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Zero repulsion in families of elliptic curve L-functions and an observation of Miller

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Abstract

We provide a theoretical explanation for an observation of S. J. Miller that if $L(s, E)$ is an elliptic curve L-function for which $L(1/2, E) \neq 0$, then the lowest lying zero of $L(s, E)$ exhibits a repulsion from the critical point which is not explained by the standard Katz-Sarnak heuristics. We establish a similar result in the case of first-order vanishing.

1. *Introduction*

In the paper [**12**], S. J. Miller investigated the statistics of the zeros of various families of elliptic curve L-functions. His key observation is illustrated in Figure 2 below. It shows a histogram of the first zero above the central point for rank zero elliptic curve L-functions generated by randomly selecting the coefficients c_1 up to c_6 in the defining equation

$$y^2 + c_1xy + c_3y = x^3 + c_2x^2 + c_4x + c_6$$

for curves with conductors in the ranges indicated in the caption. The zeros are scaled by the mean density of low zeros of the L-functions and the plots are normalized so that they represent the probability density function for the first zero of L-functions from this family. Miller observes that there is clear repulsion of the first zero from the central point; that is, the plots drop to zero at the origin, indicating a very low probability of finding an L-function with a low first zero. This is quite surprising, as one would expect to be able to model the distribution of the lowest zeros by the eigenvalue distribution of matrices in $SO(2N)$, with N chosen to be equal to half the logarithm of the conductor of the curve. However, Figure 1 illustrates that the eigenvalue of a random $SO(2N)$ matrix that is closest to 1 on the unit circle (which should model the lowest zero of the L-function) exhibits no repulsion from the origin - a fact well known in random matrix theory.

In this paper, we prove a result which provides theoretical support for Miller's observation. Rather than considering algebraic families of elliptic curves, we consider quadratic twists of a fixed curve $E/\mathbb{Q}$. Let $d \neq 1$ be a fundamental discriminant which is relatively prime to the conductor M of E, and let χ_d be the associated quadratic character of modulus $|d|$. We may form the L-function $L(s, E \times \chi_d)$ of E twisted by χ_d, and the corresponding completed L-function

$$\Lambda(s, E \times \chi_d) = \Gamma(s + 1/2) \left(\frac{|d|\sqrt{M}}{2\pi} \right)^s L(s, E \times \chi_d) \tag{1.1}$$

which satisfies the functional equation

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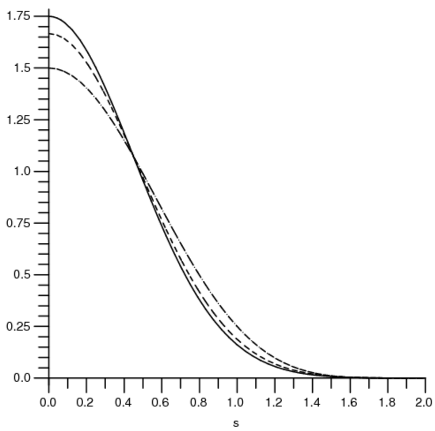

FIGURE 1. *Probability density of normalized eigenvalue closest to 1 for SO(8) (solid), SO(6) (dashed) and SO(4) (dot-dashed).*

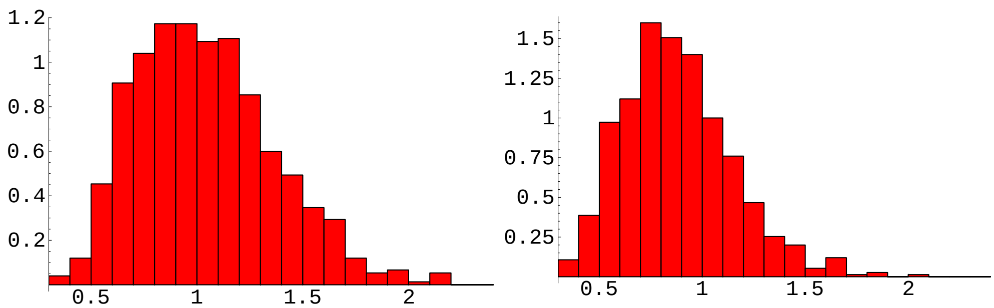

FIGURE 2. *First normalized zero above the central point: Left:* 750 *rank 0 curves from* $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, $\log(\text{cond}) \in [3.2, 12.6]$. *Right:* 750 *rank 0 curves from* $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, $\log(\text{cond}) \in [12.6, 14.9]$.

