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Theta Lifting and Cohomology Growth in p-adic Towers

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We use the theta lift to study the multiplicity with which certain automorphic representations of cohomological type occur in a family of congruence covers of an arithmetic manifold. When the family of covers is a so-called p-adic congruence tower, we obtain sharp asymptotics for the number of representations that occur as lifts. When combined with theorems on the surjectivity of the theta lift due to Howe and Li, and Bergeron, Millson, and Moeglin, this allows us to verify certain cases of a conjecture of Sarnak and Xue.

1 Introduction

The purpose of this paper is to quantify the results of Li in [11] which establish the existence of certain automorphic representations of cohomological type on the classical groups. This problem was previously considered by the first author in [4], and this paper provides a sharp form of the results proved there. To recall Li's results, let F be a totally real field and let (E, ι) be an extension of F with an involution ι of one of the following types:

$$E = \begin{cases} F, & \text{case 1,} \\ \text{a quadratic extension } E/F, & \text{case 2,} \end{cases}$$

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$$\iota = \begin{cases} \text{id}, & \text{case 1}, \\ \text{the Galois involution of } E/F, & \text{case 2}. \end{cases}$$

We shall use v and w to denote places of F and E, respectively, and denote the completion of F at v by F_v (respectively, E at w by E_w). If v is non-Archimedean, we shall say that a Hermitian space V over E is unramified at v if v does not divide 2 and the Hermitian space $V_v = V \otimes_F F_v$ over the algebra $E \otimes_F F_v$ contains a self-dual lattice L. Note that this condition is automatically satisfied in the case where v splits in E.

Let V and V' be finite-dimensional vector spaces over E with dimensions n and n', respectively, which we equip with nondegenerate sesquilinear forms $(\,,\,)$ and $(\,,\,)'$. We assume that $n' \leq n$, and that $(\,,\,)$ is η-Hermitian and $(\,,\,)'$ is $-\eta$ -Hermitian for $\eta \in \{\pm 1\}$. We may equip the space $W = V \otimes_E V'$ with the symplectic form

$$\langle , \rangle = \mathrm{tr}_{E/F}((\,,\,) \otimes (\,,\,)'),$$

and define G, G', and $\mathrm{Sp}(W)$ to be the isometry groups of $(,)$, $(\,,\,)'$, and $\langle\,,\,\rangle$, respectively, considered as algebraic groups over F. Then (G, G') is a type I irreducible dual pair inside $\mathrm{Sp}(W)$. Likewise, if $G(\mathbb{R})$ and $G'(\mathbb{R})$ are the real points of the groups obtained after restricting scalars to $\mathbb{Q}$, then $(G(\mathbb{R}), G'(\mathbb{R}))$ is a type I irreducible dual pair inside the corresponding real symplectic group. We shall assume in case 1 that both V and V' have even dimension. Let $L \subset V$ be an $\mathcal{O}_E$ lattice, whose completion in V_v we shall assume to be self-dual whenever V is unramified, and $\Gamma \subset G(\mathbb{R})$ be the arithmetic group stabilizing L. Let r be the dimension of the maximal isotropic subspace of V, and define

$$d = \begin{cases} 0, & \text{case 1}, \eta = 1, \\ 1, & \text{case 1}, \eta = -1, \\ \frac{1}{2}, & \text{case 2}, \end{cases}$$

$$\delta = \begin{cases} 0, & \text{case 1}, \\ 1, & \text{case 2}. \end{cases}$$

In [10], Li characterizes the representations π of $G(\mathbb{R})$ which are of cohomological type and are realized as the local theta lift of a discrete series representation from a smaller group $G'(\mathbb{R})$ (which must be unique). The set of such representations is quite large, and, in particular, is the entire cohomological dual in the cases, in which $G(\mathbb{R})$ is $SO(n, 1)$ or $U(n, 1)$. Moreover, in [11], he proves that if π is such a representation and $n > 2n' + 4d - 2$, then π is automorphic, that is there exists a congruence subgroup $\Gamma' \subset \Gamma$ such that π occurs discretely in $L^2(\Gamma' \backslash G(\mathbb{R}))$.

We shall state our quantification of Li's results in adelic terms in order to avoid technicalities coming from the failure of strong approximation. Let $\mathbb{A}$ be the ring of adeles of F, and $\mathbb{A}_f$ be the ring of finite adeles. If $L \subset V$ is a lattice which is self-dual in V_v whenever possible and $\mathfrak{n} \subset \mathcal{O}_F$ is an ideal, we let $K(\mathfrak{n})$ be the standard compact subgroup of $G(\mathbb{A}_f)$ defined with respect to L (notation to be defined precisely in Section 3). Let G^0 be the subgroup of elements with trivial determinant in G and $K_\infty \subset G(\mathbb{R})$ be the standard maximal compact, and define $K^0(\mathfrak{n}) = K(\mathfrak{n}) \cap G^0(\mathbb{A}_f)$ and $K^0_\infty = K_\infty \cap G^0(\mathbb{R})$. We define the adelic quotients

$$X(\mathfrak{n}) = G(F)\backslash G(\mathbb{A})/K(\mathfrak{n}),$$

$$X^0(\mathfrak{n}) = G^0(F)\backslash G^0(\mathbb{A})/K^0(\mathfrak{n}),$$

$$Y(\mathfrak{n}) = X(\mathfrak{n})/K_\infty, \quad Y^0(\mathfrak{n}) = X^0(\mathfrak{n})/K^0_\infty.$$

Define $L^2_\theta(X(\mathfrak{n})) \subset L^2(X(\mathfrak{n}))$ to be the subspace generated by twists of all automorphic forms on G of the form $\Theta(\pi')$, where Θ denotes the global theta lift and π' is a form on a group G' with n' satisfying $n > 2n' + 4d - 2$ which is in the discrete series at ∞. Let π be one of the cohomological representations of $G(\mathbb{R})$ considered by Li, which occurs as the local lift from a group $G'(\mathbb{R})$ associated to a vector space of dimension n'. Define $m_\theta(\pi, \mathfrak{n})$ to be the multiplicity with which π occurs in $L^2_\theta(X(\mathfrak{n}))$, and let $m^0_\theta(\pi, \mathfrak{n})$ be the analogous multiplicity for $X^0(\mathfrak{n})$. We aim to quantify Li's existence theorem by determining the asymptotic growth rate of $m^0_\theta(\pi, \mathfrak{n})$ for $\mathfrak{n}$ of the form $\mathfrak{c}\mathfrak{p}^k$ as $k \to \infty$, where $\mathfrak{p}$ is a sufficiently unramified prime that is inert in E and π is one of the cohomological forms considered by Li [10].

We shall state our theorem with the aid of two sets of places S and S' of F. We define S to contain the Archimedean places, the primes above those that ramify in $E/\mathbb{Q}$, and the places at which the Hermitian space V is ramified (in particular, all places above 2). S' may be any set of places containing S and such that the S'-units of F assume all possible combinations of signs at the Archimedean places. Throughout the paper, we shall use $\mathfrak{P}$ to denote the prime of E lying above $\mathfrak{p}$. Our main theorem may then be stated as follows.

