ON THE TORSION IN THE COHOMOLOGY OF ARITHMETIC HYPERBOLIC 3-MANIFOLDS

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Abstract

In this paper we consider the cohomology of a closed arithmetic hyperbolic 3*-manifold with coefficients in the local system defined by the even symmetric powers of the standard representation of* $\mathrm{SL}(2,\mathbb{C})$. *The cohomology is defined over the integers and is a finite abelian group. We show that the order of the 2nd cohomology grows exponentially as the local system grows. We also consider the twisted Ruelle zeta function of a closed arithmetic hyperbolic* 3*-manifold, and we express the leading coefficient of its Laurent expansion at the origin in terms of the orders of the torsion subgroups of the cohomology.*

1. Introduction

Let $\mathbf{G}$ be a semisimple connected algebraic group over $\mathbb{Q}$, let K be a maximal compact subgroup of its group of real points $G = \mathbf{G}(\mathbb{R})$, and let $S = G/K$ be the associated Riemannian symmetric space. Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup, and let $X = \Gamma\backslash S$ be the corresponding locally symmetric space. Let $\rho\colon \mathbf{G} \to \mathrm{GL}(V)$ be a rational representation of $\mathbf{G}$ on a $\mathbb{Q}$-vector space V. Then there exists a lattice $M \subset V$ which is stable under Γ. Let $\mathcal{M}$ be the associated local system of free $\mathbb{Z}$-modules over X defined by the Γ-module M, and let $H^*(X,\mathcal{M})$ be the corresponding sheaf cohomology. Since X has the homotopy type of a finite CW-complex, $H^*(X,\mathcal{M})$ are finitely generated abelian groups. The cohomology of arithmetic groups has important connections to the theory of automorphic forms and number theory [Sch]. In this respect, $H^*(X,\mathcal{M}\otimes\mathbb{C})$ has been studied to a great extent. Much less is known about the torsion subgroup $H^*(X,\mathcal{M})_{\mathrm{tors}}$. In a recent paper, Bergeron and Venkatesh [BV] studied the growth of $|H^j(\Gamma_N\backslash S,\mathcal{M})_{\mathrm{tors}}|$ as $N\to\infty$, where $\{\Gamma_N\}$ is a decreasing sequence of normal subgroups of finite index of Γ with trivial intersection. This is motivated by conjectures that torsion classes in the cohomology of arithmetic groups should have arithmetic significance (see [AS], [ADP]). In this paper, we consider a

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similar problem in a different aspect. We fix the discrete group and vary the local system. We also restrict attention to the case of hyperbolic 3-manifolds. However, we expect that the results hold in greater generality.

Now we explain our results in more detail. Let $\mathbb{H}^3 = \mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ be the 3-dimensional hyperbolic space, and let $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ be a discrete torsion-free cocompact subgroup. Then $X = \Gamma\backslash\mathbb{H}^3$ is a compact oriented hyperbolic 3-manifold. We are interested in arithmetic subgroups Γ which are derived from a quaternion division algebra D over an imaginary quadratic number field F. The division algebra D determines an algebraic group $\mathrm{SL}_1(D)$ over F which is an inner form of SL_2/F. Moreover, the group of its F-rational points $\mathrm{SL}_1(D)(F)$ is equal to

$$D^1 := \{x \in D \colon N(x) = 1\},$$

where $N(x) = x\overline{x}$ denotes the norm of $x \in D$. Let $\mathfrak{o} \subset D$ be an order in D, and let $\mathfrak{o}^1 = \mathfrak{o} \cap D^1$. Then $\mathfrak{o}^1$ is a discrete subgroup of $D^1 = \mathrm{SL}_1(D)(F)$. The quaternion division algebra D splits over $\mathbb{C}$; that is, there is an isomorphism $\varphi\colon D \otimes_F \mathbb{C} \to M_2(\mathbb{C})$ of $\mathbb{C}$-algebras. Let $\Gamma := \varphi(\mathfrak{o}^1)$. Then Γ is a cocompact arithmetic subgroup of $\mathrm{SL}(2,\mathbb{C})$. For $n \in \mathbb{N}$ let $V(n) = S^n(F^2)$ be the nth symmetric power of F^2, and let Sym^n be the nth symmetric power representation of SL_2/F on $V(n)$. It follows from Galois descent that for every even n there is a rational representation

$$\mu_n\colon \mathrm{SL}_1(D)/F \to \mathrm{GL}(V(n))$$

which is equivalent to Sym^n over $\overline{F}$. Using this representation it follows that for each even n there is a lattice $M_n \subset V(n)$ which is stable under Γ with respect to Sym^n. Let $\mathcal{M}_n$ be the associated local system of free $\mathbb{Z}$-modules over X. Then by [BW, Chapter VII, Theorem 6.7] we have $H^*(X, \mathcal{M}_n \otimes \mathbb{R}) = 0$. Thus $H^*(X, \mathcal{M}_n)$ is a torsion group. Let $|H^p(X, \mathcal{M}_n)|$ denote the order of $H^p(X, \mathcal{M}_n)$. The purpose of this paper is to study the behavior of $\log|H^p(X, \mathcal{M}_n)|$ as $n \to \infty$. Our main result is the following theorem.

THEOREM 1.1
For every choice of Γ-stable lattices M_{2k} in $S^{2k}(\mathbb{C}^2)$ we have

$$\lim_{k\to\infty} \frac{\log|H^2(X, \mathcal{M}_{2k})|}{k^2} = \frac{2}{\pi}\operatorname{vol}(X). \tag{1.1}$$

Furthermore, for $p = 1, 3$ we have

$$\log|H^p(X, \mathcal{M}_{2k})| \ll k \log k \tag{1.2}$$

uniformly over all choices of lattices M_{2k}.

Note that the left-hand side of (1.1) is a pure combinatorial invariant of X. So at first sight it looks surprising that the volume appears on the right-hand side. However, this is no contradiction, since we know by the Mostow–Prasad rigidity theorem that the volume of a hyperbolic manifold is a topological invariant.

The proof of (1.1) is a consequence of the following theorem combined with (1.2).

THEOREM 1.2
The alternating sum of $\log|H^p(X, \mathcal{M}_{2k})|$ *is independent of the choice of a* Γ*-stable lattice* M_{2k} *in* $S^{2k}(\mathbb{C}^2)$*, and we have*

$$\sum_{p=1}^{3}(-1)^p \log|H^p(X, \mathcal{M}_{2k})| = \frac{2}{\pi}\operatorname{vol}(X)k^2 + O(k) \tag{1.3}$$

as $k \to \infty$.

The fact that $\sum_{p=1}^{3}(-1)^p \log|H^p(X, \mathcal{M}_{2k})|$ is independent of the choice of the Γ-stable lattice M_{2k} follows from the proof of (1.3), but it can also be seen in a more elementary way (see the remark following (5.33)).

More generally for $m, n \in \mathbb{N}$ even, we may consider the local system

$$\mathcal{M}_{n,m} = \mathcal{M}_n \otimes \overline{\mathcal{M}}_m,$$

where $\overline{\mathcal{M}}_n$ is the local system attached to the complex conjugated lattice $\overline{M}_n$ of M_n. If $n \neq m$, then nothing changes. We still have $H^*(X, \mathcal{M}_{n,m} \otimes \mathbb{R}) = 0$. Thus for $n \neq m$, $H^*(X, \mathcal{M}_{n,m})$ is a torsion group, and there is an asymptotic formula similar to (1.1) as $m \to \infty$ or $n \to \infty$.

In [BV], Bergeron and Venkatesh established results of a similar nature but in a different aspect. They studied the growth of the torsion in the cohomology for a fixed local system as the lattice varies in a decreasing sequence of congruence subgroups. Again the volume of the locally symmetric space appears as the main ingredient of the asymptotic formulas.

Our next result is related to the Ruelle zeta function of X. In [De], Deninger discussed a geometric analogue of Lichtenbaum's conjectures in the context of Ruelle zeta functions attached to certain dynamical systems. We establish a result of a similar nature for the Ruelle zeta function of a compact arithmetic hyperbolic 3-manifold.

The Ruelle zeta function is a dynamical zeta function attached to the geodesic flow on the unit tangent bundle of X. We recall its definition. Let $\chi\colon \Gamma \to \mathrm{GL}(V)$ be a representation on a finite-dimensional complex vector space. Given $\gamma \in \Gamma$, denote by $[\gamma]$ the Γ-conjugacy class of γ. For $\gamma \in \Gamma \setminus \{e\}$ let $\ell(\gamma)$ be the length of the unique closed geodesic that corresponds to $[\gamma]$. Then the twisted Ruelle zeta function is defined as

$$R(s;\chi) := \prod_{\substack{[\gamma]\neq e \\ \text{prime}}} \det\big(\mathrm{I}-\chi(\gamma)e^{-s\ell(\gamma)}\big)^{-1}. \tag{1.4}$$

The product runs over all nontrivial primitive conjugacy classes. The infinite product converges in some half-plane $\mathrm{Re}(s) > c$ and admits a meromorphic extension to $\mathbb{C}$ [Fr, Section 7]. We note that the definition (1.4) differs from the usual one by the exponent -1.

