The Undecidability of the Π_4–theory for the r.e. wtt- and Turing–degrees

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We show that the Π_4–theory of the partial order of recursively enumerable weak truth-table degrees is undecidable, and give a new proof of the similar fact for r.e. T–degrees. This is accomplished by introducing a new coding scheme which consists in defining the class of finite bipartite graphs with parameters.

1. Introduction. The standard method for proving undecidability of the elementary theory of a structure, used e.g. in [A,S ta] for the r.e. T–degrees and in [A,N,S92] for the r.e. wtt–degrees, actually shows undecidability of the set of sentences in the theory with a bounded number of quantifier alternations. So as a further question one can ask for an optimal bound of this kind, namely one can ask for a number k such that the Π_{k-1}–theory (or, equivalently, the Σ_{k-1}–theory) of the structure is decidable, but the Π_k–theory is undecidable. This question is of interest, since determining such a k gives more information about the theory than a straight undecidability proof, and also since mathematically relevant first-order sentences about a degree structure usually have a small number of quantifier alternations. The undecidability proofs cited above do not give an optimal bound; in particular, from [A,N,S92] only a bound of 12 on k can be derived for Th($\mathbf{R}_{\text{wtt}}$). Here we prove a bound of 4 for the p.o. $\mathbf{R}_{\text{wtt}}$ of r.e. wtt–degrees, and by an extension of the methods, for the p.o. $\mathbf{R}_{\text{T}}$ of r.e. T–degrees. For $\mathbf{R}_{\text{T}}$ this also follows from an unpublished result of Harrington and Shelah, ([Ha,Sh82]) as was observed in [A,S ta].

The standard method to prove undecidability of the elementary theory of a structure D is indirect: roughly speaking, a class **C** of structures whose theory is known to be hereditarily undecidable (say the class of finite p.o.) is defined with parameters in D. This gives an interpretation of the theory of a class $\mathbf{C}'\supseteq\mathbf{C}$ in the theory of D, which increases the number of quantifier alternation of sentences by a constant c. Since Th(**C'**) is undecidable by hereditary undecidability, Th(**D**) must be undecidable. Let Π_r–Th(**C**) be the set of sentences in Th(**C**) of the form $(\forall\ldots\forall)(\exists\ldots\exists)(\forall\ldots\forall)(\ldots)\psi$,

[1] Supported by NSF grant DMS-9100114
[2] Supported by DFG grant Ni 400/1-1

with k–1 quantifier alternations. If the fragment Π_r–Th(**C**) is known to be hereditarily undecidable (h.u.), Π_{r+k}–Th(D) must be undecidable. In [N ta], a refinement of this method to prove hereditary undecidability of fragments is described, which makes it possible to save one quantifier alternation. The notion "**C** is Σ_p–elementarily definable with parameters (Σ_p–e.d.p.) in D" is introduced for a class **C** of relational structures and p≥1, meaning that the universe of a structure in **C**, as well as its relations and their complements are uniformly definable in D with a fixed set of Σ_p–formulas. Using ideas in [Ler83], it is shown that if **C** is Σ_p–e.d.p. in D, then

(1) Π_{r+1}–Th(**C**) h.u. ⇒ Π_{r+p}–Th(**D**) h.u.

This also holds if the language of **C** does not contain equality. We view the constant p as a measure for the efficiency of the coding scheme. Then finding a small bound on the level k where the theory of the degree structure becomes undecidable is also important from the point of view of definability, since it makes it necessary to find an efficient coding scheme.
The known proofs of undecidability for fragments of the theory of a degree structure D have two components, one algebraic and the other recursion theoretic:

(A) Find a suitable class **C** of finite structures such that Π_{r+1}–Th(**C**) h.u. for some small r
(R) Define **C** with parameters in D by an efficient coding scheme.