Theorem 1.1. For $\mathfrak{p} \notin S$ which is inert in E and any ideal $\mathfrak{c} \subset \mathcal{O}_F$, we have

$$m_\theta(\pi, \mathfrak{c}\mathfrak{p}^k) \ll (N\mathfrak{P})^{(nn'+\delta)k/2}. \quad (1)$$

Moreover, this upper bound is sharp after multiplying $\mathfrak{c}$ by a suitable product of primes in S'. □

Corollary 1.2. With the notation above, we have

$$m^0_\theta(\pi, \mathfrak{c}\mathfrak{p}^k) \ll (N\mathfrak{P})^{nn'k/2}.$$

Moreover, this upper bound is sharp in case 1 with $\eta = -1$ and case 2 after multiplying $\mathfrak{c}$ by a suitable product of primes in S'. □

Proof. This follows from Theorem 1.1 and the discussion of Section 3.2, combined with Remark 4 in case 2. ■

Remark 1. G^0 satisfies strong approximation in case 1 with $\eta = -1$ and case 2, so $Y^0(\mathfrak{n})$ is connected in these cases. This is not true in case 1 with $\eta = 1$, and so $Y^0(\mathfrak{n})$ will generally not be connected in this case. However, the number of connected components of $Y(\mathfrak{c}\mathfrak{p}^k)$ and $Y^0(\mathfrak{c}\mathfrak{p}^k)$ will be bounded as $k \to \infty$, and, moreover, any two of these components will have a common finite cover whose index above both may be bounded independently of k. We do not deal with the question of which components the forms occur on; however, [2, Section 8] contains a discussion of how to produce forms on the connected component of the identity in $Y^0(\mathfrak{n})$. □

Remark 2. It will be seen in the course of the proof that the upper bound in Theorem 1.1 continues to hold in case 1 if $\eta = 1$ and n is odd. The only point at which we use our parity assumption is when we split the metaplectic cover on restriction to the dual pair, so that we may produce forms on $G(\mathbb{A})$ and apply the multiplicity results of Savin [18] to $G'(\mathbb{A})$ in Section 3.2. In case 1 with $\eta = 1$ and n odd, the Weil representation has a very simple restriction to $\tilde{G}(\mathbb{A})$ so that we may still produce forms on $G(\mathbb{A})$. Moreover, the forms on $\tilde{G}'(F)\backslash\tilde{G}'(\mathbb{A})$ that we are lifting still have their Archimedean components in the discrete series, and so have nonzero $(\mathfrak{g}, K)$ cohomology. Proposition 4.1 is therefore still valid, and the proof of the upper bound goes through as before. The lower bound relies on Savin's results, and we do not know whether these continue to hold for the metaplectic quotient $\tilde{G}'(F)\backslash\tilde{G}'(\mathbb{A})$. □

Remark 3. It was stated in [4, Section 2.5] that the lower bounds on m_θ given there only hold under the additional assumption that $n > 5n'/2 + 1$. This is in fact unnecessary, and its only purpose was to guarantee the stable range assumption $n > 2n' + 4d - 2$ while being simpler in form. □

1.1 The conjecture of Sarnak and Xue

For any representation π of $G(\mathbb{R})$, let $p(\pi)$ denote the infimum over the set of $p \geq 2$ such that all K-finite matrix coefficients of π lie in $L^p(G(\mathbb{R}))$. There is a conjecture of Sarnak and Xue [17] which asserts that

$$m^0(\pi, \mathfrak{n}) \ll_\epsilon \mathrm{Vol}(X^0(\mathfrak{n}))^{2/p(\pi)+\epsilon}, \tag{2}$$

and this may be proved in a number of cases by combining Theorem 1.1 with results of Howe and Li [7, 13], and Bergeron et al. [2] on the surjectivity of the theta lift. If π occurs in the Archimedean theta correspondence for the pair $(G(\mathbb{R}), G'(\mathbb{R}))$ as the lift of a discrete series representation, then one may show, using [9, Theorem 3.2], that

$$\frac{2}{p(\pi)} \geq \frac{n'}{n-2+2d}. \tag{3}$$

Note that this is known to be sharp when $G(\mathbb{R})$ is an orthogonal group of split rank 1. In comparison, Theorem 1.1 may be expressed in terms of the volume of $X^0(\mathfrak{c}\mathfrak{p}^k)$ as

$$m^0_\theta(\pi, \mathfrak{c}\mathfrak{P}^k) \ll \begin{cases} \mathrm{Vol}(X^0(\mathfrak{c}\mathfrak{p}^k))^{n'/(n-\eta)}, & \text{case 1,} \\ \mathrm{Vol}(X^0(\mathfrak{c}\mathfrak{p}^k))^{nn'/(n^2-1)}, & \text{case 2.} \end{cases}$$

It can be seen that these exponents are always strictly less than the RHS of (3), so that Theorem 1.1 provides a strengthening of the Sarnak–Xue conjecture when the theta lift is surjective. We now describe the two cases in which this is known.

The results of Howe and Li [7, 13] state that $m_\theta(\pi, \mathfrak{n}) = m(\pi, \mathfrak{n})$ if π occurs in the Archimedean correspondence with a representation of $G'(\mathbb{R})$ with $G'(\mathbb{R})$ compact and $n' < r$. Note that this forces us to either be in case 1 with $\eta = -1$ or in case 2, so that G is symplectic or unitary. In the case of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$, the representations that satisfy this condition are exactly the singular holomorphic representations of rank $< n$. Note that the rank of a representation of $\mathrm{Sp}_{2n}(\mathbb{R})$ must be either n or an even integer $<n$; see [7, 13] for more information about these singular forms. As an example, we may apply Theorem 1.1 to the lattice $\Gamma = \mathrm{Sp}_{2n}(\mathbb{Z})$ and its principal congruence subgroups $\Gamma(N)$ to deduce the following.

Corollary 1.3. If π is a singular holomorphic representation of $\mathrm{Sp}_{2n}(\mathbb{R})$ of rank $n' < n$ and $p \neq 2$, we have

$$m(\pi, cp^k) \ll p^{nn'k}.$$

Moreover, this asymptotic bound is sharp if c is divisible by a sufficiently high power of 2. □

The singular holomorphic representations of rank n' are the ones that contribute via Matushima's formula to the L^2 cohomology of $Y(cp^k) = \Gamma(cp^k)\backslash \mathrm{Sp}_{2n}(\mathbb{R})/U(n)$ in bidegree $(n'(4n - n' + 2)/8, 0)$ (note that this corrects an error in [4, Theorem 0.1]). We may therefore rephrase Corollary 1.3 as stating that

$$\dim H^{t(2n-t+1)/2}_{(2)}(Y(cp^k), \mathbb{C}) \ll \mathrm{Vol}(Y(cp^k))^{t/(2n+1/2)}$$

for all $t < n/2$, and we may deduce similar results in the case of $\mathrm{Sp}_{2n}(\mathcal{O}_F)$ for arbitrary F. In the case of singular forms of rank 2 on $\mathrm{Sp}_{2n}(\mathbb{Z})$, $m(\pi, N)$ was exactly determined by Li [12]. In particular, he proves that $m(\pi, N) = 0$ unless $4|N$ or $p|N$ with $p \equiv -1$ (4), which demonstrates that the upper bound in our theorem is not always attained.

The second surjectivity theorem that we shall apply is due to Bergeron et al. [2], and is presently conditional on the stabilization of the trace formula for the (disconnected) groups $\mathrm{GL}_n \rtimes \langle \theta \rangle$ and $\mathrm{SO}_{2n} \rtimes \langle \theta' \rangle$, where θ and θ' are the outer automorphisms. They consider congruence arithmetic lattices in $\mathrm{SO}(d, 1)$ which arise from a nondegenerate quadratic form over a totally real number field, and prove that $m_\theta(\pi, \mathfrak{n}) = m(\pi, \mathfrak{n})$ for all π that contribute to the cohomology of $Y(\mathfrak{n})$ in degree $i < \lfloor d/2 \rfloor /2$. Note that they place no restriction on the parity of d. When we combine this result with Theorem 1.1 and the subsequent remark in the case of odd-dimensional orthogonal spaces, this implies the following corollary.