We consider the special case where χ is the restriction to Γ of a finite-dimensional representation ρ of $\mathrm{SL}(2,\mathbb{C})$. We denote the associated twisted Ruelle zeta function by $R(s;\rho)$. Note that the irreducible finite-dimensional representations of $\mathrm{SL}(2,\mathbb{C})$, regarded as a real Lie group, are given by

$$\rho_{m,n} := \mathrm{Sym}^m \otimes \overline{\mathrm{Sym}^n}, \quad m,n \in \mathbb{N} \tag{1.5}$$

(see [Kn, p. 32]). Our interest is in the behavior of $R(s;\rho)$ at $s = 0$. To describe it we need to introduce the regulator associated to the free part of the cohomology of a local system of free $\mathbb{Z}$-modules.

Let ρ: $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(V)$ be a finite-dimensional real representation of $\mathrm{SL}(2,\mathbb{C})$, and assume that there exists a lattice $M \subset V$ which is stable under Γ. Let $\mathcal{M}$ be the associated local system. Let E be the flat vector bundle over X attached to the restriction of ρ to Γ. By [MM, Lemma 3.1], the bundle E can be equipped with a canonical fiber metric. Let $\mathcal{H}^*(X;E)$ be the space of E-valued harmonic forms on X with respect to this metric in E and the hyperbolic metric on X. There is a canonical isomorphism

$$\mathcal{H}^*(X;E) \cong H^*(X,\mathcal{M}\otimes\mathbb{R}).$$

We equip $H^*(X;\mathcal{M}\otimes\mathbb{R})$ with the inner product $\langle\cdot,\cdot\rangle$ induced by the L^2-metric on $\mathcal{H}^*(X;E)$. Let $H^*(X;\mathcal{M})_{\mathrm{free}} = H^*(X;\mathcal{M})/H^*(X;\mathcal{M})_{\mathrm{tors}}$. We identify $H^*(X;\mathcal{M})_{\mathrm{free}}$ with a subgroup of $H^*(X;\mathcal{M}\otimes\mathbb{R})$. For each $p = 0,1,2,3$ choose a basis $a_1,\dots,a_{r_p}$ of $H^p(X;\mathcal{M})_{\mathrm{free}}$, and let $G_p(\mathcal{M})$ be the Gram matrix of this basis with respect to the inner product $\langle\cdot,\cdot\rangle$. Put $R_p(\mathcal{M}) := \sqrt{|\det G_p(\mathcal{M})|}$; this is called a "regulator." This is not the standard notion of a regulator. For a discussion in a more general context see [BV]. Put

$$R(\mathcal{M}) := \prod_{p=0}^{3} R_p(\mathcal{M})^{(-1)^p}.$$

We can now state our result which describes the behavior of the Ruelle zeta function at the origin.

THEOREM 1.3
Let $X = \Gamma\backslash\mathbb{H}^3$ *be a compact oriented arithmetic hyperbolic* 3*-manifold. Let* $\rho\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(V)$ *be an irreducible finite-dimensional representation of* $\mathrm{SL}(2,\mathbb{C})$. *Let* $M \subset V$ *be a lattice which is stable under* Γ, *and denote by* $\mathcal{M}$ *the associated local system of free* $\mathbb{Z}$*-modules over* X. *Let* $R(s;\rho)$ *be the twisted Ruelle zeta function. Then we have the following results.*

(1) *If* $\rho \neq 1$, *then the order of* $R(s;\rho)$ *at* $s = 0$ *is given by*

$$\operatorname{ord}_{s=0} R(s;\rho) = \sum_{q=1}^{3} (-1)^q q \operatorname{rk} H^q(X;\mathcal{M}). \tag{1.6}$$

If $\rho = 1$, *then the order equals* $2\dim H^1(X,\mathbb{R}) - 4$.

(2) *Let* $R^*(0;\rho)$ *be the leading coefficient of the Laurent expansion of* $R(s;\rho)$ *at* $s = 0$. *We have*

$$|R^*(0;\rho)| = R(\mathcal{M})^{-1} \cdot \prod_{q=0}^{3} |H^q(X;\mathcal{M})_{\mathrm{tors}}|^{(-1)^q}. \tag{1.7}$$

Moreover, if $\rho = \rho_{m,m}$ *for some* $m \in \mathbb{N}_0$, *then* (1.7) *holds for* $R^*(0;\rho)$.

Recall that the irreducible finite-dimensional representations of $\mathrm{SL}(2,\mathbb{C})$ are given by (1.5). For each even n there is a Γ-stable lattice in $S^n(\mathbb{C}^2)$. Therefore, if $\rho = \rho_{m,n}$ with m and n even, then there exist Γ-stable lattices in the space of the representation.

We note that there is a formal analogy of (1.6) and (1.7) to Lichtenbaum's conjectures on special values of Hasse–Weil zeta functions of algebraic varieties (see [Li1], [Li2]). To make this statement more transparent, we recall some details. Let $\mathcal{X}$ be a regular scheme which is separated and of finite type over $\mathrm{Spec}(\mathbb{Z})$. Let $\zeta_{\mathcal{X}}(s)$ be the Hasse–Weil zeta function of $\mathcal{X}$. It is given by an Euler product that converges in some half-plane $\mathrm{Re}(s) > c$. The Euler product is expected to have a meromorphic extension to the whole complex plane. This is known in some cases. Lichtenbaum's conjectures are concerned with the behavior of $\zeta_{\mathcal{X}}(s)$ at $s = 0$. First of all, Lichtenbaum conjectures the existence of a certain new cohomology theory for schemes over $\mathbb{Z}$, called "Weil–étale" cohomology. Let $H_c^p(\mathcal{X},\mathbb{Z})$ be the pth Weil–étale cohomology group of $\mathcal{X}$ with compact supports and coefficients in $\mathbb{Z}$. Then the conjectures of Lichtenbaum are the following statements.

(1) The groups $H_c^p(\mathcal{X},\mathbb{Z})$ are finitely generated and vanish for $p > 2\dim\mathcal{X} + 1$.

(2) The order of $\zeta_{\mathcal{X}}(s)$ at $s = 0$ is given by

$$\operatorname{ord}_{s=0} \zeta_{\mathcal{X}}(s) = \sum_p (-1)^p p \operatorname{rk} H_c^p(\mathcal{X},\mathbb{Z}).$$

(3) The leading coefficient $\zeta^*_{\mathcal{X}}(0)$ of the Laurent expansion of $\zeta_{\mathcal{X}}(s)$ satisfies

$$\zeta^*_{\mathcal{X}}(0) = R^{-1} \cdot \prod_p |H^p_c(\mathcal{X}, \mathbb{Z})_{\mathrm{tors}}|^{(-1)^p},$$

where R is the Reidemeister torsion of some acyclic complex associated to $H^*_c(\mathcal{X}, \mathbb{R})$, equipped with volume forms determined by the isomorphism $H^*_c(\mathcal{X}, \mathbb{R}) = H^*_c(\mathcal{X}, \mathbb{Z}) \otimes \mathbb{R}$ and a basis of $H^*_c(\mathcal{X}, \mathbb{Z})_{\mathrm{free}}$. We note that Denniger [De] first discussed a geometric analogue of the Lichtenbaum conjectures in the context of dynamical systems.

Our approach to proving our main results is based on the study of the Reidemeister torsion of X. Let ρ_{2k} be the representation of Γ obtained by the restriction of $\mathrm{Sym}^{2k}\colon \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(S^{2k}(\mathbb{C}^2))$ to Γ. Let $\rho^{\mathbb{R}}_{2k}$ be the underlying real representation. Denote by $\tau_X(\rho^{\mathbb{R}}_{2k})$ the Reidemeister torsion of X with respect to $\rho^{\mathbb{R}}_{2k}$ (see Section 2 for its definition). Then the Reidemeister torsion satisfies

$$\tau_X(\rho^{\mathbb{R}}_{2k}) = \prod_{p=1}^{3} |H^p(X, \mathcal{M}_{2k})|^{(-1)^{p+1}}.$$

This equality was first noted by Cheeger [Ch, (1.4)]. Now we apply [Mu2, Corollary 1.2], which describes the asymptotic behavior of $\log \tau_X(\rho^{\mathbb{R}}_{2k})$ as $k \to \infty$. This implies Theorem 1.2. To estimate $\log |H^p(X, \mathcal{M}_{2k})|$ for $p = 1, 3$, we use that $H^3(X, \mathcal{M}_{2k})$ is isomorphic to the space $(M_{2k})_\Gamma$ of coinvariants. To bound $(M_{2k})_\Gamma$ we can work locally. This leads to (1.2). Together with Theorem 1.2 we obtain (1.1), which proves Theorem 1.1.

To prove Theorem 1.3, we use [Mu2, Theorem 1.5]. Using this theorem we obtain the statement about the order of $R(s; \rho)$ at $s = 0$. Moreover, it follows that the leading coefficient of the Laurent expansion of $R(s; \rho)$ at $s = 0$ equals $T_X(\rho; h)^{-2}$, where $T_X(\rho; h)$ is the Ray–Singer analytic torsion of X and $\rho|_\Gamma$ with respect to the canonical fiber metric in the flat bundle E mentioned above. By [Mu1, Theorem 1], the analytic torsion equals the Reidemeister torsion $\tau_X(\rho; h)$. If there exists a Γ-stable lattice in V, then the Reidemeister torsion satisfies

$$\tau_X(\rho; h) = R(\mathcal{M})^{-1} \prod_{p=0}^{3} |H^p(X, \mathcal{M})_{\mathrm{tors}}|^{(-1)^{p+1}}. \tag{1.8}$$

Combining these facts, we obtain Theorem 1.3.