For two degree structures, namely the structure $\mathbf{D}_T$(≤Ø') of T–degrees below Ø' and the structure $\mathbf{R}_m$ of r.e. m–degrees, it is known that the Π_3–theory is undecidable [Ler 83] and [N ta]. For the r.e. btt and tt–degrees, the best known bound is 4. In all cases, **C** is a class of finite lattices, viewed as p.o.: for $\mathbf{D}_T$(≤Ø'), the class of all finite lattices (r=2 by [Ler 83]), for $\mathbf{R}_m$ the class of finite distributive lattices (r=2 by [N ta]) and for $\mathbf{R}_{btt}$ and $\mathbf{R}_{tt}$ the class of finite partition lattices with reverse inclusion (r=3 by [N ta]). The coding scheme is as simple as it can be: represent the finite lattice as an interval in the degree structure. Thus c=1 and the Π_{r+1}–theory of the degree structure is undecidable with the same value for r. For the component (R), in the case of the r.e. degree structure one can rely on the constructions in the original undecidability proofs ([La72], [Ht,S89] and [N 92] for the r.e. m–,tt– and btt–degrees, resp.). However, for the dense degree structure $\mathbf{R}_{wtt}$ and $\mathbf{R}_T$, the stress must necessarily be more on the component (R), since one cannot expect to define an appropriate class **C** as directly. We show that the class of finite bipartite graphs, which has h.u. Σ_2– (and hence Π_3–)theory is Σ_2–e.d.p. in $\mathbf{R}_{wtt}$, which gives a bound of 4. The methodological advantage of this class is that one can first define the left and right domains separately, and then define the edge relation between them with additional parameters. (In fact the undecidability results for fragments in [N ta] for the classes of lattices mentioned above were obtained in the same way.) A bipartite graph is a structure for the language L(Le,Ri,E) where Le,Ri are unary and E is a binary predicate symbol, which satisfies the axioms

$$(\forall x)[(\mathrm{Le}\ x \leftrightarrow \neg \mathrm{Ri}\ x)] \text{ and}$$

$$(\forall x)(\forall y)[Exy \rightarrow (\mathrm{Le}\ x \leftrightarrow \mathrm{Ri}\ y)].$$

The predicates Le and Ri denote the left and the right domain of the graph. In our applications, it will be the case that $\mathrm{Le}^G=\{1,\dots,n\}$ and $\mathrm{Ri}^G=\{1',\dots,m'\}$ for some copy $\{1',\dots,m'\}$ of the numbers $\leq m$.

The coding scheme for defining finite bipartite graphs can be described in the context of uppersemilattices (u.s.l.) with least element $(P,\leq,\vee,0)$. Given a finite bipartite graph $(\{1,\dots,n\},\{1',\dots,m'\},E\}$, suppose we already know how to define some disjoint sets $A=\{a_1,\dots,a_n\}$ and $B=\{b_1,\dots,b_m\}$ corresponding to the left and right domain, and suppose that $0<d_{i,j}\leq a_i,b_j$ (the elements $d_{i,j}$ represent pairs). Moreover, suppose that, if $\hat{d}_{i,j}=\sup\{d_{r,s}:\langle r,s\rangle\neq\langle i,j\rangle\}$ then

(2) $\inf(a_i,b_j,\hat{d}_{i,j})=0$.

In this case we can code the edge relation E (and in fact arbitrary edge relations) by the parameter $c_E=\sup\{d_{i,j}:Eij'\}$ via a Σ_1–formula, since

$$Eij' \Leftrightarrow (\exists u\leq c_E)[u\neq 0 \wedge u\leq a_i,b_j].$$

To define sets A,B corresponding to the domains of the graph in $\mathbf{R}_{wtt}$, we use an algebraic notion from [A,N,S92]. For an element p of P write ncl(p) if $p\neq 0$ and there are no $r,s\in P-\{0\}$ such that $r\vee s=p$ and $r\wedge s=0$. If $\mathrm{ncl}(a_i)$ and $a_i\wedge a_j=0$ $(i\neq j)$, then by the distributivity of the u.s.l. $\mathbf{R}_{wtt}$, the complemented elements in $[0,\sup(A)]$ are exactly the suprema of subsets of A. Then A is definable with parameter $s_A=\sup(A)$ as the set of minimal complemented elements in $[0,s_A]$. If we proceed similarly for B and also build degrees $d_{i,j}$ such that $0<d_{i,j}\leq a_i,b_j$, then (2) holds automatically by the distributivity of $\mathbf{R}_{wtt}$, because $\inf(a_i,b_j,d_{r,s})=0$ for $\langle r,s\rangle\neq\langle i,j\rangle$. In [A,N,S92], to ensure $\mathrm{ncl}(a_i)$, the degrees a_i are chosen as degrees which satisfy the stronger property not to bound a minimal pair. In our case, a_i bounds a minimal pair if $m\geq 2$, since it is the case that $d_{i,1},d_{i,2}\leq a_i$. So we need a more flexible strategy for constructing degrees $\mathbf{x}$ satisfying $\mathrm{ncl}(\mathbf{x})$.