$$\Lambda(s, E \times \chi_d) = \chi_d(-M)\omega(E)\Lambda(1 - s, E \times \chi_d).$$

Here, $\omega(E) = \pm 1$ is a constant depending on E. We shall refer to $\chi_d(-M)\omega(E) = \pm 1$ as the sign of the functional equation of $L(s, E \times \chi_d)$, and say that the L-function is even or odd depending on whether this sign is $+1$ or -1. Note that the parity of the order of vanishing of $L(s, E \times \chi_d)$ at $s = 1/2$ is determined by the sign of its functional equation. We shall consider the families $\mathcal{F}^+(E)$ and $\mathcal{F}^-(E)$ of all the L-functions $L(s, E \times \chi_d)$ which are even and odd respectively, and define $\mathcal{D}^+(E)$ and $\mathcal{D}^-(E)$ to be the corresponding sets of fundamental discriminants. In addition, we define the subfamilies $\mathcal{F}^0(E) \subset \mathcal{F}^+(E)$ and $\mathcal{F}^1(E) \subset \mathcal{F}^-(E)$ by

$$\mathcal{F}^i(E) = \{L(s, E \times \chi_d) \mid \text{ord}_{s=1/2} L(s, E \times \chi_d) = i\}, \quad i = 0, 1,$$

together with the associated sets of discriminants

$$\mathcal{D}^i(E) = \{d \mid \text{ord}_{s=1/2} L(s, E \times \chi_d) = i\}.$$

For any d, we define $\gamma_{0,d}$ to be the height of the lowest non-real zero of $L(s, E \times \chi_d)$. Our main result is the following.

Theorem 1. *Assume that all the L-functions $L(s, E \times \chi_d)$ satisfy the Riemann hypothesis. For all $d \in \mathcal{D}^0(E)$, we have*

$$\frac{\log |\gamma_{0,d}|}{\log |d|} \geq -1/4 + \left(\frac{\log 2}{2} + o(1) \right) \frac{1}{\log \log |d|}. \tag{1.2}$$

In the odd case, there exists a nonzero integer d_0 satisfying $(d_0, 2M) = 1$ such that if we define

$$\mathcal{D}^-_{d_0}(E) = \{ d \mid (d, 2Md_0) = 1, d_0 d < 0, d_0 d \text{ is square mod } 4M \}$$

then $\mathcal{D}^-_{d_0}(E) \subseteq \mathcal{D}^-(E)$, and for all $d \in \mathcal{D}^1(E) \cap \mathcal{D}^-_{d_0}(E)$ we have

$$\frac{\log |\gamma_{0,d}|}{\log |d|} \geq -1/4 + \left(\frac{\log 2}{2} + o(1) \right) \frac{1}{\log \log |d|}.$$

1.1. *Relation to the Katz-Sarnak heuristics*

We shall first discuss the relationship between Theorem 1 and the Katz-Sarnak heuristics in the simpler case of even functional equation. While the zero repulsion of Theorem 1 is only on a scale of $d^{-1/4+o(1)}$, which is very small in comparison with the mean zero spacing of $\sim (\log |d|)^{-1}$, the fact that it holds for all members of the (conjecturally) large family $\mathcal{F}^0(E)$ means that one can infer properties of the distribution of rescaled lowest zeros from it under some natural assumptions. Let $D > 0$ be given, and let $\mathcal{L}_D$ be the multiset of rescaled lowest zeros,

$$\mathcal{L}_D = \{ \gamma_{0,d} \log |d| \mid d \in \mathcal{D}^0(E), D/2 \leq |d| \leq D \}.$$

