Universality for optimal train fares (near the axis)

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Set-up

Consider a long train line where fares between stations are priced dynamically, varying by day. Let

$$F_{k,t} = \text{Fare for } \mathrm{station}_k \to \mathrm{station}_{k+1} \text{ on day } t.$$

Assume $F_{k,t} \sim \lambda$ are i.i.d.

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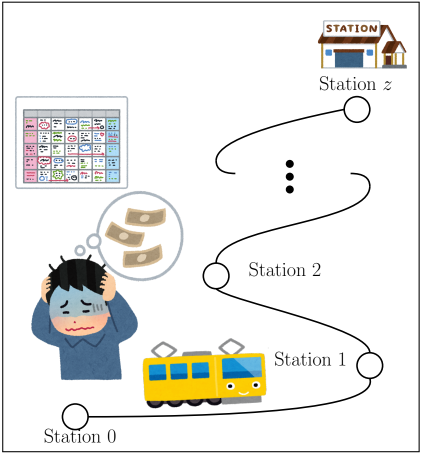
Suppose we know all of the prices in advance. We're interested in the quantity

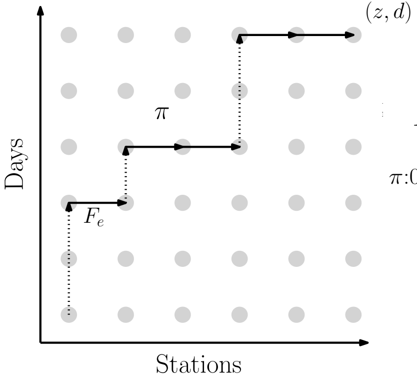
$$L(z,d) = \min_{0 \le t_1 \le \cdots \le t_s \le d} \sum_{k=1}^{s} F_{k,t_k},$$

which is the optimal price to get to station z in at most d days.



As an FPP problem

This is secretly a variant of directed first passage percolation:



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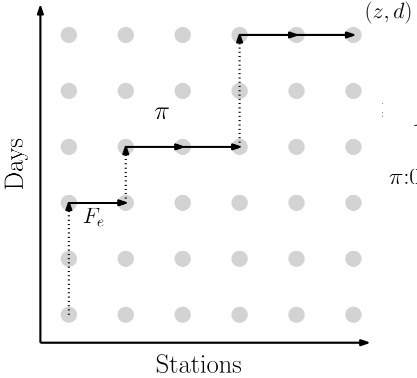

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The existence of a deterministic *time constant*, how much we can expect to pay for each fare on the optimal route, is well known:

$$L(nz, nd) = n\ell(z,d) + o(n) \text{ as } n \to \infty.$$

Many-station regime

In general we can't say much about $\ell(z, d)$, but two limiting regimes are tractable. Bodineau and Martin [Martin 2004; Bodineau and Martin 2005] found the leading terms of ℓ and Tracy-Widom fluctuations in the *many-station* regime:

Theorem

When d *is small,*

$$\ell(z, d) = z(\mu - 2\sigma\sqrt{d}) + o(\sqrt{d}),$$

and for $0 < \alpha < 3/7$*,*

$$\frac{L(n, n^{\alpha}) - n\lambda + 2\sigma n^{(1+\alpha)/2}}{\sigma n^{1/2-\alpha/6}} \Rightarrow TW_{GUE}.$$

Their proof of the latter goes through a Gaussian approximation and uses the KMT embedding.

Many-day regime

The *many-days* regime is also interesting and depends on the shape of λ near its minimum. Assume:

- $\min \operatorname{supp} \lambda = 0$,
- λ has a density f near 0, that $f(0) = 1$, and that f is Lipschitz on a small interval $[0, \epsilon)$.

For example, $\lambda = \mathrm{Unif}[0, 1]$ or $\lambda = \mathrm{Exp}(1)$.