To express that x is a minimal complemented degree in $[0,s_A]$ needs a $\Sigma_2\wedge\Pi_2$–formula in the language of p.o. However, for our result we need a Σ_2–definition of A in order to apply (1) with p=2. Suppose that A corresponds in the same way as above to the left domain of the "inequality graph" $(\{1,\dots,n\},\{1',\dots,n'\},E')$, where $E'=\{\langle i,j'\rangle:i\neq j\}$. Then, by distributivity, A can be defined by a Σ_2–formula as the set of complemented degrees x in $[0,s_A]$ such that $\inf(a,y,c_{E'})=0$ for some complemented degree y in

$[0,s_B]$. To make this compatible with the coding of the given finite bipartite graph, we restrict the class of finite bipartite graphs defined: let F–BiGraphs$_1$ denote the class of finite bipartite graphs (Le,Ri,E) such that $|Le|=|Ri|$. The fact that Σ_2–Th(F–BiGraphs) is h.u. is shown in Cor. 4.5 of [N ta] by coding converging computations in finite bipartite graphs. The graphs used for coding can be expanded without effect on the computations coded by adding new isolated points to the left and right domains. This makes it possible to achieve $|Le|=|Ri|$. Hence Σ_2–Th(F–BiGraphs$_1$) is also h.u.

To extend the proof to the r.e. T–degrees, we construct all the relevant degrees as contiguous degrees, i.e. degrees which contain only one r.e. wtt–degree. This makes it possible to carry out the algebraic arguments as for $\mathbf{R}_{wtt}$. Moreover it shows that the undecidability of Th($\mathbf{R}_T$) can be obtained by a coding which is compatible with distributivity. Analyzing the coding scheme shows that in fact the Π_3–theory of the two degree structures in the language of p.o. augmented by ternary relation symbols for "$x\vee y=z$" and "$x\wedge y=0$" is undecidable.

2. The algebraic part

We now carry out the algebraic ideas introduced above in detail.

2.1 Theorem. *Let* $\mathbf{P}=(P,\leq,\vee,0)$ *be an uppersemilattice with least element* 0. *Suppose that for each* $n\geq 1$, *there exist elements* a_i,b_j *and* $d_{i,j}$ *of* $\mathbf{P}$ $(1\leq i,j\leq n)$ *such that*

(i) $0<d_{i,j}\leq a_i,b_j$ *for each* i,j, *and if* $\hat{d}_{i,j}=\sup\{d_{i',j'}:<i',j'>\neq<i,j>\}$, *then* $\inf(a_i,b_j,\hat{d}_{i,j})=0$.

(ii) *the sets* $A=\{a_1,...,a_n\}$ *and* $B=\{b_1,...,b_n\}$ *are definable from parameters via a fixed* Σ_2*–formula in the language of p.o..*

Then Π_4–Th(P,≤) *is undecidable.*

Proof. We show that the class F–BiGraphs$_1$ is Σ_2–e.d.p. in (P,≤). Then, by the fact that the Σ_2–theory and hence the Π_3–theory of F–BiGraphs$_1$ is h.u. and by (ii) of the Transfer Lemma 3.1 in [N ta], Π_4–Th(P,≤) is undecidable. Suppose a graph (Le,Ri,E) in F–BiGraphs$_1$ is given, w.l.o.g. Le={1,…,n) and Ri={1',…,n'). We let A correspond to Le via the map $i\rightarrow a_i$ and let B correspond to Ri via $j'\rightarrow b_j$. By (i), it then suffices to give a Σ_2–definition with parameters of the relations on A×B corresponding to E and $\bar{E}$=Le×Ri–E. Let

$$c_E=\sup\{d_{i,j}:Eij'\}\text{ and}$$
$$c_{\bar{E}}=\sup\{d_{i,j}:\bar{E}ij'\}.$$

Then, by (ii),

(3) $\langle i,j'\rangle \in E \Leftrightarrow (\exists u \le c_E)[u \ne 0 \wedge u \le a_i, b_j]$.

