The Busemann process in planar directed first-passage percolation

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Contents

We will use the results of [GJR23] to establish the existence of the Busemann process for fairly general directed edge weight percolation models on the plane.

Write $E(\mathbb{Z}^2)$ to denote the collection of nearest-neighbour edges on the integer lattice, and take a collection of weights $(\omega(e))_{e\in E(\mathbb{Z}^2)} \in \Omega = \mathbb{R}^{E(\mathbb{Z}^2)}$ indexed by the edges. For brevity, we will write $\omega(\{x, x+e_i\}) = \omega(x, x+e_i) = \omega_i(x)$. If x, $y \in \mathbb{Z}^2$ are points with $x \leq y$ and $\pi = (x = \pi_0, \pi_1, \ldots, \pi_n = y)$ is a nearest neighbour up-right path between them, we define the passage time of the path to be

$$G(\pi) = \sum_{i=0}^{n-1} \omega(\pi(i), \pi(i+1)). \tag{0.1}$$

Let $\Pi(x, y)$ be the set of such paths. When the first vertex is zero we abbreviate to $\Pi(y) = \Pi(0, y)$. The (first-)passage time from x to y is the minimal passage time among directed paths:

$$G(x, y) = \min_{\pi\in\Pi(x,y)} G(\pi). \tag{0.2}$$

Note that last passage percolation is recovered by negating the signs of the weights. We will primarily be concerned with passage times from the origin, and so we abbreviate $G((0, 0), x) = G(x)$.

So as to have uniform convergence to the limit shape and to fit under Condition 3.2(c) of [GJR23], we make the following assumption on the weights:

Assumption 0.1. *Assume that the pairs* $\{(\omega_1(x), \omega_2(x))\}_{x\in\mathbb{Z}^2}$ *are i.i.d, and that* $\mathbb{E}[|\omega_i(x)|^{2+\epsilon}] < \infty$ *for* $i = 1, 2$ *and some* $\epsilon > 0$.

The assumption will be implicit in every statement below. Note that we don't need the weights on edges originating from a given vertex to be independent of one another.

0.1 Uniform convergence to the limit shape

Uniform convergence to the limit shape will be needed later in establishing the Busemann limits. The statement for the edge weight model is a small extension of Theorem 5.1 in [Mar04]. Here g_{pp} is the limit shape, which exists very generally, as in [GRS16, Theorem 2.4].

Theorem 0.2. *Almost surely*

$$\lim_{n\to 0}\frac{1}{n}\max_{x\in\mathbb{Z}_+^2,|x|_1=n}|G(0,x)-g_{pp}(0,x)|=0. \tag{0.3}$$

The argument in [Mar04] deals with vertex weights, but can be carried out in the same fashion for edge weights. It relies essentially on only two facts: that in a directed model the number of steps (in each direction) in a geodesic is deterministic; and that one has a powerful concentration inequality for passage times of bounded weights[Tal95, Theorem 8.1.1].

The proof goes through a number of lemmas, which we restate for our edge-weight FPP setup. The arguments for the most part require very little modification. The only subtlety is in Martin's Lemma 3.1, which is the statement of the concentration inequality relied on . The statement in our context is:

Lemma 0.3. *Let X_i, $i\in I$, be a finite collection of independent random variables taking values in $[0,L]^d$ and write $X_{i,k}$ for the k-th component of X_i. Let $\mathcal{C}$ be a set of subsets of $I\times[d]$, such that*

$$\max_{C\in\mathcal{C}}|C|\le R, \tag{0.4}$$

and additionally such that if $C\in\mathcal{C}$ and $(i,k_1),(i,k_2)\in C$, then $k_1=k_2$. Set

$$Z=\max_{C\in\mathcal{C}}\sum_{(i,k)\in C}X_{i,k}. \tag{0.5}$$

Then for any $u>0$,

$$\mathbb{P}(|Z-\mathbb{E}[Z]|>u)\le\exp\left(-\frac{u^2}{64RL^2}+64\right). \tag{0.6}$$

The proof in [Mar02, Lemma 5.1] is a direct application of an inequality due to Talagrand's inequality[Tal95, Theorem 8.1.1] on the concentration of passage times, which itself is a quick consequence of his isoperimetric inequality, as stated in [Tal95, Theorem 4.1.1]. An appropriate vector-valued version of the isoperimetric inequality, such as the one found in Section 7.6 of [AS16], gives the corresponding vector-valued version of his passage time inequality.

Theorem 0.4. *Let $(X_i)_{i\le N}$ be a collection of independent random variables with $X_i\in[0,1]^{d_i}$, and for $1\le k\le d_i$ write $X_{i,k}$ for the k-th component of X_i. Consider a family $\mathcal{F}$ of N-tuples of pairs $(\alpha_i,k_i)_{i\le N}$, where $\alpha_i\ge 0$ is a non-negative coefficient and k_i is an index with $1\le k_i\le d_i$. Set $\sigma=\sup_{(\alpha,k)\in\mathcal{F}}\|\alpha\|_2$. Define a maximum over this family*

$$Z=\sup_{(\alpha,k)\in\mathcal{F}}\sum_{i\le N}\alpha_i X_{i,k_i}. \tag{0.7}$$

If M is the median of Z, then for all $u>0$ we have

$$\mathbb{P}(|Z-M|\ge u)\le 4\exp\left(-\frac{u^2}{4\sigma^2}\right). \tag{0.8}$$

Now Lemma 0.3 can be proved in precisely the same way as in [Mar02, Lemma 5.1].

One can use the strong control in the bounded case to prove Theorem 0.2 for these weights. We follow Martin and begin with continuity of the limit shape.