The paper is organized as follows. In Section 2 we discuss the relation between Reidemeister torsion and cohomology if the chain complex is defined over the integers. In particular, we prove (1.8). In Section 3 we collect a number of facts about cocompact arithmetic subgroups of $\mathrm{SL}(2, \mathbb{C})$ which are derived from quaternion division algebras. In particular, we prove that the even symmetric powers contain Γ-stable lattices. In Section 4 we prove (1.2). In the final Section 5 we prove our main results.

2. Reidemeister torsion and cohomology

We recall some facts about Reidemeister torsion. Let V be a finite-dimensional real vector space of dimension n. Set $\det V = \Lambda^n(V)$. A volume element in V is a nonzero element $\omega \in \det V$. Any volume element determines an isomorphism $\det V \cong \mathbb{R}$. Furthermore, note that a volume element is given by $\omega = e_1 \wedge e_2 \wedge \cdots \wedge e_n$ for some basis $e_1, \dots, e_n$ of V.

Let

$$C^*\colon 0 \to C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C^n \to 0 \tag{2.9}$$

be a complex of finite-dimensional $\mathbb{R}$-vector spaces. Let

$$B^j := d_{j-1}(C^{j-1}), \qquad Z^j := \ker(d_j),$$

and denote by $H^j(C^*) := Z^j/B^j$ the jth cohomology group of C^*. Assume that for each j we are given volume elements $\omega_j \in \det C^j$ and $\mu_j \in \det H^j(C^*)$. Let $\omega = (\omega_0, \dots, \omega_n)$, and let $\mu = (\mu_0, \dots, \mu_n)$. Then the Reidemeister torsion $\tau(C^*, \omega, \mu) \in \mathbb{R}^+$ of the complex C^* is defined as a certain ratio of volumes (see [Mi]). In [RS], Ray and Singer gave an equivalent definition in terms of the combinatorial Laplacian, which we recall next. In each C^j we choose an inner product $\langle\cdot,\cdot\rangle_j$ with volume element ω_j. Let

$$d^*_{j+1}\colon C^{j+1} \to C^j$$

be the adjoint operator to d_j with respect to the inner products in C^j and C^{j+1}, respectively. Define the combinatorial Laplacian by

$$\Delta_j^{(c)} = d^*_{j+1} d_j + d_{j-1} d^*_j. \tag{2.10}$$

Then $\Delta_j^{(c)}$ is a symmetric nonnegative operator in C^j. We have the combinatorial Hodge decomposition

$$C^j = \ker(\Delta_j^{(c)}) \oplus d_{j-1}(C^{j-1}) \oplus d^*_{j+1}(C^{j+1}). \tag{2.11}$$

It implies that there is a canonical isomorphism

$$\ker(\Delta_j^{(c)}) \xrightarrow{\cong} H^j(C^*). \tag{2.12}$$

The inner product in C^j restricts to an inner product in $\ker(\Delta_j^{(c)})$. Using the isomorphism (2.12) we get an inner product $\langle\cdot,\cdot\rangle$ in $H^j(C^*)$. Let $h_1, \dots, h_{d_j} \in H^j(C^*)$ be a basis such that $\mu_j = h_1 \wedge \cdots \wedge h_{d_j}$. Let G_j be the Gram matrix with entries $\langle h_k, h_l\rangle_j$, $1 \le k, l \le d_j$. Put

$$V(\mu_j) = \sqrt{|\det G_j|}.$$

Denote by $\det' \Delta_j^{(c)}$ the product of the nonzero eigenvalues of $\Delta_j^{(c)}$. Then a slight generalization of [RS, Proposition 1.7] gives the following.

LEMMA 2.1
We have

$$\tau(C^*, \omega, \mu) = \prod_{j=0}^{n} V(\mu_j)^{(-1)^j} \cdot \prod_{j=1}^{n} (\det' \Delta_j^{(c)})^{(-1)^{j+1} j/2}. \tag{2.13}$$

Now let

$$A^*\colon 0 \longrightarrow A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} A^n \longrightarrow 0 \tag{2.14}$$

be a complex of free finite-rank $\mathbb{Z}$-modules, and set $C^j := A^j \otimes \mathbb{R}$. Then we get a complex (2.9) of $\mathbb{R}$-vector spaces such that each C^j has a preferred equivalence class of bases coming from bases of A^j. For any two such bases the matrix of change from one to the other is an invertible matrix with integral entries. Thus C^j is endowed with a canonical volume element ω_j. Furthermore, with respect to any of the bases coming from A^j, the coboundary operator d_j is represented by a matrix with integral entries. Let $H^j(A^*)$ denote the jth cohomology group of the complex (2.14). Let $H^j(A^*)_{\text{tors}}$ be the torsion subgroup of $H^j(A^*)$, and let $H^j(A^*)_{\text{free}} = H^j(A^*)/H^j(A^*)_{\text{tors}}$. We identify $H^j(A^*)_{\text{free}}$ with a subgroup of $H^j(C^*)$. Let μ_j be a volume element of $H^j(C^*)$. Put

$$R(\mu_j) := \operatorname{vol}\big(H^j(C^*)/H^j(A^*)_{\text{free}}\big), \tag{2.15}$$

where the volume is computed with respect to μ_j. Denote by $\tau(C^*, \mu)$ the Reidemeister torsion of $C^* = A^* \otimes \mathbb{R}$ with respect to the canonical volume element ω, which is determined by A^*, and the volume is the volume element $\mu \in \prod_{j=0}^{n} \det H^j(C^*)$. Finally, denote by $|H^j(A^*)_{\text{tors}}|$ the order of the finite group $H^j(A^*)_{\text{tors}}$. Then the following elementary lemma, which describes the relation between Reidemeister torsion and the torsion of the cohomology of the complex A^*, is proved in [BV, Section 2].

LEMMA 2.2
We have

$$\tau(C^*, \mu) = \prod_{j=0}^{n} R(\mu_j)^{(-1)^j} \cdot \prod_{j=0}^{n} |H^j(A^*)_{\text{tors}}|^{(-1)^{j+1}}. \tag{2.16}$$

If μ_j is the volume element associated to the equivalence class of bases of $H^j(C^*)$ coming from bases of the lattice $H^j(A^*)_{\text{free}}$, we have $R(\mu_j) = 1$. In this case, (2.16) was first stated in [Ch, (1.4)] and proved for the case where A^* is acyclic.

We now turn to the geometric situation. Let X be a compact n-dimensional Riemannian manifold. Choose a base point $x_0 \in X$. Let $\Gamma := \pi_1(X, x_0)$ be the fundamental group of X with respect to x_0, acting on the universal covering $\tilde{X}$ of X as deck transformations. Let V be a finite-dimensional real vector space, and let

$$\chi\colon \Gamma \to \mathrm{GL}(V)$$

be a representation of Γ on V. It defines a flat vector bundle E over X.

Fix a smooth triangulation K of X. Let $\tilde{K}$ be the lift of K to a smooth triangulation of $\tilde{X}$. We think of K as being embedded as a fundamental domain in $\tilde{K}$, so that $\tilde{K}$ is the union of the translates of K under Γ. Let $C^q(\tilde{K};\mathbb{Z})$ be the cochain group generated by the q-simplexes of $\tilde{K}$. Then $C^q(\tilde{K};\mathbb{Z})$ is a $\mathbb{Z}[\Gamma]$-module. Define the twisted cochain group $C^q(K;V)$ by

$$C^q(K;V) := C^q(\tilde{K};\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} V.$$

From the cochain complex $C^*(\tilde{K},\mathbb{Z})$ we get the twisted cochain complex

$$C^*(K;V)\colon 0 \to C^0(K;V) \xrightarrow{d_0} C^1(K;V) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C^n(K;V) \to 0. \tag{2.17}$$