Suppose that the set $\mathcal{L}_D$ has a limiting distribution of the form $\rho(x)dx$ as $D \to \infty$, where $\rho(x)$ is a smooth density on $[0, \infty)$. If $\rho(x)$ vanishes to order r at the origin and we make the natural assumption that $|\mathcal{L}_D| \gg D$ as $D \to \infty$, then we have

$$P(\exists\, x \in \mathcal{L}_D | x \leq D^{-1/(r+1)}) \geq \delta > 0$$

for D sufficiently large. Therefore, if $\mathcal{L}_D$ has a limiting distribution of the type described for which $r \leq 2$, we obtain a contradiction to Theorem 1. This is evidence that any limiting distribution of $\mathcal{L}_D$ has to vanish to order at least three at the origin, in agreement with Miller's data. Moreover, if one believes the 'minimalist conjecture' that $\mathcal{F}^0(E)$ makes up a density one subset of $\mathcal{F}^+(E)$, then our result shows that the lowest zeros of the family $\mathcal{F}^+(E)$ do not obey the standard Katz-Sarnak heuristics. For reasons which will become apparent in the course of the proof of Theorem 1, we believe that any such discrepancy should be viewed as a consequence of the special value formula

$$L(1/2, E \times \chi_d) = \kappa_E \frac{c_E(|d|)^2}{|d|^{1/2}}, \quad c_E(|d|) \in \mathbb{Z}, \tag{1.3}$$

of Waldspurger [**15**], Kohnen-Zagier [**11**], and Baruch-Mao [**2**].

The tension between Miller's data and the standard $SO(2N)$ model for even twists of a fixed elliptic curve has also been considered by Dueñez, Huynh, Keating, Miller and Snaith in [**6**]. They propose a modification to the $SO(2N)$ model which they term an *excised orthogonal ensemble*, and which exhibits the observed repulsion from the critical point. Their excised ensemble is the subset of $SO(2N)$ consisting of matrices g whose characteristic polynomial $P(g,x)$ is not small at 1, which reflects the inequality

$$L(1/2, E \times \chi_d) \gg_E |d|^{-1/2}, \quad d \in \mathcal{D}^0(E) \tag{1.4}$$

that follows from the special value formula (1.3). We note that this gap is also the basis of our proof of Theorem 1 in the even case. The natural choice of cut-off in the excised ensemble of [**6**] is $|P(g,1)| \geq C|d|^{-1/2}$ for some constant C, and with this choice the authors prove that their probability distribution, denoted $R_1^{T\mathcal{X}}(\theta)d\theta$, exhibits a hard gap around the origin similar to that of Theorem 1. More precisely, they show that $R_1^{T\mathcal{X}}(\theta) = 0$ for $|\theta| \ll d^{-1/4+\delta}$ (Theorem 1.3, [**6**]), where $\delta > 0$ depends on fitted parameters and may be made arbitrarily small. Moreover, it seems likely that they would have vanishing of $R_1^{T\mathcal{X}}(\theta)$ for $|\theta| \ll |d|^{-1/4+o(1)}$ under some equidistribution assumption on the zeros of the matrices in their ensemble. This is analogous to the way that obtaining an exponent of $-1/4$ in Theorem 1 requires controlling the bulk of the zeros of $L(s, E \times \chi_d)$.

In the case of odd functional equation, we now define $\mathcal{L}_D$ to be

$$\mathcal{L}_D = \{\gamma_{0,d} \log |d| \mid d \in \mathcal{D}^1(E) \cap \mathcal{D}^-_{d_0}(E), D/2 \leq |d| \leq D\}.$$

The minimalist conjecture would again imply that $\mathcal{D}^1(E) \cap \mathcal{D}^-_{d_0}(E)$ is a density one subsequence of $\mathcal{D}^-_{d_0}(E)$, and because this has $\gg D$ elements of size at most D, we may again combine the odd case of Theorem 1 with the above argument to deduce that any smooth limiting distribution of $\mathcal{L}_D$ must vanish to order at least three. Applying the minimalist conjecture once more, we obtain the same result for the lowest zeros in the family $\mathcal{F}^-_{d_0}$ corresponding to $\mathcal{D}^-_{d_0}$. The Katz-Sarnak heuristics predict that these zeros should have the same distribution as the first nontrivial eigenvalue of a random matrix in $SO(2N+1)$, but this distribution only vanishes to second order at the origin (see pages 10-11 and 411-416 of [**9**], or page 10 of [**10**]). Similarly to the even case, we shall see that this discrepancy may be viewed as a consequence of the formula of Gross-Zagier [**7**] for the central derivative of $L(s, E \times \chi_d)$, together with a lower bound for the height of a nontorsion point on an elliptic curve as in Anderson-Masser [**1**].