Lemma 0.5. *Let X_i, $i \in I$, be a finite collection of independent random variables taking values in $[0, L]^d$ and take $\epsilon > 0$. Then there is $\delta > 0$ such that if $x \in \mathbb{R}_+^d$ and $\|x\| \leq 1$, with $x_1 = 0$, then*

$$|g(x + he_1) - g(x)| < \epsilon \tag{0.9}$$

for all $0 \leq h \leq \delta$.

Lemma 0.6. *Suppose $|\omega_i(x)| < L$ for some $L > 0$. Then g is continuous on $\mathbb{R}_+^2$.*

The proofs of [Mar04, Lemmas 3.2, 3.3] go through word-for-word[1].

We now give Theorem 0.2 for bounded weights. This is a combination of This is a combination of Lemmas 5.3, 5.4 in [Mar04]. The proofs go through word-for-word[2].

Lemma 0.7. *Suppose $|\omega_i(x)| < L$ for some $L > 0$, and let $\epsilon > 0$ be given. Then almost surely, we have for all but finitely many $z \in \mathbb{Z}_+^2$ that*

$$|G(z) - g(z)| \leq \epsilon\|z\|. \tag{0.10}$$

Having proved these lemmas for bounded weights, Martin proceeds to generalise to unbounded distributions satisfying a certain decay assumption. Namely, we need $\int_0^\infty (1 - F(s))^{1/2}\, ds < \infty$, where F is the distribution of the vertex weights in the LPP model Martin considers. After taking negatives to bring us into FPP, the condition becomes $\int_{-\infty}^0 F(s)^{1/2}\, ds < \infty$. This condition is automatically implied by the existence of $2 + \epsilon$ moments.

As our inequalities do not have to be especially sharp, we bound our edge-weight model between vertex-weight models and apply the Martin's results to these. Given a directed path $\pi = (x = \pi_0, \pi_1, \ldots, \pi_n = y)$, define upper and lower vertex passage times

$$\bar{G}(\pi) = \sum_{i=0}^{n-1} \omega(\pi_i)_1 \vee \omega(\pi_i)_2, \tag{0.11}$$

$$\underline{G}(\pi) = \sum_{i=0}^{n-1} \omega(\pi_i)_1 \wedge \omega(\pi_i)_2. \tag{0.12}$$

Then $\bar{G}(x, y)$, $\underline{G}(x, y)$ are defined as minimums over admissible paths, as before.

Lemma 0.8. *In the notation above,*

$$\underline{G}(x, y) \leq G(x, y) \leq \bar{G}(x, y). \tag{0.13}$$

Proof. Let π be a minimising path for $\bar{G}(x, y)$. Then

$$\bar{G}(\pi) = \sum_{i=0}^{n-1} \omega(\pi_i)_1 \vee \omega(\pi_i)_2 \geq \sum_{i=0}^{n-1} \omega(\pi(i), \pi(i+1)) = G(\pi), \tag{0.14}$$

and in turn $G(\pi) \geq G(x, y)$. Similarly for $\underline{\mathrm{G}}(x, y)$. □

Let F_i be the distribution function for $\omega(0)_i$. Write $\bar{F}, \underline{F}$ for the distributions of $\omega(0)_1 \vee \omega(0)_2$ and $\omega(0)_1 \wedge \omega(0)_2$, respectively. These again have finite $2 + \epsilon$ moments.

Lemma 0.9. *There exists c (independent of the weight distribution) such that:*

[1] Martin uses T for the passage times in place of G and has weights $X(v)$ rather than $\omega(e)$.

[2] With the same transpositions as before.

(i) for all $z \in \mathbb{Z}_+^2$,

$$\mathbb{E}[G(z)] \geq -c\|z\| \int_{-\infty}^{0} \underline{F}(s)^{1/2}\, ds. \tag{0.15}$$

(ii) with probability 1,

$$\liminf_{n\to\infty} \frac{1}{n} \min_{\|z\|_1 \leq n} G(z) \geq -c \int_{-\infty}^{0} \underline{F}(s)^{1/2}\, ds. \tag{0.16}$$

(iii) for all $x \in \mathbb{R}_+^2$,

$$\sum_{i=1}^{2} \langle x, e_i \rangle \mathbb{E}[\omega(0)_i] \geq g(x) \geq -c\|x\| \int_{-\infty}^{0} \underline{F}(s)^{1/2}\, ds. \tag{0.17}$$

Proof. The lower bounds all follow from taking negatives in [Mar04, Lemma 3.5] and applying the statement to the lower vertex passage times $\underline{G}$. For the upper bound in (iii) we need only follow Martin's calculation. Let $\tilde{\pi}$ be some path connecting z to the origin. Then

$$\begin{aligned}
\mathbb{E}[G(z)] &= \mathbb{E}[\max_{\pi} \sum_{i=0}^{\|z\|-1} \omega(\pi_i, \pi_{i+1})] && (0.18)\\
&\leq \mathbb{E}[\sum_{i=0}^{\|z\|-1} \omega(\tilde{\pi}_i, \tilde{\pi}_{i+1})] && (0.19)\\
&= \sum_{\pi_{i+1}-\pi(i)=e_1} \mathbb{E}[\omega(0)_1] + \sum_{\pi_{i+1}-\pi(i)=e_2} \mathbb{E}[\omega(0)_2] && (0.20)\\
&= \langle z, e_1 \rangle \mathbb{E}[\omega(0)_1] + \langle z, e_2 \rangle \mathbb{E}[\omega(0)_2]. && (0.21)
\end{aligned}$$

□

Take $L \geq 0$ and consider the environment of truncated weights $\{\omega^{(L)}(x)_i\}$, where $\omega^{(L)}(x)_i = (\omega(x)_i \vee (-L)) \wedge L$. Let $G^{(L)}, g^{(L)}$ be the passage times and limit shape under the truncated weights. The next lemma quantifies the rate at which $g^{(L)} \to g$ as $L \to \infty$.