This makes it possible to define the relation corresponding to E by a Σ_2–formula. Similarly, proceed for the relation corresponding to $\bar{E}$. □

We now consider relative complements in u.s.l. Let

$$\mathrm{Compl}(x,d) \Leftrightarrow x \ne 0 \wedge (\exists y)[x \wedge y = 0 \wedge x \vee y = d].$$

Note that the binary predicate Compl can be expressed by a Σ_2–formula in the language of p.o. (and by a Σ_1–formula in the language with additional ternary supremum and infimum relations).

2.2 Lemma. *Let* **P** *be a distributive u.s.l. with* 0. *Suppose* $b, c, y_1, \ldots, y_m$, $a_1, \ldots, a_n$ *are elements of* **P**, *and let* $s = \sup_i a_i$.

(i) *If* $b \wedge y_i = 0$ *for each* i, *then* $b \wedge \sup_i y_i = 0$. *If* $\inf(b,c,y_i) = 0$ *for each* i, *then* $\inf(b,c,\sup_i y_i) = 0$.

(ii) *Suppose that* $\mathrm{ncl}(a_i)$ *and* $a_i \wedge a_j = 0$ *for* $i \ne j$. *Then*

$$\mathrm{Compl}(x,s) \Leftrightarrow x = a_F := \sup\{a_i : i \in F\} \text{ for some nonempty } F \subseteq \{1,\ldots,n\}.$$

Proof. (i) By distributivity, $0 \ne x \le \sup_i y_i \Rightarrow \neg x \wedge y_j = 0$ for some j. Then, if $0 \ne x \le b, \sup_i y_i$, $0 \ne r \le b, y_j$ for some j and r. The second part is proved similarly..

(ii). If $x = a_F$ for some $F \ne \emptyset$, then, by (i), Compl(x,s) holds via $a_{\bar{F}}$. Now suppose that $x \ne 0$, $x \wedge y = 0$ and $x \vee y = s$. Then, by distributivity, for each i, $a_i = x_1 \vee y_1$ for some $x_1 \le x, y_1 \le y$. Since a_i is not the supremum of a minimal pair, $x_1 = 0$ or $y_1 = 0$. In the first case, $a_i \wedge x \le y \wedge x = 0$, in the second case $a_i \le x$. Let $F = \{i : a_i \le x\}$. Then $a_F \le x$. But also $x \le a_F$, since $x = \sup_i b_i$ for some $b_i \le a_i$ and $b_i = 0$ if $i \notin F$. Thus $F \ne \emptyset$ and $x = a_F$. □

3. The Π_4–theory of the p.o. of r.e. wtt–degrees is undecidable

If C is an r.e. set, we write Ncl(C) if $\mathrm{ncl}(\deg_{wtt}(C))$ holds, i.e.

$\mathrm{Ncl}(C) \Leftrightarrow (\forall \text{r.e. } X,Y)[C \equiv_T X \oplus Y \Rightarrow X \text{ recursive} \vee Y \text{ recursive} \vee$

$(\exists \text{nonrecursive } S)[S \le_{wtt} X, Y]]$

3.1 Main Lemma. *Let* $n, m \ge 1$. *Then there exist r.e. sets* A_i, B_j *and nonrecursive r.e. sets* $D_{i,j} \le_{wtt} A_i, B_j$. $(1 \le i \le n, 1 \le j \le m)$ *such that* $\mathrm{Ncl}(A_i)$ and $\mathrm{Ncl}(B_j)$ *holds, and for all distinct* i, i' $(1 \le i, i' \le n)$ *and all distinct* j, j' $(1 \le j, j' \le m)$ $A_i, A_{i'}$ *as well as* $B_j, B_{j'}$ *form a T–minimal pair.*

3.2 Theorem. Π_4–$\mathrm{Th}(\mathbf{R}_{wtt}, \le)$ *is undecidable.*

Proof. Let $\mathbf{x} = \deg_{wtt}(X)$ for each set X mentioned in the Lemma above. We show that the wtt–degrees $\mathbf{a}_i,\mathbf{b}_i$ and $\mathbf{d}_{i,j}$ satisfy the hypotheses of Theorem 2.1. For (i), note that, by distributivity, $\inf(\mathbf{a}_i,\mathbf{b}_j,\widehat{\mathbf{d}}_{i,j})=\mathbf{0}$ follows from the fact that $\inf(\mathbf{a}_i,\mathbf{b}_j,\mathbf{d}_{i',j'})=\mathbf{0}$ ($<i',j'>\neq<i,j>$) and (i) of Lemma 2.2. For (ii), to show the definability of $\mathbf{A}=\{\mathbf{a}_1,...,\mathbf{a}_n\}$ and $\mathbf{B}=\{\mathbf{b}_1,...,\mathbf{b}_n\}$ via a fixed Σ_2–formula, we make again use of the possibility to code bipartite graphs with left domain **A** and right domain **B** as in (3) (as described in the introduction): The parameter