Let $H^*(K;V)$ be the cohomology groups. They are independent of K and will be denoted by $H^*(X;V)$. The cohomology can also be computed by a different complex. Let $C^q(K;E)$ be the set of E-valued q-cochains [Wh, Chapter VI]. An element of $C^q(K;E)$ is a function that assigns to each q-simplex a section of E on that simplex. The corresponding cochain complex of finite-dimensional $\mathbb{R}$-vector spaces computes the cohomology groups $H^*(K;E) = H^*(X;E)$. By [Wh, p. 278] there is canonical isomorphism

$$H^*(X;V) \cong H^*(X;E).$$

Assume that a volume element $\theta \in \det V$ is given. Let σ_j^q, $j = 1,\dots,r_q$, be the oriented q-simplexes of K considered as a preferred basis of the $\mathbb{Z}[\Gamma]$-module $C^q(\tilde{K};\mathbb{Z})$. Let $e_1,\dots,e_m$ be a basis of V such that $\theta = \pm e_1 \wedge \cdots \wedge e_n$. Then $\{\sigma_j^q \otimes e_k \colon j = 1,\dots,r_q, k = 1,\dots,m\}$ is a preferred basis of $C^q(K;V)$. It defines a volume element $\omega_q \in \det C^q(K;V)$. The volume element depends on several choices (see [Mu2, p. 727]). Now assume that χ is unimodular; that is, we have

$$|\det \chi(\gamma)| = 1, \quad \forall \gamma \in \Gamma.$$

Then ω is unique up to sign. Let $\mu \in \det H^*(X;V)$ be a volume element. Then we can define the Reidemeister torsion $\tau(C^*(K;V);\omega,\mu)$. It still depends on $\theta \in \det V$. However, if the Euler characteristic of X vanishes, then it is also independent of θ (see [Mu1, p. 727]). It is well known that $\tau(C^*(K;V);\omega,\mu)$ is invariant under subdivision (see [Wh]). Since any two smooth triangulations of X have a common subdivision, $\tau(C^*(K;V);\omega,\mu)$ is independent of K. Therefore we may put

$$\tau_X(\chi;\mu) := \tau\big(C^*(K;V);\omega,\mu\big).$$

Now pick a fiber metric h on E. Let $\mathcal{H}^*(X;E)$ be the space of E-valued harmonic forms on X. Then we have the Hodge–de Rham isomorphism

$$\mathcal{H}^*(X;E) \cong H^*(X;E) \cong H^*(X;V).$$

Using this isomorphism, we get an inner product in $H^*(X;V)$ and a corresponding volume element $\mu_h \in \det H^*(X;V)$. Let

$$\tau_X(\chi;h) := \tau_X(\chi,\mu_h).$$

Now assume that there exists a lattice $M \subset V$ which is invariant under Γ; that is, M is a free abelian subgroup of V such that $V = M \otimes \mathbb{R}$ and M is invariant under Γ. Thus M is a finitely generated $\mathbb{Z}[\Gamma]$-module. It defines a local system $\mathcal{M}$ of free $\mathbb{Z}$-modules on X. Set

$$C^q(K;M) = C^q(\tilde{K};\mathbb{Z}) \otimes_{\mathbb{Z}[\Gamma]} M, \quad q = 0,\dots,n, \tag{2.18}$$

and let $H^*(K;M) = H^*(X;M)$ denote the cohomology of the corresponding complex. Now $C^*(K;M)$ is a complex of finitely generated free $\mathbb{Z}$-modules, and we have

$$C^*(K;V) = C^*(K;M) \otimes \mathbb{R}.$$

So Lemma 2.2 applies to this situation. As above, we may also consider the set $C^q(K,\mathcal{M})$ of $\mathcal{M}$-valued q-cochains (see [Wh, Chapter VI]). The complex $C^*(K;\mathcal{M})$ computes the cohomology $H^*(K;\mathcal{M}) = H^*(X;\mathcal{M})$, and we have a canonical isomorphism

$$H^*(X,M) \cong H^*(X,\mathcal{M}).$$

Each $H^q(X,\mathcal{M})$ is a finitely generated $\mathbb{Z}$-module. Let $H^q(X,\mathcal{M})_{\text{tors}}$ be the torsion subgroup, and

$$H^q(X;\mathcal{M})_{\text{free}} = H^q(X,\mathcal{M})/H^q(X,\mathcal{M})_{\text{tors}}.$$

We identify $H^q(X,\mathcal{M})_{\text{free}}$ with a subgroup of $H^q(X,E)$. Let $\langle\cdot,\cdot\rangle_q$ be the inner product in $H^q(X,E)$ induced by the L^2-metric on $\mathcal{H}^q(X,E)$. Let $e_1,\dots,e_{r_q}$ be a basis of $H^q(X,\mathcal{M})_{\text{free}}$, and let G_q be the Gram matrix with entries $\langle e_k,e_l\rangle$. Put

$$R_q(\chi, h) = \sqrt{|\det G_q|}, \quad q = 0, \dots, n.$$

Define the "regulator" $R(\chi, h)$ by

$$R(\chi, h) = \prod_{q=0}^{n} R_q(\chi, h)^{(-1)^q}.$$

PROPOSITION 2.3

Let χ be a unimodular representation of Γ on a finite-dimensional $\mathbb{R}$-vector space E. Let $M \subset E$ be a Γ-invariant lattice, and let $\mathcal{M}$ be the associated local system of finitely generated free $\mathbb{Z}$-modules on X. Let h be a fiber metric in the flat vector bundle $E = \mathcal{M} \otimes \mathbb{R}$. Then we have

$$\tau_X(\chi, h) = R(\chi, h) \cdot \prod_{q=0}^{n} |H^q(X, \mathcal{M})_{\mathrm{tors}}|^{(-1)^{q+1}}.$$

We also consider complex representations $\chi\colon \pi_1(X) \to \mathrm{GL}(V)$ of $\pi_1(X)$. If χ is unimodular, the Reidemeister torsion $\tau_X(\chi) \in \mathbb{R}^+$ is defined. In fact, one can define the complex Reidemeister torsion $\tau_X^c(\chi) \in \mathbb{C}^*/\{\pm 1\}$ (see [Mi]) in the same way as in the real case. Then $\tau_X(\chi) = |\tau_X^c(\chi)|$. On the other hand, let $V^{\mathbb{R}}$ be the underlying real vector space. Let $i\colon\ \mathrm{GL}(V) \to \mathrm{GL}(V^{\mathbb{R}})$ be the canonical embedding. Let

$$\chi^{\mathbb{R}} := i \circ \chi$$

be the associated real representation. If χ is unimodular, then $\chi^{\mathbb{R}}$ is also unimodular. Thus $\tau_X(\chi^{\mathbb{R}})$ is defined, and the relation with $\tau_X(\chi)$ is described by the following lemma.

LEMMA 2.4

Let $\chi\colon \Gamma \to \mathrm{GL}(V)$ be a unimodular, acyclic representation in a finite-dimensional complex vector space V. Then we have

$$\tau_X(\chi^{\mathbb{R}}) = \tau_X(\chi)^2.$$

Proof

The p-simplexes of K form a basis of $C^p(\tilde{K}; \mathbb{Z})$ as a $\mathbb{Z}[\Gamma]$-module. With respect to these bases, the coboundary operator $\tilde{d}_p\colon C^p(\tilde{K}; \mathbb{Z}) \to C^{p+1}(\tilde{K}; \mathbb{Z})$ is given by a matrix (a_{ij}) with $a_{ij} \in \mathbb{Z}[\Gamma]$. Let $\chi\colon \mathbb{Z}[\Gamma] \to \mathrm{End}(V)$ be defined by

$$\chi\Big(\sum_j n_j \gamma_j\Big) = \sum_j n_j \chi(\gamma_j), \quad n_j \in \mathbb{Z}, \gamma_j \in \Gamma.$$

It follows that the coboundary operator $d_p^\chi\colon C^p(K;V) \to C^{p+1}(K;V)$ is given by the matrix $(\chi(a_{ij}))$. Choose an inner product in V. This gives rise to an inner product in each $C^p(K;V)$. Let

$$\Delta_p^\chi = (d_p^\chi)^* \circ d_p^\chi + d_{p-1}^\chi \circ (d_{p-1}^\chi)^*.$$

Since χ is acyclic, Δ_p^χ is invertible. By a formula analogous to (2.13), the Reidemeister torsion is given by

$$\tau_X(\chi) = \prod_{j=1}^{n} (\det \Delta_j^\chi)^{(-1)^{j+1} j/2}.$$

A similar formula holds for $\tau_X(\chi^{\mathbb{R}})$. Thus to prove the lemma it suffices to prove that $\det \Delta_j^{\chi^{\mathbb{R}}} = (\det \Delta_j^\chi)^2$, $j = 1, \ldots, n$. For a linear operator $A\colon W \to W$ in a complex vector space W let $A^{\mathbb{R}}$ denote the corresponding real operator in $W^{\mathbb{R}}$. By construction we have $\Delta_j^{\chi^{\mathbb{R}}} = (\Delta_j^\chi)^{\mathbb{R}}$. Thus it suffices to prove that for any linear operator A in a complex vector space we have $\det A^{\mathbb{R}} = |\det A|^2$. Using the Jordan normal form of A, the problem is reduced to the 1-dimensional case. Let $A\colon \mathbb{C} \to \mathbb{C}$ be the multiplication by $z = u + iv$. Then $A^{\mathbb{R}} = \left(\begin{smallmatrix} u & -v \\ v & u \end{smallmatrix}\right)$ and therefore $\det A^{\mathbb{R}} = u^2 + v^2 = |z|^2 = |\det A|^2$, which proves the lemma. □

3. Arithmetic groups

In this section we recall some facts about quaternion algebras and arithmetic subgroups of $\mathrm{SL}(2,\mathbb{C})$ derived from quaternion algebras.

Let F be a field of characteristic zero. Every quaternion algebra over F can be described as a 4-dimensional F-vector space with basis $1, i, j, k$ satisfying the following relations:

$$i^2 = a, \qquad j^2 = b, \qquad ij = k, \qquad ji = -k,$$

where $a, b \in F^\times$. This algebra is denoted by $H(a,b;F)$ or simply $H(a,b)$. Let $x \in H \mapsto \overline{x} \in H$ be the involution defined by

$$\overline{x_0 + x_1 i + x_2 j + x_3 k} = x_0 - x_1 i - x_2 j - x_3 k.$$

Then the norm and the trace in H are defined by

$$N(x) = x\overline{x}, \qquad \mathrm{Tr}(x) = x + \overline{x}, \quad x \in H.$$

It follows from Wedderburn's structure theorem for simple algebras that a quaternion algebra H over F is either a division algebra or is isomorphic to $M_2(F)$ (see [Lam, Theorem II.2.7], [MR, Theorem 2.1.7]). In the latter case H is called split over F.