Lemma 0.10. *For any $x \in \mathbb{R}_+^2$,*

$$g^{(L)}(x) - c\|x\| \int_{-\infty}^{-L} \underline{F}(s)^{1/2}\, ds \leq g(x) \leq g^{(L)}(x) + \|x\| \int_{L}^{\infty} 1 - \bar{F}(s)\, ds. \tag{0.22}$$

In particular, for any $R > 0$,

$$\sup_{x \in \mathbb{R}_+^2, \|x\| \leq R} |g(x) - g^{(L)}(x)| \underset{L\to\infty}{\longrightarrow} 0. \tag{0.23}$$

Proof. The argument is largely identical to [Mar04, Lemma 3.6]. For lower bound, take $x \in \mathbb{R}_+^2$.

We have

$$\begin{aligned}
g(x) - g^{(L)}(x) &= \lim_{n\to\infty} n^{-1}\mathbb{E}[G(\lfloor nx\rfloor)] - \lim_{n\to\infty} n^{-1}\mathbb{E}[G^{(L)}(\lfloor nx\rfloor)] \\
&= \lim_{n\to\infty} n^{-1}\mathbb{E}[\min_{\pi\in\Pi(0,\lfloor nx\rfloor)} \sum_{e\in\pi}\omega(e) - \min_{\pi\in\Pi(0,\lfloor nx\rfloor)}] \sum_{e\in\pi}\omega^{(L)}(e) \\
&\geq \lim_{n\to\infty} n^{-1}\mathbb{E}[\min_{\pi\in\Pi(0,\lfloor nx\rfloor)} \omega(e) - \omega^{(L)}(e)] \\
&= \lim_{n\to\infty} n^{-1}\mathbb{E}[\min_{\pi\in\Pi(0,\lfloor nx\rfloor)} (\omega(e) - L)_+] \\
&\geq -c\|x\| \int_{-\infty}^{-L} \underline{F}(s)^{1/2}\, ds.
\end{aligned}$$

The last inequality comes from applying Lemma 0.9.(iii) to the weights $\{(\omega(x)_i - L)_+\}$.

We need an auxiliary calculation before the upper bound. For the other side, take $z \in \mathbb{Z}_+^2$ and let π^* be the rightmost geodesic for $G^{(L)}(z)$. One sees that the presence of an edge $e \in E(\mathbb{Z}_+^2)$ in π^* is a nonincreasing function of $\omega(e)$. The probability that $\omega(e)$ is truncated above can only decrease when we condition on it belonging to π^*:

$$\mathbb{P}(\omega(e) \geq L \mid e \in \pi^*) \leq \mathbb{P}(\omega(e) \geq L). \tag{0.24}$$

Conditional on $\{\omega(e) \geq L\}$, the event $e \in \pi^*$ is independent of $\omega(e)$ (since under this conditioning, $\omega^{(L)}(e) = L$ is constant). So

$$\begin{aligned}
\mathbb{E}[(\omega(e) - L)_+ \mid e \in \pi^*] &= \mathbb{E}[(\omega(e) - L)_+ \mid \omega(e) \geq L]\mathbb{P}(\omega(e) \geq L \mid e \in \pi^*) && (0.25) \\
&\leq \mathbb{E}[(\omega(e) - L)_+ \mid \omega(e) \geq L]\mathbb{P}(\omega(e) \geq L) && (0.26) \\
&= \mathbb{E}[(\omega(e) - L)_+] && (0.27) \\
&\leq \int_L^\infty 1 - \bar{F}(s)\, ds. && (0.28)
\end{aligned}$$

Now

$$\begin{aligned}
\mathbb{E}[G(z)] &= \mathbb{E}[\min_\pi \sum_{e\in\pi} \omega(e)] && (0.29) \\
&\leq \mathbb{E}[\min_\pi \sum_{e\in\pi} \omega^{(L)}(e) + (\omega(e) - L)_+] && (0.30) \\
&\leq \mathbb{E}[\sum_{e\in\pi^*} \omega^{(L)}(e)] + \mathbb{E}[\sum_{e\in\pi^*} (\omega(e) - L)_+] && (0.31) \\
&= \mathbb{E}[G^{(L)}(z)] - \sum_{e\in E(\mathbb{Z}_+^2)} \mathbb{P}(e \in \pi^*)\mathbb{E}[(\omega(e) - L)_+ \mid e \in \pi^*] && (0.32) \\
&\leq \mathbb{E}[G^{(L)}(z)] + \int_L^\infty 1 - \bar{F}(s)\, ds \sum_{e\in E(\mathbb{Z}_+^2)} \mathbb{P}(e \in \pi^*) && (0.33) \\
&= \mathbb{E}[G^{(L)}(z)] + \|z\| \int_L^\infty 1 - \bar{F}(s)\, ds. && (0.34)
\end{aligned}$$

Putting $z = \lfloor nx \rfloor$ and taking the limit, we arrive at the upper bound. □

There are two final lemmas establishing uniform convergence of the truncated passage times and limit shape to the untruncated counterparts.

Lemma 0.11. *Let $\epsilon > 0$ be given. Then there is L large enough such that almost surely, we have for all but finitely many $z \in \mathbb{Z}_+^2$ that*

$$|G(z) - G^{(L)}(z)| \le \epsilon \|z\|. \tag{0.35}$$

Lemma 0.12. *Let $\epsilon > 0$ be given. Then there is L large enough such that almost surely, we have for all but finitely many $z \in \mathbb{Z}_+^2$ that*

$$|g(z) - g^{(L)}(z)| \le \epsilon \|z\|. \tag{0.36}$$

Of these, Lemma 0.12 is immediate from Lemma 0.10.

