Decidability and Undecidability in the Enumeration Degrees

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(joint work with Ted Slaman and Mariya Soskova)

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  We often also consider *local r-degree structures*

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  for a (usually countable) subfamily $\mathcal{S} \subset \mathcal{P}(\omega)$.

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- Most degree structures support a "jump" operation $\mathbf{a} \mapsto \mathbf{a}'$ such that $\mathbf{a} < \mathbf{a}'$, and $\mathbf{a} \leq \mathbf{b}$ implies $\mathbf{a}' \leq \mathbf{b}'$.

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If a class $\mathcal{C}$ of finite structures is ∃-definable with parameters in a degree structure $\mathcal{D}$, and the common ∀∃∀-theory of $\mathcal{C}$ is hereditarily undecidable, then the ∃∀∃-theory of $\mathcal{D}$ is undecidable.

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Deciding the ∀∃-theory of $\mathcal{D}$ amounts to giving a uniform decision procedure to the following

Problem (for deciding the ∀∃-theory of $\mathcal{D}$)

Given finite partial orders $\mathcal{P}$ and $\mathcal{Q}_i \supseteq \mathcal{P}$ (for $i < n$), does every embedding of $\mathcal{P}$ into $\mathcal{D}$ extend to an embedding of $\mathcal{Q}_i$ into $\mathcal{D}$ for some $i < n$ (where i may depend on the embedding of $\mathcal{P}$)?

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For the Δ^0_2-Turing degrees, they embedded $\mathcal{L}$ both as an initial segment; and also $\mathcal{L} - \{1\}$ as an initial segment, mapping 1 to $\mathbf{0}'_T$.

Two major subproblems of the $\forall\exists$-theory are the following:

Extension of Embeddings Problem

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The LE problem remains open for the c.e. Turing degrees, but is decidable for the enumeration degrees and for the Σ^0_2-enumeration degrees (Lempp/Sorbi 2002: all finite lattices embed).

Our main technical result extends the following earlier results:

Theorem (Cooper 1990; Calhoun, Slaman 1996)

The (Π^0_2-)enumeration degrees are not dense, i.e., there are $\mathbf{a} < \mathbf{b}$ such that there is no enumeration degree $\mathbf{x}$ with $\mathbf{a} < \mathbf{x} < \mathbf{b}$.
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Theorem (Slaman, Sorbi 2014)

For any enumeration degrees $\mathbf{a} < \mathbf{b}$ where $\mathbf{a}$ is a total degree, any finite partial order embeds into the interval $(\mathbf{a}, \mathbf{b})$.

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For any finite distributive lattice $\mathcal{L}$, there is an embedding ι of $\mathcal{L}$ into the Π^0_2-enumeration degrees with $\iota(0) = \mathbf{a}$ and $\iota(1) = \mathbf{b}$, say, such that every enumeration degree $\mathbf{x} \leq \mathbf{b}$ is in the range of ι or is $< \mathbf{a}$. (Call such an embedding a *strong embedding* of $\mathcal{L}$.)

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This would have put us closer to a decision procedure for $\mathrm{Th}_{\forall\exists}(\mathcal{D}_e)$.

However, this conjecture is false:

Theorem (Jacobsen-Grocott, M. Soskova)

If $\mathbf{a}$ and $\mathbf{b}$ are incomparable enumeration degrees such that any enumeration degree $\mathbf{x} < \mathbf{a} \vee \mathbf{b}$ is $\leq \mathbf{a}$ or $\leq \mathbf{b}$, then $\mathbf{a}$ and $\mathbf{b}$ do not form a minimal pair.

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This conjecture seems to hold for a larger collection of finite lattices (the "interval dismantlable" lattices) but it is not clear, e.g., for the 5-element modular lattice M_3.

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Conjecture

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This conjecture seems to hold for a larger collection of finite lattices (the "interval dismantlable" lattices) but it is not clear, e.g., for the 5-element modular lattice M_3.

We currently have no good conjecture for deciding $\mathrm{Th}_{\forall\exists}(\mathcal{D}_e)$.

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For any join-irreducible $j \in \mathcal{L}$ (including 0), we build a degree $\mathbf{a}_j$ (represented by a Π^0_2-set A_j) and define, for $k \in \mathcal{L}$,

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We need to ensure the following requirements:

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Resolving the conflicts between these requirements uses a $\mathbf{0}'''$-priority argument.

Thanks!

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Stay safe until we are all vaccinated!