Solvability in a restricted first passage percolation

Sam McKeown (University of Wisconsin-Madison)

Participant Talk
Virginia Integrable Probability Summer School, July 2024

Strict-weak FPP

Strict-weak first passage percolation (SWFPP) is an FPP model where we consider only directed paths and fix edge weights in one direction to be zero.

On the plane, let $(\omega_x)_{x\in\mathbb{Z}^2}$ be i.i.d weights. If $x, y \in \mathbb{Z}^2$, $x \geq y$, and π is an up-right path $x \to y$, define the weight of the path as

$$G(\pi) = \sum_{z\in\pi} \omega_z \mathbb{1}\{\pi \text{ has horizontal edge at } z\}.$$

The *SWFPP passage time* between x and y is the minimum weight over such π and is denoted $G(x, y)$. Minimising paths are called *geodesics*.

This model (with certain weights) has been shown to have the directed landscape as a scaling limit (Dauvergne-Virág '21).

Explicit limit shapes

The passage times in these models obey a law of large numbers. For a fixed direction $(x, y) \in (\mathbb{R}^{\geq 0})^2$:

$$\lim_{n\to\infty} \frac{G(0,(\lfloor nx \rfloor, \lfloor ny \rfloor))}{n} = g(x, y) \text{ a.s.}$$

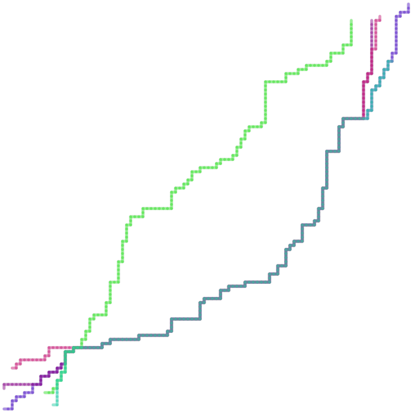
This $g(x, y)$ is called the *time constant*. The level sets are called the *limit shape*.

The time constant exists with mild assumptions on the weight distribution, but for certain distributions we have explicit formulas. The simplest is $\omega_x \sim \mathrm{Exp}(1)$, where

$$g(x, y) = (\sqrt{x + y} - \sqrt{y})^2.$$

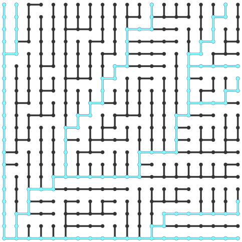
The other distributions for which g is known are Bernoulli, geometric, and products of Bernoulli and exponential / geometric variables. We call these choices *solvable*.

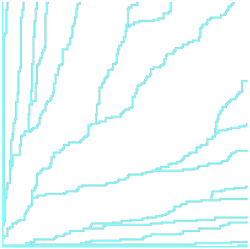
Coalescence of geodesics



Pairs of geodesics with nearby endpoints tend to overlap for much of their length. Geodesics seek to join *highways* going in the correct direction, then exit near their destination.

Geodesic trees





We can study the highway phenomenon through the graph of geodesics with a fixed start point. When the weight distribution is continuous, this is a tree.

Of interest are those geodesics which extend to infinity, the *(semi-)infinite* geodesics. Typical questions concern their existence, directedness, and whether two-sided infinite geodesics exist.

Busemann functions

The main tool in answering these questions is the *Busemann process*, the collection of (random) limits

$$B^{\xi}(x,y) = \lim_{n\to\infty} G(y, v_n) - G(x, v_n),$$

where $(v_n)_{n\in\mathbb{N}}$ is a sequence of vertices with $\lim_n v_n/|v_n|$ parallel to $(1,\xi)$. With the Busemann process in hand, one can produce infinite geodesics with arbitrary starting point and direction.

These limits have been shown to exist in planar last passage percolation (LPP) for general weight distributions and are well understood in exponential LPP.

Busemann functions for SWFPP

The same techniques used in LPP can be adapted to SWFPP[1]. A key input is the existance of “generalised Busemann functions” established by Groathouse-Janjigian-Rassoul-Agha ’23. There are immediate consequences for the existance of infinite geodesics.

Theorem

Suppose $\mathbb{E}[|\omega|^{2+\epsilon}] < \infty$ *and let* $\mathscr{D}$ *be the set of directions at which the limit shape is strictly convex. Then almost surely, there is for each* $\xi \in \mathscr{D}$ *at least one* ξ*-directed infinite geodesic starting at* 0*.*

When the weight distribution is continuous, the geodesic for a given direction is almost surely unique (though there is a countable set of exceptional direction).

[1] Or directed planar FPP in general.

In the solvable cases

Theorem

Suppose the weight distribution in solvable and take $\xi \in \mathscr{D}$ (which here is equivalent to $g(1,\xi) > 0$). Then the variables

$$\{B^{\xi}((n,0),(n+1,0)) : n \in \mathbb{Z}\}$$

are i.i.d. The same is true for

$$\{B^{\xi}((0,n),(0,n+1)) : n \in \mathbb{Z}\}.$$

When $\omega_x \sim \mathrm{Exp}(1)$,

$$B^{\xi}(x, x+e_1) \sim \mathrm{Exp}(1+\rho_{\xi}),$$
$$B^{\xi}(x, x+e_2) \sim -Ber((1+\rho_{\xi})^{-1})\mathrm{Exp}(\rho_{\xi}),$$

with $\rho_{\xi} = \sqrt{\xi}(\sqrt{\xi+1} + \sqrt{\xi})$.

Highways and byways

The highways and byways problem asks about the density of highways among the lattice points.

Let S_x be the event that x lies on an infinite geodesic beginning at 0. It has been shown for FPP (Ahlberg-Hanson-Hoffman '22) and (under more restrictive assumptions) for LPP (Coupier '15) that

$$\lim_{x \to \infty} \mathbb{P}(S_x) = 0.$$

As a consequence, the lower density of the set of highways is 0 almost surely.

Highways near the axis (1)

The independence property of Busemann functions tells us in particular:

Corollary

Let $u(\xi)$ be the point at which the ξ-directed infinite geodesic from the origin leaves the e_2-axis. Under a solvable weight distribution, $u(\xi) \sim \mathrm{Geom}(\alpha_\xi)$ (where α_ξ can be written explicitly).

By optimising ξ we get (under exponential weights):

$$\mathbb{P}(\text{The path } (0,0) \to (0,n) \to (1,n) \text{ is part of an infinite geodesic}) \geq \frac{e^{-1} + o(1)}{n+1}.$$

The above path is already a geodesic with probability $1/(n+1)$, of the same order.

Highways near the axis (2)

This observation leads to:

Lemma

Under exponential weights we have $\mathbb{P}(S_{(1,n)}) \geq c$, for some $c > 0$ uniform in n.

Fix $k \geq 1$ and rectangles $A_n = [1, k] \times [1, n]$. Write $D(A) = |A|^{-1} \sum_{x \in A} \mathbb{1}_{S_x}$.

Theorem

Under exponential weights, we have almost surely that $\liminf_{n\to\infty} D(A_n) = 0$ and with positive probability that $\limsup_{n\to\infty} D(A_n) > 0$.

In particular, with positive probability the set of highways fails to have a natural density with respect to the A_n.