Proof of Lemma 0.11. Except for swapping signs and the need to involve $\underline{F}, \bar{F}$ due to their appearance in Lemma 0.9, we can largely follow Martin's argument unchanged. Choose L so that $c\int_{-\infty}^{-L} \underline{F}^{1/2}\, ds < \epsilon$ and $c\int_L^{\infty}(1-\bar{F})^{1/2}\, ds < \epsilon$.

Take $z \in \mathbb{Z}_+^2$. There is some path $\pi \in \Pi(z)$ which is a geodesic under both $\{\omega(e)\}$ and $\{\omega^{(L)}(e)\}$:

$$G(z) - G^{(L)}(z) = \sum_{e\in\pi^*} \omega(e) - \omega^{(L)}(e). \tag{0.37}$$

Then we can estimate the difference by

$$|G(z) - G^{(L)}(z)| \le \sum_{e\in\pi^*} (\omega(e) - L)_+ + \sum_{e\in\pi^*} (-L - \omega(e))_+ \tag{0.38}$$

$$\le -\min_{\pi\in\Pi(z)} \sum_{e\in\pi} V^{(L)}(e) - \min_{\pi\in\Pi(z)} \sum_{e\in\pi} W^{(L)}(e). \tag{0.39}$$

Here we set $V^{(L)}(e) = -(\omega(e) - L)_+$ and $W^{(L)}(e) = -(-L - \omega(e))_+$.

Observe that the $\{V^{(L)}(e)\}$ and $\{W^{(L)}(e)\}$ fall under Assumption 0.1. Write $F_{V,i}^{(L)}$ for the distribution of $V^{(L)}(0)_i$, and $\underline{F}_V^{(L)}$ for the distribution of $V^{(L)}(0)_1 \wedge V^{(L)}(0)_2$. Then $\underline{F}_V^{(L)}(s) = 1 - \bar{F}(L-s)$ on $s \le 0$, and $\underline{F}_V^{(L)}(s) = 1$ elsewhere. We apply Lemma 0.9.(ii) to find that almost surely

$$\liminf_{n\to\infty} \frac{1}{n} \min_{\|z\|_1 \le n} \min_{\pi\in\Pi(z)} \sum_{e\in\pi} V^{(L)}(e) \ge -c\int_{-\infty}^{0} \underline{F}_V^{(L)}(s)^{1/2}\, ds \tag{0.40}$$

$$= -c\int_{-\infty}^{0} (1 - \bar{F}(L-s)^{1/2}\, ds \tag{0.41}$$

$$= -c\int_{L}^{\infty} (1 - \bar{F}(s))^{1/2}\, ds \tag{0.42}$$

$$\ge -\epsilon/2. \tag{0.43}$$

Then there are only finitely many z for which

$$-\min_{\pi\in\Pi(z)} \sum_{e\in\pi} V^{(L)}(e) \ge \frac{\epsilon}{2}\|z\|. \tag{0.44}$$

We can do the same for the $\{W^{(L)}(e)\}$ to find

$$\liminf_{n\to\infty} \frac{1}{n} \min_{\|z\|_1 \le n} \min_{\pi\in\Pi(z)} \sum_{e\in\pi} W^{(L)}(e) \ge -c \int_{-\infty}^{0} \underline{F}_W^{(L)}(s)^{1/2}\, ds \tag{0.45}$$

$$= -c \int_{-\infty}^{-L} \underline{F}(s)^{1/2}\, ds \tag{0.46}$$

$$\ge -\epsilon/2, \tag{0.47}$$

so that there are only finitely many z with

$$-\min_{\pi\in\Pi(z)} \sum_{e\in\pi} W^{(L)}(e) \ge \frac{\epsilon}{2}\|z\|. \tag{0.48}$$

Looking back now at (0.38), we find that only finitely many z have

$$|G(z) - G^{(L)}(z)| \ge \epsilon\|z\|. \tag{0.49}$$

This is what we wanted. □

Combining Lemmas 0.7, 0.11 and 0.12 gives Theorem 0.2.

0.2 Generalised Busemann functions

Under Assumption 0.1, the results of [GJR23] apply to give the existence of *generalised Busemann functions*. In what follows, let $\mathscr{U} = \{(t, 1-t) : 0 < t < 1\}$ be the set of directions into the first quadrant, and let $\mathscr{U}_0$ be some countable subset (which can be assumed to be dense). Denote by T_x translations of the environment $T_x(\omega_y) = \omega_{y-x}$.

Below is essentially a restatement of Theorem 4.4 of [GJR23], but specialised to $\beta = \infty$ and the face $\mathcal{A} \in \mathbb{A}$ being the entire limit shape. The m in their statement is a member of a superdifferential of the limit shape, and here the role is taken by $(\zeta, \pm)$.

Theorem 0.13. *There exists a probability space $(\hat{\Omega}, \hat{\mathscr{B}}, \hat{\mathbb{P}})$ with a measurable projection onto Ω, and real-valued measurable functions $B^{\xi}(\hat{\omega}, x, y)$ of $(\hat{\omega}, \xi, x, y) \in \hat{\Omega} \times \mathscr{U}_0 \times \mathbb{Z}^2 \times \mathbb{Z}^2$ and a translation invariant Borel probability measure $\hat{\mathbb{P}}$ on $(\hat{\Omega}, \hat{\mathscr{B}})$, such that the following properties hold:*

(i) (Consistency) Under $\hat{\mathbb{P}}$, the marginal distribution of the configuration ω is i.i.d with the specified marginals. For each $\xi \in \mathscr{U}_0$, the $\mathbb{R}^3$-valued process $\{\psi_x^{\xi}\}_{x\in\mathbb{Z}^2}$ defined by

$$\psi_x^{\xi}(\hat{\omega}) = (\omega_x, B^{\xi}(\hat{\omega}, x, x+e_1), B^{\xi}(\hat{\omega}, x, x+e_2)) \tag{0.50}$$

is stationary under translations T_x. For any $I \subseteq \mathbb{Z}^2$, the variables

$$\left\{(\omega_x, B^{\xi}(\hat{\omega}, x, x+e_i)) : x \in I, \xi \in \mathscr{U}_0, i \in \{1,2\}\right\} \tag{0.51}$$

are independent of $\{\omega_x : x \in I^<\}$.