Corollary 1.4. Assume the results set out in [2, Section 1.18]. Let $Y(\mathfrak{n})$ be compact arithmetic hyperbolic manifolds of dimension d that arise from a quadratic form over a totally real field. If $\mathfrak{c}$ and $\mathfrak{p}$ are as in Theorem 1.1 and $i < \lfloor d/2 \rfloor /2$, we have

$$\dim H^i(Y(\mathfrak{c}\mathfrak{p}^k), \mathbb{C}) \ll \mathrm{Vol}(Y(\mathfrak{c}\mathfrak{p}^k))^{2i/d}. \qquad \square$$

The structure of the paper is as follows. We shall begin by recalling the construction of the global theta lift and its dependence on certain auxiliary data in Section 2. In Section 3, we shall introduce our notation and review the methods developed by the first author in [4], and Section 4 presents the modifications that we have made to the arguments of Cossutta [4] in order to make them sharp.

2 Review of the Theta Lift

2.1 The local theta lift

Let $G(F_v)$, $G'(F_v)$, and $\mathrm{Sp}(W_v)$ denote the F_v-valued points of the groups G, G', and $\mathrm{Sp}(W)$, respectively. Define $\mathrm{Mp}(W_v)$ to be the metaplectic cover of $\mathrm{Sp}(W_v)$ satisfying

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Mp}(W_v) \longrightarrow \mathrm{Sp}(W_v) \longrightarrow 1,$$

and let $\tilde{G}(F_v)$ and $\tilde{G}'(F_v)$ be the inverse images of $G(F_v)$ and $G'(F_v)$ in $\mathrm{Mp}(W_v)$, respectively. Because we are not in case 1 with one of n and n' odd, the covering $\tilde{G}(F_v) \times \tilde{G}'(F_v) \to G(F_v) \times G'(F_v)$ is trivial. In particular, Kudla [8] has given an explicit section

$$i_\chi : G(F_v) \times G'(F_v) \longrightarrow \tilde{G}(F_v) \times \tilde{G}'(F_v)$$

attached to any character χ of $(E \otimes_F F_v)^\times$ whose restriction to F_v is the quadratic character of the extension E_w/F_v.

For each choice of nontrivial additive character ψ of F_v, there exists a Weil representation ω_ψ of $\mathrm{Mp}(W_v)$ in which S^1 acts via the standard character; see [6, 19]. We denote the restriction of ω_ψ to $G(F_v) \times G'(F_v)$ under i_χ by ω_χ; we shall discuss the dependence of this representation on ψ and χ in Section 2.2. If π_v is an irreducible admissible representation of $G'(F_v)$, we define

$$\Theta(\pi_v, V_v) = (\pi_v \otimes \omega_\chi)_{G'(F_v)}.$$

Here $\Theta(\pi_v, V_v)$ is a representation of $G(F_v)$, and in [6] it was conjectured by Howe that $\Theta(\pi_v, V_v)$ admits a unique irreducible quotient. This is known in almost every case by the work of Howe [6], Li [9], and Waldspurger et al. [15].

Theorem 2.1. Howe's conjecture is true if:

(1) v is Archimedean (Howe [6]);
(2) v is non-Archimedean and does not divide 2 (Waldspurger et al. [15]);
(3) v is non-Archimedean and the pair $(G'(F_v), G(F_v))$ is in the 'stable range'. If v splits in E, this means that $n > 2n'$, and otherwise that V_v contains an isotropic subspace of dimension at least n' (Li [9]). Both of these are guaranteed by the inequality $n > 2n' + 4d - 2$. □

In the cases where it is known to exist, we shall denote the unique irreducible quotient of $\Theta(\pi_v, V_v)$ by $\theta(\pi_v, V_v)$.

2.2 The global theta lift

Let $\mathbb{A}$ and $\mathbb{A}_E$ denote the rings of adeles of F and E, respectively. Let $\mathrm{Sp}(W, \mathbb{A})$, $G(\mathbb{A})$, and $G'(\mathbb{A})$ denote, respectively, the adelic points of the groups $\mathrm{Sp}(W)$, G, and G', and define $\mathrm{Mp}(W, \mathbb{A})$ to be the metaplectic cover of $\mathrm{Sp}(W, \mathbb{A})$ satisfying

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Mp}(W, \mathbb{A}) \longrightarrow \mathrm{Sp}(W, \mathbb{A}) \longrightarrow 1.$$

The restriction of this covering to $\mathrm{Sp}(W, F)$ has a unique section i_F, and we shall frequently consider $\mathrm{Sp}(W, F)$ as a subgroup of $\mathrm{Mp}(W, \mathbb{A})$ by means of this section. Let $\tilde{G}(\mathbb{A})$ and $\tilde{G}'(\mathbb{A})$ be the inverse images of $G(\mathbb{A})$ and $G'(\mathbb{A})$ in $\mathrm{Mp}(W, \mathbb{A})$. As in the local case, we may use the results of Kudla to associate a section

$$i_\chi : G(\mathbb{A}) \times G'(\mathbb{A}) \longrightarrow \tilde{G}(\mathbb{A}) \times \tilde{G}'(\mathbb{A}) \tag{4}$$

to any character χ of $\mathbb{A}_E^\times / E^\times$ whose restriction to $\mathbb{A}^\times$ is the quadratic character of the extension E/F. It is known that the sections i_F and i_χ agree on $G(F) \times G'(F)$. In the symplectic-orthogonal case, this follows from the simple form of the Weil representation on the orthogonal member of the pair, and the fact that the symplectic group has no nontrivial characters. In the unitary case, this is proved in [5, Section 3.1].

If ψ is a nontrivial additive character of $\mathbb{A}/F$, we let ω_ψ be the global Weil representation of $\mathrm{Mp}(W, \mathbb{A})$. If $W = X \oplus Y$ is a complete polarization of W, we may realize ω_ψ on the space $L^2(X(\mathbb{A}))$, in which the space of smooth vectors is the Bruhat–Schwarz space $S(X(\mathbb{A}))$. On $S(X(\mathbb{A}))$, there is a distribution θ defined by

$$\theta(\phi) = \sum_{\xi \in X(F)} \phi(\xi),$$

which satisfies

$$\theta(\omega_\psi(\gamma)\phi) = \theta(\phi), \quad \gamma \in \mathrm{Sp}(W, F).$$

For each $\psi \in S(X(\mathbb{A}))$, we set

$$\theta_\phi(g, h) = \theta(\omega_\psi(gh)\phi).$$

It is known [6, 19] that θ_ϕ is a smooth, slowly increasing function on $G'(F)\backslash\tilde{G}'(\mathbb{A}) \times G(F)\backslash\tilde{G}(\mathbb{A})$. Note that our earlier remark that $i_F = i_\chi$ implies that there is no ambiguity in the definition of $G'(F) \times G(F) \subset \tilde{G}'(\mathbb{A}) \times \tilde{G}(\mathbb{A})$.

Let π be a cusp form contained in $L^2(G'(F)\backslash\tilde{G}'(\mathbb{A}))$ which transforms under S^1 according to the inverse of the standard character. Because of the splitting i_χ, this is equivalent to giving a cusp form on $G'(F)\backslash G'(\mathbb{A})$. For each $f \in \pi$, we may then define

$$\theta_\phi(f, h) = \int_{G'(F)\backslash\tilde{G}'(\mathbb{A})} \theta_\phi(g, i_\chi(h)) f(g)\, \mathrm{d}g, \quad h \in G(\mathbb{A}).$$

In general, $\theta_\phi(f, h)$ is a smooth function of moderate growth on $G(F)\backslash G(\mathbb{A})$. However, if we assume in addition that $n > 2n' + 4d - 2$, then the following theorem was proved by Li [11].

