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TRIPLE PRODUCT L-FUNCTIONS AND QUANTUM CHAOS ON $SL(2,\mathbb{C})$

BY

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ABSTRACT

Let Y be an arithmetic hyperbolic 3-manifold. We establish a link between quantum unique ergodicity for sections of automorphic vector bundles on Y and subconvexity for the triple product L-function, which extends a result of Watson in the case of hyperbolic 2-manifolds. The proof combines the representation theoretic microlocal lift for bundles developed by Bunke and Olbrich with the triple product formula of Ichino. A key step is determining the asymptotic behaviour of the local integrals at complex places that appear in Ichino's formula.

1. Introduction

If M is a compact Riemannian manifold, it is a central problem in quantum chaos to understand the behaviour of high energy Laplace eigenfunctions on M. If $\{\psi_n\}$ is a sequence of such eigenfunctions with eigenvalues λ_n tending to ∞, a natural question that one may ask is whether ψ_n are becoming approximately constant. This may be asked either in a pointwise sense, by showing that certain L^p norms of ψ_n are small, or on average, by showing that the probability measures $\mu_n = |\psi_n(x)|^2 dv$ tend weakly to the Riemannian volume dv of M. There is a conjecture of Rudnick and Sarnak [22] known as the quantum unique ergodicity conjecture, or QUE, which predicts the equidistribution of ψ_n in this weak-* sense when M is negatively curved, and in this paper we shall be

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interesed in a case of this conjecture in which M is an arithmetic hyperbolic three manifold.

The QUE conjecture predicts not just the equidistribution of μ_n, which can be thought of as the positions of the quantum states ψ_n, but of a semiclassical analogue of the combined position and momentum called the microlocal lift. This is most naturally described in terms of the correspondence between the geodesic flow on S^*M and the space $L^2(M)$ with the unitary Schrödinger evolution. The observables on the classical phase space are functions $a \in C^\infty(S^*M)$, and after making a choice of quantisation scheme it is possible to associate to each classical observable a a self-adjoint operator $\mathrm{Op}(a)$, which may be thought of as a quantum observable taking the value $\langle \mathrm{Op}(a)\psi, \psi\rangle$ on a wavefunction $\psi \in L^2(M)$. For each fixed ψ the map

$$\widetilde{\mu}_\psi(a) : a \mapsto \langle \mathrm{Op}(a)\psi, \psi\rangle \tag{1}$$

can be shown to be a distribution on $C^\infty(S^*M)$, and this is defined to be the microlocal lift of ψ. This construction is due to Šnirel'man, Colin de Verdière [5], and Zelditch [30], who also proved that any high energy weak limit of the $\widetilde{\mu}_\psi$ is a measure invariant under the geodesic flow. The microlocal form of the QUE conjecture then predicts that the only limit of $\{\widetilde{\mu}_n\}$ is Liouville measure, or that all derivatives of ψ_n are behaving randomly.

The fact that weak limits of the $\widetilde{\mu}_n$ are flow invariant measures makes it possible to apply ergodic techniques to the QUE conjecture, which has lead to an essentially complete solution in the case of arithmetic quotients of $\mathbb{H}^2$ and $(\mathbb{H}^2)^n$ by Lindenstrauss [15, 14] (with contributions by Soundararajan [27] to deal with the noncompact case), and compact quotients of $GL(n, \mathbb{R})$ for n prime by Silberman and Venkatesh [24, 25]. In the case of arithmetic quotients of $\mathbb{H}^2$, a second approach based on the triple product L-function was developed by Watson [28]. To give an illustration of his results, let $X = SL(2, \mathbb{Z})\backslash\mathbb{H}^2$ and let ϕ_i be three L^2 normalised Hecke–Maass cusp forms on X with associated representations π_i. Watson then proves the beautiful identity

$$\left| \int_X \phi_1\phi_2\phi_3 dv \right|^2 = \frac{1}{8} \frac{\Lambda(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3)}{\prod \Lambda(1, \mathrm{Ad}\pi_i)}. \tag{2}$$

A consequence of this formula is that the coarse form of the QUE conjecture would be implied by a subconvex bound for the triple product L-function in the eigenvalue aspect, and similar formulae for vectors of higher weight in π_i would allow one to deduce the full microlocal version. The purpose of this paper is to prove the same implication for a standard collection of arithmetic quotients of $\mathbb{H}^3$ called the Bianchi manifolds.

Let $G = PSL(2,\mathbb{C})$ and $K = PSU(2)$. If $\Gamma \subset G$ is a lattice, spherical automorphic representations $\pi \subset L^2(\Gamma\backslash G)$ are equivalent to Laplace eigenfunctions on $Y = \Gamma\backslash\mathbb{H}^3$, while nonspherical π are equivalent to sections of certain automorphic vector bundles on Y. If we ignore the complimentary series and trivial representation, the unitary dual of G is indexed by a weight $k \in 2\mathbb{Z}$ and a spectral parameter $r \in \mathbb{R}$, with the spherical representations being those with $k = 0$ (see Section 2.3 for details). It is therefore natural to study the QUE conjecture for the eigensections associated to each of these spectral families.

Let $\{\pi_n\}$ be a sequence of automorphic representations of fixed weight and growing spectral parameter, to which are associated sections s_n of a principal K-bundle over $Y = \Gamma\backslash G/K$. Our first step in studying the quantum ergodic behaviour of s_n is to apply a construction of Bunke and Olbrich [4], which produces a microlocal lift of s_n in terms of the representation π_n. As this lift is defined in terms of automorphic forms, we may test its convergence to the expected limit by integrating against an automorphic basis of $L^2(\Gamma\backslash G)$ and applying the triple product formula of Ichino [10]. The expected equivalence between QUE and subconvexity follows from Ichino's formula once we have made it sufficiently quantitative, which requires the estimation of certain archimedean local integrals using the Whittaker function formulas of Jacquet [11] and a formula appearing in a paper of Michel and Venkatesh [18].

The structure of the paper is as follows. We introduce our notation in Section 2, before defining the microlocal lift we shall use and stating our main theorem in Section 3. We analyse the global and local integrals involved in the proof of the theorem in Sections 4 and 5 respectively, and finish Section 5 with a calculation in the weight aspect which is needed in the paper [17].

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2. Notation

2.1. Groups. Let $G = PGL(2,\mathbb{C})$, which we shall implicitly identify with $PSL(2,\mathbb{C})$ throughout. Let $K = PSU(2)$ be the maximal compact subgroup, and $M \subset K$ be the subgroup of diagonal matrices. We define A and N to be the usual subgroups of G, with the parameterisations

$$a(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}, \quad y \in \mathbb{C}^\times, \qquad n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{C}.$$

Let dy be Lebesgue measure on $\mathbb{C}$, and $dy^\times = |y|^{-2}dy$ be the multiplicative measure on $\mathbb{C}^\times$. We give G the Haar measure which is the product of the probability Haar measure on K and the hyperbolic volume on $G/K \simeq \mathbb{H}^3$. We define ψ to be the standard additive character $\psi(z) = \exp(2\pi i \mathrm{tr}(z))$ of $\mathbb{C}$.

2.2. Automorphic forms. Let F be an imaginary quadratic field, which we assume for simplicity to have class number one, with ring of integers $\mathcal{O}$, adele ring $\mathbb{A}$, and group of units μ. If $k \in \mathbb{Z}$ is divisible by $|\mu|$, we define the character χ_k of $F^\times \backslash \mathbb{A}^\times$ by requiring that its restriction to $\mathbb{C}^\times$ is $\chi_k(z) = (z/|z|)^k$, and that it is unramified at all finite places.

Define $\Gamma \subset G$ to be the lattice $PGL(2,\mathcal{O})$, and let $X = \Gamma\backslash G$ and $Y = \Gamma\backslash G/K$ be the associated arithmetic quotients; Y is a Bianchi orbifold. We shall implicitly deal with all technicalities arising from the orbifold singularities of Y by passing to a smooth finite index cover $\widetilde{Y}$, and thinking of objects on Y as those on $\widetilde{Y}$ that are invariant under the deck group. By a cuspidal automorphic representation π on X we shall mean an irreducible representation of G on the cuspidal subspace $L^2_{\mathrm{cusp}}(X)$ that is also stable under the Hecke operators. Such objects are equivalent to cuspidal automorphic representations on $L^2(PGL(2,F)\backslash PGL(2,\mathbb{A}))$ having full level.