(ii) (Adaptedness) For a fixed $\xi \in \mathscr{U}_0$, the process $B^{\xi} = \{B^{\xi}(x,y)\}_{x,y\in\mathbb{Z}^2}$ is a stationary $L^1(\hat{\mathbb{P}})$ cocycle (in the sense of [GRS16]) that recovers the potential:

$$\min_{i\in\{1,2\}} B^{\xi}(x, x+e_i) - \omega(x, x+e_i) = 0. \tag{0.52}$$

(iii) (Distinct means) The mean vectors $h(\xi) = h(B^\xi)$ defined by

$$h(\xi) \cdot e_i = \mathbb{E}[B^\xi(0, e_i)] \tag{0.53}$$

satisfy

$$h(\xi) = \nabla g_{pp}(\xi). \tag{0.54}$$

If $h(\xi) = h(\zeta)$ then $B^\xi(x, y) = B^\zeta(x, y)$ a.s.

Remark 0.14. The generalised Busemann functions are related to the underlying environment only through the adaptedness property. Thus, any proof which takes the above as its starting point and produces these objects as functions of the environment must use adaptedness in an essential way. The utility of this property was identified in [GRS16], where it is connected to maximisers of variational formulas for the limit shape.

The property has been called *recovery* in the context of vertex-weight LPP[Sep18], referring to the fact that one can completely recover the weight configuration from the collection of Busemann functions in a fixed direction. In edge weight LPP we have only this weaker statement: if $\mathscr{U}_0$ is dense, then almost surely the weight configuration is determined by the full collection of generalised Busemann functions.

We are interested in showing that these functions arise as limiting differences of the passage times, and moreover that they extend to a full-fledged *Busemann process* indexed by $\mathscr{U}$. Specifically, for a direction $\xi \in \mathscr{U}$ and $x, y \in \mathbb{Z}^2$, we look at limits

$$B^\xi(x, y) = \lim_{n\to\infty} G(x, v_n) - G(y, v_n), \tag{0.55}$$

where $(v_n)_{n\in\mathbb{Z}_{\geq 0}} \subset \mathbb{Z}^2$ is a sequence of vertices with limiting direction ξ and $|v_n| \to \infty$.

This task has been carried out in [JR20] for planar edge weight models. The extension to the remaining directions relies on the "path crossing trick" to give monotonicity, after which limits can be taken. The relevant statement for our setup is below.

Lemma 0.15 (Path-crossing trick)**.** *Suppose $|u|_1 = |v|_1$ and $u_1 \leq v_1$. Then*

$$G(0, u) - G(e_1, u) \leq G(0, v) - G(e_1, v) \tag{0.56}$$

and

$$G(0, u) - G(e_2, u) \geq G(0, v) - G(e_2, v), \tag{0.57}$$

whenever these passage times are defined.

Proof. We prove (0.56). This holds trivially if $u_1 = v_1$, so assume $u_1 < v_1$. This implies $u_2 > v_2$. Fix a geodesic connecting e_1 to u and 0 to v. The relative positions of u and v ensure that the geodesic (or any other directed path) connecting e_1 to u must cross the geodesic connecting 0 to v. Let x be the first point of intersection. Passage times are sub-additive, hence

$$G(0, x) + G(x, u) \geq G(0, u), \quad G(e_1, x) + G(x, v) \geq G(e_1, v). \tag{0.58}$$

These can be combined and rearranged to give

$$G(0, u) - G(e_1, x) - G(x, u) \leq G(0, x) + G(x, v) - G(e_1, v). \tag{0.59}$$

That x lies on both geodesics means $G(e_1, x) + G(x, u) = G(e_1, u)$ and $G(0, x) + G(x, v) = G(0, v)$. Thus we arrive at

$$G(0, u) - G(e_1, u) \leq G(0, v) - G(e_1, v). \tag{0.60}$$

□

It is then quite transparent from the construction of the generalised Busemann functions in [GJR23] that we have the following additional property:

Lemma 0.16. *There exists an event $\hat{\Omega}_0$ with $\hat{\mathbb{P}}(\hat{\Omega}_0) = 1$ and such that if $\xi, \zeta \in \mathscr{U}$ with $\xi \cdot e_1 < \zeta \cdot e_1$, then*

$$B^{\xi}(\hat{\omega}, x, x + e_1) \le B^{\zeta}(\hat{\omega}, x, x + e_1) \tag{0.61}$$

and

$$B^{\xi}(\hat{\omega}, x, x + e_2) \ge B^{\zeta}(\hat{\omega}, x, x + e_2). \tag{0.62}$$

It now makes sense to define for $\xi \in \mathscr{U}$

$$B^{\xi+}(\hat{\omega}, x, y) = \lim_{\zeta \cdot e_1 \searrow \xi \cdot e_1} B^{\zeta}(\hat{\omega}, x, y), \tag{0.63}$$

$$B^{\xi-}(\hat{\omega}, x, y) = \lim_{\zeta \cdot e_1 \nearrow \xi \cdot e_1} B^{\zeta}(\hat{\omega}, x, y). \tag{0.64}$$

The limits are taken through $\zeta \in \mathscr{U}_0$. That these limits exist follows from monotonicity and the cocycle property.