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2. *Proof of Theorem 1 in the even case*

To prove Theorem 1 in the even case, we must find a relation between the size of $L(1/2, E \times \chi_d)$ and the distribution of its zeros near the critical point. There are number of ways in which this can be done. The simplest is to observe that the completed L-function $\Lambda(s, E \times \chi_d)$ has vanishing central derivative, and (after suitable normalisation) its central value satisfies (1.4) while its second derivative on the critical line is $\ll |d|^\epsilon$ by Lindelöf. This would establish Theorem 1 with an error term of $o(1)$. One may also apply Jensen's formula to $L(s, E \times \chi_d)$ in a ball around the critical point.

The approach we shall take combines the Hadamard factorization formula with the explicit formula for $L(s, E \times \chi_d)$, and is adopted from a paper of Chandee and Soundararajan [**5**] where it is used to bound $|\zeta(\frac{1}{2} + it)|$ on the Riemann hypothesis (and in a sense optimally). We are grateful to K. Soundararajan for making us aware of this technique, and describing how to adapt it to the problem at hand. We shall prove the following result.

Proposition 2. *Suppose that $L(s, E \times \chi_d)$ satisfies the Riemann hypothesis. Then there is a constant $C > 0$ such that, for $|t| \leq 1$, we have*

$$|L(\tfrac{1}{2} + it, E \times \chi_d)| \ll \left(\prod_{|t-\gamma_d| \leq 1/(5 \log\log |d|)} (C|t-\gamma_d| \log\log |d|) \right) \exp\left(\left(\frac{\log 2}{2} + o(1) \right) \frac{\log |d|}{\log\log |d|} \right).$$

Note that Proposition 2 makes no assumption on the sign of the functional equation. Theorem 1 follows from Propositon 2 in the even case by setting $t = 0$ and discarding all but the smallest pair of zeros in the product. As in [**5**], the constant $(\log 2)/2$ is the best possible that may be obtained using the method we present, though of course this is not crucial for our application to explaining zero repulsion.

Proof.

Let $\Lambda(s, E \times \chi_d)$ be the completed L-function associated to $L(s, E \times \chi_d)$, which was defined in (1.1). Recall (see for instance Theorem 5.6, Chapter 5 of [**8**]) Hadamard's factorization formula

$$\Lambda(s, E \times \chi_d) = e^{A+Bs} \prod_\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

where ρ runs over the non-trivial zeros of $\Lambda(s, E \times \chi_d)$, and $\mathrm{Re}(B) = -\sum_\rho \mathrm{Re}(1/\rho)$. (Note that $\mathrm{Re}(1/\rho)$ is positive and $\sum_\rho \mathrm{Re}(1/\rho)$ converges, and we know that B is real as our L-function is self dual.) If we substitute $s = \frac{1}{2} + it$ and $s = -\frac{3}{2} + it$ and take the quotient of the two equations, we obtain

$$\left| \frac{\Lambda(\frac{1}{2} + it, E \times \chi_d)}{\Lambda(-\frac{3}{2} + it, E \times \chi_d)} \right| = e^{2B} \prod_\rho \left| \frac{i(\gamma_d - t)}{2 + i(\gamma_d - t)} \right| e^{\mathrm{Re}(2/\rho)} = \prod_\rho \left| \frac{(\gamma_d - t)^2}{4 + (\gamma_d - t)^2} \right|.$$