$$\mathbf{u}=\sup\{\mathbf{d}_{i,j}:i\neq j\}$$

codes the relation $\{<\mathbf{a}_i,\mathbf{b}_j>:i\neq j\}$. Let $\mathbf{s}_A=\sup_i\mathbf{a}_i$ and $\mathbf{s}_B=\sup_i\mathbf{b}_i$. We claim that

$$\mathbf{a}\in\mathbf{A} \Leftrightarrow \mathrm{Compl}(\mathbf{a},\mathbf{s}_A) \wedge (\exists\mathbf{y})[\mathrm{Compl}(\mathbf{y},\mathbf{s}_B) \wedge \inf(\mathbf{a},\mathbf{y},\mathbf{u})=\mathbf{0}].$$

Since $\inf(\mathbf{a},\mathbf{y},\mathbf{u})=\mathbf{0}$ is expressible by the Π_1–formula $(\forall z)(\forall w)[z\leq a,y,u \Rightarrow z\leq w]$, this gives a Σ_2–formula which defines **A** with the parameters $\mathbf{s}_A,\mathbf{s}_B$ and **u**. By symmetry, the same formula defines **B** with parameters $\mathbf{s}_B,\mathbf{s}_A$ and **u**.
For the direction from left to right, if $\mathbf{a}=\mathbf{a}_i$, let $\mathbf{y}=\mathbf{b}_i$. For the other direction, suppose that the right hand side holds. If $\mathbf{a}\notin\mathbf{A}$, then by (ii) of Lemma 2.2 $\mathbf{a}_i,\mathbf{a}_j\leq\mathbf{a}$ for some $i\neq j$ and $\mathbf{b}_k\leq\mathbf{y}$ for some k. But then $k\neq i$ or $k\neq j$, so $\mathbf{d}_{i,k}$ or $\mathbf{d}_{j,k}$ is below $\mathbf{a},\mathbf{y}$ and **u**.

Proof of the Main Lemma 3.1. We use a finitely branching tree **T** of strategies. Each node on **T** is an R–strategy for some requirement R. The tree **T** is defined inductively as a set of strings of possible outcomes of a strategy. The outcomes are linearly ordered with rightmost element <r>, and the ordering on **T** is given by

$$\alpha\leq\beta \Leftrightarrow \alpha\subseteq\beta \vee \alpha<_L\beta.$$

As usual, during stage s inductively we define an approximation δ_s to the true path. At substage p<s of stage s, the R–strategy σ, $\sigma=\delta_s|p$ becomes accessible, performs some action and determines its outcome <X>. We then let $\delta_s(p)=$<X>. If $\sigma\subseteq\delta_s$ or s=0, s is called a *σ–stage*. Since **T** is finitely branching, there exists a *true path,* namely a path f through **T** such that, for each e, if $\sigma=f|e$,

$$(a.e.s)[\sigma\leq\delta_s] \text{ and}$$
$$(\exists^\infty s)[\sigma\subseteq\delta_s].$$

We build r.e. sets A_i,B_j ($1\leq i\leq n$, $1\leq j\leq m$) and nonrecursive r.e. sets $D_{i,j}\leq_{wtt}A_i,B_j$. The requirements fall into three groups. To make $D_{i,j}$ nonrecursive, we satisfy

$$P_e^{i,j}: D_{i,j}\neq\{e\}$$

by the standard strategy. For $D_{i,j}\leq_{wtt}A_i,B_j$, if a number x is enumerated into $D_{i,j}$, we enumerate numbers ≤x into A_i and into B_j.