Theorem 2.2. $\theta_\phi(f, h)$ is square integrable, and nonzero for some choice of $f \in \pi$ and $\phi \in S(X(\mathbb{A}))$. If we define $\Theta(\pi, V)$ to be the subspace of $L^2(G(F)\backslash G(\mathbb{A}))$ generated by the functions $\theta_\phi(f, h)$ under the action of $G(\mathbb{A})$, then $\Theta(\pi, V)$ is an irreducible automorphic representation that is isomorphic to $\bigotimes_v \theta(\pi_v, V_v)$. □

We shall refer to the automorphic representation $\Theta(\pi, V)$ as the global theta lift of π. We complete this section by recalling how $\Theta(\pi, V)$ depends on the choices of ψ and χ we have made. Let ψ_0 be the standard additive character of the adeles of $\mathbb{Q}$, and choose ψ to be the pullback of ψ_0 to $\mathbb{A}$ via the trace. Any other nontrivial character of $\mathbb{A}/F$ is of the form $\psi(ax)$ for some $a \in F^\times$, and we denote this character by ψ_a. Denote the Weil representation ω_{ψ_a} by ω_a. It is known that ω_a and ω_b have no irreducible component in common if ab is not a square in $F^\times$, and that the duality correspondence between $\tilde{G}'(\mathbb{A})$ and $\tilde{G}(\mathbb{A})$ depends on a. However, it is well known that this dependence can be transferred from ψ_a to the quadratic form $(\, , \,)'$, in the sense that the correspondence with respect to ω_a for the pair (G', G) considered as the isometry groups of $(\, , \,)'$ and $(\, , \,)$ is the same as the correspondence with respect to ω_ψ for (G', G) considered as the isometry groups of $a(\, , \,)'$ and $(\, , \,)$. Because we shall be taking a sum over all quadratic forms $(\, , \,)'$ in this paper, we therefore only need to consider the character ψ which is unramified away from the different of $F/\mathbb{Q}$.

We only need to consider one choice of section i_χ for a similar reason. Define $\mathbb{A}_E^1$ by

$$\mathbb{A}_E^1 = \{z \in \mathbb{A}_E | z z^\iota = 1\}.$$

If χ_1 and χ_2 are two characters of $\mathbb{A}_E^\times / E^\times$ that agree on $\mathbb{A}^\times$, we may define a character α_{χ_1,χ_2} of $\mathbb{A}_E^1 / E^1$ as follows. If $z \in \mathbb{A}_E^1$, we may choose $u \in \mathbb{A}_E$ such that $z = u^\iota / u$, and define $\alpha_{\chi_1,\chi_2}(z)$ to be

$$\alpha_{\chi_1,\chi_2}(z) = \chi_1 \chi_2^{-1}(u).$$

It can be seen that this is independent of u, and factors through E^1. The sections i_{χ_1} and i_{χ_2} then differ by twisting by α_{χ_1,χ_2}, that is,

$$i_{\chi_2}(g) = \alpha_{\chi_1,\chi_2}(\det g) i_{\chi_1}(g), \quad g \in G'(\mathbb{A}) \times G(\mathbb{A}).$$

Note that we are identifying the S^1 subgroup of Mp with the norm 1 complex numbers. This relation between i_{χ_1} and i_{χ_2} implies that the two representations $\Theta_{\chi_1}(\pi, V)$ and $\Theta_{\chi_2}(\pi, V)$ that they provide are twists of one another. Because we shall be considering all twists of these representations in any case, we only need to consider the lift constructed with the use of one character χ, which we may take to be unramified away from the different of $E/\mathbb{Q}$.

3 Construction of Cohomological Automorphic Representations

3.1 Notation and definitions

We shall now introduce our notation and give a precise definition of $L^2_\theta(X(\mathfrak{n}))$. Let $\mathcal{O}_v$ and $\mathcal{O}_w$ denote the rings of integers of F_v and E_w for v and w finite, respectively, and define

$$\hat{\mathcal{O}}_E = \prod_{w \nmid \infty} \mathcal{O}_w.$$

Let L be an $\mathcal{O}_E$ lattice in V, and let

$$\hat{L} = \prod_{w \nmid \infty} L_w = L \otimes \hat{\mathcal{O}}_E$$

be its adelic closure. If $\mathfrak{n} \subset \mathcal{O}_F$ is an ideal, define the principal congruence subgroup $K(\mathfrak{n}) \subset G(\mathbb{A})$ by

$$K(\mathfrak{n}) = \{g \in G(\mathbb{A}_f) | (g-1)\hat{L} \subset \mathfrak{n}\hat{L}\}.$$

We shall write $K(\mathfrak{n})$ as a product of the form

$$K(\mathfrak{n}) = \prod_{v \nmid \infty} K_v(\mathfrak{n}).$$

Similarly, if ϖ_v denotes a uniformizer of $\mathcal{O}_v$, then, for v a non-Archimedean place of F and any choice of $w|v$, define

$$K_v(\varpi_v^k) = \{g \in G(F_v) | (g-1)L_w \subset \varpi_v^k L_w\}.$$

We shall use K and K_v to denote the compact subgroup corresponding to the choice $\mathfrak{n} = \mathcal{O}_F$. Recall that we have defined $X(\mathfrak{n})$ and $X^0(\mathfrak{n})$ to be the quotients

$$X(\mathfrak{n}) = G(F)\backslash G(\mathbb{A})/K(\mathfrak{n}),$$
$$X^0(\mathfrak{n}) = G^0(F)\backslash G^0(\mathbb{A})/K^0(\mathfrak{n}), \quad K^0(\mathfrak{n}) = K(\mathfrak{n}) \cap G^0(\mathbb{A}).$$

We define $\alpha(V)$ to be

$$\begin{array}{ll} n(n-\eta)/2, & \text{case 1}, \\ n^2/2, & \text{case 2}, \end{array}$$

so that we have

$$\mathrm{Vol}(X(\mathfrak{c}\mathfrak{p}^k)) \sim N\mathfrak{P}^{\alpha(V)k}$$

(and likewise for $\alpha(V')$).

We define $L^2_\theta(G(F)\backslash G(\mathbb{A})) \subset L^2(G(F)\backslash G(\mathbb{A}))$ to be the $G(\mathbb{A})$-invariant subspace

$$\bigoplus_{\alpha, V'} \bigoplus_{\pi} \alpha(\det g) \otimes \Theta(\pi, V).$$

The first sum is indexed by the $-\eta$ -Hermitian spaces V' over E in the stable range, and the set of characters α of $\mu(F)\mu(\mathbb{R})\backslash\mu(\mathbb{A})$, where μ is the algebraic group

$$\begin{array}{ll} \mathrm{id}, & \text{case } 1, \eta = -1, \\ \{\pm 1\}, & \text{case } 1, \eta = 1, \\ U(1), & \text{case } 2. \end{array}$$

The second sum is taken over the set of cuspidal automorphic representations of $G'(\mathbb{A})$ that are in the discrete series at all Archimedean places. We denote the $K(\mathfrak{n})$-fixed subspace of $L^2_\theta(G(F)\backslash G(\mathbb{A}))$ by $L^2_\theta(X(\mathfrak{n}))$, and the set of restrictions of functions in this space to $X^0(\mathfrak{n})$ by $L^2_\theta(X^0(\mathfrak{n}))$.

