ACKNOWLEDGMENTS. I would like to thank Wilhelm K. Essler, who familiarized me with the problem years ago.

REFERENCES

1. A. Tarski, *Introduction to Logic and to the Methodology of Deductive Sciences*, Oxford University Press, New York, 1961.

Institut fuer Philosophie, Fachbereich 8, Johann Wolfgang Goethe-Universitaet,
Grueneburgplatz 1, 60629 Frankfurt am Main, Germany
Stefanie.Ucsnay@web.de

Another Simple Proof of the High Girth, High Chromatic Number Theorem

Simon Marshall

1. INTRODUCTION. We begin with a little graph theoretic terminology. A k-colouring of a graph is an assignment of a number between 1 and k to each of its vertices in such a way that any two vertices joined by an edge have different labels, and the chromatic number of a graph is the smallest k such that this can be achieved. The girth of a graph is the number of edges in the shortest cycle. It seems reasonable to expect that if the girth of a graph is high it should be easy to colour. However this is not the case: in 1959 Erdős [**1**], in a celebrated paper, proved that there exist graphs with arbitrarily large chromatic number and girth. The central idea of Erdős' proof was as follows: we consider random graphs on n vertices where edges occur with some probability p. If p is low the graph will have few short cycles, and if p is high we may expect graphs in which any large set of vertices has at least one edge between them. The right choice of p ensures that both happen at the same time. We can then choose an appropriate graph and delete a vertex from each short cycle while leaving the 'large vertex collection' property intact. The resulting graph will have large girth and cannot be coloured with few colours. We will describe a proof of Erdős' theorem which, while similar to an existing proof, is significantly simpler and independently discovered. Rather than considering random graphs where the edges are chosen with some carefully adjusted probability, we use a simple counting argument on a set of graphs with bounded vertex degree.

There have also been a number of constructive approaches to this problem. Perhaps the most remarkable is due to Müller [**4**], who has proven the following: let natural numbers k and g and a set A be given, and let $P_1, \dots, P_r$ be distinct partitions of A into at most k classes. Then there exists a k-colourable graph of girth greater than g with A as a subset of its vertex set such that each P_i may be extended to a colouring of G, and these are the only possible colourings. Genuinely constructive proofs of Erdős' result can be found in [**3**] and [**5**], among other places.

An alternative proof of the equivalent result for hypergraphs is given by Erdős and Lovász in [**2**], which works by bounding vertex degrees (the degree of a vertex is the number of edges meeting it) and adding edges without forming short cycles. Enough edges can be added to ensure that every set of more than $1/k$ of the vertices contains an edge, so the graph cannot be k-coloured. The proof we give shares some ideas, most notably a restriction to graphs of bounded degree, with that given in [**2**]. However we believe its simplicity makes it an effective illustration of the underlying methods of that proof.

2. THE PROOF. Let g and k be positive integers. For some n and d, denote by Γ the set of graphs on n vertices with at most nd edges and degree at most d^2, and whose girth is at least g. Similarly, denote by Ψ the set of graphs on n vertices with at most nd edges and degree at most d^2 which are k-colourable. Throughout the proof, g and k will be fixed while n and d are made large enough to satisfy certain inequalities. In particular, we may assume that $d > 4$, $n \geq 2(g+1)d^{2g}$. In both Γ and Ψ we consider the edges and vertices of our graphs to be numbered, so that we consider the same graph with the edges and vertices in a different order to be different. Let Γ_m and Ψ_m be the subsets of Γ and Ψ consisting of graphs with exactly m edges.

The idea of the proof is as follows: because the degrees of our graphs are bounded, the requirement that the girth is at least g becomes a local condition. By this we mean that if we go about constructing graphs in Γ by sucessively adding edges, only a bounded number of choices (as a function of n) are excluded at each step. On the other hand, a given vertex colouring excludes some constant proportion of the edges we may use in our graphs. Therefore for n sufficiently large the k-colourabilty requirement excludes more graphs than the girth requirement and we will have $|\Gamma_{nd}| > |\Psi_{nd}|$, that is, there is a graph in our class which has girth at least g but is not k-colourable.