Finally, we summarise the properties of the collection of these extended generalised Busemann functions.

Theorem 0.17. *Let $(\hat{\Omega}, \hat{\mathscr{B}}, \hat{\mathbb{P}})$ be as in Theorem 0.13. There are functions $B^{\xi\pm}(\hat{\omega}, x, y)$ of $(\hat{\omega}, \xi, x, y) \in \hat{\Omega} \times \mathscr{U} \times \mathbb{Z}^2 \times \mathbb{Z}^2$, such that the following properties hold:*

(i) (Consistency) Under $\hat{\mathbb{P}}$, the marginal distribution of the configuration ω is i.i.d with the specified marginals. For each $\xi \in \mathscr{U}$, the $\mathbb{R}^3$-valued process $\{\psi_x^{\xi\pm}\}_{x \in \mathbb{Z}^2}$ defined by

$$\psi_x^{\xi\pm}(\hat{\omega}) = (\omega_x, B^{\xi\pm}(\hat{\omega}, x, x + e_1), B^{\xi\pm}(\hat{\omega}, x, x + e_2)) \tag{0.65}$$

is stationary under translations T_x. For any $I \subseteq \mathbb{Z}^2$, the variables

$$\{(\omega_x, B^{\xi+}(\hat{\omega}, x, x + e_i), B^{\xi-}(\hat{\omega}, x, x + e_i)) : x \in I, \xi \in \mathscr{U}_0, i \in \{1, 2\}\} \tag{0.66}$$

are independent of $\{\omega_x : x \in I^<\}$.

(ii) (Adaptedness) For a fixed $\xi \in \mathscr{U}$, the process $B^{\xi\pm} = \{B^{\xi\pm}(x, y)\}_{x,y \in \mathbb{Z}^2}$ is a stationary $L^1(\hat{\mathbb{P}})$ cocycle satisfying

$$\max_{i \in \{1,2\}} B^{\xi}(x, x + e_i) - \omega(x, x + e_i) = 0. \tag{0.67}$$

(iii) There exists an event $\hat{\Omega}_0$ with $\hat{\mathbb{P}}(\hat{\Omega}_0) = 1$ and such that the following hold for all $\hat{\omega} \in \hat{\Omega}_0$, $x, y \in \mathbb{Z}^2$ and $\xi, \zeta \in \mathscr{U}$.

(a) (Monotonicity) If $\xi \cdot e_1 < \zeta \cdot e_1$, then

$$B^{\xi-}(\hat{\omega}, x, x + e_1) \le B^{\xi+}(\hat{\omega}, x, x + e_1) \le B^{\zeta-}(\hat{\omega}, x, x + e_1) \tag{0.68}$$

and

$$B^{\xi-}(\hat{\omega}, x, x + e_2) \ge B^{\xi+}(\hat{\omega}, x, x + e_2) \ge B^{\zeta-}(\hat{\omega}, x, x + e_2). \tag{0.69}$$

(b) (One-sided continuity) If $\xi_n \cdot e_1 \searrow \zeta_n \cdot e_1$, then

$$\lim_{n \to \infty} B^{\xi_n \pm}(\hat{\omega}, x, y) = B^{\zeta+}(\hat{\omega}, x, y). \tag{0.70}$$

Similarly, if $\xi_n \cdot e_1 \nearrow \zeta_n \cdot e_1$, then

$$\lim_{n \to \infty} B^{\xi_n \pm}(\hat{\omega}, x, y) = B^{\zeta-}(\hat{\omega}, x, y). \tag{0.71}$$

(iv) (Distinct means) The mean vectors $h(\xi\pm) = h(B^{\xi\pm})$ defined by

$$h(\xi\pm)\cdot e_i = \mathbb{E}[B^{\xi\pm}(0,e_i)] \tag{0.72}$$

satisfy

$$h(\xi\pm) = \nabla g_{pp}(\xi\pm). \tag{0.73}$$

If $h(\xi+) = h(\zeta-)$ then $B^{\xi+}(x,y) = B^{\zeta-}(x,y)$ a.s. Similarly for $h(\xi+) = h(\zeta+)$ and $h(\xi-) = h(\zeta-)$.

0.3 As gradients of passage times

It remains to see that these objects are genuine Busemann functions given by the limits of (0.55). We will go through the lemmas of Section 6 in [GRS17], noting where the details differ. To that end, fix a $v \in \mathbb{Z}^2$ and for $x \le v - e_1$, $y \le v - e_2$, define increments

$$I(x,v) = G(x,v) - G(x+e_1,v) \tag{0.74}$$

$$\text{and } J(y,v) = G(y,v) - G(y+e_2,v). \tag{0.75}$$

The path crossing trick applies to give relations:

Lemma 0.18.