In [10], Li determines those cohomological unitary representations of $G(\mathbb{R})$ which occur in the Archimedean theta correspondence for (G', G) as the lift of a discrete series representation. We shall refer to the cohomological representations that are obtained in this way from a group $G'(\mathbb{R})$ with $n > 2n' + 4d - 2$ as being of discrete type. See [4, Section 2.2] for a description of these representations, including the degrees in which they contribute to cohomology. Suppose that $A_\mathfrak{q}$ is a cohomological automorphic representation of $G(\mathbb{R})$ of discrete type, so that there is a quadratic space $V'_\mathfrak{q}$ of dimension n' over $E \otimes_\mathbb{Q} \mathbb{R}$ and a discrete series representation $\pi_\mathfrak{q}$ of the isometry group $U(V'_\mathfrak{q})$ of $V'_\mathfrak{q}$ such that $A_\mathfrak{q}$ is realized as the lift of $\pi_\mathfrak{q}$ from $U(V'_\mathfrak{q})$. We denote the multiplicity with which $A_\mathfrak{q}$ occurs in $L^2_\theta(X(\mathfrak{n}))$ (respectively, $L^2_\theta(X^0(\mathfrak{n}))$) by $m_\theta(A_\mathfrak{q}, \mathfrak{n})$ (respectively, $m^0_\theta(A_\mathfrak{q}, \mathfrak{n})$).

3.2 Reduction to local representation theory

We shall assume for the rest of the paper that $\mathfrak{n} = \mathfrak{c}\mathfrak{p}^k$ for $\mathfrak{c}$ and $\mathfrak{p}$ as in Theorem 1.1. The multiplicity $m_\theta(A_\mathfrak{q}, \mathfrak{c}\mathfrak{p}^k)$ is bounded above as follows:

$$\begin{aligned} m_\theta(A_\mathfrak{q}, \mathfrak{c}\mathfrak{p}^k) &\leq \bigoplus_{\substack{\alpha, \\ V' \in \mathcal{V}'}} \bigoplus_{\substack{\pi = \pi_\infty \otimes \pi_f \\ \pi_\infty \simeq \pi_\mathfrak{q}}} \dim(\alpha(\det g) \otimes \Theta(\pi, V)_f)^{K(\mathfrak{c}\mathfrak{p}^k)} \\ &= \bigoplus_{\substack{\alpha, \\ V' \in \mathcal{V}'}} \bigoplus_{\substack{\pi = \pi_\infty \otimes \pi_f \\ \pi_\infty \simeq \pi_\mathfrak{q}}} \prod_{v \nmid \infty} \dim(\alpha_v(\det g) \otimes \Theta(\pi, V)_v)^{K_v(\mathfrak{c}\mathfrak{p}^k)}. \end{aligned} \quad (5)$$

Here, $\mathcal{V}'$ denotes the set of Hermitian spaces V' such that $V' \otimes_{\mathbb{Q}} \mathbb{R} \simeq V'_{\mathfrak{q}}$, and $\Theta(\pi, V)_f$ denotes the finite components of the representation $\Theta(\pi, V)$. We shall begin to prove bounds for m_θ by first controlling the Hermitian spaces V' and the level of the automorphic representations π which may contribute to the sum. For each $V' \in \mathcal{V}'$, choose an $\mathcal{O}_E$ lattice L' which is self-dual in V'_v whenever possible, and define compact subgroups $K'(\mathfrak{n})$ and $K'_v(\varpi_v^k)$ of $G'(\mathbb{A})$ with respect to L' in the same way as with $G(\mathbb{A})$. In [4, Lemma 2.9], the first author proves the following proposition.

Proposition 3.1. For all places $v \notin S$:

(1) if $(\alpha_v \otimes \Theta(\pi, V)_v)^{K_v(\varpi_v^k)} \neq 0$, then $\alpha_v|_{\det K_v(\varpi_v^k)} = 1$;

(2) $\Theta(\pi, V)_v^{K_v} \neq 0$ if and only if V'_v is unramified and $\pi_v^{K'_v} \neq 0$. □

Proposition 3.1 implies that the only spaces V' that can contribute to the sum (5) must be unramified outside S and the places dividing $\mathfrak{c}$, and it is known that the number of these is finite. Indeed, when V' is orthogonal, this follows from the simple observation that the existence of a self-dual lattice at a place v implies that V'_v has trivial Hasse invariant and that its discriminant has even valuation. Therefore, there are only finitely many square classes that can be assumed by the discriminant of V', and finitely many possibilities for its local Hasse invariants, and these uniquely determine V' by the classification of quadratic forms over a number field. The case where V' is Hermitian reduces to the orthogonal case, as Hermitian forms are uniquely determined by their trace forms, and the symplectic case is trivial.

As a result, when proving the upper bound (1), we may assume that $\mathcal{V}'$ consists of a single space V'. We may control the level of π at the places dividing $\mathfrak{c}$ and those in S using the following result, stated as [4, Proposition 2.11].

Proposition 3.2. Let v be any non-Archimedean place of F, and assume that

$$(\alpha_v \otimes \theta_v(\pi_v, V_v))^{K_v(\varpi_v^t)} \neq 0$$

for some t. Then there exists t' depending only on t such that

$$\alpha_v|_{\det K'_v(\varpi_v^{t'})} = 1, \quad \pi_v^{K'_v(\varpi_v^{t'})} \neq 0.$$

Moreover, there exists $C(L, L', v, t)$ such that

$$\dim(\alpha_v \otimes \theta_v(\pi_v, V_v))^{K_v(\varpi_v^t)} \le C(L, L', v, t) \dim \pi_v^{K'_v(\varpi_v^{t'})}. \qquad \square$$

(Note that the result stated in [4] does not include the statement about the ramification of α_v, but this follows easily in the same way as the first point of Proposition 3.1.) Propositions 3.1 and 3.2 together imply that the conductor of the character α must be bounded away from $\mathfrak{p}$, and so the number of characters that may contribute to the sum is on the order of $N\mathfrak{p}^{\delta k}$. Furthermore, there exists an ideal $\mathfrak{c}'$ of $\mathcal{O}_F$ which is divisible only by primes in S and those dividing $\mathfrak{c}$ such that

$$\begin{aligned} m_\theta(A_\mathfrak{q}, \mathfrak{c}\mathfrak{p}^k) &\ll \bigoplus_\alpha \bigoplus_{\substack{\pi=\pi_\infty\otimes\pi_f \\ \pi_\infty\simeq\pi_\mathfrak{q}}} (\alpha_\mathfrak{p} \otimes \theta(\pi_\mathfrak{p}, V_\mathfrak{p}))^{K_\mathfrak{p}(\mathfrak{p}^k)} \prod_{v\nmid\infty\mathfrak{p}} \dim(\pi_v)^{K'_v(\mathfrak{c}')} \\ &\ll N\mathfrak{p}^{\delta k} \bigoplus_{\substack{\pi=\pi_\infty\otimes\pi_f \\ \pi_\infty\simeq\pi_\mathfrak{q}}} \theta_\mathfrak{p}(\pi_\mathfrak{p}, V_\mathfrak{p})^{K_\mathfrak{p}(\mathfrak{p}^k)} \prod_{v\nmid\infty\mathfrak{p}} \dim(\pi_v)^{K'_v(\mathfrak{c}')}. \end{aligned}$$

Factorize $K'(\mathfrak{c}')$ as $K'_\mathfrak{p}(\mathfrak{c}')K'^{\mathfrak{p}}(\mathfrak{c}')$, and let $L^2_{\text{cusp}}(X'(\mathfrak{c}'\mathfrak{p}^\infty))$ be the space of $K'^{\mathfrak{p}}(\mathfrak{c}')$-fixed vectors in $L^2_{\text{cusp}}(G'(F)\backslash G'(\mathbb{A}))$. This space decomposes discretely with finite multiplicities under the action of $G'(\mathbb{R}) \times G'(F_\mathfrak{p})$, and we write the $\pi_\mathfrak{q}$-isotypic component of this space as

$$\pi_\mathfrak{q} \otimes \bigoplus_{i=1}^\infty \pi_{\mathfrak{p},i} = \pi_\mathfrak{q} \otimes \Pi_\mathfrak{p}.$$

We may now reformulate the problem of bounding m_θ in local terms. It follows from the trace formula (see Savin [18]) that

$$\dim \Pi_\mathfrak{p}^{K_\mathfrak{p}(\mathfrak{p}^k)} \sim N\mathfrak{P}^{k\alpha(V')}. \tag{6}$$

In fact, we shall use a slight refinement of this upper bound, proved in Section 4. By studying the local theta correspondence, we will prove that (6) implies

$$\sum_{i=1}^\infty \dim(\theta(\pi_{\mathfrak{p},i}, V_\mathfrak{p}))^{K_\mathfrak{p}(\mathfrak{p}^k)} \sim N\mathfrak{P}^{nn'k/2}, \tag{7}$$

and the required upper bound on m_θ^0 follows from this and Remark 4.