2.3. Representation theory. For $m \geq 0$ even, let (ρ_m, V_m) denote the irreducible $m+1$ dimensional representation of K with Hermitian inner product $\langle \, , \rangle$. We shall denote the conjugate linear map taking a vector v to the functional $\langle \cdot, v\rangle$ by $v \mapsto v^*$. We choose an orthonormal basis $\{v_t\}$ $(t = m, m-2, \dots, -m)$ for V_m and a dual basis $\{v_t^*\}$ for V_m^* which consist of eigenvectors of M satisfying

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} v_t = e^{it\theta} v_t, \quad \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} v_t^* = e^{-it\theta} v_t^*.$$

If $r \in \mathbb{C}$ and $k \in 2\mathbb{Z}$, let $\pi_{(k,r)}$ be the representation of G unitarily induced from the character

$$\chi : \begin{pmatrix} z & x \\ 0 & 1 \end{pmatrix} \mapsto (z/|z|)^{k/2}|z|^{2ir}.$$

These are unitarisable for (k,r) in the set

$$U = \{(k,r)|r \in \mathbb{R}\} \cup \{(k,r)|k = 0, r \in i(-1/2,1/2)\},$$

and two such representations $\pi_{(k,r)}$, $\pi_{(l,s)}$ are equivalent iff $(k,r) = \pm(l,s)$. Furthermore, these are all the irreducible unitary representations of G other than the trivial representation. We choose a set $U' \subset U$ representing every equivalence class in U to be

$$U' = \{(k,r)|r \in (0,\infty)\} \cup \{(k,r)|r = 0, k \geq 0\} \cup \{(k,r)|k = 0, r \in i(0,1/2)\}.$$

Given $\pi \in \widehat{G}$ nontrivial, we shall say π has weight k and spectral parameter r if it is isomorphic to $\pi_{(k,r)}$ with $(k,r) \in U'$. It is proven in [13] (see also [1]) that if π is a cuspidal automorphic representation on X, then its archimedean component π_∞ is either tempered or spherical with spectral parameter $r \in i(0, 7/64]$.

For fixed k, all representations of weight k may be realised on the space

$$W = \{f \in L^2(K)|f(mg) = \chi_k(m)f(g), m \in M\}.$$

We let W_K denote the subspace of K-finite vectors. If $|k| \leq m$, $\pi_{(k,r)}$ will contain (ρ_m, V_m) (or (ρ_m^*, V_m^*), as they are isomorphic) as a subrepresentation with multiplicity one, and we shall work with the unitary embeddings $V_m \to W$ and $V_m^* \to W$ given by

$$v \mapsto (m+1)^{1/2}\langle \rho_m(k)v, v_k\rangle, \tag{3}$$

$$v^* \mapsto (m+1)^{1/2}\langle \rho_m^*(k)v^*, v_{-k}^*\rangle. \tag{4}$$

If $v \in W$ we shall often think of v as a vector in all representations $\pi_{(k,r)}$ of weight k simultaneously.

If π is an automorphic representation, we shall let $R_\pi : W \to L^2(X)$ be the associated unitary embedding. When working with various triple product formulas, it will be necessary to commute complex conjugation past the embedding R_π. To do this, we note that because the π we consider are automorphic representations on $GL_2(\mathbb{A})$ with trivial central character, they are isomorphic to their contragredients $\widetilde{\pi}$ (see for instance [3], Theorem 4.2.2). For each π, we may therefore define $\sigma : W \to W$ to be a conjugate linear isomorphism that satisfies $\overline{R_\pi(v)} = R_\pi(\sigma(v))$.

As we are only interested in the absolute values of various quantities in this paper, we will frequently ignore scalar factors of absolute value 1, and make statements such as "let $v \in \rho_m$ be the unique unit vector of weight w…".

2.4. Principal bundles. Recall that if (τ, V_τ) is a representation of K, the principal bundle $X \times_K V_\tau$ is the quotient of $X \times V_\tau$ by the right K-action

$$(x, v)k = (xk, \tau(k)^{-1}v), \tag{5}$$

so that sections of $X \times_K V_\tau$ are equivalent to sections of $X \times V_\tau$ satisfying

$$\tau(k)v(xk) = v(x).$$

Fix an m, let $(\rho, V_\rho) = (\rho_m, V_m)$, and let $(\sigma, V_\sigma) = (\rho \otimes \rho^*, V_\rho \otimes V_\rho^*) = \mathrm{End}(V_\rho)$. We define the bundles B and E on Y by

$$B = X \times_K V_\rho, \qquad E = \mathrm{End}(B) = X \times_K V_\sigma,$$

with Hermitian structures coming from the one on ρ. There is a natural identification of X with the orthonormal frame bundle of Y and of S^*Y with X/M, and so if $\pi : S^*Y \to Y$ is the projection this induces isomorphisms of $\pi^*(B)$ and $\pi^*(E)$ with $X \times_M V_\rho$ and $X \times_M V_\sigma$.

There is an equivalence between square integrable sections $s \in L^2(Y, B)$ of B and K-homomorphisms $V_\rho^* \to L^2(X)$ via the map

$$s \mapsto (v \mapsto (s(x), v)), \tag{6}$$

and so the decomposition of $L^2(X)$ as a direct integral of automorphic representations induces one of $\mathrm{Hom}_K(\rho^*, L^2(X))$ and $L^2(Y, B)$. Elements of $L^2(Y, B)$ corresponding to the discrete spectrum will be called automorphic sections, and these are the analogues of Laplace eigenfunctions for which our lift will be defined. In particular, we ignore the continuous spectrum of X; see [12, 16, 20] for results in that case. If $\pi \subset L^2(X)$ occurs discretely, the definition of the L^2 normalised section s associated to π by (6) may be unwound to give

$$s = \frac{1}{\sqrt{m+1}} \sum_{i=0}^{m} R_\pi(v_{m-2i}^*) v_{m-2i}, \tag{7}$$

where v_{m-2i}^* are embedded in W via (4).

3. Statement of results

Fix a non-negative even integer m, a representation $(\rho, V_\rho) = (\rho_m, V_m)$ with associated bundles B and E as above, and a weight k with $|k| \leq m$, and let $s \in L^2(Y, B)$ be the automorphic section of B associated to a representation π of weight k by (7). It can be shown that the microlocal lift of s defined by the analogue of equation (1) is a distribution on the space of symbols of pseudodifferential endomorphisms of B, which is $C_0^\infty(S^*Y, \pi^*(E))$ (see also the discussion of Section 2 of [4]). Note that we take our symbols to be compactly supported on account of the noncompactness of Y.

Bunke and Olbrich construct a distribution of this kind using the representation π, and prove that this representation theoretic lift coincides with the microlocal lift in the high energy limit. To state their definition, let δ be the infinite formal sum of K-types representing the delta distribution at the identity in W_K, so that $\langle f, \delta \rangle = f(e)$ for $f \in W_K$. For the rest of the paper, v_w will always denote the unit vector of weight w in V_ρ, and $R_\pi(v_w)$ will denote the composition of R_π with the embedding (3). We then have:

Definition 1: We define the distribution ν_s on $C_{0,K}^\infty(X, V_\sigma)$ by

$$\nu_s(a) = \int_X \langle a R_\pi(\delta) v_{-k}, s \rangle dg. \tag{8}$$

It follows from the transformation properties of $R_\pi(\delta)$ and s under M that ν_s descends naturally to a distribution on $C_{0,K}^\infty(S^*Y, \pi^*(E))$. Note that the K-finiteness of s and a means that we only need to consider finitely many terms in the formal sum of K-types defining δ in the integral (8), so that ν_s is well defined. We shall sometimes denote it by $s^* \otimes R_\pi(\delta) v_{-k}$, and use similar notation for other distributions of this form. The key properties of ν_s are summarised in the following proposition.

Proposition 2: *Let $\{s_n\}$ be a sequence of L^2 normalised automorphic sections of B with fixed weight k and spectral parameter tending to ∞. Then, after replacing $\{s_n\}$ by an appropriate subsequence and denoting ν_{s_n} by ν_n, there exist sections $\tilde{s}_n$ in $L^2(S^*Y, \pi^*(B))$ such that:*

(1) *The projection of ν_n to Y coincides with the element $s_n^* \otimes s_n$ of $C_0^\infty(Y, E)'$.*
(2) *For every $a \in C_{0,K}^\infty(S^*Y, \pi^*(E))$ we have $\lim_{n \to \infty} (\nu_n(a) - \tilde{s}_n^* \otimes \tilde{s}_n(a)) = 0$.*

(3) *Every weak-* limit of the states* $\tilde{s}_n^* \otimes \tilde{s}_n$ *is A-invariant.*

(4) $\langle Op(a)s_n, s_n\rangle = \nu_n(a) + o(1)$ *for all* $a \in C^\infty_{0,K}(S^*Y, \pi^*(E))$.