Let D be a division algebra over F of degree d^2, $d \leq 2$. Let $m = 2/d$. There is an associated almost simple semisimple algebraic group $\mathrm{SL}_m(D)$. It is defined as the functor from F-algebras to groups which assigns to an F-algebra R the group

$$\mathrm{SL}_m(D)(R) := \{x \in M_m(D \otimes_F R)\colon N(x) = 1\}.$$

The algebraic groups $\mathrm{SL}_m(D)$ are the forms of SL_2 over F (see [PR, Section 2.3]). If $d = 1$, we have $D = F$ and $\mathrm{SL}_2(D) = \mathrm{SL}_2$. If $d = 2$, D is a quaternion division algebra.

We consider SL_2 as an algebraic group over F. For $n \in \mathbb{N}$ let $V(n) := S^n(F^2)$ be the nth symmetric power of F^2, and let $\mathrm{Sym}^n\colon \mathrm{SL}_2 \to \mathrm{GL}(V(n))$ be the nth symmetric power of the standard representation $\rho\colon \mathrm{SL}_2 \to \mathrm{GL}_2$ of SL_2. The following lemma is a special case of [Tit, Theorem 3.3]. For the convenience of the reader we include the proof, which was kindly communicated to us by Skip Garibaldi.

LEMMA 3.1
Let G' be a form of SL_2/F. For every even n there exists an F-rational representation of G' on $V(n)$ which is equivalent to Sym^n over $\overline{F}$.

Proof
First we observe that the Dynkin diagram A_1 has no nontrivial automorphisms. Therefore SL_2 has no outer automorphims and

$$\mathrm{Aut}(\mathrm{SL}_2) = \mathrm{PGL}_2\,. \tag{3.19}$$

Hence the forms of SL_2 are classified by the Galois cohomology set $H^1(\overline{F}/F, \mathrm{PGL}_2)$ (see [PR, Section 2.2, Theorem 2.9]). Let $\mu\colon \mathrm{SL}_2(\overline{F}) \to G'(\overline{F})$ be an isomorphism (which exists by assumption). For $\sigma \in \mathrm{Gal}(\overline{F}/F)$ put

$$\varphi_\sigma = \mu^{-1} \circ \sigma \circ \mu \circ \sigma^{-1}. \tag{3.20}$$

Then φ_σ is an automorphism of $\mathrm{SL}_2/\overline{F}$. By (3.19) there exists $a_\sigma \in \mathrm{PGL}_2(\overline{F})$ such that $\varphi_\sigma = \mathrm{Int}(a_\sigma)$. As in [PR, Section 2.2.1], it follows that $\sigma \in \mathrm{Gal}(\overline{F}/F) \mapsto a_\sigma \in \mathrm{PGL}_2(\overline{F})$ is the cocycle that represents G'. Let $a'_\sigma \in \mathrm{SL}_2(\overline{F})$ be any lift of a_σ. Let $n \in \mathbb{N}$ be even. Then Sym^n is trivial on the center of SL_2, and therefore the map

$$\sigma \in \mathrm{Gal}(\overline{F}/F) \mapsto \mathrm{Sym}^n(a'_\sigma) \in \mathrm{GL}\big(V(n) \otimes \overline{F}\big)$$

is a well-defined cocycle. By Hilbert's Theorem 90, we have $H^1(\overline{F}/F, \mathrm{GL}_N) = 1$ for every N. Hence there exists $x \in \mathrm{GL}(V(n) \otimes \overline{F})$ such that

$$\mathrm{Sym}^n(a'_\sigma) = x^{-1}\sigma(x), \quad \forall \sigma \in \mathrm{Gal}(\overline{F}/F). \tag{3.21}$$

Define the map $\rho_n\colon G' \to \mathrm{GL}(V(n))$ by

$$\rho_n := \mathrm{Int}(x) \circ \mathrm{Sym}^n \circ \mu. \tag{3.22}$$

This is a representation of G' over $\overline{F}$ which over $\overline{F}$ is equivalent to Sym^n. By (3.20), the action of $\mathrm{Gal}(\overline{F}/F)$ on $G'(\overline{F})$ is given by

$$\mu\big(\sigma(g)\big) = \mathrm{Int}(a_\sigma)\cdot\sigma\big(\mu(g)\big), \quad g \in G'(\overline{F}),\ \sigma \in \mathrm{Gal}(\overline{F}/F). \tag{3.23}$$

Using (3.21), (3.22), and (3.23), it follows that for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$ and $g \in G'(\overline{F})$ we get

$$\begin{aligned}\rho_n\big(\sigma(g)\big) &= \mathrm{Int}(x)\,\mathrm{Sym}^n\big(\mathrm{Int}(a'_\sigma)\sigma(\mu(g))\big)\\ &= \mathrm{Int}\big(x\,\mathrm{Sym}^n(a'_\sigma)\big)\sigma\big(\mathrm{Sym}^n(\mu(g))\big)\\ &= \mathrm{Int}\big(\sigma(x)\big)\sigma\big(\mathrm{Sym}^n(\mu(g))\big)\\ &= \sigma\big(\mathrm{Int}(x)\,\mathrm{Sym}^n(\mu(g))\big) = \sigma\big(\rho_n(g)\big).\end{aligned}$$

Thus ρ_n commutes with the action of $\mathrm{Gal}(\overline{F}/F)$ and hence is defined over F. □

Now let F be an imaginary quadratic number field, and let $\mathcal{O}_F$ be the ring of integers of F. Fix an embedding $F \subset \mathbb{C}$. As explained above, every quaternion algebra over F, which is not isomorphic to $M_2(F)$, is a division algebra. This is the case which is of interest for us. So let D be a quaternion division algebra over F. Put

$$D^1 = \big\{x \in D\colon N(x) = 1\big\}, \qquad D^0 = \big\{x \in D\colon \mathrm{Tr}(x) = 0\big\}.$$

Let

$$G = R_{F/\mathbb{Q}}\,\mathrm{SL}_1(D)$$

be the algebraic group obtained from $\mathrm{SL}_1(D)$ by restriction of scalars. Then G is defined over $\mathbb{Q}$. We have

$$G(\mathbb{Q}) \cong D^1, \qquad G(\mathbb{R}) \cong \mathrm{SL}_1(D)(F \otimes_{\mathbb{Q}} \mathbb{R}) = (D \otimes_F \mathbb{C})^1, \tag{3.24}$$

and there is an isomorphism of $\mathbb{C}$-algebras

$$\varphi\colon (D \otimes_F \mathbb{C})^1 \xrightarrow{\cong} \mathrm{SL}(2,\mathbb{C}). \tag{3.25}$$

Let $\mathfrak{o}$ be an order in D. Recall that this means that $\mathfrak{o}$ is a finitely generated $\mathcal{O}_F$-module which contains an F-basis of D and which is also a subring of D. Let

$$\mathfrak{o}^1 = \big\{x \in \mathfrak{o}\colon N(x) = 1\big\}, \qquad \mathfrak{o}^0 = \big\{x \in \mathfrak{o}\colon \mathrm{Tr}(x) = 0\big\}.$$

Then $\mathfrak{o}^1 \subset D^1$, and $\mathfrak{o}^0 \subset D^0$ is a lattice which is invariant under $\mathfrak{o}^1$ with respect to the adjoint action of $\mathrm{SL}_1(D)$ on D^0. Thus by (3.24), it follows that $\mathfrak{o}^1$ corresponds to an arithmetic subgroup $\Gamma^1 \subset G(\mathbb{Q})$. Put

$$\Gamma = \varphi(\mathfrak{o}^1).$$

Then Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{C})$. Such a group is called an arithmetic subgroup derived from a quaternion algebra. The following lemma is a consequence of [EGM, Theorem 10.1.2].

LEMMA 3.2
Let $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ *be derived from a quaternion division algebra over* F. *Then the quotient* $\Gamma\backslash \mathrm{SL}(2,\mathbb{C})$ *is compact.*

By imposing further conditions on D, one can achieve that Γ is torsion-free (see [EGM, Theorem 10.1.2]). Alternatively, we can pass to a normal subgroup of Γ of finite index which is torsion-free.

Let $V_1(n) = R_{F/\mathbb{Q}}(V(n))$. Then $V_1(n)$ is a $\mathbb{Q}$-vector space, and it follows from Lemma 3.1 that for every even n there exists a $\mathbb{Q}$-rational representation

$$\rho_n \colon G \to \mathrm{GL}\big(V_1(n)\big)$$

which over $\mathbb{R}$ is equivalent to Sym^n of SL_2. Since Γ^1 is an arithmetic subgroup of $G(\mathbb{Q})$, there exists a lattice $L \subset V_1(n)$ which is stable under Γ^1 (see [PR, Chapter 4, Theorem 4.2]). Using the isomorphism (3.25) this implies the following proposition.