Fix a set C among the sets A_i and B_j. To ensure that Ncl(C) holds, we adapt A.□Lachlan's proof of the Non–Diamond Theorem ([La□66]). A strategy of the type used here relies on the hypothesis

(4) $C=\Phi(X\oplus Y) \wedge X\oplus Y=\Psi(C)$,

where X,Y are given r.e. sets and Φ,Ψ are T-functionals, and has the goal to build a nonrecursive r.e. set S wtt–below X and Y, or to show that X or Y are recursive. Let L=<C,Φ,Ψ,X,Y>. This goal is accomplished collectively by a strategy Q_L, substrategies $Q_{L,k}$ and strategies $Q_{L,k,h}$ which are substrategies of $Q_{L,k}$ (k,h∈ω). The candidates for S are sets E_L and $F_{L,k}$. The strategy Q_L builds a wtt–reduction of E_L to X and Y. For each k, $Q_{L,k}$ tries to show that E≠{k} or that X is recursive. There is also a marginal case where $Q_{L,k}$ acts only finitely often, in which case Y must be recursive. If all these fail for some k, $Q_{L,k}$ builds a wtt–reduction of $F_{L,k}$ to X,Y, and, for each h, $Q_{L,k,h}$ succeeds in showing that $F_{L,k}\neq\{h\}$ or that Y is recursive. Thus the requirements in the second group are

Q_L: $C=\Phi(X\oplus Y) \wedge X\oplus Y=\Psi(C) \Rightarrow E_L\leq_{wtt}X,Y$

$Q_{L,k}$: $C=\Phi(X\oplus Y) \wedge X\oplus Y=\Psi(C) \Rightarrow E_L\neq\{k\}$ ∨ X recursive ∨ Y recursive ∨ $F_{L,k}\leq_{wtt}X,Y$

$Q_{L,k,l}$: $C=\Phi(X\oplus Y) \wedge X\oplus Y=\Psi(C) \wedge E_L=\{k\} \Rightarrow F_k\neq\{h\}$ ∨ Y recursive

In the proof of the Non–Diamond Theorem, L is fixed (where C is replaced by the creative set K and a K-change is forced indirectly by using the recursion theorem). In our adapted strategy, one works with a triple of numbers m,x,y such that $\Phi(X\oplus Y)(m)=0$ and $x,y>\phi(m)$. The number x is a candidate for showing E≠{k}, and y is a candidate for showing $F_k\neq\{h\}$. If X has changed below x and Y has changed below y, then m is enumerated into C. Then X or Y must change below min(x,y) again; in the first case y is enumerated into F_k, in the second, x into E. The reductions of E to Y and of F_k to X are built by usual permitting, while $E\leq_{wtt}X$ and $F_k\leq_{wtt}Y$ are built by delayed permitting: for the moment call the reduction of E to X, which we build, Θ. It will be the case that the use of $\Theta^X(z)$ is z, so Θ is a bounded reduction. We can only enumerate z into E while $\Theta^X(z)$ is undefined. If $\Theta^X(z)[s]$ is defined and X|z changes, we may declare $\Theta^X(z)$ to be undefined, but we must redefine it at a later stage t to the value $E_t(z)$ and such a stage t must be bounded by stage g(s) for a recursive function g. Then if X|z has settled down at s,

$E(z)=\Theta^X(z)[g(s)]$, so $E\leq_{wtt}X$. In the following we will not name the wtt–functionals we build explicitly, but rather say that "$E\leq_{wtt}X(z)$ is declared to be undefined" etc.

We discuss how the strategies are implemented in our tree construction. A Q_L–strategy α guesses at (4) in the following way. Let

$$\text{length}_{L,1}(s)=\max\{x:(\forall y<x)[C(y)=\Phi(X\oplus Y)(y)[s]]\} \text{ and}$$

$$\text{length}_{L,2}(s)=\max\{x:(\forall y<x)[X\oplus Y(y)=\Psi(C)(y)[s]]\}.$$

If t is the greatest α^<∞> stage <s, then α gives outcome <∞> if both $\text{length}_{L,1}$ and $\text{length}_{L,2}$ have a bigger value than at t. Below α^<∞>, there are nodes β working on $Q_{L,k}$, which have the possible outcomes <E≠{k}>,<E={k}>,<X rec> and <r>. Below the nodes β^<E={k}>, there are nodes γ working on $Q_{L,k,h}$, which have the outcomes <F_k≠{h}>, <Y rec> and <r>.