We will approach the lower bound in the same way, after first choosing a suitable Hermitian space V' from which to lift our forms. In case 1 with $\eta = 1$, we may take the standard symplectic form, while in the other cases we choose the diagonal form

$$(\, , \,)' = \sum_{i=1}^{n'} a_i x_i x_i^{\iota},$$

where a_i are S'-units with the required signs at Archimedean places. It can be seen that V' is then unramified away from S'. We let $L' \subset V'$ be a lattice that is self-dual away from S', which we use to define families of compact subgroups of $G'(\mathbb{A})$.

When proving a lower bound for m_θ, there is a technical issue that arises related to twisting and the disconnectedness of our adelic quotients, which is dealt with by the following lemma.

Lemma 3.3. Suppose that we are in case 2. Let V' be fixed, let π_1 and π_2 be two orthogonal cuspidal automorphic representations of $G'(\mathbb{A})$, and let α be an automorphic character of $G(\mathbb{A})$. Then the representations $\Theta(\pi_1, V)$ and $\Theta(\pi_1, V) \otimes \alpha$ are orthogonal. □

Proof. Let v be a place of F that splits in E and at which all data are unramified. Denote the local components of π_i and α by $\pi_{1,v}$, $\pi_{2,v}$, and α_v, so that $\pi_{i,v}$ are representations of $GL(n', F_v)$. The local theta correspondence for unramified representations of the pair $(\mathrm{GL}(n', F_v), \mathrm{GL}(n, F_v))$ may be computed explicitly in the stable range $n > 2n'$; see, for instance, [1, 14, 16]. We find that the set of Satake parameters of $\Theta(\pi_1, V)_v$ must contain a subset of the form $\{q^{(a-1)/2}\eta, \dots, q^{-(a-1)/2}\eta\}$, where $a = n - n'$, q is the order of the residue field of F_v, and η is a complex number of modulus 1 determined by the choice of section $G' \times G \to \mathrm{Mp}$. Likewise, the Satake parameters of $\Theta(\pi_2, V)_v \otimes \alpha_v$ must contain a subset of the form $\{\alpha_v(\varpi) q^{(a-1)/2}\eta, \dots, \alpha_v(\varpi) q^{-(a-1)/2}\eta\}$, where $\varpi \in F_v$ is a uniformizer. Because $n > 2n'$, these subsets cannot be disjoint, which forces $\alpha_v(\varpi) = 1$. This proves that α must be trivial at all but finitely many split places of E/F, and it must also be trivial at all but finitely many inert places. As α was automorphic, this means it must be trivial. The result now follows from the Rallis scalar product formula as in [4]. ■

In case 2, Lemma 3.3 implies that

$$m_\theta(A_{\mathfrak{q}}, \mathfrak{c}\mathfrak{p}^k) \geq \bigoplus_{\alpha} \bigoplus_{\substack{\pi = \pi_\infty \otimes \pi_f \\ \pi_\infty \simeq \pi_{\mathfrak{q}}}} \dim(\alpha(\det g) \otimes \Theta(\pi, V)_f)^{K(\mathfrak{c}\mathfrak{p}^k)},$$

while in case 1 we have

$$m_\theta(A_{\mathfrak{q}}, \mathfrak{c}\mathfrak{p}^k) \geq \bigoplus_{\substack{\pi=\pi_\infty\otimes\pi_f \\ \pi_\infty\simeq\pi_{\mathfrak{q}}}} \dim(\alpha(\det g) \otimes \Theta(\pi, V)_f)^{K(\mathfrak{c}\mathfrak{p}^k)}.$$

As the number of characters that are trivial on $\det(K(\mathfrak{c}\mathfrak{p}^k))$ is $\sim N\mathfrak{p}^{\delta k}$, these inequalities may be combined to give

$$m_\theta(A_{\mathfrak{q}}, \mathfrak{c}\mathfrak{p}^k) \gg N\mathfrak{p}^{\delta k} \bigoplus_{\substack{\pi=\pi_\infty\otimes\pi_f \\ \pi_\infty\simeq\pi_{\mathfrak{q}}}} \prod_{v\nmid\infty} \dim(\alpha_v(\det g) \otimes \Theta(\pi, V)_v)^{K_v(\mathfrak{c}\mathfrak{p}^k)}.$$

We may reduce the proof of a lower bound for m_θ to a local problem using the following counterpart of Proposition 3.2, stated in [4, Proposition 2.13].

Proposition 3.4. For every non-Archimedean place v and compact open subgroup K'_v, there exists a t such that

$$\dim \theta(\pi_v, V_v)^{K_v(\varpi_v^t)} \geq \dim \pi_v^{K'_v}. \qquad \square$$

As in our approach to the upper bound, Propositions 3.1 and 3.4 imply that there exists an ideal $\mathfrak{c}'$ divisible only by the primes in S' such that

$$m_\theta(A_{\mathfrak{q}}, \mathfrak{c}'\mathfrak{p}^k) \gg N\mathfrak{p}^{\delta k} \bigoplus_{\substack{\pi=\pi_\infty\otimes\pi_f \\ \pi_\infty\simeq\pi_{\mathfrak{q}}}} \theta(\pi_{\mathfrak{p}}, V_{\mathfrak{p}})^{K_{\mathfrak{p}}(\mathfrak{p}^k)} \prod_{v\nmid\infty\mathfrak{p}} \dim(\pi_v)^{K'_v},$$

and hence the lower bound in Theorem 1.1 would also follow from the asymptotic (7).

Remark 4. If we define $m_\theta(V')$ to be the multiplicity with which $A_{\mathfrak{q}}$ occurs in the space of lifts from a fixed Hermitian space V', Lemma 3.3 implies that the action of characters on $m_\theta(V')$ by twisting is free in case 2. As $Y_0(\mathfrak{c}\mathfrak{p}^k)$ is the kernel of all characters, and the number of Hermitian spaces that must be considered is finite, this implies that $m_\theta \sim N\mathfrak{p}^{\delta k} m_\theta^0$ so that Corollary 1.2 follows from Theorem 1.1. $\square$

4 Application of the Local Theta Correspondence

We have reduced the proof of Theorem 1.1 to showing that an asymptotic such as (6) for the dimension of the $K_{\mathfrak{p}}(\mathfrak{p}^k)$-fixed subspace of a direct sum of irreducible admissible representations of $G'(F_{\mathfrak{p}})$ implies the corresponding asymptotic (7) for their set of local theta lifts. We shall in fact use a slightly refined form of the upper bound in (6), which takes into account the decomposition of $\Pi_{\mathfrak{p}}$ into irreducibles under the action of $K'_{\mathfrak{p}}$.