PROPOSITION 3.3
Let $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ *be an arithmetic subgroup derived from a quaternion algebra over an imaginary quadratic number field. Then for each even* $n \in \mathbb{N}$ *there exists a lattice* $M_n \subset S^n(\mathbb{C}^2)$ *which is stable under* Γ *with respect to* Sym^n.

We add some remarks about the classification of quaternion algebras over a number field F (for details see [Vig, Chapter III]). Let H be a quaternion algebra over F. Given a place v of F, let

$$H_v = H \otimes_F F_v.$$

Over $\mathbb{C}$ every quaternion algebra splits; that is, there is an $\mathbb{C}$-algebra isomorphism

$$\varphi \colon H \otimes_F \mathbb{C} \xrightarrow{\cong} M_2(\mathbb{C}).$$

By the Frobenius theorem, a quaternion algebra over $\mathbb{R}$ is either split or isomorphic to the Hamiltonian quaternions. If $\mathfrak{p}$ is a finite place of F, then, up to isomorphism, there is a unique quaternion division algebra over $F_{\mathfrak{p}}$ (see [Vig, Theorem II.1.1]). We call H ramified at a given place v, if H_v is a division algebra. If $H = H(a,b;F)$, $a,b \in F^\times$, the behavior at v is determined by the Hilbert symbol $(a,b)_v \in \{\pm 1\}$. We have that H_v is split if and only if $(a,b)_v = 1$. It follows from the Hasse–Minkowski principle that H splits over F if and only if H_v splits for every place v of F (see [MR, Theorem 2.9.6]). For $H = H(a,b;F)$ this is equivalent to $(a,b)_v = 1$ for all places v. Furthermore the number of places where H is ramified is even. Denote by $V_f(F)$ and $V_{\mathbb{R}}(F)$ the set of finite places and real places of F, respectively. Then for every finite set $S \subset V_f(F) \cup V_{\mathbb{R}}(F)$ of even cardinality there is a quaternion algebra H over F which is unique up to F-algebra isomorphism, such that the set of places where H is ramified is equal to S.

Thus for an imaginary quadratic number field F, we can pick any nonempty finite set S of finite places $\mathfrak{p}$ with $|S|$ even. Then up to isomorphism, there is a unique quaternion division algebra D over F which is ramified exactly at the places $\mathfrak{p} \in S$.

4. Bounds for torsion in cohomology

We now prove (1.2) of Theorem 1.1, which states that the contribution of the terms with cohomological degree 1 and 3 to the alternating sum (1.3) is small. To begin with the H^3 term, it is required to show that

$$\log |H^3(X, \mathcal{M}_{2k})| \ll k \log k \tag{4.26}$$

uniformly over all choices of lattice M_{2k}. We first apply the isomorphism $H^3(X, \mathcal{M}_{2k}) \simeq (M_{2k})_\Gamma$, which follows from computing H^3 by using a triangulation of X and then observing that this is equivalent to computing $H_0(\Gamma, M_{2k})$ by using the dual triangulation. We then bound $(M_{2k})_\Gamma$ by working locally. Let $V(2k) = S^{2k}(F^2)$. For each prime $\mathfrak{p}$ of F, let $V_{\mathfrak{p}}(2k)$ be the completion of $V(2k)$ at $\mathfrak{p}$, and let $M_{2k,\mathfrak{p}}$ be the completion of the image of M_{2k}. $M_{2k,\mathfrak{p}}$ is a Γ-stable lattice, and we have

$$\log |(M_{2k})_\Gamma| = \sum_{\mathfrak{p}} \log |(M_{2k,\mathfrak{p}})_\Gamma|. \tag{4.27}$$

We divide the primes of F into two sets, which we call unramified and ramified. The first set contains all primes with odd residue characteristic at which the division algebra D is unramified and the closure of Γ in $D^1_{\mathfrak{p}} \simeq \mathrm{SL}_2(F_{\mathfrak{p}})$ is isomorphic to the standard maximal compact, and the second contains the remainder. The following lemma implies that the unramified primes whose residue characteristic is greater than $2k$ make no contribution to the sum.

LEMMA 4.1

If $\mathbb{F}_q$ is the field with $q = p^j$ elements, then the dth symmetric power representation of $\mathrm{SL}_2(\mathbb{F}_q)$ over $\mathbb{F}_q$ is irreducible for $d < p$.

Proof
Denote this representation by (ρ, V_d), and let N and $\overline{N}$ be nontrivial upper and lower unipotent elements of $\mathrm{SL}_2(\mathbb{F}_q)$. When $\rho(N) - I$ and $\rho(\overline{N}) - I$ are expressed in the standard monomial basis of V_d they are strictly upper (resp., lower) triangular, and all their entries in the spaces lying immediately above (resp., below) the diagonal are nonzero. It follows that the only subspaces of V_d which are invariant under both of these operators are $\{0\}$ and V_d. □

Because F was imaginary quadratic, Lemma 4.1 implies that $(M_{2k,\mathfrak{p}})_\Gamma$ is trivial when $N\mathfrak{p} > 4k^2$, and so we may assume that $N\mathfrak{p} \le 4k^2$ in (4.27). The following proposition gives a bound for $|(M_{2k,\mathfrak{p}})_\Gamma|$ at unramified primes which is uniform in $\mathfrak{p}$ and M_{2k} and gives us the required bound (4.26) when summed over those $\mathfrak{p}$ with $N\mathfrak{p} \le 4k^2$. For ramified primes we adapt the proof of the proposition to give bounds which are only uniform in the lattice, but this will be sufficient for our purposes as the number of such primes is bounded independently of k.

PROPOSITION 4.2
Let K be a p-adic field, let $\mathcal{O} \subset K$ be the ring of integers in K, let ϖ be a uniformizer of $\mathcal{O}$, and let $q = |\mathcal{O}/\varpi|$. Consider the $2k$th symmetric power representation of $G =$ $\mathrm{SL}(2, \mathcal{O})$ *on a vector space V of dimension $2k+1$. If $L \subset V$ is any G-stable lattice and L/L' is the largest G-invariant quotient of L, then we have*

$$\log_q |L : L'| \ll k/q,$$

where the implied constant is absolute, that is, independent of L and K.

Proof
The proof proceeds in two steps. First, choose $a \in \mathcal{O}^\times$ to have the highest possible order in all the quotient groups $\mathcal{O}^\times/(1 + \varpi^t \mathcal{O})$, or equivalently so that a is a primitive root in $(\mathcal{O}/\varpi)^\times$ and $a^{q-1} \notin 1 + \varpi^2\mathcal{O}$. Define T to be the diagonal matrix

$$T = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$

We know that $(T-1)L \subset L'$, so it would suffice to bound the order of $L/(T-1)L$. This is infinite, however, because T fixes the vector $(xy)^k$. Instead, we shall prove that if V_0 is the space spanned by $(xy)^k$ and we define J and J' to be the intersections of V_0 with L and L', then we have

$$\log_q |L : L'| \ll \frac{k}{q} + \log_q |J : J'|. \tag{4.28}$$

The second step is to bound $|J : J'|$, for which we need to consider the action of the unipotents on V.

We carry out the first step by decomposing V into a direct sum of subspaces $\mathcal{V}_i$, for $i \in \mathbb{Z}_{\geq 0}$ and $i = \infty$. For $i \in \mathbb{Z}_{\geq 0}$, we define $\mathcal{V}_i$ to be the span of the eigenspaces of $T - 1$ with eigenvalue λ satisfying $\operatorname{ord}_{\varpi} \lambda = i$, and we define $\mathcal{V}_\infty = \operatorname{span}\{(xy)^k\}$. In other words, if V_t is the span of the monomial $x^{k+t} y^{k-t}$, we define

$$\mathcal{V}_0 = \bigoplus_{2t \not\equiv 0 (q-1)} V_t,$$

$$\mathcal{V}_i = \bigoplus_{\substack{2t = j(q-1)p^{i-1} \\ j \not\equiv 0 (p)}} V_t,$$

$$\mathcal{V}_\infty = V_0.$$

Note that for $1 \leq i < \infty$ we have the bound

$$\dim \mathcal{V}_i \leq Ck/qp^{i-1}$$

for some absolute constant C. Let Π_i be the projection onto $\mathcal{V}_i$, and define $\mathcal{V}_i'$ by

$$\mathcal{V}_i' = \bigoplus_{\substack{j > i, \\ j = \infty}} \mathcal{V}_j$$

so that $\mathcal{V}_i'$ is the span of the eigenspaces of $T - 1$ with eigenvalue λ satisfying $\operatorname{ord}_{\mathfrak{p}} \lambda > i$. Let $L_i = L \cap \mathcal{V}_i'$, and let $L_i' = L' \cap \mathcal{V}_i'$. We prove (4.28) by using induction on L_i, by establishing the inequalities

$$|L : L'| = |L_0 : L_0'|, \tag{4.29}$$

$$\log_q |L_i : L_i'| \leq Ck(i+1)/qp^i + \log_q |L_{i+1} : L_{i+1}'| \tag{4.30}$$

for all $i \geq 0$.