We begin by estimating $|\Psi_m|$ from above. For a given colouring C, denote by Ψ_m^C the set of graphs from Ψ_m compatible with C, that is, those for which C is a valid colouring. If the number of vertices in C with colour i is denoted n_i, it follows from the convexity of the function $\binom{n}{2}$ that the set of allowed edges E_C has maximal cardinality when all n_i are as close to being equal as possible, that is,

$$|E_C| \leq \binom{n}{2} - k\binom{n/k}{2},$$

$$|E_C| \leq \frac{1}{2}n^2\left(1 - \frac{1}{k}\right).$$

Therefore

$$|\Psi_m^C| \leq \left(\frac{1}{2}n^2\left(1 - \frac{1}{k}\right)\right)^m.$$

As there are k^n possible colourings,

$$|\Psi_m| \leq k^n\left(\frac{1}{2}n^2\left(1 - \frac{1}{k}\right)\right)^m.$$

We now recursively estimate $|\Gamma_m|$ from below, by adding edges to graphs in Γ_{m-1} and giving a lower bound on the proportion of the resulting graphs which are valid. At each step we have no more than nd edges in our graph, which means there exist at most $2n/d$ vertices which will exceed the degree bound of d^2 if we add another edge to them. Therefore there are at least $n(1 - 2/d)$ vertices between which we may add an edge at each step of our graph construction, which is nonnegative as $d > 2$. In order to ensure we do not create a cycle of length g or shorter when we add an edge, the edge may not end at any vertex within distance g of the initial vertex. Because all vertices have degree at most d^2, this excludes no more than $1 + d^2 + d^4 + \cdots + d^{2g} \le (g+1)d^{2g}$ choices. We then have at least

$$\frac{1}{2}n\left(1-\frac{2}{d}\right)\left(n\left(1-\frac{2}{d}\right)-(g+1)d^{2g}\right)$$

choices at each step of our construction, and our bounds on d and n ensure that this quantity is nonnegative. Our labelling of the vertices and edges means that all choices give us distinct graphs. This gives us the estimate

$$|\Gamma_m| \ge \left(\frac{1}{2}n\left(1-\frac{2}{d}\right)\left(n\left(1-\frac{2}{d}\right)-(g+1)d^{2g}\right)\right)^m.$$

Suppose now that $|\Gamma_{nd}| \le |\Psi_{nd}|$ for all n and d satisfying the required inequalities, that is, that

$$\left(\frac{1}{2}n\left(1-\frac{2}{d}\right)\left(n\left(1-\frac{2}{d}\right)-(g+1)d^{2g}\right)\right)^{nd} \le k^n\left(\frac{1}{2}n^2\left(1-\frac{1}{k}\right)\right)^{nd}$$

or

$$\frac{1}{2}n\left(1-\frac{2}{d}\right)\left(n\left(1-\frac{2}{d}\right)-(g+1)d^{2g}\right) \le k^{1/d}\left(\frac{1}{2}\right)n^2\left(1-\frac{1}{k}\right).$$

As n may be made arbitrarily large with d fixed, the coefficient of n^2 on the left hand side must be less than or equal to the coefficient on the right hand side. Therefore

$$\left(1-\frac{2}{d}\right)^2 \le k^{1/d}\left(1-\frac{1}{k}\right)$$

for all d. However, as $d \to \infty$ the right hand side approaches $1 - 1/k$ while the left hand side approaches 1. Therefore for some n and d we have $|\Gamma_{nd}| > |\Psi_{nd}|$, so there exists a graph with girth at least g which is not k-colourable. It may also be shown that a choice of $d = k^3$, $n = 9gk^{6g+1}$ suffices for all $k \ge 3$, while $d = k^2$, $n = 2gk^{4g+1}$ suffices for $k \ge 144$.

REFERENCES

1. P. Erdős, Graph theory and probability, *Canad. J. Math.* **11** (1959) 34–38.
2. P. Erdős and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, *Colloquia Mathematica Societatis János Bolyai* **10** (1973) 609–627.
3. L. Lovász, On chromatic number of finite set-systems, *Acta Math. Acad. Sci. Hungary* **19** (1968) 59–67.
4. V. Müller, On colourings of graphs without short cycles, *Discrete Mathematics* **26** (1979) 165–176.
5. J. Nešetřil and V. Rödl, A short proof of the existence of highly chromatic hypergraphs without short cycles, *J. Combin. Theory Ser. B* **27** (1979) 225–227.

Department of Mathematics, Fine Hall, Washington Road, Princeton, NJ 08544
slm@math.princeton.edu