Lemma 4.1. If ρ is an irreducible, finite-dimensional complex representation of $K'_{\mathfrak{p}}$ on a vector space F_1, we have

$$\dim \mathrm{Hom}_{K'_{\mathfrak{p}}}(\rho, \Pi_{\mathfrak{p}}) \ll \dim F_1. \qquad \square$$

Proof. Let K'_∞ be a maximal compact subgroup of $G'(\mathbb{R})$, and define $Y'(\mathfrak{c})$ to be the locally symmetric space $X'(\mathfrak{c})\backslash K'_\infty$. The lemma will follow easily from the fact that π_q belongs to the discrete series, and therefore contributes to the cohomology of certain automorphic vector bundles over $Y'(\mathfrak{c}')$. More precisely, if (τ, F_2) is an irreducible, finite-dimensional representation of $G'(\mathbb{R})$ with the same infinitesimal character as $\pi_{\mathfrak{q}}$ and $q = (\dim G'(\mathbb{R}) - \dim K'_\infty)/2$, it is well known (see [3, Chapter 2, Section 5]) that

$$H^q(\mathfrak{g}, K; \pi_{\mathfrak{q}} \otimes F_2) \simeq \mathbb{C}. \tag{8}$$

Let $\tilde{\mathcal{F}}$ be the trivial vector bundle over $G'(\mathbb{A})$ with fiber $F_1 \otimes F_2$. Let $G'(F)$ act on $\tilde{\mathcal{F}}$ from the left by

$$\gamma(g, v_1 \otimes v_2) = (\gamma g, v_1 \otimes \tau(\gamma) v_2),$$

where $\tau(\gamma)$ denotes the composition of τ with the natural embedding of $G'(F)$ into $G'(\mathbb{R})$. Let $K'_{\mathfrak{p}}$ act on $\tilde{\mathcal{F}}$ from the right by

$$(g, v_1 \otimes v_2)k = (gk, \rho(k^{-1}) v_1 \otimes v_2),$$

and let $K'^{\mathfrak{p}}$ and K'_∞ act on the right through the first factor alone. Define $\mathcal{F}$ to be the quotient of $\tilde{\mathcal{F}}$ by the left and right actions of $G'(F)$ and $K'_\infty K'$; it is a flat bundle over $Y'(\mathfrak{c}')$, and it follows from (8) that

$$\dim H^q_{\mathrm{cusp}}(Y'(\mathfrak{c}'), \mathcal{F}) \geq \dim \mathrm{Hom}_{K'_{\mathfrak{p}}}(\rho, \Pi_{\mathfrak{p}}).$$

Because the cuspidal cohomology is a subspace of the total cohomology, it therefore suffices to prove that

$$\dim F_1 \gg \dim H^q(Y'(\mathfrak{c}'), \mathcal{F}).$$

Fix a triangulation of $Y'(\mathfrak{c}')$, and let N be the number of q-dimensional cells. If we compute $H^q(Y'(\mathfrak{c}'), \mathcal{F})$ using this triangulation, the dimension of the space of q-chains is $N \dim F_1 \otimes F_2$ so that

$$\begin{aligned}\dim H^q(Y'(\mathfrak{c}'), \mathcal{F}) &\leq N \dim F_1 \otimes F_2 \\ &\ll \dim F_1,\end{aligned}$$

as required. ■

4.1 The upper bound

To deduce the upper bound in (7) from Lemma 4.1, we shall use the lattice model of the oscillator representation of $\mathrm{Mp}(W_\mathfrak{p})$. Note that from now on all groups and vector spaces will be assumed to be local at $\mathfrak{p}$, and so we shall drop the subscript from our notation and also use $\mathfrak{p}$ to denote a uniformizer of $F_\mathfrak{p}$. We recall that we have chosen self-dual lattices $L' \subset V'$ and $L \subset V$, and define $J \subset W$ to be the self-dual lattice $L' \otimes L$. Let $S(W)$ be the lattice model of the Weil representation of $\mathrm{Mp}(W)$, consisting of compactly supported functions on W that satisfy

$$f(w + j) = \psi(-\tfrac{1}{2}\langle j, w\rangle f(w)), \quad \forall j \in J.$$

Here, ψ is the local component at $\mathfrak{p}$ of the additive character introduced in Section 2, which we have assumed to be unramified. If $H \subset \mathrm{Sp}(W)$ is the stabilizer of J, the action of H in $S(W)$ may be written (up to a character) as

$$\omega(h)\phi(x) = \phi(h^{-1}x). \tag{9}$$

Let $S(W)[r]$ be the subspace of functions which are supported on $\mathfrak{p}^{-r}J$, which is fixed by $K(\mathfrak{p}^{2r}) \times K'(\mathfrak{p}^{2r})$ by virtue of (9). By the definition of the theta correspondence, if π is an irreducible admissible representation of G', then there is a G'-covariant surjection

$$\pi \otimes S(W) \to \theta(\pi, V).$$

Let $K(\mathfrak{p}^a)$ and $K'(\mathfrak{p}^b)$ be compact open subgroups of G and G', respectively. In the course of the proof of the first author [4, Proposition 3.2] it is shown that if $S_0(W) \subset S(W)$ is a finite-dimensional subspace such that

$$S(W)^{1\times K(\mathfrak{p}^a)} = \omega_\chi(\mathcal{H}' \times 1)S_0(W),$$

where $\mathcal{H}'$ denotes the Hecke algebra of G', and $K'(\mathfrak{p}^b)$ fixes each element of $S_0(W)$, then there exists a K'-covariant surjection

$$\varphi : \pi^{K'(\mathfrak{p}^b)} \otimes S_0(W) \to \theta(\pi, V)^{K(\mathfrak{p}^a)}. \tag{10}$$

Moreover, it is proved in [15, Chapter 5, Section I.4] that these conditions are satisfied when $a=b=k$ and $S_0(W)=S(W)[\frac{k}{2}]$ or $S(W)[\frac{k+1}{2}]$ depending on the parity of k.

Write the decompositions of $\Pi_\mathfrak{p}^{K(\mathfrak{p}^k)}$ and $S_0(W)$ into irreducible representations of K' as

$$\Pi_\mathfrak{p} = \sum_\rho \dim \operatorname{Hom}_{K'}(\rho, \Pi_\mathfrak{p})\rho,$$

$$S_0(W) = \sum_\rho n(\rho, k)\rho.$$

The existence of the K'-covariant surjection (10) with $a=b=k$ and $S_0(W)=S(W)[\frac{k}{2}]$ or $S(W)[\frac{k+1}{2}]$ implies that

$$\dim \theta(\pi_{\mathfrak{p},i}, V)^{K(\mathfrak{p}^k)} \ll \sum_\rho n(\rho, k) \dim \operatorname{Hom}_{K'}(\rho, \pi_{\mathfrak{p},i}),$$

$$\begin{aligned}\sum_{i=1}^{\infty} \dim \theta(\pi_{\mathfrak{p},i}, V)^{K(\mathfrak{p}^k)} &\ll \sum_\rho n(\rho, k) \dim \operatorname{Hom}_{K'}(\rho, \Pi_\mathfrak{p}) \\ &\ll \sum_\rho n(\rho, k) \dim \rho \\ &= \dim S_0(W) \\ &\ll N\mathfrak{P}^{nn'k/2}.\end{aligned}$$

This completes the proof of the upper bound in (7), and hence in Theorem 1.1.