To prove (4.29), note that because T commutes with all coordinate projections we have

$$(T - 1)\Pi_0 L \subseteq \Pi_0 L' \subseteq \Pi_0 L,$$

and that all the above inclusions must be equalities because $\operatorname{Im}(\Pi_0) = \mathcal{V}_0$ and the determinant of $T - 1$ restricted to $\mathcal{V}_0$ is a unit of $\mathcal{O}$. The equality $\Pi_0 L = \Pi_0 L'$ then implies that the inclusion of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_0' & \longrightarrow & L' & \xrightarrow{\Pi_0} & \Pi_0 L' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_0 & \longrightarrow & L & \xrightarrow{\Pi_0} & \Pi_0 L & \longrightarrow & 0 \end{array}$$

induces an isomorphism $L_0/L_0' \simeq L/L'$. Likewise, we have

$$(T-1)\Pi_{i+1}L_i \subseteq \Pi_{i+1}L_i' \subseteq \Pi_{i+1}L_i,$$

and taking determinants gives

$$\begin{aligned}\log_q |\Pi_{i+1}L_i : \Pi_{i+1}L_i'| &\le \log_q |\Pi_{i+1}L_i : (T-1)\Pi_{i+1}L_i| \\ &\le (i+1)\dim \mathcal{V}_{i+1} \le Ck(i+1)/qp^i,\end{aligned}$$

where $i+1$ appears because it is the ϖ-adic valuation of the eigenvalues of $T-1$ on $\mathcal{V}_{i+1}$. The inequality (4.30) follows by combining this with the snake lemma for the inclusion

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_{i+1}' & \longrightarrow & L_i' & \xrightarrow{\Pi_{i+1}} & \Pi_{i+1}L_i' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_{i+1} & \longrightarrow & L_i & \xrightarrow{\Pi_{i+1}} & \Pi_{i+1}L_i & \longrightarrow & 0. \end{array}$$

On summing (4.29) and (4.30) we obtain

$$\begin{aligned}\log_q |L : L'| &\ll \sum_{t=1}^{i} \frac{kt}{qp^{t-1}} + \log_q |L_i : L_i'| \\ &\ll \frac{k}{q} + \log_q |L_i : L_i'|,\end{aligned}$$

from which (4.28) follows by letting $i \to \infty$.

To bound $|J : J'|$, we may assume that J is generated by $(xy)^k$, and we act on this monomial by an upper triangular element $N \in \mathrm{SL}(2, \mathcal{O})$. Then $(N-1)(xy)^k$ will contain a nonzero term of the form $x^{k+1}y^{k-1}$, and using the endomorphism algebra generated by T, we may in fact show that L' contains a monomial $\varpi^s x^{k+1}y^{k-1}$ with s small. Repeating this argument with a lower triangular matrix $\overline{N}$ gives the required bound on $|J : J'|$. The statement we require about the endomorphism algebra generated by T is the following.

LEMMA 4.3

For $\gamma \in \mathcal{O}^\times$, let E be the ring of endomorphisms of $\mathcal{O}^{2k+1}$ generated over $\mathcal{O}$ by the diagonal matrix A with entries $1, \gamma, \dots, \gamma^{2k}$. Then for $0 \le j \le 2k$, E contains $p_j(\gamma^j)\pi_j$ where π_j is the projection onto the jth coordinate and $p_j(z)$ is the polynomial

$$p_j(z) = \prod_{\substack{0 \le i \le 2k, \\ i \ne j}} (z - \gamma^i).$$

Proof
This follows easily by considering the matrix $p_j(A)$, whose only nonzero entry is $p_j(\gamma^j)$ in the (j, j)th coordinate. ☐

Consider the polynomial $x^k(y+x)^k - (xy)^k \in L'$. Applying the lemma with the choice of $\gamma = a^2$ and the projection onto the space spanned by $x^{k+1}y^{k-1}$, we see that L' contains

$$k \prod_{\substack{-2k-2 \leq 2i \leq 2k-2, \\ i \neq 0}} (1 - a^{2i}) x^{k+1} y^{k-1} = u \varpi^\alpha x^{k+1} y^{k-1},$$

where u is a unit of $\mathcal{O}$ and $\alpha \in \mathbb{Z}$. In the product above, at most Ck/q terms are divisible by ϖ, at most Ck/qp are divisible by ϖ^2 and so on, so that we may bound α by

$$\begin{aligned} \alpha &\ll \log_q k + \frac{k}{q} + \frac{k}{qp} + \cdots \\ &\ll \frac{k}{q}. \end{aligned}$$

By applying the same argument to the monomial $\varpi^\alpha x^{k+1} y^{k-1}$, but now with an element of the opposite unipotent, we see that $\varpi^\beta (xy)^k \in L'$ with $\beta \ll k/q$. This gives the required bound on $|J : J'|$ and completes the proof. ☐

Using Proposition 4.2, we see that the contribution to the order of $\log |(M_{2k})_\Gamma|$ from unramified primes is at most a constant times

$$\sum_{N\mathfrak{p} \leq 4k^2} \frac{k \log N\mathfrak{p}}{N\mathfrak{p}} \ll k \log k.$$

As the number of ramified primes is bounded independently of k, in the remaining cases it suffices to prove bounds of the form $\log |(M_{2k,\mathfrak{p}})_\Gamma| \ll k$, where the constant is allowed to depend on $\mathfrak{p}$.

First, consider a prime $\mathfrak{p}$ at which D is split but such that either the residue characteristic of $\mathfrak{p}$ is 2 or the closure of Γ in $D^1_\mathfrak{p} \simeq \mathrm{SL}_2(F_\mathfrak{p})$ is not isomorphic to $\mathrm{SL}(2, \mathcal{O})$, and denote this closure by G. As in the unramified case, let V be the $2k$th degree symmetric power representation of $\mathrm{SL}_2(F_\mathfrak{p})$, let $L \subset V$ be any G-stable lattice, and let L/L' be its largest G-invariant quotient. As G is open, we know it contains an element

$$T = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

for some $a \in \mathcal{O}^\times$. As in the unramified case, define $\mathcal{V}_i$ to be the sum of the eigenspaces of $T-1$ with eigenvalue λ satisfying $\operatorname{ord}_\varpi \lambda = i$, let $\mathcal{V}_i'$ be the sum of the eigenspaces with $\operatorname{ord}_\varpi \lambda > i$, and let $J = L \cap \mathcal{V}_\infty$ and $J' = L' \cap \mathcal{V}_\infty$. Because we have inequalities

$$\dim \mathcal{V}_i \leq C(a)\frac{k}{p^i}$$

uniformly in i and k, we may use these subspaces to perform the same reduction argument which bounds $|L : L'|$ in terms of $|J : J'|$ to obtain

$$\log_q |L : L'| \leq \log_q |J : J'| + O_a(k).$$

G also contains upper and lower unipotent elements

$$N = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \overline{N} = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$$

for some $n \in \mathcal{O}$, and arguing as in the unramified case, we may use these to show that

$$\begin{aligned}\log_q |J : J'| &\ll_n \log_q k + \sum_{\substack{-2k-2 \leq 2i \leq 2k+2, \\ i \neq 0}} \operatorname{ord}_\varpi(1 - a^{2i}) \\ &\ll_{n,a} \log_q k + \sum_{i \geq 0} \frac{k}{p^i} \\ &\ll_{n,a} k.\end{aligned}$$

We therefore have an upper bound of $\log_q |L : L'| \ll k$, where the constant may depend on $\mathfrak{p}$.

Finally, suppose that D is ramified at $\mathfrak{p}$. We continue to denote the closure of Γ in $D^1_\mathfrak{p}$ by G. Using the adjoint representation, we may realize $D^1_\mathfrak{p}$ as an algebraic subgroup of $\mathrm{GL}_3(F_\mathfrak{p})$. There is an open subgroup U of $\mathrm{GL}_3(F_\mathfrak{p})$ such that the power series $\log(1 + A)$ converges p-adically for $1 + A \in U$, from which we see that there is an open subgroup $U' \subset D^1_\mathfrak{p}$ on which we may define an inverse exponential map. By applying this map to $U' \cap G$ we obtain a lattice N in the Lie algebra of $D^1_\mathfrak{p}$ which annihilates L/L'. We may then extend scalars to a field over which $D_\mathfrak{p}$ splits so that we are again dealing with SL_2, and apply the method of Proposition 4.2 to obtain an upper bound of $\log_q |L : L'| \ll k$. This completes the proof of (4.26).

In the case of H^1, we may apply the long exact sequence in cohomology associated to the short exact sequence

$$0 \longrightarrow M_{2k} \xrightarrow{\times d} M_{2k} \longrightarrow M_{2k}/d \longrightarrow 0,$$

of which one segment is

$$H^0(\Gamma, M_{2k}/d) \longrightarrow H^1(\Gamma, M_{2k}) \xrightarrow{\times d} H^1(\Gamma, M_{2k}).$$

Therefore the torsion in $H^1(\Gamma, M_{2k})$ is bounded by the limit of the Γ-invariants in M_{2k}/d as d becomes divisible by arbitrarily high powers of every prime. We may again bound this by working locally, and so wish to bound the order of the module of Γ-invariants $(M_{2k,\mathfrak{p}}/\varpi^t M_{2k,\mathfrak{p}})^\Gamma$ for each $\mathfrak{p}$ and arbitrary t. If we let L be the inverse image of $(M_{2k,\mathfrak{p}}/\varpi^t M_{2k,\mathfrak{p}})^\Gamma$ in $V_\mathfrak{p}(2k)$, we see that Γ acts trivially on $L/\varpi^t M_{2k,\mathfrak{p}}$. We may therefore apply our bounds for L_Γ to obtain

$$\log |H^1(\Gamma, M_{2k})| \ll k \log k$$

as required.