(5) *Let* $T \subset End_G(C^\infty(X, \rho))$ *be a* $\mathbb{C}$ *subalgebra of bounded automorphisms of* $C^\infty(X \times \rho)$ *commuting with the G action and with the right action of K on* $X \times \rho$. *Then each* $t \in T$ *induces an automorphism of* $C^\infty(Y, B)$, *and we may suppose that* s_n *is an eigenfunction of T. Then we may choose* $\tilde{s}_n$ *to be an eigenfunction with the same eigenvalues as* s_n.

Proof. This is a summary of the results of [4]; we indicate where the proof of each assertion may be found there. The distribution ν_n we have defined is denoted in [4] by $\sigma^\xi_{\psi_T,\delta_T}$. Assertion (1) is Lemma 3.15. Assertion (2) is proven in Proposition 3.10, and the sections $\tilde{s}_n$ are denoted $\xi(f_j\psi_T)$. Note that, a priori, $\xi(f_j\psi_T)$ lie in $L^2(X, V_\rho)$, but they will belong to $L^2(S^*Y, \pi^*(B))$ if the functions f_j are chosen to be invariant under M. Assertion (3) is proven in Section 4. Assertion (4) is proven in Proposition 3.12; the reason the statements are slightly different is that Bunke and Olbrich have passed to a subsequence for which ν_n are weakly converging to a limiting distribution denoted σ_T. Finally, assertion (5) is Theorem 3.14. ∎

We should briefly explain our use of the term 'state' in assertion (3) of the proposition. The distributions

$$a \mapsto \int_X \langle a\tilde{s}_n, \tilde{s}_n\rangle$$

extend to positive linear functionals on $C(S^*Y, \pi^*(E))$, the C^*-algebra of continuous endomorphisms of $\pi^*(B)$, and $\tilde{s}_n$ are normalised so that $\tilde{s}_n^* \otimes \tilde{s}_n$ evaluates to 1 on the identity section. These functionals play the role of probability measures in this context. We should also note that Proposition 2 is valid without any arithmeticity assumptions on either π_n or Γ.

In this paragraph, we allow G to be an arbitrary semisimple Lie group. Representation theoretic lifts such as that of Proposition 2 have already been constructed for functions on an arbitrary locally symmetric space $Y = \Gamma\backslash G/K$, which was first carried out by Zelditch [29, 31] for $G = SL(2,\mathbb{R})$, before being extended to $SL(2,\mathbb{R})^n$ by Lindenstrauss [14] and to arbitrary semisimple Lie groups by Silberman and Venkatesh [24]. These require the function to be an eigenfunction of the full ring of invariant differential operators on Y rather than just the Laplacian, and as a result produce lifts whose weak limits are invariant under a maximal $\mathbb{R}$-split torus of G rather than just the geodesic flow. They are most naturally thought of as distributions on $C_0^\infty(\Gamma\backslash G)$, but the standard lift may be recovered from them in the large eigenvalue limit via a correspondence between S^*Y and a K-principal bundle over $\Gamma\backslash G/K$.

3.1. A quantum unique ergodicity conjecture. The most natural question one may ask about the vector valued lifts we have defined is what their high energy limits should be, and simple heuristics lead us to conjecture the following answer in a generalisation of QUE for functions.

Conjecture 1: *Let ν be the state*

$$\nu(a) = \frac{1}{\mathrm{Vol}(X)} \int_X \langle a v_{-k}, v_{-k}\rangle dx.$$

If $\{s_n\}$ is a sequence of L^2 normalised automorphic sections of B of weight k with microlocal lifts ν_n, and $a \in C_{0,K}^\infty(S^*Y, \pi^*(E))$, then we have

$$\lim_{n\to\infty} \nu_n(a) - \nu(a) = 0.$$

To demonstrate why this should be true, let π_n be the representations associated to s_n, and denote R_{π_n} by R_n. If

$$a = \sum_{i,j=0}^{m} f(m-2i, m-2j) v_{m-2i} \otimes v_{m-2j}^*$$

is an element of $C_{0,K}^\infty(S^*Y, \pi^*(E))$, we shall calculate $\nu_n(a)$ using the formal expansion of δ. For $l \geq |k|$, let u_l be the unit vector of weight k in (ρ_l, V_l), which we embed in W via (3). Then

$$\delta = \sum_{l=|k|}^{\infty} (l+1)^{1/2} u_l,$$

and $\nu_n(a)$ may be evaluated by the following (finite) sum:

$$\nu_n(a) = (m+1)^{-1/2} \sum_{l=|k|}^{\infty} \sum_{i=0}^{m} \int_X R_n((l+1)^{1/2}u_l)\overline{R_n(v_{m-2i}^*)} f(m-2i, -k) dx.$$

If $v_{m-2i}^* = u_l \in W$, that is $l = m$ and $m - 2i = -k$, we expect

$$\begin{aligned} \lim_{n\to\infty}{}^* \; R_n((l+1)^{1/2}u_l)\overline{R_n(v_{m-2i}^*)}/(m+1)^{1/2} &= \lim_{n\to\infty}{}^* \; |R_n(u_m)|^2 \\ &= \|R_n(u_m)\|^2/\mathrm{Vol}(X) \\ &= 1/\mathrm{Vol}(X), \end{aligned}$$

while all other terms should be tending weakly to 0 (here $\lim^*$ denotes weak limit of functions on X). This would imply that

$$\lim_{n\to\infty} \nu_n(a) = \frac{1}{\mathrm{Vol}(X)}\langle f(-k,-k),1\rangle,$$

which is the assertion of Conjecture 1.

Note that Conjecture 1 may be interpreted as a statement about various differential geometric objects on Y. For instance, using the isomorphism $T^*Y \simeq X \times_K V_2$, one obtains a conjecture about the quantum limits of differential 1-forms on Y. In particular, because Laplace eigensections of $X \times_K V_2$ that are exact and coclosed are the same as automorphic sections with $k=0$ and $k=2$ respectively, one sees that these two kinds of 1-forms are expected to have different quantum limits.

3.2. Relations with subconvexity. Our main theorem provides support for Conjecture 1 by relating it to a subconvex bound for certain triple product L-functions in the eigenvalue aspect.

Theorem 3: *Let $\{s_n\}$ be a sequence of L^2 normalised automorphic sections of B, with associated representations π_n having spectral parameters r_n. If the asymptotics*

$$\frac{L(1/2,\pi_n\otimes\pi_n\otimes\pi')}{L(1,\mathrm{sym}^2\pi_n)^2} = o_{\pi'}(r_n^2) \tag{9}$$

and

$$\frac{|L(1/2+it,\pi_n\otimes\pi_n\otimes\chi_k)|}{L(1,\mathrm{sym}^2\pi_n)} = o_k(r_n t^A) \tag{10}$$

hold for all cuspidal automorphic representations π' of $PGL_2(\mathbb{A})$ that are unramified at all finite places, all characters χ_k, and some $A>0$, then Conjecture 1 is true for $\{s_n\}$. Moreover, if Conjecture 1 holds for $\{s_n\}$ then (9) holds for these π'.

As the analytic conductors of the L-functions in (9) and (10) behave like r_n^8 and r_n^4 with respect to r_n, and the factors $L(1,\mathrm{sym}^2\pi_n)$ are bounded above and below by powers of $\log r_n$ (see for instance [6, 8] and the generalisation to a number field in Section 2.9 of [2]), these asymptotics represent modest savings over the convexity bound. A consequence of this is that the GRH for the triple product L-function implies the equidistribution of ν_n at the optimal rate, which is $\nu_n(a) = O_\epsilon(r_n^{-1+\epsilon})$ for $a \in C^\infty_{0,K}(X, V_\sigma)$ of mean 0. It also illustrates that the phenomenon studied by Milićević [19] of base change forms becoming large at CM points of Y does not interfere with their weak equidistribution. See [9, 12, 16, 20, 23, 28] for other results on QUE on $SL_2(\mathbb{Z})\backslash\mathbb{H}^2$ and $SL_2(\mathcal{O})\backslash\mathbb{H}^3$ that are obtained via relations with special L-values.

