# Nondensity of double bubbles in the d.c.e. degrees

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(Joint work with Andrews, Kuyper, M. Soskova and Yamaleev)

January 6, 2017

## Definition

For $n \in \omega$, call a set $A \subseteq \omega$ *n-c.e.* if there is a computable approximation $\{A_s\}_{s\in\omega}$ to $A$ such that

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We define, for a d.c.e. degree $\mathbf{d}$, the set $\mathbf{R}(\mathbf{d})$ of c.e. Turing degrees $\mathbf{a}$ such that $\mathbf{d}$ is CEA in $\mathbf{a}$.

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Ishmukhametov also observed that unlike $\mathbf{R}(\mathbf{d})$, $L(D)$ is badly behaved in the sense that its degree depends not only on the degree of $D$: For d.c.e. sets $D_0$ and $D_1$ of the same degree, $L(D_0)$ and $L(D_1)$ can have the same degree, incomparable degree or strictly comparable degree!

As mentioned above, the c.e. degrees and the d.c.e. degrees do not coincide, but both are downward dense.

There are important elementary differences, however:

**Theorem**

- (Arslanov 1988) Every noncomputable d.c.e. degree cups to $\mathbf{0}'$. (This fails in the c.e. degrees by Yates/Cooper (1973).)

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- (Cooper/Harrington/Lachlan/Lempp/Soare 1991) There is a maximal incomplete d.c.e. degree. (This fails for the c.e. degrees by the Sacks Density Theorem (1964).)

The seemingly hopeless search for similar elementary differences led Downey to the following

**Conjecture (Downey 1989)**

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This conjecture, more intended to spur further research, was refuted in the following

**Theorem (Arslanov, Kalimullin, Lempp 2010)**

The following statement holds in the 3-c.e. degrees but fails in the 2-c.e. degrees: There are degrees $\mathbf{f} > \mathbf{e} > \mathbf{d} > \mathbf{0}$ such that any degree $\mathbf{a} \leq \mathbf{f}$ is comparable with $\mathbf{e}$ iff it is comparable with $\mathbf{d}$.

More generally, consider the following

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An *n-bubble* of degrees is a sequence $\mathbf{d}_n > \mathbf{d}_{n-1} > \ldots \mathbf{d}_1 > \mathbf{d}_0 = \mathbf{0}$ such that any degree $\mathbf{a} \leq \mathbf{d}_n$ is contained in an interval $[\mathbf{d}_i, \mathbf{d}_{i+1}]$ for some $i < n$.

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Ishmukhametov's theorem already implies that there is a 2-bubble ("double bubble") in the d.c.e. degrees. Wu and Yamaleev (2012) improved our result for 3-c.e. degrees by showing that there is a 3-bubble ("triple bubble") in the 3-c.e. degrees.

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A pairwise elementary difference between all degree structures of $n$-c.e. degrees would be given by proving the following

### Conjecture

For any $n > 3$, there is an $n$-bubble in the $n$-c.e. degrees but not in the $(n-1)$-c.e. degrees.

It is easy to see (by the Sacks Splitting Theorem) that if $\mathbf{b} > \mathbf{a} > \mathbf{0}$ forms a double bubble in the d.c.e. degrees, then $\mathbf{a}$ must be c.e., and in fact must be the greatest c.e. degree $< \mathbf{b}$ (and $\mathbf{b}$ is exact).

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Surprisingly, we were able to show that their result does not extend to double bubbles:

**Theorem (Andrews, Kuyper, Lempp, M. Soskova, Yamaleev)**

The tops of double bubbles are *not* downward dense in the d.c.e. degrees.

