Solvability in a restricted first passage percolation

Sam McKeown

University of Wisconsin-Madison

Short Talk
Cornell Probability Summer School, July 2024

Strict-weak first passage percolation

Strict-weak first passage percolation (SWFPP) is an FPP model where we consider only directed paths and fix edge weights in one direction to be zero.

On the plane, let $(\omega_x)_{x\in\mathbb{Z}^2}$ be i.i.d weights. If $x, y \in \mathbb{Z}^2$, $x \geq y$, and π is an up-right path $x \to y$, define the weight of the path as

$$G(\pi) = \sum_{z\in\pi} \omega_z \mathbb{1}\{\pi \text{ has horizontal edge at } z\}.$$

The *SWFPP passage time* between x and y is the minimum weight over such π and is denoted $G(x, y)$. Minimising paths are called *geodesics*.

Explicit limit shapes

The passage times in these models obey a law of large numbers. For a fixed direction $(x, y) \in (\mathbb{R}^{\geq 0})^2$:

$$\lim_{n\to\infty} \frac{G(0, (\lfloor nx \rfloor, \lfloor ny \rfloor))}{n} = g(x, y) \text{ a.s.}$$

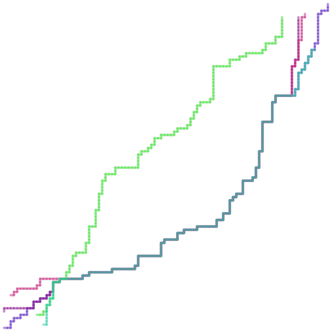
This $g(x, y)$ is called the *time constant*. The level sets are called the *limit shape*.

For certain distributions we have explicit formulas. The simplest is $\omega_x \sim \mathrm{Exp}(1)$, where

$$g(x, y) = (\sqrt{x + y} - \sqrt{y})^2.$$

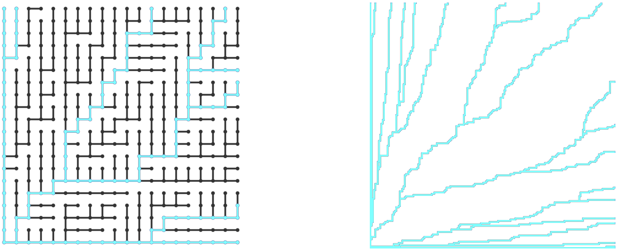
We call these special choices *solvable*.

Coalescence of geodesics



Geodesics tend to overlap for much of their run. Geodesics join *highways* going in their direction, then exit near their destination.

Geodesic trees



We can study the highway phenomenon through the graph of geodesics with a fixed start point. When the weight distribution is continuous, this is a tree.

We are interested in the *(semi-)infinite geodesics*.

Busemann functions

The main tool in answering these questions is the *Busemann process*, the collection of (random) limits

$$B^{\xi}(x, y) = \lim_{n\to\infty} G(y, v_n) - G(x, v_n),$$

where $(v_n)_{n\in\mathbb{N}}$ is a sequence of vertices with $\lim_n v_n/|v_n|$ parallel to $(1, \xi)$.

With the Busemann process in hand, one can produce infinite geodesics with *arbitrary starting point and direction*.

Busemann functions for SWFPP

The same techniques used in LPP can be adapted to SWFPP. A key input is the existance of "generalised Busemann functions" established by Groathouse-Janjigian-Rassoul-Agha '23. There are immediate consequences for the existance of infinite geodesics.

Theorem

Suppose $\mathbb{E}[|\omega|^{2+\epsilon}] < \infty$ *and let* $\mathscr{D}$ *be the set of directions at which the limit shape is strictly convex. Then almost surely, there is for each* $\xi \in \mathscr{D}$ *at least one* ξ*-directed infinite geodesic starting at* 0*.*

In the solvable models, Busemann functions along a line are *independent*. Computations become tractable.

Highways and byways

The highways and byways problem asks about the density of highways among the lattice points.

Let S_x be the event that x lies on an infinite geodesic beginning at the origin. It has been shown for FPP (Ahlberg-Hanson-Hoffman '22) and (under more restrictive assumptions) for LPP (Coupier '15) that

$$\lim_{x \to \infty} \mathbb{P}(S_x) = 0.$$

As a consequence, the lower density of the set of highways is 0 almost surely.

Highways near the axis (1)

The Busemann functions in the exponential model have exponential marginals, so we can easily compute:

$$\mathbb{P}(\text{The path } (0,0) \to (0,n) \to (1,n) \text{ is part of an infinite geodesic}) \geq \frac{e^{-1} + o(1)}{n+1}.$$

The above path is already a geodesic with probability $1/(n+1)$, of the same order.

Highways near the axis (2)

Lemma

Under exponential weights we have $\mathbb{P}(S_{(1,n)}) \geq c$, *for some* $c > 0$ *uniform in* n.

Fix $k \geq 1$ and rectangles $A_n = [1, k] \times [1, n]$. Write $D(A) = |A|^{-1} \sum_{x \in A} \mathbb{1}_{S_x}$.

Theorem

Under exponential weights, we have almost surely that $\liminf_{n\to\infty} D(A_n) = 0$ *and with positive probability that* $\limsup_{n\to\infty} D(A_n) > 0$.

In particular, with positive probability the set of highways fails to have a natural density with respect to the A_n.