4. Proof of Theorem 3: Global calculations

We may assume without loss of generality that the archimedean components of all the representations π_n are tempered, so that $r_n \in \mathbb{R}$. It suffices to deduce Conjecture 1 from the asymptotics (9) and (10) when a is of the form $a = \phi v_w \otimes v^*_{-k}$ with $\phi \in C^\infty_{0,K}(X)$. We define the coordinate distribution $\nu_{n,w}$ of ν_n by

$$\nu_{n,w}(f) = \int_X R_n(\delta)\overline{R_n(v_w^*)} f\, dx, \quad f \in C^\infty_{0,K}(X),$$

so that we have $\nu_{n,w}(\phi) = \sqrt{m+1}\nu_n(a)$. We define ν_w to be the corresponding coordinate distribution of $\sqrt{m+1}\nu$, i.e.,

$$\nu_w(f) = \delta_{w,-k}\frac{\sqrt{m+1}}{\mathrm{Vol}(X)}\langle f, 1\rangle.$$

Let $C_{00,K}(X)$ denote the space of continuous K-finite functions on X decaying in the cusp with the L^∞ norm. Let $\mathcal{R} \subset \widehat{K}$ be the set of K-types occurring in ϕ, and define $C_{00,\mathcal{R}}(X)$ and $C^\infty_{0,\mathcal{R}}(X)$ to be the subspaces of $C_{00,K}(X)$ and $C^\infty_{0,K}(X)$ respectively that contain only K-types in $\mathcal{R}$. If we replace δ with a sufficiently large truncation δ_N and replace $\overline{R_n(v_w^*)}$ with $R_n(v_w)$, we have

$$\nu_{n,w}(f) = \int_X R_n(\delta_N) R_n(v_w) f\, dx$$

for all $f \in C^\infty_{0,\mathcal{R}}(X)$, so that

$$|\nu_{n,w}(f)| \le \|f\|_\infty \|\delta_N\|_2 \|v_w\|_2.$$

This imples that $\nu_{n,w}$ extends uniquely to a continuous functional on $C_{00,\mathcal{R}}(X)$ whose norm is bounded independently of n, and so Conjecture 1 is equivalent to proving that

$$\lim_{n\to\infty} \nu_{n,w}(f) = \nu_w(f), \quad f \in C_{00,\mathcal{R}}(X), \tag{11}$$

for all fixed $\mathcal{R}$ and w. Moreover, as the span of the cusp forms and incomplete Eisenstein series is dense in $C_{00,\mathcal{R}}(X)$, it suffices to prove (11) when f is one of these automorphic forms, which we denote ϕ. In this section we shall use the triple product formulas of Ichino and Rankin–Selberg to express $\nu_{n,w}(\phi)$ in terms of central L-values and a local archimedian integral, and Theorem 3 will follow from this once we have established asymptotics for these local integrals in Section 5.

4.1. The cuspidal case. Let π' be cuspidal of weight k' and spectral parameter r', and let $\phi \in \pi'$ be K-finite. We shall evaluate $\nu_{n,w}(\phi)$ using a formula of Ichino [10], which we state below in the case under consideration in which all three automorphic forms have full level on the split group PGL_2.

Theorem 4: *Let π_i be three cuspidal automorphic representations on $\Gamma\backslash G$, and $\phi_i \in \pi_i$ be three L^2 normalised K-finite vectors. Then there is a constant C depending only on F such that*

$$\left| \int_X \phi_1\phi_2\phi_3 dx \right|^2 = C \int_G \langle \pi_1(g)\phi_1, \phi_1\rangle \langle \pi_2(g)\phi_2, \phi_2\rangle \langle \pi_3(g)\phi_3, \phi_3\rangle dg \times \frac{L(1/2, \pi_1 \otimes \pi_2 \otimes \pi_3)}{\prod L(1, \mathrm{sym}^2 \pi_i)}. \tag{12}$$

Note that Lemma 2.1 of [10] and the bounds of [13] imply that the archimedean integral appearing in (12) is always absolutely convergent. If we apply Ichino's formula to the integral

$$\nu_{n,w}(\phi) = \int_X R_n(\delta_N) R_n(v_w) \phi \, dx$$

we obtain

$$|\nu_{n,w}(\phi)|^2 = C\mathcal{S} \frac{L(1/2, \pi_n \otimes \pi_n \otimes \pi')}{L(1, \mathrm{sym}^2 \pi_n)^2 L(1, \mathrm{sym}^2 \pi')},$$

where $\mathcal{S}$ is defined by

$$\mathcal{S} = \int_G \langle \pi_n(g)\delta_N, \delta_N\rangle \langle \pi_n(g) v_w, v_w\rangle \langle \pi'(g)\phi, \phi\rangle dg.$$

To show that (9) implies $\nu_{n,w}(\phi) \to 0$, we must show that $\mathcal{S}$ satsfies $\mathcal{S} \ll r_n^{-2}$ when all other parameters are fixed, while to deduce (9) from $\nu_{n,w}(\phi) \to 0$ we need to show that for all k, k' and r' there is some choice of w and ϕ such that $\mathcal{S} \gg r_n^{-2}$. We prove both of these in Section 5. The gamma factors of $L(s, \pi_n \otimes \pi_n \otimes \pi')$ are

$$L_\infty(s, \pi_n \otimes \pi_n \otimes \pi') = \prod_\pm \Gamma\left(s \pm 2ir_n \pm ir' + \tfrac{|k|}{2} + \tfrac{|k'|}{4}\right) \times \prod_\pm \Gamma\left(s \pm ir' + \tfrac{|k'|}{4}\right)^2,$$

so that the analytic conductor behaves like r_n^8 in the eigenvalue aspect as remarked in Section 3.2

4.2. The Eisenstein case. We now consider the case of ϕ an incomplete Eisenstein series. Let k' be an integer divisible by $2|\mu|$, let $\chi = \chi_{k'/2}$, and let $\{f(s) \in C^\infty(G) | s \in \mathbb{C}\}$ be a family of functions satisfying

$$f(s)(na(y)g) = \chi(y)|y|^{2s} f(s)(g)$$

and whose restrictions to K are fixed and K-finite. We let $E_f(s,g)$ be the complete Eisenstein series associated to $f(s)$, which is defined by

$$E_f(s,g) = \sum_{\Gamma_\infty \backslash \Gamma} f(s)(\gamma g)$$

for $\mathrm{Re}(s) > 1$ and analytically continued to $\mathbb{C}$. Note that our divisibility assumption on k' is necessary to ensure that the functions $f(s)$ are invariant under Γ_∞, and our convention for k' means that the functions $f(1/2+it)$ lie in the principal series representation $\pi_{(k',t)}$, in order to agree with the parameters used in $\mathcal{S}$. Our assumption that the class number of F is one implies that X has only one cusp, and so the functions $E_f(1/2+it,g)$ provide the entire continuous spectrum of X.

To construct the incomplete Eisenstein series, let $h \in C_0^\infty(0,\infty)$ and let

$$H(s) = \int_0^\infty y^{s-1} h(y) dy$$

be its Mellin transform, and define

$$E_h(g) = \int_{(2)} H(s) E_f(s,g) ds.$$

We shall express $\nu_{n,w}(E_h(\cdot))$ in terms of the values of $L(s, \pi_n \otimes \pi_n \otimes \chi)$ on the critical line, and an archimedean integral $\mathcal{T} = \mathcal{T}(r_n, s)$ which we now define. Let $\mathcal{W}(\pi_n, \psi)$ be the Whittaker model of π_n with respect to ψ, which we equip with the inner product

$$\langle W_1, W_2 \rangle = \frac{1}{2\pi} \int_{\mathbb{C}^\times} W_1(a(y)) \overline{W_2(a(y))} dy^\times. \tag{13}$$

Fix a unitary isomorphism between π_n and $\mathcal{W}(\pi_n, \psi)$, and let W_1 be the function corresponding to δ_N. We likewise let $W_2 \in \mathcal{W}(\pi_n, \overline{\psi})$ be the L^2 normalised function corresponding to v_w. We define $\mathcal{T}$ to be

$$\mathcal{T}(r_n, s) = \int_0^\infty \int_K W_1(a(y)k)W_2(a(y)k)f(s)(a(y)k)y^{-2}dy^\times dk.$$

Our assumption that $r_n \in \mathbb{R}$, and the formulas of Proposition 8, imply that $\mathcal{T}(r_n, s)$ converges absolutely for $\mathrm{Re}(s) > 0$, and it has an analytic continuation to $\mathbb{C}$ by Theorem 18.1 of [11]. We then have

