Universality for optimal train fares (near the axis)

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InterPlay between Integrable Probability and Interacting
Particle systems
Université d'Angers

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Set-up

Consider a long train line where fares between stations are priced dynamically, varying by day. Let

$$F_{k,t} = \text{Fare for } \text{station}_k \to \text{station}_{k+1} \text{ on day } t.$$

Assume $F_{k,t} \sim \lambda$ are i.i.d.

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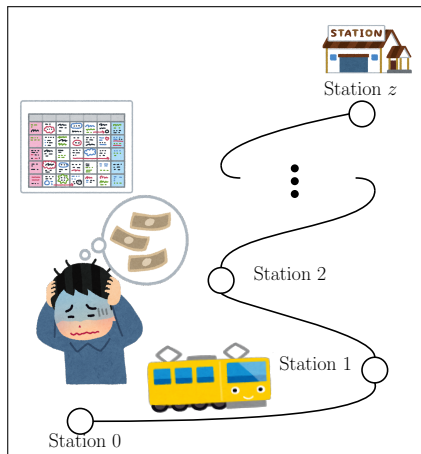Consider a long train line where fares between stations are priced dynamically, varying by day. Let

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Suppose we know all of the prices in advance. We're interested in the quantity

$$L(z, d) = \min_{0 \le t_1 \le \cdots \le t_s \le d} \sum_{k=1}^{s} F_{k,t_k},$$

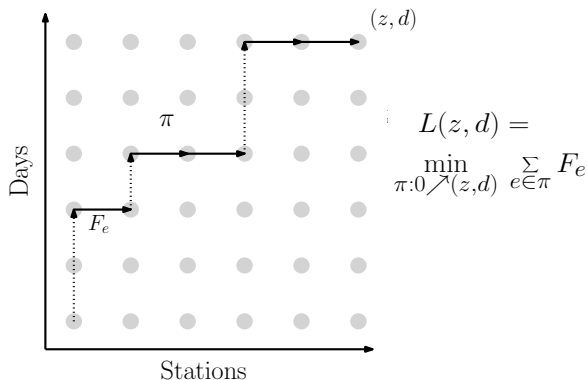which is the optimal price to get to station z in at most d days.

STATION

Station z

Station 2

Station 1

Station 0

As an FPP problem

This is secretly a variant of directed first passage percolation:

$$L(z, d) =$$
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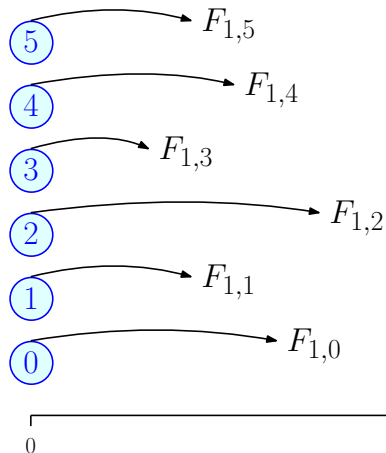$$L(z, d) = \min_{\pi: 0 \nearrow (z,d)} \; \sum_{e \in \pi} F_e$$
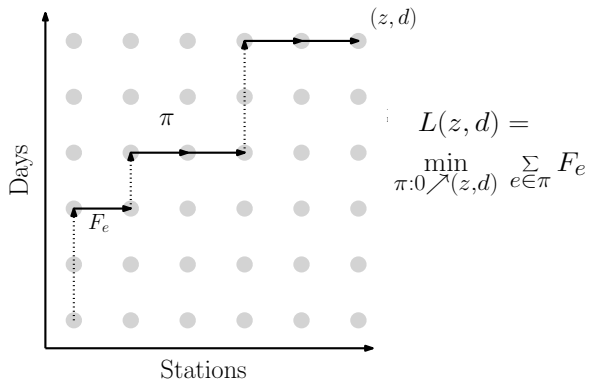
The existence of a deterministic *time constant*, how much we can expect to pay for each fare on the optimal route, is well known:

$$L(nz, nd) = n\ell(z, d) + o(n) \text{ as } n \to \infty.$$

As an IPS

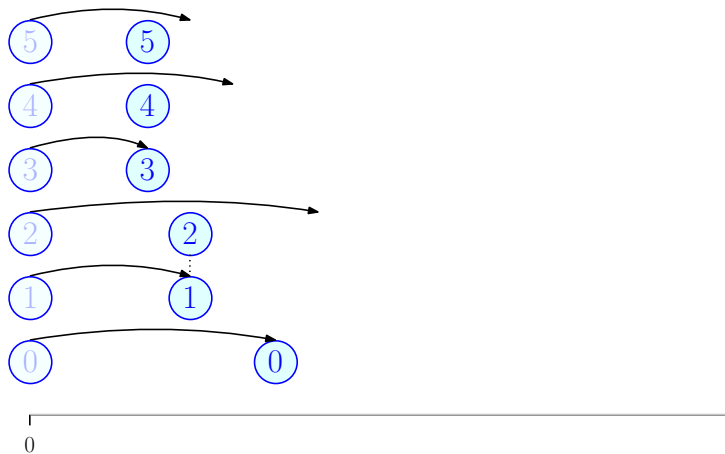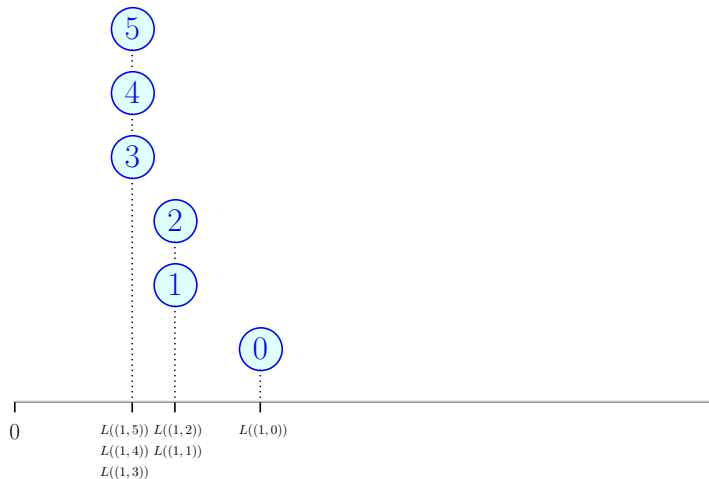There's also a mapping to the following system (which I won't explain here):

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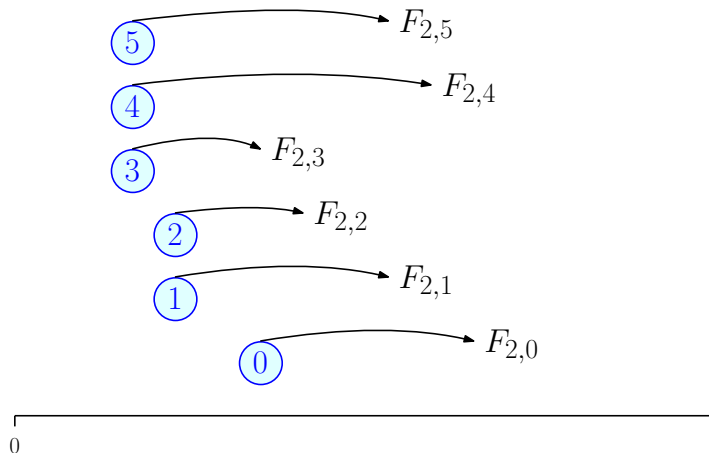There's also a mapping to the following system (which I won't explain here):

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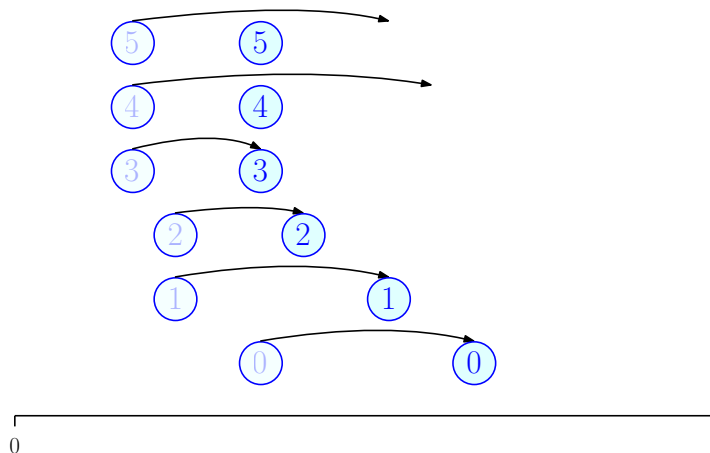There's also a mapping to the following system (which I won't explain here):
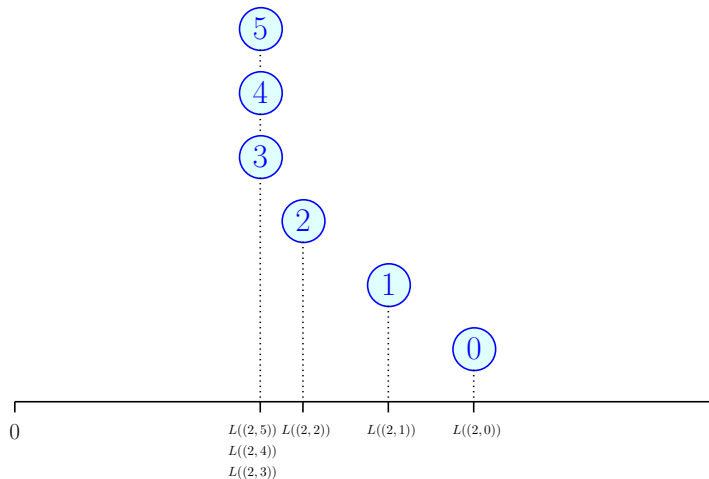
Many-station regime

In general we can't say much about $\ell(z, d)$, but two limiting regimes are tractable. Bodineau and Martin [Martin 2004; Bodineau and Martin 2005] found the leading terms of ℓ and Tracy-Widom fluctuations in the *many-station* regime:

Theorem
When d is small,

$$\ell(z, d) = z(\mu - 2\sigma\sqrt{d}) + o(\sqrt{d}),$$

and for $0 < \alpha < 3/7$,

$$\frac{L(n, n^\alpha) - n\mu + 2\sigma n^{(1+\alpha)/2}}{\sigma n^{1/2 - \alpha/6}} \Rightarrow TW_{GUE}.$$

Their proof of the latter goes through a Gaussian approximation and uses the KMT embedding.