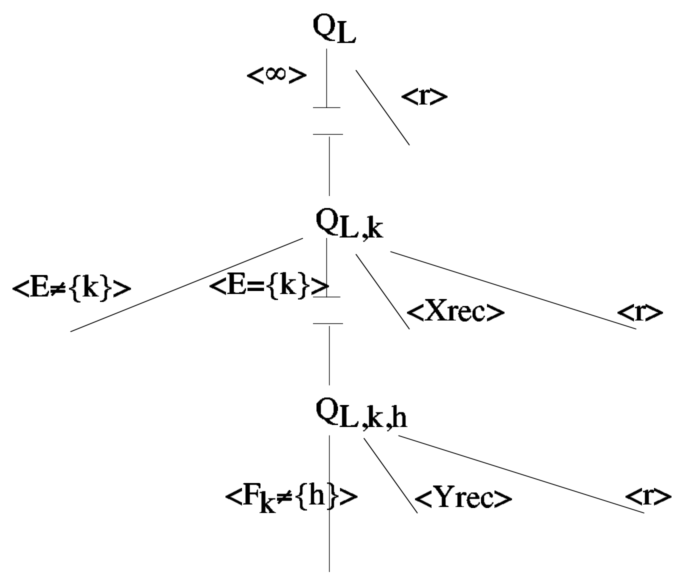

Fig. 1. A cooperating triple of Q–strategies

We call nodes $\alpha\subset\beta\subset\gamma$ as above a *cooperating triple of Q–strategies.* We drop the subscript L. The $Q_{k,h}$–strategy γ works with a fixed small number $m<\text{length}_1(s)$, and assumes that the computation $\Phi(X\oplus Y)(m)=0$ does not change unless m is enumerated into C. At γ–stages s, $Q_{k,h}$ appoints larger and larger numbers y. At β–stages s, the Q_k–strategy checks whether, since the preceding β–stage t, there was a $Y|y$ change for some number $y<\text{length}_2(s)$ already appointed at t such that $\{h\}_t(y)=0$, in which case β is activated by the $Q_{k,h}$–strategy. (Actually, β is activated by the highest priority such $Q_{k,h'}$–strategy for all h'.) It declares $F_k\leq_{wtt}Y(y)$ to be undefined and starts appointing larger and larger x at β–stages s. We say that the $Q_{k,h}$–strategy γ is now *run by node* β. Now, similarly, at

α^<∞>–stages s the Q–strategy checks whether, since the last α^<∞>–stage t, there was an X|x change for some number x<$length_2$(s) already appointed at t by the Q_k–strategy such that $\{k\}_t(x)=0$. In this case, the Q–strategy is activated by Q_k: Q enumerates m into C and declares $E\leq_{wtt}X(x)$ to be undefined. We call this a secondary C–enumeration, since it was induced by C–enumerations which allowed Y|y and X|x to change. The strategy γ is now *run by node* α. At the next α^<∞>–stage t, $\Phi(X\oplus Y)(m)$ is defined again, now with value 1, so there was a change of X|min(x,y) or of Y|min(x,y). In the first case y is enumerated into F_k, in the second case, x into E. Q also redefines $E\leq_{wtt}X(x)$ and $F_k\leq_{wtt}Y(y)$ to the correct values. Note that redefining a functional always is carried out by a fixed node in order to make the stage when it is redefined recursively bounded. This is why we need the activation procedures. The conditions x<$length_2$(s) and y<$length_2$(s) above are needed to meet the minimal pair requirements.

The outcomes of the strategies β,γ are determined as follows: if, at s, the strategy α enumerates y into F_k, it sends an instruction to the Q_k–strategy β to give <E={k}> as outcome at the next β–stage, and it sends an instruction to the $Q_{k,h}$–strategy γ to give <$F_k\neq$\{h\}> as outcome from the next γ–stage on. If x is enumerated into E, strategy α sends an instruction to Q_k to give <E$\neq$\{k\}> from now on. If Q_k has found a new realized candidate x, i.e. an already appointed number x<$length_2$(s)such that {k}(x)=0, but does not receive a message from Q, it gives <Xrec> as outcome. In the similar situation, $Q_{k,h}$ gives <Yrec> as outcome.

We call the $Q_{L,k,h}$– and the $P_e^{i,j}$–strategies *primary strategies*. A strategy is *initialized* by setting its program to the initial state, declaring all its parameters undefined, cancelling all instructions it may have received and redefining all values of functionals the strategy may have declared undefined. For a $Q_{L,k,h}$–strategy we cancel a possible run on a node higher up. Candidates chosen by a primary strategy must be bigger than the stage number s_{init} when it was initialized the last time. So the enumeration of such a candidate cannot destroy any computation that already existed at stage s_{init}.