Since $\Lambda(-\frac{3}{2} + it, E \times \chi_d) = \pm\Lambda(\frac{5}{2} - it, E \times \chi_d)$, and $\Lambda(\frac{1}{2} + it, E \times \chi_d) \sim |d|^{5/2}$, we deduce that

$$\log \left| L(\tfrac{1}{2} + it, E \times \chi_d) \right| = 2 \log |d| + O(1) - \tfrac{1}{2} \sum_{\gamma_d} f(t - \gamma_d), \tag{2.1}$$

where we have set

$$f(x) = \log \frac{4 + x^2}{x^2}.$$

The method of Chandee and Soundararajan now proceeds by finding a minorant for f whose Fourier transform is compactly supported in a suitable interval $[-\Delta, \Delta]$. We shall denote it by $g_\Delta(x)$ as in their paper. Its properties are summarised in the following proposition, which is Proposition 2.1 of [**5**]

Proposition 3. *Let Δ be a positive real number. There is an entire function g_Δ which satisfies the following properties.*

(i) *There is a positive constant C such that*

$$-C \frac{1}{1 + x^2} \leq g_\Delta(x) \leq f(x)$$

for all real x.

(ii) *The Fourier transform of g_Δ, namely*

$$\hat{g}_\Delta(\xi) = \int_{-\infty}^{\infty} g_\Delta(x) e^{-2\pi i \xi} dx,$$

is real-valued, equals zero for $|\xi| \geq \Delta$, and satisfies $|\hat{g}_\Delta(\xi)| \ll 1$.

(iii) *The L^1 distance between g_Δ and f equals*

$$\int_{-\infty}^{\infty} (f(x) - g_\Delta(x)) dx = \frac{1}{\Delta}(2 \log 2 - 2 \log(1 + e^{-4\pi\Delta})).$$

Explicitly, we have (see equation (8) of [**5**], and the penultimate line of page 247 there)

$$g_\Delta(x) = \sum_{n=-\infty}^{\infty} \left(\frac{\sin(\pi(\Delta x - n + 1/2))}{\pi(\Delta x - n + 1/2)} \right)^2 \left(f\left(\frac{n-1/2}{\Delta}\right) + \frac{\Delta x - n + 1/2}{\Delta} f'\left(\frac{n-1/2}{\Delta}\right) \right).$$

The function g_Δ is optimal, in the sense that it minimizes the L^1 distance between f and g_Δ subject to the restrictions on its Fourier transform. Its construction is due to work of Carneiro and Vaaler [**4**], which generalizes special cases discovered by Beurling and Selberg.

Returning to (2.1), we have for any positive Δ

$$\sum_{\gamma_d} f(t - \gamma_d) \geq g_\Delta(t - \gamma_d).$$

We now invoke the explicit formula relating the zeros of $L(s, E \times \chi_d)$ to the coefficients of its Dirichlet series (see Theorem 5.12 of [**8**]).

Lemma 4. *Let $h(s)$ be analytic in the strip $|Im(s)| \leq 1/2 + \epsilon$ for some $\epsilon > 0$, and such that $|h(s)| \ll (1 + |s|)^{-(1+\delta)}$ for some $\delta > 0$ when $|Re(s)| \to \infty$. Let $h(w)$ be real-valued for real w, and set $\hat{h}(x) = \int_{-\infty}^{\infty} h(w) e^{-2\pi i w} dw$. Then*

$$\sum_\rho h(\gamma_d) = \frac{1}{2\pi} \hat{h}(0)(\log M + 2 \log |d| - 2 \log 2\pi) + \frac{1}{2\pi} \int_{-\infty}^{\infty} h(u) \mathrm{Re} \frac{\Gamma'}{\Gamma}(1 + iu) du$$
$$- \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{\Lambda_E(n)}{\sqrt{n}} \left(\hat{h}\left(\frac{\log n}{2\pi}\right) + \hat{h}\left(\frac{-\log n}{2\pi}\right) \right),$$

where $\Lambda_E(n)$ are defined by $-L'/L(s, E \times \chi_d) = \sum_n \Lambda_E(n)/n^s$.