Proposition 5: *Suppose that the asymptotic* (10) *holds, and that*

$$|\mathcal{T}(r_n, 1/2 + it)| \ll r_n^{-1} t^A$$

when all other parameters are held fixed, for some $A > 0$ possibly depending on these parameters. Then

$$\lim_{n\to\infty} \nu_{n,w}(E_h(\cdot)) - \nu_w(E_h(\cdot)) = 0. \tag{14}$$

Proof. Because the functions $R_n(\delta_N)$ and $R_n(v_w)$ are rapidly decaying in the cusp of X and $H(s)$ decays rapidly in vertical strips, we may interchange the order of integration in $\nu_n(E_h(\cdot))$ to obtain

$$\nu_{n,w}(E_h(\cdot)) = \int_{(2)} H(s)\nu_{n,w}(E_f(s,\cdot))ds. \tag{15}$$

We shall evaluate the integral

$$\nu_{n,w}(E_f(s,\cdot)) = \int_X R_n(\delta_N)(g)R_n(v_w)(g)E_f(s,g)dg \tag{16}$$

for $\mathrm{Re}(s) = 2$ by unfolding. The Fourier expansions of $R_n(\delta_N)$ and $R_n(v_w)$ with respect to ψ and $\overline{\psi}$ are

$$R_n(\delta_N) = \sum_{\xi\in\mathcal{O}_{\neq 0}} a_\xi W_1(a(\xi\kappa)g), \tag{17}$$

$$R_n(v_w) = \sum_{\xi\in\mathcal{O}_{\neq 0}} a_\xi W_2(a(\xi\kappa)g). \tag{18}$$

Here, κ is a generator of the inverse different $\mathcal{O}^*$ of $\mathcal{O}$, and the Fourier coefficients a_ξ satisfy

$$a_\xi = a_1 N\xi^{-1/2}\lambda_n(\xi)$$

where λ_n are the automorphically normalised Hecke eigenvalues of π_n. Note that $\lambda_n(\xi)$ are all real, so that we do not have to take complex conjugates in

(18). The L^2 normalisations of R_n and W_i imply that

$$|a_1|^2 = \frac{4\pi}{|D|L(1, \mathrm{sym}^2\pi_n)},$$

where D is the discriminant of F.

Unfolding (16) gives

$$\begin{aligned}\nu_{n,w}(E_f(s,\cdot)) =& \int_{\Gamma_\infty\backslash G} R_n(\delta_N)(g)R_n(v_w)(g)f(s)(g)dg \\ =& \int_{\mathbb{C}/\mu\mathcal{O}}\int_0^\infty\int_K R_n(\delta_N)(n(x)a(y)k)R_n(v_w)(n(x)a(y)k) \\ & \times f(s)(a(y)k)y^{-2}dxdy^\times dk.\end{aligned}$$

Here, $\mathbb{C}/\mu\mathcal{O}$ denotes the quotient of $\mathbb{C}/\mathcal{O}$ by multiplication by the group of units. Substituting the Fourier expansions (17) and (18) gives

$$\begin{aligned}\nu_{n,w}(E_f(s,\cdot)) =& \frac{\sqrt{|D|}}{2|\mu|}\sum_{\xi\in\mathcal{O}_{\neq 0}}|a_\xi|^2 \\ & \times \int_0^\infty\int_K W_1(a(\xi\kappa y)k)W_2(a(\xi\kappa y)k)f(s)(a(y)k)y^{-2}dy^\times dk \\ =& |a_1|^2\frac{\sqrt{|D|}}{2|\mu|}N\kappa^{1-s}\chi(\kappa)^{-1}\sum_{\xi\in\mathcal{O}_{\neq 0}}N\xi^{-s}\chi(\xi)^{-1}\lambda_n(\xi)^2 \\ & \times \int_0^\infty\int_K W_1(a(y)k)W_2(a(y)k)f(s)(a(y)k)y^{-2}dy^\times dk \\ =& \frac{|a_1|^2}{2}N\kappa^{1/2-s}\chi(\kappa)^{-1}\frac{L(s,\pi_n\otimes\pi_n\otimes\chi^{-1})}{L(2s,\chi^{-2})}\mathcal{T}(r_n,s).\end{aligned}$$

Note that the factor of $|\mu|$ vanished because $\mathcal{O}_{\neq 0}$ counts every ideal with this multiplicity. Applying the normalisation of $|a_1|^2$, this becomes

$$\nu_{n,w}(E_f(s,\cdot)) = \frac{2\pi N\kappa^{3/2-s}\chi(\kappa)^{-1}}{L(2s,\chi^{-2})}\frac{L(s,\pi_n\otimes\pi_n\otimes\chi^{-1})}{L(1,\mathrm{sym}^2\pi_n)}\mathcal{T}(r_n,s).$$

We now substitute this into (15) and analytically continue to the line $\mathrm{Re}(s) = 1/2$. The only pole we may pass over is at $s = 1$, and the residue from this pole makes a contribution of

$$2\pi i\mathrm{Res}_{s=1}H(s)\nu_{n,w}(E_f(s,\cdot)) = 2\pi i H(1)\nu_{n,w}(\mathrm{Res}_{s=1}E_f(s,\cdot))$$

to the integral. By considering the spectral decomposition of $L^2(X)$ as the direct sum of the cusp forms, constant function, and integrals of unitary Eisenstein series, it follows that

$$2\pi i H(1)\mathrm{Res}_{s=1}E_f(s,\cdot) = \frac{1}{\mathrm{Vol}(X)}\langle E_f(\cdot),1\rangle,$$

and because $\nu_{n,w}$ and ν_w agree on constant functions this implies that the contribution from the pole is

$$\nu_w(\langle E_f(\cdot),1\rangle)/\mathrm{Vol}(X) = \nu_w(E_f(\cdot)).$$

We therefore have

$$\nu_{n,w}(E_h(\cdot)) - \nu_w(E_f(\cdot)) = \int_{(1/2)} H(s)\frac{2\pi N\kappa^{3/2-s}\chi(\kappa)^{-1}}{L(2s,\chi^{-2})}\frac{L(s,\pi_n\otimes\pi_n\otimes\chi^{-1})}{L(1,\mathrm{sym}^2\pi_n)}\mathcal{T}(r_n,s)ds.$$

If we assume the asymptotics (10) and $|\mathcal{T}(r_n,1/2+it)| \ll r_n^{-1}t^A$ for some $A>0$, the rapid decay of $H(1/2+it)$ and the moderate growth of the other factors in the integral imply that $\nu_{n,w}(E_h(\cdot)) - \nu_w(E_f(\cdot)) = o(1)$ as required. ∎

To deduce the asymptotic (14) from (10) it therefore suffices to prove that $|\mathcal{T}(r_n,t)| \ll r_n^{-1}t^A$, which is done in Section 5. The gamma factors of $L(s,\pi_n\otimes\pi_n\otimes\chi^{-1})$ are

$$L_\infty(s,\pi_n\otimes\pi_n\otimes\chi^{-1}) = \Gamma\Big(s+\tfrac{|k'|}{4}\Big)^2\prod_{\pm}\Gamma\Big(s\pm 2ir_n+\tfrac{|k|}{2}+\tfrac{|k'|}{4}\Big),$$

so that the analytic conductor behaves like r_n^4 in the eigenvalue aspect as remarked in Section 3.2.

5. Proof of Theorem 3: Local calculations

For the remainder of the paper, we will denote $\mathcal{T}(r_n,1/2+it)$ by $\mathcal{T}(r_n,t)$. We have shown that Theorem 3 is implied by the following proposition.

Proposition 6: *We have $\mathcal{S} \ll r_n^{-2}$ when all other parameters are held fixed, and for each k, k' and r' there exist w and ϕ such that $\mathcal{S} \gg r_n^{-2}$. In addition, we have $|\mathcal{T}(r_n,t)| \ll r_n^{-1}t^A$ when all other parameters are held fixed, for some A possibly depending on these parameters.*

We drop the subscript n for the remainder of the section. We shall prove Proposition 6 by applying a relation between $\mathcal{S}$ and $\mathcal{T}$ due to Michel and

Venkatesh, which reduces the problem to the calculation of a Whittaker integral similar to $\mathcal{T}$. We calculate this integral in Section 5.3, by applying the formulas of Jacquet [11] for the Whittaker functions of representations of G which we recall in Section 5.2. Note that the upper bounds we require may also be proven quite easily by applying stationary phase to $\mathcal{S}$.