Given a c.e. set $A$, meet for all $\Psi$ and $\Phi$ and all d.c.e. sets $D$ the **Requirements:**

$$\mathcal{P}_\Psi\colon A \neq \Psi;$$

$$\mathcal{R}_{\Phi,D}\colon D = \Phi^A \Rightarrow \exists E = \Lambda^D\,[E\mid_T L(D)] \vee D \leq_T L(D) \vee L(D) \leq_T \emptyset.$$

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**Basic Strategies:**

$\mathcal{P}_{\Psi}$: trivial

$\mathcal{T}_{\Phi,D,\Psi}$: code $D$ into $E$

$\mathcal{S}_{\Phi,D,\Omega}$: restrain $E$ and build $\Delta$

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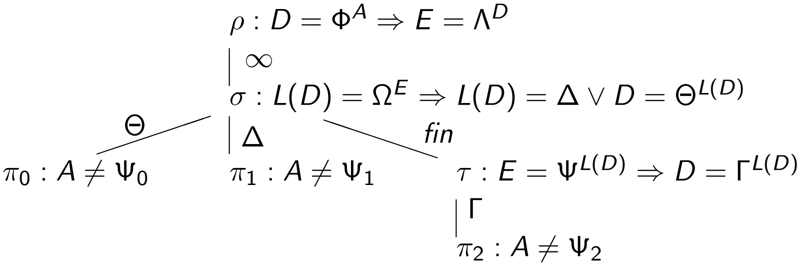

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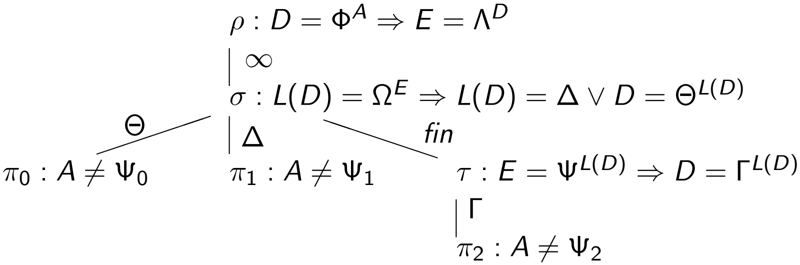

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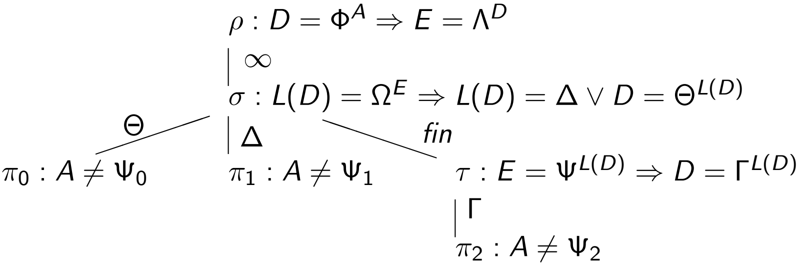

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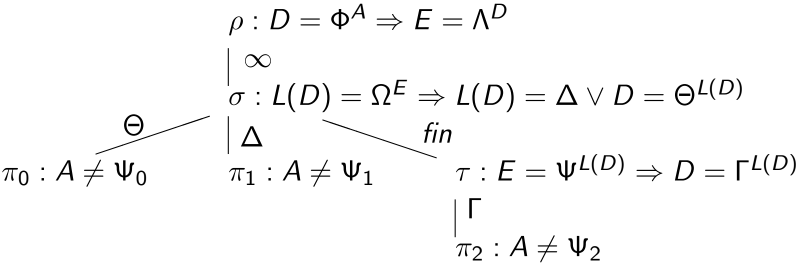

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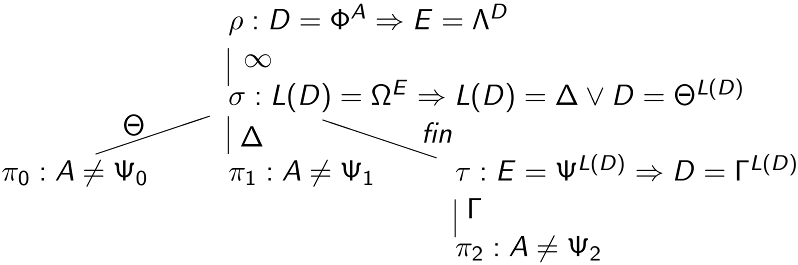

