SOME EISENSTEIN SERIES MOTIVATED BY STRING THEORY

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1. Introduction

Let $k \geq 2$ be an even integer. For z in the complex upper half-plane we let $\Lambda_z \subset \mathbb{C}$ be the lattice $\{az+b | a, b \in \mathbb{Z}\}$, and define the function $E_\theta(k,z)$ by the absolutely convergent sum

$$E_\theta(k,z) = \sum_{\substack{(\lambda_1,\lambda_2,\lambda_3)\in\Lambda_z^3 \\ \lambda_i\neq 0,\, \Sigma\lambda_i=0}} (\lambda_1\lambda_2\lambda_3)^{-k}.$$

If we let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be elements of $\mathbb{Z}^3$, we may rewrite this as

$$E_\theta(k,z) = \sum_{\Sigma a_i=0} \sum_{\Sigma b_i=0}{}' \prod_{j=1}^{3} (a_j z + b_j)^{-k},$$

where the ' on the second sum denotes that we do not allow a_i and b_i to both be zero. It may be seen that $E_\theta(k,z) \in S_{3k}$, where S_l is the space of modular forms of weight l for $SL_2(\mathbb{Z})$. We denote the rational subspace of forms with rational Fourier coefficients by $S_l(\mathbb{Q})$. We shall prove the following result:

Proposition 1. *We have $\pi^{-3k}E_\theta(k,z) \in S_{3k}(\mathbb{Q})$. Moreover, for any k and n there is an algorithm that calculates the first n Fourier coefficients of $E_\theta(k,z)$.*

Applying this to the constant term of $E_\theta(k,z)$, we recover the fact that

$$\sum_{\substack{\Sigma b_i=0 \\ b_i\neq 0}} (b_1b_2b_3)^{-k} \in \pi^{3k}\mathbb{Q}^\times.$$

These Eisenstein series are essentially equal to a Feynman graph integral of the Green's function of the operator $(\partial/\partial z)^k$ on a complex torus, where the graph is taken to be the θ-graph consisting of two points joined by three edges. It should be possible to prove Proposition 1 for the series associated to any graph.

2. Proof of Proposition 1

Because $S_{3k}(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = S_{3k}$, we may prove that $\pi^{-3k}E_\theta(k,z) \in S_{3k}(\mathbb{Q})$ by showing that all of its nonzero Fourier coefficients are rational. The nth Fourier coefficient of $E_\theta(k,z)$ is equal to

$$\sum_{\Sigma a_i=0} \int_0^1 \sum_{\Sigma b_i=0}{}' \prod_{j=1}^{3} (a_j z + b_j)^{-k} e^{-2\pi i n x} dx,$$

and the following proposition implies that this lies in $\pi^{3k}\mathbb{Q}$. Moreover, it will be seen how to calculate this coefficient in the course of the proof.

Proposition 2. *For any $n \geq 1$ and $\mathbf{a} \neq 0$, the expression*

$$I(n, \mathbf{a}) = \int_0^1 \sum_{\Sigma b_j = 0}{}' \prod_{j=1}^{3} (a_j x + b_j + i a_j y)^{-k} e^{-2\pi i n x} dx \tag{1}$$

lies in $\pi^{3k} e^{-2\pi n y}\mathbb{Q}$. Moreover, for each fixed n it is nonzero for only finitely many $\mathbf{a}$.

Proof. We first consider the case in which $a_i \neq 0$ for all i. Define the function $f : \mathbb{R}^3 \to \mathbb{C}$ by

$$f(\mathbf{x}) = \prod_{j=1}^{3} (x_j + iy)^{-k}.$$

The Fourier transform of f, defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^3} f(\mathbf{x}) e^{2\pi i \langle \mathbf{x}, \xi \rangle} d\mathbf{x},$$

is equal to

$$\widehat{f}(\xi) = \begin{cases} (2\pi i)^{3k} \xi_1^{k-1} \xi_2^{k-1} \xi_3^{k-1} e^{-2\pi y \Sigma \xi} & \text{if all } \xi_i \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Define $\Lambda_{\mathbf{a}}$ to be the discrete subgroup $\{(b_1/a_1, b_2/a_2, b_3/a_3) | \Sigma b_i = 0\}$ of $\mathbb{R}^3$, and define $\Lambda_{\mathbf{a}} f$ by

$$\Lambda_{\mathbf{a}} f(x) = \sum_{\lambda \in \Lambda_{\mathbf{a}}} f(x + \lambda).$$