5.1. An equivalence of integrals. There is a simple relation between $\mathcal{S}$ and $\mathcal{T}$ due to Michel and Venkatesh ([18], Lemma 3.4.2) which will be of great use to us. To state it, let $v_i \in \pi_i$ be three vectors in nontrivial irreducible unitary representations of G. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be the Whittaker models for π_1 and π_2 with respect to ψ and $\overline{\psi}$, and let $\mathcal{I}_3$ be the induced model of π_3. We equip $\mathcal{W}_i$ with the inner product (13), and $\mathcal{I}_3$ with the inner product

$$\langle f, f\rangle = \int_K |f(k)|^2 dk.$$

Fix unitary equivalences between π_i and their respective models, under which v_i correspond to W_1, W_2 and f_3. Michel and Venkatesh then prove

Proposition 7 (Lemma 3.4.2 of [18]): *We have*

$$\int_G \langle \pi_1(g)v_1, v_1\rangle\langle \pi_2(g)v_2, v_2\rangle\langle \pi_3(g)v_3, v_3\rangle dg = \frac{1}{8\pi}\left|\int_0^\infty \int_K W_1(a(y)k)W_2(a(y)k)f_3(a(y)k)y^{-2}dy^\times dk\right|^2 \tag{19}$$

whenever both integrals are absolutely convergent.

Note that the constant factor in (19) is different from that appearing in [18], because the measure dg chosen there is four times ours. Proposition 7 implies that if we choose the parameters r' and t appearing in $\mathcal{S}$ and $\mathcal{T}$ to be equal, and let $\phi \in \pi'$ correspond to the function $f \in \pi_{(k',t)}$, then we have $\mathcal{S} = |\mathcal{T}|^2/8\pi$. It therefore suffices to prove Proposition 6 for $\mathcal{S}$ alone, i.e. that $\mathcal{S}(r,r') \ll r^{-2}r'^A$ when all other parameters are fixed, and that for each k, k' and r' there exist w and ϕ such that $\mathcal{S}(r) \gg r^{-2}$.

5.2. Whittaker functions. We now give formulas for the Whittaker functions of representations of G. Let π be a nontrivial irreducible unitary representation of G with weight k and spectral parameter r, and let $\mathcal{W}(\pi,\psi)$ be the Whittaker model of π with respect to ψ. We change our convention for these parameters when π is not spherical, by requiring that $k \geq 0$ while r may be negative. In the remainder of the paper, y will always denote a positive real number.

PROPOSITION 8: *For $m \geq k$ even and $w \in \{-m, -m+2, \ldots, m\}$, let $v \in \pi$ be the unique normalised vector with K-type ρ_m and weight w. Let $\mathcal{A} \subset \mathbb{Z}^2$ be the set of (p, q) satisfying*

$$\begin{gathered} m \geq p \geq 0, \quad m \geq q \geq 0, \quad (m-k)/2 \geq p+q, \\ p \geq w/2 - k/2, \quad q \geq -w/2 - k/2. \end{gathered} \tag{20}$$

There exist constants $C_{p,q}$ depending on k, m, w, p, and q (but not r) such that the function $W \in \mathcal{W}(\pi, \psi)$ corresponding to v satisfies

$$W(a(y)) = \frac{y^{k/2+1}}{|\Gamma\left(1 + m/2 + 2ir\right)|} \sum_{(p,q) \in \mathcal{A}} C_{p,q} y^{p+q} K_{2ir+p-q-w/2}(4\pi y). \tag{21}$$

Moreover, when $k = m$ we have

$$W(a(y)) = \frac{(2\pi)^{k/2}(k+1)^{1/2} y^{k/2+1}}{|\Gamma(1 + k/2 + 2ir)|} \binom{k}{(k-w)/2}^{1/2} K_{2ir-w/2}(4\pi y). \tag{22}$$

Proof. Aside from the assertion about unitary normlisation, formula (21) follows from the results in Section 18 of [11], in particular by inverting the Mellin transform given there on p. 106. We therefore know that there are constants $C_{p,q}$ as described in the Proposition and a constant D depending only on π and m such that

$$W(a(y)) = D y^{k/2+1} \sum_{(p,q) \in \mathcal{A}} C_{p,q} y^{p+q} K_{2ir+p-q-w/2}(4\pi y). \tag{23}$$

We may calculate D by setting $w = m$ and considering the equation $\|W\| = 1$. In this case we have $\mathcal{A} = \{(m/2 - k/2, 0)\}$, and we may assume without loss of generality that

$$W(a(y)) = D y^{m/2+1} K_{2ir-k/2}(4\pi y).$$

We therefore have

$$\begin{aligned}\langle W, W\rangle =&\frac{1}{2\pi}\int_{\mathbb{C}^\times} |W(a(z))|^2 dz^\times\\ =&\int_0^\infty |W(a(y))|^2 dy^\times\\ =&D^2\int_0^\infty y^{m+2}|K_{2ir-k/2}(4\pi y)|^2 dy^\times\\ =&(4\pi)^{-m-2}D^2\int_0^\infty y^{m+2}K_{2ir-k/2}(y)K_{-2ir-k/2}(y)dy^\times.\end{aligned}$$

Applying the formula

$$\int_0^\infty y^\lambda K_\mu(y)K_\nu(y)dy = \frac{2^{\lambda-2}}{\Gamma(\lambda+1)}\prod_{\pm}\Gamma\left(\frac{1+\lambda\pm\mu\pm\nu}{2}\right) \tag{24}$$

from Gradshteyn and Ryzhik ([7], 6.576) gives

$$\langle W, W\rangle = (4\pi)^{-m-2}D^2\frac{2^{m-1}}{\Gamma(m+2)}\prod_{\pm}\Gamma(1+m/2\pm k/2)\Gamma\left(1+m/2\pm 2ir\right).$$

We may therefore choose $D = |\Gamma\left(1+m/2+2ir\right)|^{-1}$ and absorb all factors that depend only on k and m into $C_{p,q}$, which completes the proof of the first assertion.

For the second assertion, we observe that when $m = k$ we have $\mathcal{A} = \{(0,0)\}$ for all w, and the result follows from the explicit formula of [11] combined with a calculation of $\|W\|^2$ as above (see also the calculation in Section 5 of [21]). ∎

5.3. Computation in the eigenvalue aspect. We will establish Proposition 6 for $\mathcal{S}$, which we recall is given by

$$\mathcal{S} = \int_G \langle\pi_n(g)\delta_N, \delta_N\rangle\langle\pi_n(g)v_w, v_w\rangle\langle\pi'(g)\phi,\phi\rangle dg. \tag{25}$$

Let $\mathcal{I}$ be the induced model of π, and $f\in\mathcal{I}$ be the vector corresponding to δ_N, which is given by

$$f(na(y)k) = y^{1+2ir}\delta_N(k). \tag{26}$$

Let $W_1\in\mathcal{W}(\pi,\psi)$ be the function corresponding to v_w. As ϕ was arbitrary, we may replace ϕ with $\sigma(\phi)$ in (25), and let $W_2\in\mathcal{W}(\pi',\psi)$ be the function corresponding to ϕ so that $\overline{W_2}\in\mathcal{W}(\pi',\overline{\psi})$ is the function corresponding to $\sigma(\phi)$. By linearity, we may assume that $\phi\in\rho_{m'}$ and has weight w' under M.

If we define

$$\mathcal{T}' = \int_0^\infty \int_K W_1(a(y)k)\overline{W_2(a(y)k)} f(a(y)k) y^{-2} dy^\times dk,$$

then $\mathcal{T}'$ converges absolutely by the formulas of Proposition 8, and Proposition 7 implies that $\mathcal{S} = |\mathcal{T}'|^2/8\pi$. If we substitute the formula (26) into this integral, we obtain

$$\mathcal{T}' = \int_0^\infty \int_K W_1(a(y)k)\overline{W_2(a(y)k)} \delta_N(k) y^{-1+2ir} dy^\times dk.$$

For fixed y, consider $W_1(a(y)k)$, $\overline{W_2(a(y)k)}$ and $\delta_N(k)$ as elements in $C^\infty(K)$. They are eigenvectors under the right action of M with weights w, $-w'$ and k respectively, so for this integral to be nonzero we must have $w - w' + k = 0$. Now δ_N is also an eigenvector under the left M action of weight k, and its integral against functions $f \in C^\infty(K)$ of left weight $-k$ is $f(e)$. Because $W_1\overline{W_2}$ has weight $-k$ under the right action of M, all its components with weight other than $-k$ under the left action must vanish at the identity. Therefore the inner integral in K reduces to evaluation of the first two terms at the identity, and our formula simplifies to

$$\mathcal{T}' = \int_0^\infty W_1(a(y))\overline{W_2(a(y))} y^{-1+2ir} dy^\times.$$