Many-day regime

The *many-days* regime is also interesting and depends on the shape of λ near its minimum. To state things more easily, suppose $\lambda = \mathrm{Unif}[0,1]$[1]

Theorem
When z is small,

$$\ell(z, d) = \frac{z^2}{4d} + o(z^2),$$

and for $0 \leq \alpha < 3/5$,

$$\frac{L(n^\alpha, n) - n^{2\alpha-1}/4}{cn^{4\alpha/3-1}} \Rightarrow TW_{GUE}.$$

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[1]Or any distribution with min supp $\lambda = 0$, a density f near 0, with $f(0) = 1$ and f Lipschitz on a small interval $[0, \epsilon)$

The integrable scaling limit

It's easy to check that

$$n \cdot \left(\min_{0 \leq t \leq n} F_{0,t} \right) \Rightarrow \mathrm{Exp}(1).$$

This extends to a continuous time process level limit

$$\{ nL(z, \lfloor nt \rfloor) \} \Rightarrow \{ M(z, t) \},$$

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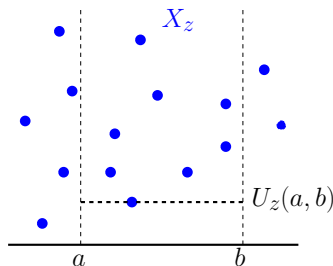
where $M(z, t)$ is defined in terms of homogeneous Poisson processes.

Namely, for each z let X_z be an intensity 1 PPP on $\mathbb{R} \times [0, \infty)$, and let

$$U_z(a, b) = \mathrm{Bottom}(X_z \cap [a, b) \times \mathbb{R}).$$

Then define

$$M(z, t) = \min_{0 \leq t_1 \leq \cdots \leq t_z \leq d} \sum_{k=1}^{z} U_z(t_{k-1}, t_k).$$

RMT inputs

This $M(z, t)$ is scale invariant with $M(z, at) \stackrel{d}{=} a^{-1} M(z, t)$ and has RMT marginals:

$$\mathbb{P}(M(z, 1) \geq x/4) = \det(I - K^{(z)} \mid_{L^2(0,x)}),$$

where $K^{(z)}$ is the Bessel kernel with parameter z. This follows from the results of [Draief, Mairesse, and O'Connell 2005; Forrester 1993].

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where $K^{(z)}$ is the Bessel kernel with parameter z. This follows from the results of [Draief, Mairesse, and O'Connell 2005; Forrester 1993]. Let $m(z, t)$ be the time constant here. We can borrow RMT results to get:

Proposition

For all $z \geq 0$,

$$m(z, t) = \frac{z^2}{4t}$$

and

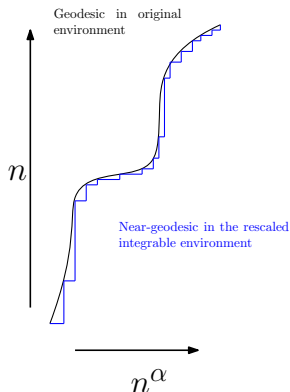$$\frac{L(n, n) - n/4}{cn^{1/3}} \Rightarrow TW_{GUE}.$$

The line ensemble, Busemann process, etc. associated to $\{M(z, t)\}$ have nice descriptions.

Control of the geodesics

We prove the near-edge results for our train problem by producing a strong coupling to the M process which works well on thin rectangles $[n^\alpha] \times [n]$, $0 \le \alpha < 3/5$.

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Geodesic in original environment

Near-geodesic in the rescaled integrable environment

n

n^α

Actually, if $\alpha < 1/5$, we get control of the optimal paths: with high probability they stay close to geodesics of the coupled M.

We should be able to leverage this to get the correct order of transversal fluctuations, and perhaps the Directed Landscape limit, once these are known for M.

Thanks for listening!

References

📄 Bodineau, T. and J. Martin (2005). "A universality property for last-passage percolation paths close to the axis". In: *Electron. Comm. Probab.* 10, pp. 105–112.

📄 Draief, M., J. Mairesse, and N. O'Connell (2005). "Queues, stores, and tableaux". In: *J. Appl. Probab.* 42.4, pp. 1145–1167.

📄 Forrester, P. J. (1993). "The spectrum edge of random matrix ensembles". In: *Nuclear Phys. B* 402.3, pp. 709–728.

📄 Martin, J. (2004). "Limiting shape for directed percolation models". In: *Ann. Probab.* 32.4, pp. 2908–2937.