We apply Lemma 4 with $h(z)$ chosen to be $g_\Delta(t - z)$, so that $\hat{h}(x) = \hat{g}_\Delta(-x) e^{-2\pi i x t}$. Part (iii) of Proposition 3 and the fact that $\int_{-\infty}^{\infty} f(x) dx = 4\pi$ imply that

$$\hat{h}(0) = 4\pi - \frac{1}{\Delta}(2 \log 2 - 2 \log(1 + e^{-4\pi\Delta})),$$

and Stirling's formula together with (i) of the proposition imply that

$$\int_{-\infty}^{\infty} h(u) \mathrm{Re} \frac{\Gamma'}{\Gamma}(1 + iu) du = O(1).$$

If, as in [**5**], we choose Δ so that $\pi\Delta = (1 - \epsilon) \log \log |d|$, then we may estimate the sum over primes trivially to obtain

$$\left|\sum_{n=1}^{\infty}\frac{\Lambda_E(n)}{\sqrt{n}}\left(\hat{h}\left(\frac{\log n}{2\pi}\right)+\hat{h}\left(\frac{-\log n}{2\pi}\right)\right)\right| \ll (\log|d|)^{1-\epsilon}.$$

Note that we have used the Ramanujan bound here, but removing this would only affect the constant $\frac{\log 2}{2}$ in Proposition 2. Combining these bounds gives

$$2\log|d| - \frac{1}{2}\sum_{\gamma_d} g_\Delta(t-\gamma_d) = \left(\frac{\log 2}{2}+o(1)\right)\frac{\log|d|}{\log\log|d|}. \tag{2.2}$$

As in [**5**], we may now combine $f(t-\gamma_d) \ge g_\Delta(t-\gamma)$ with (2.1) and use the asymptotic (2.2) to obtain a precise Lindelöf bound for $|L(s, E\times\chi_d)|$. To prove Proposition 2, instead of using $f(t-\gamma_d) \ge g_\Delta(t-\gamma)$ for all γ_d, we shall keep track of the information from those γ_d that are close to t. From (2.1) and (2.2) we have, with $\pi\Delta = (1-\epsilon)\log\log|d|$,

$$\begin{aligned}\log|L(1/2+it, E\times\chi_d)| &= -\frac{1}{2}\sum_{\gamma_d}(f(t-\gamma_d)-g_\Delta(t-\gamma_d)) + 2\log|d| + O(1) - \frac{1}{2}\sum_{\gamma_d} g_\Delta(t-\gamma_d)\\ &= -\frac{1}{2}\sum_{\gamma_d}(f(t-\gamma_d)-g_\Delta(t-\gamma_d)) + \left(\frac{\log 2}{2}+o(1)\right)\frac{\log|d|}{\log\log|d|}. \end{aligned} \tag{2.3}$$

Proposition 2 will follow from this and a bound for $f(t-\gamma_d) - g_\Delta(t-\gamma_d)$ when $|t-\gamma_d|$ is small. We shall establish such a bound using the following lemma.

Lemma 5. *If $|x| \le 1/(2\Delta)$, then for all $n\in\mathbb{Z}$ we have*

$$\left|f\left(\frac{n-1/2}{\Delta}\right)+\frac{\Delta x - n + 1/2}{\Delta}f'\left(\frac{n-1/2}{\Delta}\right)\right| \le 4+\log(1+16\Delta^2).$$

Proof.

Because $f(x)$ decreases away from the origin,

$$\left|f\left(\frac{n-1/2}{\Delta}\right)\right| \le f(1/(2\Delta)) = \log(1+16\Delta^2).$$

We have $|xf'(x)| = \frac{8}{4+x^2} \le 2$, so that

$$\left|\frac{n-1/2}{\Delta}f'\left(\frac{n-1/2}{\Delta}\right)\right| \le 2.$$

Finally,

$$\left|xf'\left(\frac{n-1/2}{\Delta}\right)\right| \le \frac{1}{2\Delta}f'\left(\frac{1}{2\Delta}\right) \le 2,$$

which completes the proof.