We now substitute the formulas for W_1 and W_2 from Proposition 8. If we denote the sets of indices appearing in the Proposition by $\mathcal{A}$ and $\mathcal{B}$, we have

$$W_1(a(y)) = \frac{y^{k/2+1}}{|\Gamma(1+m/2+2ir)|} \sum_{(p,q)\in\mathcal{A}} C_{p,q} y^{p+q} K_{2ir+p-q-w/2}(4\pi y),$$

$$W_2(a(y)) = \frac{y^{k'/2+1}}{|\Gamma(1+m'/2+2ir')|} \sum_{(p',q')\in\mathcal{B}} C_{p',q'} y^{p'+q'} K_{2ir'+p'-q'-w'/2}(4\pi y).$$

We may ignore all factors in $\mathcal{T}'$ whose absolute value does not depend on r or r', and write $\propto$ to indicate that two quantities are proportional up to such a factor. Ignoring constant factors, the contribution to $\mathcal{T}'$ from a given p, q, p' and q' is

$$\frac{1}{|\Gamma(1+m/2+2ir)\Gamma(1+m'/2+2ir')|} \int_0^\infty y^{a+2ir} K_{b+2ir}(4\pi y) K_{c-2ir'}(4\pi y) dy^\times,$$

where

$$\begin{aligned} a =& 1 + (k+k')/2 + p + q + p' + q', \\ b =& p - q - w, \end{aligned}$$

and

$$c = p' - q' - w'.$$

Evaluating the integral using (24), this becomes (up to constant factors)

$$\frac{\prod_{\pm} \Gamma((a-b\pm c)/2 \mp ir')\Gamma((a+b\pm c)/2 \mp ir' + 2ir)}{\Gamma(a+2ir)|\Gamma(1+m/2+2ir)\Gamma(1+m'/2+2ir')|}.$$

We have assumed that $r \in \mathbb{R}$, and we first consider the case in which $r' \in \mathbb{R}$ also. We further assume without loss of generality that r, $r' \geq 0$. If $r' \geq r$, it suffices to prove that $\mathcal{T}(r, r') \ll r'^A$, and this follows immediately from Stirling's formula. We may therefore assume that $r > r'$. Stirling's formula gives the asymptotic

$$\begin{aligned} &\frac{\prod_{\pm} \Gamma((a-b\pm c)/2 \mp ir')\Gamma((a+b\pm c)/2 \mp ir' + 2ir)}{\Gamma(a+2ir)|\Gamma(1+m/2+2ir)\Gamma(1+m'/2+2ir')|} \\ &\qquad \sim r^{-m/2-a}(r')^{-m'/2+a-b-3/2}(2r+r')^{(a+b-c-1)/2}(2r-r')^{(a+b+c-1)/2}, \end{aligned}$$

and because $2r - r' > r$ this may be simplified to

$$\frac{\prod_{\pm} \Gamma((a-b\pm c)/2 \mp ir')\Gamma((a+b\pm c)/2 \mp ir' + 2ir)}{\Gamma(a+2ir)|\Gamma(1+m/2+2ir)\Gamma(1+m'/2+2ir')|} \sim r^{\sigma}(r')^{-m'/2+a-b-3/2}$$

where

$$\sigma = b - m/2 - 1 = p - q - w - m/2 - 1.$$

To establish the upper bound in Proposition 6 it therefore suffices to show that $\sigma \leq 1$. Adding the constraints $(m-k)/2 \geq p+q$ and $q \geq -w/2 - k/2$ from (20) gives $m/2 + w/2 \geq p$, so

$$p - q - w/2 - m/2 - 1 \leq m/2 + w/2 - w/2 - m/2 - 1 = -1 \tag{27}$$

and $\sigma \leq -1$ as required.

In the case when $r' \in i(0, 1/2)$, we continue to have the asymptotic

$$\frac{\prod_{\pm} \Gamma((a-b\pm c)/2 \mp ir')\Gamma((a+b\pm c)/2 \mp ir' + 2ir)}{\Gamma(a+2ir)|\Gamma(1+m/2+2ir)\Gamma(1+m'/2+2ir')|} \sim r^{\sigma},$$

and the result again follows from the bound $\sigma \leq -1$.

To establish the lower bound, we begin by determining those p and q for which equality can hold in (27). We must have $q = 0$ and $p = m/2 + w/2$, so that the third and fifth inequalities of (20) become

$$\begin{aligned}(m-k)/2 \geq& m/2 + w/2,\\ -k \geq& w,\end{aligned}$$

and

$$\begin{aligned}0 \geq& -w/2 - k/2,\\ w \geq& -k.\end{aligned}$$

Therefore we have $\sigma = 1$ if and only if $w = -k$, $p = (m-k)/2$ and $q = 0$. Let $w = -k$, and let $\phi \in \pi'$ be the vector of K-type $\rho_{k'}$ and weight $w' = 0$, so that $\mathcal{B} = \{(0,0)\}$. We have shown that all terms other than $(p,q) = ((m-k)/2, 0) \in \mathcal{A}$ make a contribution of $O(r^{-2})$ to $\mathcal{T}'$, while $((m-k)/2, 0)$ makes a contribution asymptotic to r^{-1}. Therefore $\mathcal{T}' \gg r^{-1}$ for this choice of w and ϕ, which completes the proof of Proposition 6.

5.4. Computation in the weight aspect. We finish this section by computing two triple product integrals which we will need for a paper on QUE in the weight aspect [17]. Let π be the representation of G with weight $k \geq 0$ and spectral parameter 0, and $v_{\pm k} \in \pi$ be the two unit vectors of K-type ρ_k and weight $\pm k$. Let π' be the spherical representation with spectral parameter r', and let $u \in \pi'$ be the unit K-fixed vector. The first calculation we require is the integral

$$\mathcal{S}_1 = \int_G \langle \pi(g) v_k, v_k \rangle \langle \pi(g) v_{-k}, v_{-k} \rangle \langle \pi'(g) u, u \rangle dg. \tag{28}$$

Proposition 9: *We have*

$$\mathcal{S}_1 = \frac{|\Gamma((1+k)/2 + ir')\Gamma(1/2 + ir')|^4}{2^{11}\pi^3 \Gamma(1+k/2)^4 |\Gamma(1+2ir')|^2}.$$

Proof. We transfer v_k to the function f in the induced model of π, and let W_1 and W_2 be the images of u and v_k in $\mathcal{W}(\pi', \psi)$ and $\mathcal{W}(\pi, \psi)$ respectively, so that $\mathcal{S}_1$ is determined by the Whittaker integral $\mathcal{T}_1$ given by

$$\mathcal{T}_1 = \int_0^\infty \int_K W_1(a(y)k) \overline{W_2(a(y)k)} f(a(y)k) y^{-2} dy^\times dk.$$

(Note that we are conjugating W_2 and replacing v_{-k} with $\sigma(v_{-k}) = v_k$ as before.) f is given by

$$f(a(y)k) = (k+1)^{1/2} y \langle \rho(k) v_k, v_k \rangle,$$

and by (22) of Proposition 8, W_1 satisfies

$$W_1(a(y)k) = |\Gamma(1+2ir')|^{-1} y K_{2ir'}(4\pi y).$$

Moreover, using (22) for all $v_i \in \rho_k$ and the K-covariance of the Whittaker embedding, it follows that

$$W_2(a(y)k) = \frac{(2\pi)^{k/2}(k+1)^{1/2}}{\Gamma(1+k/2)} \langle \rho(k) v_k, \mathbf{W}_2(y) \rangle, \tag{29}$$

where

$$\mathbf{W}_2(y) = y^{k/2+1} \sum_{i=0}^{k} \binom{k}{i}^{1/2} K_{k/2-i}(4\pi y) v_{k-2i}. \tag{30}$$

Substituting these into $\mathcal{T}_1$ gives

$$\begin{aligned}\mathcal{T}_1 =& \frac{(2\pi)^{k/2}(k+1)}{\Gamma(1+k/2)|\Gamma(1+2ir')|} \\ &\times \int_0^\infty \int_K y K_{2ir'}(4\pi y) \overline{\langle \rho(k) v_k, \mathbf{W}_2(y) \rangle} y \langle \rho(k) v_k, v_k \rangle y^{-2} dy^\times dk.\end{aligned}$$