We may rewrite the integral $I(\mathbf{a}, n)$ as

$$\begin{aligned} I(\mathbf{a}, n) &= (a_1 a_2 a_3)^{-k} \int_0^1 \sum_{\Sigma b_j = 0} \prod_{j=1}^{3} (x + b_j/a_j + iy)^{-k} e^{-2\pi i n x} dx \\ &= (a_1 a_2 a_3)^{-k} \int_0^1 \Lambda_{\mathbf{a}} f((x, x, x)) e^{-2\pi i n x} dx. \end{aligned} \tag{2}$$

We shall establish the proposition by examining the Fourier transform of $\Lambda_{\mathbf{a}} f$. This function descends to $\mathbb{R}^3/\Lambda_{\mathbf{a}}$, and the group of characters of this quotient is the closed subgroup

$$\Lambda_{\mathbf{a}}^{\perp} = \{x \in \mathbb{R}^3 | \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in \Lambda_{\mathbf{a}}\}$$

of $\mathbb{R}^3$. The connected component of the identity in $\Lambda_{\mathbf{a}}^{\perp}$ is a line which we give the parametrization $t(a_1, a_2, a_3)$, and the quotient by this line is isomorphic to $\mathbb{Z}^2$. The measure on $\Lambda_{\mathbf{a}}^{\perp}$ dual to the Lebesgue measure on $\mathbb{R}^3/\Lambda_{\mathbf{a}}$ is the Haar measure that is equal to $a_1 a_2 a_3 dt$ on the line $t(a_1, a_2, a_3)$. We denote this by $\mu_{\mathbf{a}}$. By Fourier inversion we have

$$\Lambda_{\mathbf{a}} f(\mathbf{x}) = \int_{\Lambda_{\mathbf{a}}^{\perp}} \widehat{f}(\xi) e^{2\pi i \langle \mathbf{x}, \xi \rangle} d\mu_{\mathbf{a}}(\xi),$$

and substituting this into (2) gives

$$I(\mathbf{a}, n) = (a_1 a_2 a_3)^{-k} \int_{\substack{\Lambda_{\mathbf{a}}^{\perp} \\ \Sigma\xi = n}} \widehat{f}(\xi) d\mu_{\mathbf{a}}(\xi). \tag{3}$$

We first show that there are only finitely many **a** such that (3) is nonzero. We first observe that if (3) is to be nonzero, there must be $\xi \in \Lambda_{\mathbf{a}}^{\perp}$ with $\Sigma\xi = n$ and $\widehat{f}(\xi) \neq 0$, which implies that $n \geq \xi_i > 0$ for all i. The condition $\xi \in \Lambda_{\mathbf{a}}^{\perp}$ means that we must have

$$\langle (\xi_1/a_1, \xi_2/a_2, /xi_3/a_3, \mathbf{b} \rangle \in \mathbb{Z}$$

for all **b** with $\Sigma b_i = 0$, which is equivalent to requiring that all the differences $\xi_i/a_i - \xi_j/a_j$ lie in $\mathbb{Z}$.

If we have $|a_i| > 2n$ for all i, then the condition $\xi_i < n$ means that $|\xi_i/a_i - \xi_j/a_j| < 1$ for all pairs i, j, and this implies that $\xi_1/a_1 = \xi_2/a_2 = \xi_3/a_3$. However, two of the a_i must have opposite signs while the ξ_i are all positive, which is a contradiction. If we have $|a_i|$, $|a_j| > 2n$ for distinct indices i and j, we again have $\xi_i/a_i = \xi_j/a_j$. This implies that a_i and a_j have the same sign, so the third entry of **a** must have absolutle value at least $4n$ which again leads to a contradiction.

We therefore see that the only contributions to (3) come from **a** that have two indices of absolute value at most $2n$, and the condition $\Sigma a_i = 0$ implies that the number of such **a** is finite.

To prove that $I(n, \mathbf{a}) \in \pi^{3k} e^{-2\pi n y} \mathbb{Q}$, observe that it is equal to $(a_1 a_2 a_3)^{-k}$ times a finite sum of integrals

$$\int \widehat{f}(\xi + t(a_1, a_2, a_3)) a_1 a_2 a_3 dt,$$

where ξ ranges over a set of representatives for the connected components of $\Lambda_{\mathbf{a}}^{\perp}$ with $\Sigma\xi = n$ that meet the positive quadrant. If the line $\xi + t(a_1, a_2, a_3)$ lies in the positive quadrant for $t \in [a, b]$, substituting the formula for $\widehat{f}$ gives

$$\int \widehat{f}(\xi + t(a_1, a_2, a_3)) a_1 a_2 a_3 dt = (2\pi i)^{3k} e^{-2\pi n y} \int_a^b \prod_{j=1}^{3} (\xi_j + t a_j)^{k-1} dt.$$

The remaining integral is of a polynomial with rational coefficients over a rational interval, and hence is rational. This completes the proof in the case where all a_i are nonzero.