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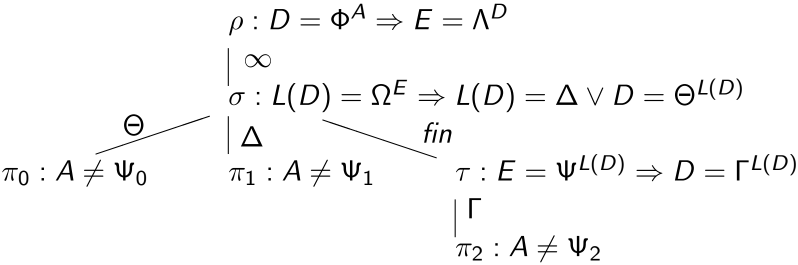

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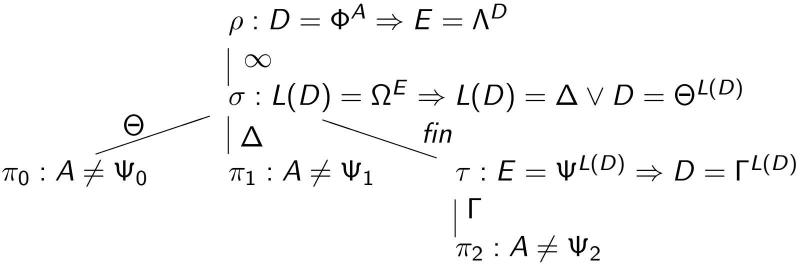

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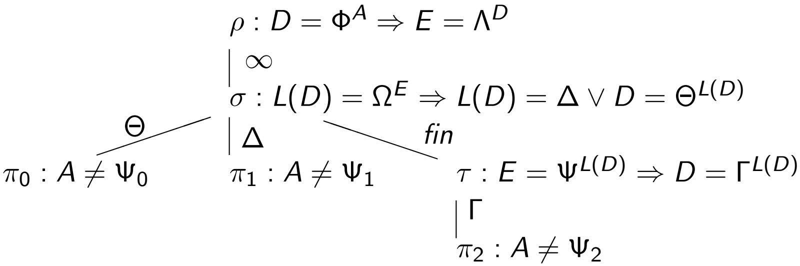
    - If such $x' \searrow D$, use it to correct $\Lambda^D(y_x) = 1$ and achieve $L(D)(s(x)) \neq \Omega^E(s(x))$.

# Main conflict between strategies:



$\rho : D = \Phi^A \Rightarrow E = \Lambda^D$

$\infty$

$\sigma : L(D) = \Omega^E \Rightarrow L(D) = \Delta \vee D = \Theta^{L(D)}$

$\Theta$ $\Delta$ $fin$

$\pi_0 : A \neq \Psi_0$ $\pi_1 : A \neq \Psi_1$ $\tau : E = \Psi^{L(D)} \Rightarrow D = \Gamma^{L(D)}$

$\Gamma$

$\pi_2 : A \neq \Psi_2$

1. $\pi_2$ puts $a_2$ into $A$, allowing $x \searrow D$, forcing $\tau$ to put $y_x \searrow E$.
2. $\pi_1$ puts $a_1 < a_2$ in $A$, allowing $x \nearrow D$, causing $s(x) \searrow L(D)$. (But now $\Lambda^D(y_x) = 0$ and $\Delta(s(x)) = 0$.)
3. $\sigma$ starts defining $\Theta^{L(D)}$.
4. $\pi_0$ puts $a_0 < a_1$ into $A$, allowing $D(x')$-change for $x' < x$:
    - If such $x' \searrow D$, use it to correct $\Lambda^D(y_x) = 1$ and achieve $L(D)(s(x)) \neq \Omega^E(s(x))$.
    - If such $x' \nearrow D$, use $s(x') \searrow L(D)$ to correct $\Theta^{L(D)}(x')$.

Thanks!

Thanks!

And early Happy Birthday, Rod!