4.2 The lower bound

To establish the lower bound, we will prove an injectivity result for the restriction of φ to some subspace of $\pi \otimes \omega$ that strengthens the one given in [4, Section 5]. Our main tool in doing so will be the local Rallis scalar product formula, which states that there is a nonzero constant c such that if $v, v' \in \pi$ and $\phi, \phi' \in \omega$, we have

$$\langle \varphi(v \otimes \phi), \varphi(v' \otimes \phi') \rangle = c \int_{G'} \langle \omega(g)\phi, \phi' \rangle \langle \pi(g)v, v' \rangle \, \mathrm{d}g. \tag{11}$$

See [4, Section 4] for a discussion and proof of this formula. We shall construct a large subspace $A \subset \omega$ that is invariant under $K'(\mathfrak{p}^k) \times K(\mathfrak{p}^k)$ and such that $\langle \omega(g)\phi, \phi' \rangle$ is supported on $K'(\mathfrak{p}^k)$ for ϕ and ϕ' in A, after which (11) will imply that $\varphi : \pi^{K'(\mathfrak{p}^k)} \otimes A \to \theta(\pi, V)^{K(\mathfrak{p}^k)}$ is an injection.

By our assumption that (V', V) lay in the stable range, we may decompose V as $V_0 \oplus V_1$, where $V_0 \perp V_1$ and $V_0 = X \oplus Y$ for X and Y two isotropic subspaces of dimension n' in duality. Furthermore, our assumption that L was self-dual implies that we can arrange for this decomposition to satisfy $L = L \cap X + L \cap Y + L \cap V_1$, and we identify V_0 with $V' \oplus V'^*$ in such a way that L' and L'^* correspond to $L \cap X$ and $L \cap Y$. Define W_0 and W_1 to be $V' \otimes V_0$ and $V' \otimes V_1$, respectively. If ω_0 and ω_1 are the Weil representations associated to the Heisenberg groups $H(W_0)$ and $H(W_1)$, respectively, then we have $\omega = \omega_0 \otimes \omega_1$, and G' acts via the product of its actions in each tensor factor. As $V' \otimes V'^* \simeq \mathrm{End}(V')$ is a maximal isotropic subspace of W_0, we may work with the Schrödinger model of ω_0 on the Schwartz space $\mathcal{S}(\mathrm{End}(V'))$. Because G' stabilizes $V' \otimes V'^* \subset W_0$, the action of G' in $\mathcal{S}(\mathrm{End}(V'))$ is (up to a character) given by

$$\omega(g)\phi(x) = \phi(g^{-1} \circ x).$$

We shall work with ω_1 in the lattice model $S(W_1)$ with respect to the lattice $L' \otimes (L \cap V_1)$.

Assume that k is even. Let $\mathcal{E}$ be a set of representatives for the equivalence classes of elements of $\mathrm{GL}(L'/\mathfrak{p}^k L')$ under right multiplication by K', and, for $t \in \mathcal{E}$, let $\phi_t \in \mathcal{S}(\mathrm{End}(V'))$ be the characteristic function of t. Suppose that t and t' are elements of $\mathcal{E}$, and T and T' are elements of $\mathrm{Supp}(\phi_t)$ and $\mathrm{Supp}(\phi_{t'})$, respectively. Because T and T' are automorphisms of L', it follows that if $gT = T'$ for $g \in G'$, then g must preserve L', so we have $g \in K'$. It then follows that

$$\langle \omega(g)\phi_t, \phi_{t'} \rangle = \delta_{t,t'} 1_{K'(\mathfrak{p}^k)}(g), \quad g \in G'.$$

If $h \in G$ is the element that multiplies $X \simeq V'^*$ by $\mathfrak{p}^{k/2}$ and $Y \simeq V'$ by $\mathfrak{p}^{-k/2}$, then $\omega(h)\phi_t$ is a multiple of the characteristic function of the set $\mathfrak{p}^{-k/2}t + \mathfrak{p}^{k/2}\mathrm{End}(L')$. We let $\phi_t^0 = \omega(h)\phi_t$. If $t, t' \in \mathcal{E}$ and $\phi, \phi' \in \omega_1^{K'(\mathfrak{p}^k)}$, we have

$$\begin{aligned}\langle \varphi(v \otimes \phi_t^0 \otimes \phi), \varphi(v' \otimes \omega(h^{-1})\phi_{t'} \otimes \phi') \rangle &= c \int_{G'} \langle \omega(g)\phi_t^0, \omega(h^{-1})\phi_{t'} \rangle \langle \omega(g)\phi, \phi' \rangle \langle \pi(g)v, v' \rangle \, \mathrm{d}g \\ &= c \int_{G'} \langle \omega(g)\phi_t, \phi_{t'} \rangle \langle \omega(g)\phi, \phi' \rangle \langle \pi(g)v, v' \rangle \, \mathrm{d}g \\ &= c\delta_{t,t'} \langle \phi, \phi' \rangle \langle v, v' \rangle.\end{aligned}$$

It follows from this that if we let $\{v_i\}$ be an orthonormal basis for $\pi^{K'(\mathfrak{p}^k)}$ and $\{\phi_i'\}$ be an orthonormal basis for $S(W_1)[\frac{k}{2}]$, then the sets

$$\{\varphi(v_i \otimes \phi_t^0 \otimes \phi_i')\} \quad \text{and} \quad \{\varphi(v_i \otimes \omega(h^{-1})\phi_t \otimes \phi_i')\}$$

are dual to each other under the inner product on $\theta(\pi, V)$. Consequently, if we define A to be the span of the vectors $\{\phi_t^0 \otimes \phi_i'\}$, then we have an injection $\pi^{K'(\mathfrak{p}^k)} \otimes A \to \theta(\pi, V)$, and it remains to show that $K(\mathfrak{p}^k)$ acts trivially on A to deduce that the image of this map lies in $\theta(\pi, V)^{K(\mathfrak{p}^k)}$.

Because we assumed that ϕ' was contained in $S(W_1)[\frac{k}{2}]$, it suffices to show that if $S(W_0)$ is the lattice model for ω_0 with respect to the lattice

$$J = L' \otimes L'^* \oplus L' \otimes L',$$

and $I : \mathcal{S}(\mathrm{End}(V')) \to S(W_0)$ is an intertwining operator between the two models, then $I(\phi_t^0)$ lies in $S(W_0)[\frac{k}{2}]$. We may choose I to be the map

$$I(\phi)(x, y) = \psi(\tfrac{1}{2}\langle x, y \rangle) \int_{L' \otimes L'^*} \psi(\langle z, y \rangle)\phi(x + z) \, \mathrm{d}z, \quad x \in V' \otimes V'^*, \quad y \in V' \otimes V',$$

where $\langle\,,\rangle$ here denotes the pairing between $V' \otimes V'^*$ and $V' \otimes V'$ given by the tensor product of the Hermitian form $(\,,)$ in the first factor and the natural pairing in the second factor. It follows easily from our description of ϕ_t^0 given earlier that $I(\phi)(x, y)$ vanishes if (x, y) lies outside $\mathfrak{p}^{-k/2}J$, and so $\phi_t^0 \otimes \phi_i'$ is fixed by $K'(\mathfrak{p}^k)$. We therefore have

$$\begin{aligned}\dim \theta(\pi, V)^{K(\mathfrak{p}^k)} &\gg \dim A \times \dim \pi^{K'(\mathfrak{p}^k)} \\ &\gg |\mathcal{E}| \dim S(W_1) \left[\frac{k}{2}\right] \times \dim \pi^{K'(\mathfrak{p}^k)} \\ &\gg N\mathfrak{P}^{(n'^2-\alpha(V'))k} N\mathfrak{P}^{n'(n-2n')k/2} \dim \pi^{K'(\mathfrak{p}^k)} \\ &\gg N\mathfrak{P}^{(nn'/2-\alpha(V'))k} \dim \pi^{K'(\mathfrak{p}^k)}.\end{aligned}$$

Because this estimate is uniform in π, we may combine it with the lower bound of (6) to deduce the lower bound in (7) and complete the proof of Theorem 1.1.

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