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Theorem

When z is small,

$$\ell(z, d) = \frac{z^2}{4d} + o(z^2),$$

and for $0 \leq \alpha < 3/5$,

$$\frac{L(n^\alpha, n) - n^{2\alpha - 1}/4}{c n^{4\alpha/3 - 1}} \Rightarrow TW_{GUE}.$$

The integrable scaling limit

It's easy to check that

$$n\left(\min_{0\le t\le n} F_{0,t}\right) \Rightarrow \mathrm{Exp}(1).$$

This extends to a continuous time process level limit

$$\{nL(z, \lfloor nt \rfloor)\} \Rightarrow \{M(z,t)\},$$

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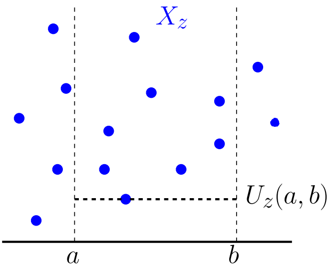
Namely, for each z let X_z be an intensity 1 PPP on $\mathbb{R}\times[0,\infty)$, and let

$$U_z(a,b) = \mathrm{Bottom}(X_z \cap [a,b)\times\mathbb{R}).$$



Then define

$$M(z,t) = \min_{0\le t_1\le\cdots\le t_z\le d} \sum_{k=1}^{z} U_z(t_{k-1},t_k).$$

RMT inputs

This $M(z,t)$ is scale invariant with $M(z,at) \stackrel{d}{=} a^{-1}M(z,t)$ and has RMT marginals:

$$\mathbb{P}(M(z,1) \geq x/4) = \det(I - K^{(z)}\mid_{L^2(0,x)}),$$

where $K^{(z)}$ is the Bessel kernel with parameter z. This follows from the results of [Draief, Mairesse, and O'Connell 2005; Forrester 1993].

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where $K^{(z)}$ is the Bessel kernel with parameter z. This follows from the results of [Draief, Mairesse, and O'Connell 2005; Forrester 1993]. Let $m(z,t)$ be the time constant here. We can borrow RMT results to get:

Proposition

For all $z \geq 0$,

$$m(z,t) = \frac{z^2}{4t}$$

and

$$\frac{L(n,n) - n/4}{cn^{1/3}} \Rightarrow TW_{GUE}.$$

The line ensemble, Busemann process, etc. associated to $\{M(z,t)\}$ have nice descriptions.

Control of the geodesics

We prove the near-edge results for our train problem by producing a strong coupling to the M process which works well on thin rectangles $[n^\alpha] \times [n]$, $0 \leq \alpha < 3/5$.

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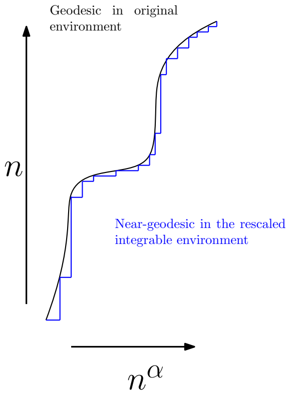

Actually, if $\alpha < 1/5$, we get control of the optimal paths: with high probability they stay close to geodesics of the coupled M.

We should be able to leverage this to get the correct order of transversal fluctuations, and perhaps the Directed Landscape limit, once these are known for M.

Thanks for listening!

References

Bodineau, T. and J. Martin (2005). "A universality property for last-passage percolation paths close to the axis". In: *Electron. Comm. Probab.* 10, pp. 105–112.

Draief, M., J. Mairesse, and N. O'Connell (2005). "Queues, stores, and tableaux". In: *J. Appl. Probab.* 42.4, pp. 1145–1167.

Forrester, P. J. (1993). "The spectrum edge of random matrix ensembles". In: *Nuclear Phys. B* 402.3, pp. 709–728.

Martin, J. (2004). "Limiting shape for directed percolation models". In: *Ann. Probab.* 32.4, pp. 2908–2937.