mathbf{b}_1, \overline{\mathbf{q}}_{11})$ and $\mathbf{x}_{22} \in \mathfrak{N}(\mathbf{b}_2, \overline{\mathbf{q}}_{22})$ represent the same natural number as $\mathbf{x}_1$ and $\mathbf{x}_2$. Since $\mathbf{b}_1 \vee \mathbf{b}_2$ is half of a $\mathcal{K}$-pair, $(\mathbf{x}_{11}, \overline{\mathbf{q}}_{11}) \approx (\mathbf{x}_{22}, \overline{\mathbf{q}}_{22})$. On the other hand $\mathbf{b}_1 \vee \mathbf{a}_1 = \mathbf{a}_1$ and $\mathbf{a}_1$ is a half of a $\mathcal{K}$-pair so that $(\mathbf{x}_1, \overline{\mathbf{q}}_1) \approx (\mathbf{x}_{11}, \overline{\mathbf{q}}_{11})$. Analogously $(\mathbf{x}_{22}, \overline{\mathbf{q}}_{22}) \approx (\mathbf{x}_2, \overline{\mathbf{q}}_2)$.

Thus we obtain that $(\mathbf{x}_1, \mathbf{a}_1, \overline{\mathbf{q}}_1) \simeq (\mathbf{x}_2, \mathbf{a}_2, \overline{\mathbf{q}}_2)$ if and only if there are $\mathbf{b}_1$, $\mathbf{b}_2$, $\overline{\mathbf{q}}_{11}$, $\overline{\mathbf{q}}_{22}$, $\mathbf{x}_{11}$ and $\mathbf{x}_{22}$ such that

(i) $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a $\mathcal{K}$-pair, $\mathbf{b}_1 \leq \mathbf{a}_1$, $\mathbf{b}_2 \leq \mathbf{a}_2$ and $\mathbf{b}_1 \vee \mathbf{b}_2$ is half of a $\mathcal{K}$-pair.
(ii) $\mathcal{G}_e \models SMA(\mathbf{b}_1, \overline{\mathbf{q}}_{11})$ and $\mathcal{G}_e \models SMA(\mathbf{b}_1, \overline{\mathbf{q}}_{11})$.
(iii) $\mathbf{x}_{11} \in \mathfrak{N}(\mathbf{b}_1, \overline{\mathbf{q}}_{11})$ and $\mathbf{x}_{22} \in \mathfrak{N}(\mathbf{b}_2, \overline{\mathbf{q}}_{22})$
(iv) $(\mathbf{x}_1, \overline{\mathbf{q}}_1) \approx (\mathbf{x}_{11}, \overline{\mathbf{q}}_{11})$, $(\mathbf{x}_{11}, \overline{\mathbf{q}}_{11}) \approx (\mathbf{x}_{22}, \overline{\mathbf{q}}_{22})$ and $(\mathbf{x}_{22}, \overline{\mathbf{q}}_{22}) \approx (\mathbf{x}_2, \overline{\mathbf{q}}_2)$.

Thus the relation $\simeq$ is definable in $\mathcal{G}_e$ and we can build a parameterless standard model of arithmetic.

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