Chains and antichains in the Weihrauch lattice

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(joint work with Alberto Marcone and Manlio Valenti)

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So in particular, each of the above degree structures has size $2^{\mathfrak{c}}$.

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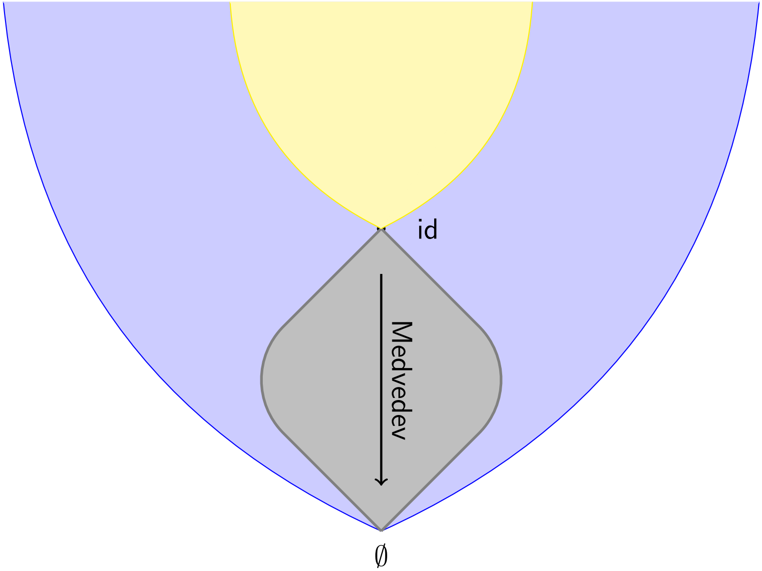
So the Medvedev degrees embed onto the initial segment $[\mathbf{0}, \deg(\mathrm{id})]$ of the Weihrauch degrees under the *order-reversing* map

$$\mathcal{A} \mapsto \mathrm{id} \restriction \mathcal{A}.$$

The (upside down) Medvedev degrees inside the Weihrauch degrees



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However, using Medvedev degrees, we can show:

Theorem (Lempp, Marcone, Valenti)

For every $\kappa \leq \mathfrak{c}$ of uncountable cofinality, there is a chain of Weihrauch degrees of order type κ with a least upper bound.

Let's sketch the proof:

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And as with $\mathcal{A}_\kappa$ above, f_κ is a least upper bound for $\{f_\alpha\}_{\alpha<\kappa}$.

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Let $\kappa \leq \mathfrak{c}$ be an cardinal of uncountable cofinality. Then:

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- There is a chain of order type κ in the Weihrauch degrees which has no upper bound.
- (essentially Shafer, 2011, for the Medvedev degrees)
  There is a chain of order type κ^* in the nonzero Weihrauch degrees which has no lower bound $> \mathbf{0}$.

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- The existence of coinitial chains in the nonzero Weihrauch degrees is equivalent to the Continuum Hypothesis.

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Theorem (Lempp, Marcone, Valenti)

- Every nonzero Weihrauch degree is contained in an antichain of size $2^{\mathfrak{c}}$.
- No antichain in the nonzero Weihrauch degrees of size $< \mathfrak{c}$ is maximal.

We do not know if an antichain of size $< 2^{\mathfrak{c}}$ can be maximal.

Thanks!