We may perform the integral over K using the inner product formula for matrix coefficients, which gives

$$\begin{aligned}\mathcal{T}_1 =& \frac{(2\pi)^{k/2}}{\Gamma(1+k/2)|\Gamma(1+2ir')|} \int_0^\infty K_{2ir'}(4\pi y) \overline{\langle v_k, \mathbf{W}_2(y) \rangle} dy^\times \\ =& \frac{(2\pi)^{k/2}}{\Gamma(1+k/2)|\Gamma(1+2ir')|} \int_0^\infty y^{k/2+1} K_{2ir'}(4\pi y) K_{k/2}(4\pi y) dy^\times \\ =& \frac{(2\pi)^{k/2}}{\Gamma(1+k/2)|\Gamma(1+2ir')|} (4\pi)^{-k/2-1} \frac{2^{k/2-2}}{\Gamma(1+k/2)} \prod_{\pm} \Gamma\left(\tfrac{1+k/2 \pm k/2 \pm 2ir'}{2}\right) \\ =& \frac{|\Gamma((1+k)/2+ir')\Gamma(1/2+ir')|^2}{16\pi\Gamma(1+k/2)^2|\Gamma(1+2ir')|}.\end{aligned}$$

Applying the relation $\mathcal{S}_1 = |\mathcal{T}_1|^2/8\pi$ completes the proof. ∎

If $\mathbf{W}_2(y)$ is as in (30), the second calculation we require is the integral

$$\mathcal{T}_2 = \frac{(2\pi)^k}{\Gamma(k/2+1)} \int_0^\infty y^{1+2ir'} \|\mathbf{W}_2(y)\|^2 y^{-2} dy^\times.$$

Note that $\mathbf{W}_2(y)$ is the same as the function denoted $\mathbf{K}_j(y)$ in [17].

PROPOSITION 10: *We have*

$$|\mathcal{T}_2| = \frac{|\Gamma\left((1+k)/2+ir'\right)\Gamma\left(1/2+ir'\right)|^2}{16\pi\Gamma(1+k/2)^2|\Gamma(1+2ir')|}.$$

Remark: Applying Proposition 10 gives a formula in disagreement with that stated in Section 8.1 of [17]. The formula given there, and the gamma factors at complex places given in Section 8.2, are incorrect; the constants $\beta(m,j)$ should be $\beta(m,j)/2$.

Proof. We first apply the identity

$$\|\mathbf{W}_2(y)\|^2 = (k+1)\int_K |\langle \rho(k)v_k, \mathbf{W}_2(y)\rangle|^2 dk,$$

which gives

$$\mathcal{T}_2 = \frac{(2\pi)^k(k+1)}{\Gamma(k/2+1)}\int_0^\infty \int_K y^{1+2ir'}|\langle \rho(k)v_k, \mathbf{W}_2(y)\rangle|^2 y^{-2}dy^\times dk.$$

If $W_2 \in \mathcal{W}(\pi,\psi)$ corresponds to v_k as in Proposition 9, then by (29) we have

$$\mathcal{T}_2 = \int_0^\infty \int_K y^{1+2ir'} W_2(a(y)k)\overline{W_2(a(y)k)}y^{-2}dy^\times dk.$$

The three vectors occurring here have norm one in their respective models, and when we convert the integral to its matrix coefficient form we obtain the integral (28). Therefore the absolute value of $\mathcal{T}_2$ is the same as that of $\mathcal{T}_1$, and the result follows from Proposition 9. ∎

References

[1] V. Blomer and F. Brumley, *On the Ramanujan conjecture over number fields*, Annals of Mathematics **174** (2011), 581–605.

[2] V. Blomer and G. Harcos, *Twisted L-functions over number fields and Hilbert's eleventh problem*, Geometric and Functional Analysis **20** (2010), 1–52.

[3] D. Bump, *Automorphic Forms and Representations*, Cambridge University Press, Cambridge, 1997.

[4] U. Bunke and M. Olbrich, *Quantum unique ergodicity for vector bundles*, Acta Applicandae Mathematicae **90** (2006), 19–41.

[5] Y. Colin de Verdière, *Ergodicité et fonctions propres du laplacien*, Communications in Mathematical Physics **102** (1985), 497–502.

[6] D. Goldfeld, J. Hoffstein and D. Lieman, *Appendix: An effective zero-free region*, Annals of Mathematics **140** (1994), 177–181.

[7] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, New York, 2000.

[8] J. Hoffstein and P. Lockhart, *Coefficients of Maass forms and the Siegel zero*, Annals of Mathematics **140** (1994), 161–181.

[9] R. Holowinsky and K. Soundararajan, *Mass equidistribution for Hecke eigenforms*, Annals of Mathematics **172** (2010), 1517–1528.

[10] A. Ichino, *Trilinear forms and the central values of triple product L-functions*, Duke Mathematical Journal **145** (2008), 281–307.

[11] H. Jacquet, *Automorphic Forms on GL_2 Part II*, Lecture Notes in Mathematics, Vol. 278, Springer-Verlag, Berlin, 1972.

[12] D. Jakobson, *Quantum unique ergodicity for Eisenstein series on $PSL(2,\mathbb{Z})\backslash PSL(2,\mathbb{R})$*, Annales de l'Institut Fourier **44** (1994), 1477–1504.

[13] H. Kim, *Functoriality for the exterior square of GL_4 and symmetric fourth of GL_2*, with appendix 1 by Dinikar Ramakrishnan and appendix 2 by Henry Kim and Peter Sarnak, Journal of the American Mathematical Society **16** (2003), 139–183.

[14] E. Lindenstrauss, *On quantum unique ergodicity for $\Gamma\backslash\mathbb{H}\times\mathbb{H}$*, International Mathematics Research Notices **17** (2001), 913–933.

[15] E. Lindenstrauss, *Invariant measures and arithmetic quantum unique ergodicity*, Annals of Mathematics **163** (2006), 705–741.

[16] W. Luo and P. Sarnak, *Quantum ergodicity of eigenfunctions on $PSL(2,\mathbb{Z})\backslash\mathbb{H}^2$*, Institut des Hautes Études Scientifiques. Publications Mathématiques **81** (1995), 207–237.

[17] S. Marshall, *Mass equidistribution for automorphic forms of cohomological type on GL_2*, Journal of the American Mathematical Society **24** (2011), 1051–1103.

[18] P. Michel and A. Venkatesh, *The subconvexity problem for GL_2*, Institut des Hautes Études Scientifiques. Publications Mathématiques **111** (2010), 171–271.

[19] D. Milićević, *Large values of eigenfunctions on arithmetic hyperbolic 3-manifolds*, Duke Mathematical Journal **155** (2010), 365–401.

[20] Y. Petridis and P. Sarnak, *Quantum unique ergodicity for $SL_2(\mathcal{O})\backslash\mathbb{H}^3$ and estimates for L-functions*, Journal of Evolution Equations **1** (2001), 277–290.

[21] A. Popa, *Whittaker newforms for archimedean representations of $GL(2)$*, Journal of Number Theory **128** (2008), 1637–1645.

[22] Z. Rudnick and P. Sarnak, *The behaviour of eigenstates of arithmetic hyperbolic manifolds*, Communications in Mathematical Physics **161** (1994), 195–213.

[23] P. Sarnak, *Estimates for Rankin-Selberg L-functions and quantum unique ergodicity*, Journal of Functional Analysis **184** (2001), 419–453.

[24] L. Silberman and A. Venkatesh, *Quantum unique ergodicity on locally symmetric spaces*, Geometric and Functional Analysis **17** (2007), 960–998.

[25] L. Silberman and A. Venkatesh, *Entropy bounds for Hecke eigenfunctions on division algebras*, preprint.

[26] A. I. Šnirel'man, *Ergodic properties of eigenfunctions*, Akademija Nauk SSSR i Moskovskoe Matematičeskoe Obščestvo. Uspehi Matematičeskih Nauk **29** (1974), 181–182.

[27] K. Soundararajan, *Quantum unique ergodicity for $SL(2,\mathbb{Z})\backslash\mathbb{H}$*, Annals of Mathematics **172** (2010), 1529–1538.

[28] T. Watson, *Rankin triple products and quantum chaos*, Ph.D. Thesis, Princeton University (2001) (eprint available at http://www.math.princeton.edu/~tcwatson/ watson_thesis_final.pdf).

[29] S. Zelditch, *Pseudodifferential analysis on hyperbolic surfaces*, Journal of Functional Analysis **68** (1986), 72–105.

[30] S. Zelditch, *Uniform distribution of eigenfunctions on compact hyperbolic surfaces*, Duke Mathematical Journal **55** (1987), 919–941.

[31] S. Zelditch, *The averaging method and ergodic theory for pseudo-differential operators*, Journal of Functional Analysis **82** (1989), 38–68.