5. Proof of the main results

Let $X = \Gamma\backslash\mathbb{H}^3$ be a compact oriented hyperbolic 3-manifold defined by a cocompact torsion-free discrete subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$. For $m \in \mathbb{N}$ let $\rho_m\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(S^m(\mathbb{C}^2))$ be the mth symmetric power of the standard representation of $\mathrm{SL}(2,\mathbb{C})$. We regard $S^m(\mathbb{C}^2)$ as an $\mathbb{R}$-vector space. Let $\rho_m^{\mathbb{R}}$ be the corresponding real representation. Let $E_m \to X$ be the flat vector bundle associated to ρ_m or $\rho_m^{\mathbb{R}}$, respectively. By [BW, Chapter VII, Theorem 6.7] we have $H^*(X, E_m) = 0$. Therefore, the Reidemeister torsions $\tau_X(\rho_m)$ and $\tau_X(\rho_m^{\mathbb{R}})$ of X with respect to the restrictions to Γ of ρ_m and $\rho_m^{\mathbb{R}}$, respectively, are well defined. By Lemma 2.4 we have

$$\tau_X(\rho_m^{\mathbb{R}}) = \tau_X(\rho_m)^2. \tag{5.31}$$

Then it follows from [Mu2, Corollary 1.2] that

$$-\log \tau_X(\rho_m^{\mathbb{R}}) = \frac{\mathrm{vol}(X)}{2\pi} m^2 + O(m) \tag{5.32}$$

as $m \to \infty$.

Now let $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$ be an arithmetic subgroup derived from a quaternion division algebra over an imaginary quadratic number field F. By Lemma 3.2 it is cocompact. By passing to a normal subgroup of finite index we can assume that Γ is torsion-free. Let $m \in \mathbb{N}$ be even. By Proposition 3.3 there exists a lattice $M_m \subset S^m(\mathbb{C}^2)$ which is stable under Γ. Let $\mathcal{M}_m$ be the associated local system of free $\mathbb{Z}$-modules over X. Let $\mathcal{M}_m(\mathbb{R}) := \mathcal{M}_m \otimes \mathbb{R}$. This is the local system associated to the restriction of ρ_m to Γ. Hence $H^*(X, \mathcal{M}_m(\mathbb{R})) \cong H^*(X, E_m)$, and it follows from the remark above that $H^*(X, \mathcal{M}_m(\mathbb{R})) = 0$. Therefore $H^*(X, \mathcal{M}_m)$ is a finite abelian group. Note that $H^0(X, \mathcal{M}_m) = M_m^\Gamma = 0$. Denote by $|H^p(X, \mathcal{M}_m)|$ the order of $H^p(X, \mathcal{M}_m)$, $p = 1, 2, 3$. Then by Proposition 2.3 we get

$$\tau_X(\rho_m^{\mathbb{R}}) = \prod_{p=1}^{3} |H^p(X, \mathcal{M}_m)|^{(-1)^{p+1}}. \tag{5.33}$$

Combining (5.32) and (5.33), we obtain Theorem 1.1.

Remark 5.1
It follows from (5.33) that the quantity $\sum_{p=1}^{3}(-1)^p \log|H^p(X, \mathcal{M}_{2k})|$ is independent of the choice of lattice $M_{2k} \subset V(2k)$, but this may also be deduced in an elementary way from the long exact sequence in cohomology associated to any inclusion of lattices $M'_{2k} \subset M_{2k}$. Let $T = M_{2k}/M'_{2k}$. The long exact sequence associated to

$$0 \longrightarrow M'_{2k} \longrightarrow M_{2k} \longrightarrow T \longrightarrow 0$$

implies that

$$\sum_{p=1}^{3}(-1)^p \log|H^p(X, \mathcal{M}_{2k})| - \sum_{p=1}^{3}(-1)^p \log|H^p(X, \mathcal{M}'_{2k})| = \sum_{p=1}^{3}(-1)^p \log|H^p(X, \mathcal{T})|. \tag{5.34}$$

The groups $H^p(X, \mathcal{T})$ are the cohomology groups of a complex $\{C^p(T)\}$, where $C^p(T)$ is the group of T-valued cochains of degree p in some triangulation of X. Because X is 3-dimensional, its Euler characteristic $\chi(X)$ is zero and so we have

$$\sum_{p=1}^{3}(-1)^p \log|C^p(T)| = \chi(X)\log|T| = 0.$$

This implies the same relation for the groups $H^p(X, \mathcal{T})$, and so (5.34) is zero as required.

The proof of Theorem 1.3 follows from [Mu2, Theorem 1.5]. Let $\rho: \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(V)$ be an irreducible finite-dimensional complex representation of $\mathrm{SL}(2,\mathbb{C})$, regarded as a real Lie group. The Ruelle zeta function $R_\rho(s)$ considered in [Mu2] is related to the zeta function $R(s;\rho)$ by

$$R(s;\rho) = R_\rho(s)^{-1}.$$

The restriction of ρ to Γ defines a flat vector bundle $E_\rho \to X$. By [MM, Lemma 3.1] it carries a canonical fiber metric h. Let $H^*(X, E_\rho)$ be the de Rham cohomology of E_ρ-valued differential forms. Let θ be the Cartan involution of $\mathrm{SL}(2,\mathbb{C})$ with respect to SU(2). Let $\rho_\theta := \tau \circ \rho$. If $\rho_\theta \ncong \rho$, then it follows from [BW, Chapter VII, Theorem 6.7] that $H^*(X, E_\rho) = 0$. Let $T_X(\rho)$ be the Ray–Singer analytic torsion of X with respect to the restriction of ρ to Γ. Note that to define the analytic torsion, we need to choose a fiber metric in E_ρ. However, since $H^*(X, E_\rho) = 0$ and the dimension of X is odd, the analytic torsion is independent of the fiber metric (see [Mu1, Corollary 2.7]), which justifies the notation. By [Mu2, Theorem 1.5, (1)] the Ruelle zeta function $R(s;\rho)$ is regular at $s = 0$, and we have

$$|R(0;\rho)| = T_X(\rho)^{-2}. \tag{5.35}$$

Furthermore, by [Mu1, Theorem 1], the analytic torsion $T_X(\rho)$ equals the Reidemeister torsion $\tau_X(\rho)$ of X and $\rho|_\Gamma$. Let $\rho^{\mathbb{R}}$ be the real representation associated to ρ. Together with Lemma 2.4 we obtain

$$|R(0;\rho)| = \tau_X(\rho^{\mathbb{R}})^{-1}. \tag{5.36}$$

Now assume that $\rho_\theta = \rho$. Then $H^*(X, E_\rho)$ may be nonzero. It follows from [Mu2, Theorem 1.5, (2)] that the order of $R(s;\rho)$ at $s = 0$ is given by

$$\operatorname{ord}_{s=0} R(s;\rho) = 2\sum_{p=0}^{3} (-1)^p p \dim H^p(X, E_\rho), \tag{5.37}$$

and the leading coefficient $R^*(0;\rho)$ of the Laurent expansion of $R(s;\rho)$ at $s = 0$ equals the analytic torsion $T_X(\rho;h)^{-2}$, where h is the canonical fiber metric h on E_ρ. Let $\tau_X(\rho;h)$ be the Reidemeister torsion with respect to the L^2-inner product in $H^*(X, E_\rho)$ defined by the isomorphism with the space $\mathcal{H}^*(X, E_\rho)$ of E_ρ-valued harmonic forms. Using again [Mu1, Theorem 1] and Lemma 2.4, we get

$$R^*(0;\rho) = \tau_X(\rho^{\mathbb{R}};h)^{-1}. \tag{5.38}$$

Now assume that $M \subset V$ is a lattice which is stable under Γ with respect to ρ. Let $\mathcal{M}$ be the associated local system of free $\mathbb{Z}$-modules, and let $\mathcal{M}(\mathbb{R}) = \mathcal{M} \otimes \mathbb{R}$. If $\rho_\theta \ncong \rho$, we have $H^*(X, \mathcal{M}(\mathbb{R})) = 0$. Combining Theorem 2.3 and (5.36) we get

$$|R(0;\rho)| = \prod_{p=0}^{3} |H^p(X, \mathcal{M})_{\mathrm{tors}}|^{(-1)^p}.$$

Now assume that $\rho_\theta = \rho$. Then $\operatorname{rank} H^p(X, \mathcal{M}) = 2\dim H^p(X, E_\rho)$. Let $\rho \neq 1$. Then it follows from (5.37) that

$$\operatorname{ord}_{s=0} R(s;\rho) = \sum_{p=1}^{3} (-1)^p \operatorname{rk} H^p(X, \mathcal{M}).$$

Using Theorem 2.3 and (5.38), we get

$$R^*(0;\rho) = R(\mathcal{M})^{-1} \prod_{p=0}^{3} |H^p(X,\mathcal{M})_{\mathrm{tors}}|^{(-1)^p}.$$

Finally, if $\rho = 1$ it follows from [Mu2, Theorem 1.5] that the order of $R(s;1)$ at $s = 0$ equals $2\dim H^1(X,\mathbb{R}) - 4$.

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