We now consider the case in which $a_i = 0$ for some i. We may assume without loss of generality that $a_3 = 0$, which imposes the condition $b_3 \neq 0$ on **b**. We begin by fixing b_3 and considering the integrals

$$I(n, \mathbf{a}, b_3) = \int_0^1 \sum_{b_1 + b_2 = -b_3} (a_1 x + a_1 i y + b_1)^{-k} (a_2 x + a_2 i y + b_2)^{-k} b_3^{-k} e^{-2\pi i n x} dx.$$

As we have $a_2 = -a_1$ and $b_2 = -b_1 - b_3$, we may rewrite this as

$$I(n, \mathbf{a}, b_3) = a_1^{-2k} b_3^{-k} \int_0^1 \sum_{b_1} (x + iy + b_1/a_1)^{-k} (x + iy + b_1/a_1 + b_3/a_1)^{-k} e^{-2\pi i n x} dx.$$

Unfolding the integral gives

$$\begin{aligned} I(n, \mathbf{a}, b_3) &= a_1^{-2k} b_3^{-k} \int_{-\infty}^{\infty} \sum_{b_1 \in \mathbb{Z}/a_1\mathbb{Z}} (x + iy + b_1/a_1)^{-k} (x + iy + b_1/a_1 + b_3/a_1)^{-k} e^{-2\pi i n x} dx \\ &= a_1^{-2k} b_3^{-k} \sum_{b_1 \in \mathbb{Z}/a_1\mathbb{Z}} e^{-2\pi i n b_1/a_1} \int_{-\infty}^{\infty} (x + iy)^{-k} (x + iy + b_3/a_1)^{-k} e^{-2\pi i n x} dx. \end{aligned}$$

The sum over b_1 vanishes unless $a_1 | n$, which proves that $I(n, \mathbf{a}, b_3)$, and hence $I(n, \mathbf{a})$, vanish for all but finitely many $\mathbf{a}$.

If $a_1 | n$, we have

$$I(n, \mathbf{a}, b_3) = a_1^{-2k+1} b_3^{-k} \int_{-\infty}^{\infty} (x + iy)^{-k} (x + iy + b_3/a_1)^{-k} e^{-2\pi i n x} dx.$$

We shall calculate this integral by moving the contour into the lower half-plane. The integrand $(w+iy)^{-k}(w+iy+b_3/a_1)^{-k}e^{-2\pi i n w}$ has poles at $-iy$ and $-iy - b_3/a_1$, and the residues there are equal to

$$\left(\frac{\partial}{\partial w}\right)^{k-1} (w + iy + b_3/a_1)^{-k} e^{-2\pi i n w} \Big|_{w=-iy} = e^{-2\pi n y} \sum_{j=0}^{k-1} c(j) (2\pi i)^j b_3^{-2k+1+j}$$

and

$$\left(\frac{\partial}{\partial w}\right)^{k-1} (w + iy)^{-k} e^{-2\pi i n w} \Big|_{w=-iy-b_3/a_1} = e^{-2\pi n y} \sum_{j=0}^{k-1} d(j) (2\pi i)^j b_3^{-2k+1+j}$$

respectively, where the coefficients $c(j)$ and $d(j)$ are rational numbers that depend on k, a_1, and j, but not b_3. We therefore have

$$I(n, \mathbf{a}, b_3) = a_1^{-2k+1} e^{-2\pi n y} \sum_{j=0}^{k-1} (c(j) + d(j)) (2\pi i)^{j+1} b_3^{-3k+1+j}.$$

Summing over $b_3 \neq 0$ now gives

$$I(n, \mathbf{a}) = a_1^{-2k+1} e^{-2\pi n y} \sum_{\substack{0 \le j \le k-1 \\ j \text{ odd}}} (c(j) + d(j)) (2\pi i)^{j+1} 2\zeta(3k - j - 1)$$

The fact that $\zeta(2k) \in \pi^{2k}\mathbb{Q}^\times$ for $k \geq 1$ completes the proof.

$\square$