$$I(x,v+e_2) \le I(x,v) \le I(x,v+e_1) \tag{0.76}$$

$$\textit{and } J(x,v+e_2) \ge J(x,v) \ge J(x,v+e_1). \tag{0.77}$$

The next lemma links the Busemann functions to limiting directions of the LPP, but requires quite a bit of notation. Recalling that g_{pp} is the limit shape of our model, set $\gamma(s) = g_{pp}(1,s)$. Note that g_{pp} won't in general be symmetric. The convexity of g_{pp} ensures the existence of one-sided derivatives for γ. Fix $\zeta \in \mathscr{U}$ and a cocycle $B = B^{\zeta\pm}$. Take the (inverse) slope $r = \zeta\cdot e_1/\zeta\cdot e_2$, so that $\alpha = \gamma'(r\pm)$ (the same choice of sign as for B) satisfies

$$\alpha = \hat{\mathbb{E}}[B(0,e_1)]. \tag{0.78}$$

This is Theorem 0.17.(iv). Define $f(\alpha)$ by

$$f(\alpha) = \hat{\mathbb{E}}[B(0,e_2)]. \tag{0.79}$$

Fix for the moment some $v \in \mathbb{Z}^2$ and define, for $u \le v$,

$$G^{\mathrm{NE}}(u,v) = \begin{cases} B(u,v), & v-u = ke_i,\ k\in\mathbb{Z}_+, i\in\{1,2\} \\ (\omega(u,u+e_1) + G^{\mathrm{NE}}(u+e_1,v)) \wedge (\omega(u,u+e_w) + G^{\mathrm{NE}}(u+e_w,v)), & \text{otherwise.} \end{cases} \tag{0.80}$$

These are the passage times with the generalised Busemann functions as initial conditions along the northeast boundary.

Write $G^{\mathrm{NE}}_{v-e_i\in\pi}(0,v)$ for the minimal passage time among paths which reach v through the edge $\{v-e_i,v\}$ (where we use in the definition the corresponding weights for G^{NE}, which can be recovered uniquely from the definition above).

Lemma 0.19. *Fix $0 < s,t < \infty$. Let $v_n \in \mathbb{Z}^2$ be such that $v_n/|v_n|_1 \to (s,t)/(s+t)$ as $n \to \infty$ and such that $|v_n| \geq \eta_0 n$ for some $\eta_0 > 0$. Then the following limits hold almost surely:*

$$|v_n|_1^{-1} G^{\mathrm{NE}}_{v_n - e_1 \in \pi}(0, v_n) \underset{n\to\infty}{\longrightarrow} (s+t)^{-1} \max_{0 \leq \tau \leq s} \{g_{pp}(s-\tau, t) + \alpha\tau\} \tag{0.81}$$

and

$$|v_n|_1^{-1} G^{\mathrm{NE}}_{v_n - e_2 \in \pi}(0, v_n) \underset{n\to\infty}{\longrightarrow} (s+t)^{-1} \min_{0 \leq \tau \leq s} \{g_{pp}(s, t-\tau) + f(\alpha)\tau\}. \tag{0.82}$$

Proof. The proof can be carried out as in [GRS17] after minor modifications to the estimates. The symmetry of the setup means that it is enough to look at the e_1-axis. Fix $\epsilon > 0$, let $M = \lfloor \epsilon^{-1} \rfloor$, and define steps

$$q_j^n = j \left\lfloor \frac{\epsilon |v_n|_1 s}{s+t} \right\rfloor, \text{ for } 0 \leq j \leq M-1, \text{ and } q_M^n = v_n \cdot e_1. \tag{0.83}$$

Notice that for n large we have $q_{M-1}^n < v_n \cdot e_1 = q_M^n$.

Suppose a minimal path for $G^{\mathrm{NE}}_{v-e_1 \in \pi}(0, v)$ enters the north boundary at the point $v_n - (l, 0)$ and choose j so that $q_j^n < l \leq q^{n_{j+1}}$. Write $m_0 = \mathbb{E}[\omega_1(x)]$. We have a bound

$$G^{\mathrm{NE}}_{v-e_1 \in \pi}(0, v) = G(0, v_n - (l,1)) + \omega_2(v_n - (l,1)) + B(v_n - (l,1), v_n) \tag{0.84}$$

$$\begin{aligned} &\geq G(0, v_n - (q_j^n, 1)) + q_j^n \alpha + \omega_2(v_n - (l,1)) + \sum_{k=q_j^n+1}^{l} (\omega_1(v_n - (k,1)) - m_0) \\ &\quad + (l - 1 - q_j^n) m_0 + (B(v_n - (l,1), v_n) - l\alpha) + (l - q_j^n)\alpha. \end{aligned} \tag{0.85}$$

Proceeding in the same way, define $F(x,y) = h(B) \cdot (x-y) - B(x,y)$, which has

$$B(v_n - (l,0), v_n) - l\alpha = F(0, v_n - (l, \alpha)) - F(0, v_n). \tag{0.86}$$

Here the adaptedness property is $0 = (B(0,e_1) - \omega_1(0)) \vee (B(0,e_2) - \omega_2(0))$. The resulting bound on the centred co-cycle is

$$F(0, e_i) \leq \alpha \wedge f(\alpha) - \omega_1(0) \vee \omega_2(0). \tag{0.87}$$

By Assumption 0.1, the variables $\{\omega_1(x) \vee \omega_2(x)\}_{x \in \mathbb{Z}^2}$ are i.i.d with $2+\epsilon$ moments, and so [GRS17, Theorem A.1] applies to F. Writing $S_{j,m}^n = \sum_{l=q_j^n+1}^{q_j^n+m} (\omega_1(v_n - (k,1)) - m_0)$ and C for some constant, then maximising over j in our bound above,

$$\begin{aligned} G^{\mathrm{NE}}_{v-e_1 \in \pi}(0, v) \geq \max_{0 \leq j \leq M-1} \Bigg\{ & G(0, v_n - (q_j^n, 1)) + q_j^n \alpha + \max_{q_j^n \leq l \leq q_{j+1}^n} |\omega_2(v_n - (l,1))| + \max_{0 \leq m \leq q_{j+1}^n - q_j^n} |S_{j,m+1}^n| \\ & + \max_{q_j^n \leq l \leq q_{j+1}^n} (|F(0, v_n - (l,0))| + |F(0, v_n)|) \Bigg\}. \end{aligned} \tag{0.88}$$