By an application of the Poisson summation formula we see that

$$\left(\frac{\cos\pi z}{\pi}\right)^2\sum_{n=-\infty}^{\infty}\frac{1}{(z-n+1/2)^2} = \sum_{n=-\infty}^{\infty}\left(\frac{\sin(\pi(z-n+1/2))}{\pi(z-n+1/2)}\right)^2 = 1,$$

and by combining this with Lemma 5 and the explicit formula for g_Δ we have

$$g_\Delta(x) \le 4 + \log(1 + 16\Delta^2)$$

when $|x| \le 1/(2\Delta)$. Hence, if $|t - \gamma_d| \le 1/(5 \log\log |d|)$, a little calculation shows that

$$-\frac{1}{2}(f(t-\gamma_d) - g_\Delta(t-\gamma_d)) \le \log(C|t-\gamma_d| \log\log |d|)$$

for some $C > 0$. Using this in (2.3) completes the proof of the Proposition.

3. *The proof in the odd case*

We begin the proof of Theorem 1 in the case of odd functional equation by applying the results of Bump-Friedberg-Hoffstein [**3**], Murty-Murty [**13**], and Waldspurger [**16**] [**17**] to deduce the existence of a quadratic character χ_{d_0} with $(2M, d_0) = 1$ such that $L(1/2, E \times \chi_{d_0}) \neq 0$. This is the d_0 that we shall take in the statement of Theorem 1. If we choose a second fundamental discriminant d satisfying

$$(d, 2Md_0) = 1, \quad d_0 d < 0, \quad d_0 d \equiv \square\ (4M),$$

then the discussion on pages 268-269 of [**7**] shows that $L(s, E \times \chi_d)$ has odd functional equation and hence $\mathcal{D}^-_{d_0}(E) \subseteq \mathcal{D}^-(E)$ as claimed. We may deduce Theorem 1 from the following lemma, by letting t approach 0 in Proposition 2 and combining the resulting upper bound on $|L'(1/2, E \times \chi_d)|$ with the lower bound it provides.

Lemma 6. *For $d \in \mathcal{D}^1(E) \cap \mathcal{D}^-_{d_0}(E)$, we have*

$$|L'(1/2, E \times \chi_d)| \gg |d|^{-1/2}.$$

Proof.

The conditions that $(d, d_0) = 1$ and $d_0 d < 0$ allow us to define a genus class character χ_{d,d_0} of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{d_0 d})$ (see for instance [**8**], chapter 22, page 510). We let E_K be the elliptic curve over K obtained from E by extension of scalars, and let $L(s, E_K \times \chi_{d,d_0})$ denote the L-function of E_K twisted by χ_{d,d_0}. As the discriminant of K is prime to $2M$ and all primes dividing M split in K, we may apply the Gross-Zagier formula [**7**] to the central derivative $L'(1/2, E_K \times \chi_{d,d_0})$ to obtain

$$L'(1/2, E_K \times \chi_{d,d_0}) = C_E h(P_{d,d_0}) |d_0 d|^{-1/2}. \tag{3.1}$$

Here, P_{d,d_0} is an algebraic point on E which is defined over $\mathbb{Q}(\sqrt{d})$ (see the discussion on pages 268-269 of [**7**]), and $h(\cdot)$ denotes the canonical height on E. The Kronecker factorisation formula and the fact that $L(1/2, E \times \chi_d) = 0$ imply that

$$L(s, E_K \times \chi_{d,d_0}) = L(s, E \times \chi_{d_0}) L(s, E \times \chi_d)$$
$$L'(1/2, E_K \times \chi_{d,d_0}) = L(1/2, E \times \chi_{d_0}) L'(1/2, E \times \chi_d). \tag{3.2}$$

Substituting (3.2) into (3.1) and using our assumption that $L(1/2, E \times \chi_{d_0}) \neq 0$, we have

$$L'(1/2, E \times \chi_d) = C_{E,d_0} h(P_{d,d_0}) |d|^{-1/2}.$$

We now apply the results of Anderson-Masser [**1**], which state that if P is a nontorsion algebraic point on E whose degree over $\mathbb{Q}$ is bounded, then the height $h(P)$ must be bounded from below.

Our assumption that $L'(1/2, E \times \chi_d) \neq 0$ implies that P_{d,d_0} is not torsion, from which we then deduce that

$$|L'(1/2, E \times \chi_d)| \gg |d|^{-1/2}$$

as required.

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