Divide through by $|v_n|_1$ and let $n \to \infty$. Each of these terms converges as in [GRS17], except that we have an additional term

$$|v_n|_1^{-1} \max_{0 \leq l \leq v_n \cdot e_1} |\omega_2(v_n - (l,1))|. \tag{0.89}$$

But the finite variance of the weights is enough to ensure that this term too goes to zero almost surely. This gives the upper bound

$$\limsup_{n\to\infty} |v_n|_1^{-1} G^{\mathrm{NE}}_{v_n-e_1\in\pi}(0,v_n) \geq (s+t)^{-1} \max_{0\leq\tau\leq s}\{g_{pp}(s-\tau,t)+\alpha\tau\} \tag{0.90}$$

The argument for the upper bound is completely identical to the one in [GRS17] (after swapping signs). □

Lemma 0.20. *Let $s \in (r,\infty)$. Let $v_n \in \mathbb{Z}^2$ be such that $v_n/|v_n|_1 \to (s,t)/(s+t)$ as $n \to \infty$ and such that $|v_n| \geq \eta_0 n$ for some $\eta_0 > 0$. Assume that $\gamma'(r+) > \gamma'(s-)$. Then $\hat{\mathbb{P}}$- a.s there exists a random $n_0 < \infty$ such that for all $n \geq n_0$*

$$G^{\mathrm{NE}}(0,v_n) = G^{\mathrm{NE}}_{v_n-e_1\in\pi}(0,v_n). \tag{0.91}$$

The proof in [GRS17] makes no mention of the weights and goes through word-for-word, so we skip it.

There are some final definitions before the theorem. Write

$$\mathscr{D} = \{\xi \in \mathscr{U} : g_{pp} \text{ is differentiable at } \xi\}. \tag{0.92}$$

For a direction $\xi \in \mathscr{U}$, consider the maximal line segments of g_{pp} to which ξ belongs:

$$\mathscr{U}_{\xi\pm} = \{\zeta \in \mathscr{U} : g_{pp}(\zeta) - g_{pp}(\xi) = \nabla g_{pp}(\xi\pm)\cdot(\zeta-\xi)\}. \tag{0.93}$$

Let

$$\mathscr{U}_\zeta = \mathscr{U}_{\xi-} \cup \mathscr{U}_{\xi+} = [\underline{\xi},\bar{\xi}], \quad \text{where } \underline{\xi}\cdot e_1 \leq \bar{\xi}\cdot e_1. \tag{0.94}$$

Theorem 0.21. *Fix a possibly degenerate segment $[\zeta,\eta] \subseteq \mathscr{U}$. Assume that either $[\zeta,\eta]$ consists of a single exposed point ξ such that $\xi = \underline{\xi} = \bar{\xi} = \zeta = \eta$, or that $[\zeta,\eta]$ is a maximal, non-degenerate linear segment of g_{pp} so that $[\zeta,\eta] = [\underline{\xi},\bar{\xi}]$ for all $\xi \in (\zeta,\eta)$. Then there exists an event $\hat{\Omega}_0$ with $\hat{\mathbb{P}}(\hat{\Omega}_0) = 1$ such that for each $\hat{\omega} \in \hat{\Omega}_0$ and for any sequence $v_n \in \mathbb{Z}^2_+$ with*

$$|v_n|_1 \to \infty \text{ and } \zeta\cdot e_1 \leq \liminf_{n\to\infty}\frac{v_n\cdot e_1}{|v_n|_1} \leq \limsup_{n\to\infty}\frac{v_n\cdot e_1}{|v_n|_1} \leq \eta\cdot e_1, \tag{0.95}$$

we have for all $x \in \mathbb{Z}^2_+$

$$B^{\zeta-}(\hat{\omega},x,x+e_1) \leq \liminf_{n\to\infty}(G(\omega,x,v_n) - G(\omega,x+e_1,v_n)) \tag{0.96}$$

$$\leq \limsup_{n\to\infty}(G(\omega,x,v_n) - G(\omega,x+e_1,v_n)) \geq B^{\eta-}(\hat{\omega},x,x+e_1) \tag{0.97}$$

and

$$B^{\eta+}(\hat{\omega},x,x+e_2) \leq \liminf_{n\to\infty}(G(\omega,x,v_n) - G(\omega,x+e_2,v_n)) \tag{0.98}$$

$$\leq \limsup_{n\to\infty}(G(\omega,x,v_n) - G(\omega,x+e_2,v_n)) \leq B^{\zeta-}(\hat{\omega},x,x+e_2). \tag{0.99}$$

After the appropriate redefinition of the G^N passage times, the proof is identical. Finally, we arrive at

Corollary 0.22. *Assume $\xi, \underline{\xi}, \bar{\xi} \in \mathscr{D}$. Then there exists an event $\hat{\Omega}_0$ with $\hat{\mathbb{P}}(\hat{\Omega}_0) = 1$ such that for each $\hat{\omega} \in \hat{\Omega}_0$, x, $y \in \mathbb{Z}_+^2$, and for any sequence $v_n \in \mathbb{Z}_+^2$ with*

$$|v_n|_1 \to \infty \text{ and } \underline{\xi} \cdot e_1 \leq \liminf_{n\to\infty} \frac{v_n \cdot e_1}{|v_n|_1} \leq \limsup_{n\to\infty} \frac{v_n \cdot e_1}{|v_n|_1} \leq \bar{\xi} \cdot e_1, \tag{0.100}$$

we have

$$B^{\xi}(\hat{\omega}, x, x + e_1) = \lim_{n\to\infty} (G(\omega, x, v_n) - G(\omega, y, v_n)). \tag{0.101}$$

The corollary shows in particular that when the limit shape is differentiable, the Busemann process is a function of the weights and so is $\mathscr{B}$-measurable.

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