We now discuss the strategies to make A_i,A_j minimal pairs for i$\neq$j. For B_i,B_j, the strategies are similar. We satisfy the requirements

$$N_e^{A_i,A_j}:\ Z=\{e\}^{A_i}=\{e\}^{A_j} \Rightarrow Z \text{ recursive}.$$

Let $length_e^{A_i,A_j}(s)=\max\{x:(\forall y<x)[\{e\}^{A_i}(y)=\{e\}^{A_j}(y)[s]]\}$

and suppose the node μ works on $N_e^{A_i,A_j}$. If at a μ–stage s>0, $length_e^{A_i,A_j}(s)$ is bigger than at the preceding μ^<∞>–stage, then μ gives <∞> as outcome and initializes all strategies $>_L\mu$^<∞>.

The standard minimal pair strategy relies on the following. Let s<t be consecutive μ^<∞> stages. Then, for k=i or k=j

(5) if $x<\mathrm{length}_e^{A_i,A_j}(s)$, then the computation $\{e\}^{A_k}(x)[s]$ is not destroyed at any stage t', $s\leq t'<t$.

Then $Z=\{e\}^{A_i}=\{e\}^{A_j}$ implies $Z(y)=\{e\}^{A_i}(y)[s]$ for the first μ^<∞>–stage s such that $\mathrm{length}_e^{A_i,A_j}(s)>y$, since from one μ^<∞>–stage s' to the next one side of $\{e\}^{A_i}(y)=\{e\}^{A_j}(y)[s']$ is preserved.

To make (5) true, we first ensure that at s, only one of the sets A_i,A_j is enumerated into , say A_i. Then, by initialization at s, no primary A_j–enumeration at a stage t', $s\leq t'<t$ can violate (5). We will be able to show that the same holds for secondary A_j–enumerations which may be carried out by strategies Q_L, $L=\langle A_j,\Phi,\Psi,X,Y\rangle$. The argument is that, since we hold $X\oplus Y_s(z)=\Psi(A_j)(z)[s]$ for each $z<\mathrm{length}_{L,2}(s)$ (by initialization at s), an activation of Q_L which might result in a secondary enumeration of a number <s into A_j cannot occur at a stage t', $s\leq t'<t$.

To make sure that only one set among $A_1,\dots,A_n$, say, is enumerated into at each stage we proceed as follows. If at substage t of stage s a primary strategy σ wants to enumerate a number z into a set S among $A_i,B_j,D_{i,j}$ ($1\leq i\leq n,1\leq j\leq m$), then instead it enumerates $\langle z,\sigma\rangle$ into an auxiliary set $\widetilde{S}$ which was empty at the beginning of the stage s. At the end of stage s, the strategy with the highest priority (i.e. the one with minimal σ) which enumerated into $\widetilde{S}$ succeeds and initializes all the others. If σ is a $P_e^{i,j}$–strategy, this still ensures that an enumeration into $D_{i,j}$ is permitted by A_i,B_j. Note that, by activation, σ may be left of δ_s.

We now describe the construction formally. Order the requirements in a priority list so that, for each L,k,h, Q_L precedes $Q_{L,k}$ and $Q_{L,k}$ precedes $Q_{L,k,h}$. By induction on n, define the n-th level $T^{[n]}$ of the tree of strategies, and what it means for a strategy to receive attention along $\xi\in T^{[n]}$. A string $\xi\in T^{[n]}$ *is an* R–strategy for the highest priority requirement R which does not receive attention along ξ.

Let $T^{[0]}=\{\lambda\}$. A requirement R *receives attention along* a string $\xi\in T^{[n]}$ if some $\eta\subset\xi$ is an R–strategy or

(i) R is $Q_{L,k}$ or $Q_{L,k,h}$ and some α, α^<r>$\subseteq\xi$, is a Q_L–strategy or

(ii) R is $Q_{L,k,h}$ and there is a $Q_{L,k}$–strategy β such that

β^<E≠{k}> or β^<Xrec> $\subseteq\xi$ or β^<r> $\subseteq\xi$

Suppose $\xi\in T^{[n]}$ is an R–strategy. The immediate successors of ξ, from left to right, are ξ^X, where X is a possible outcome of R. These possible outcomes are

<r>	if R is P_e
<∞>,<r>	if R is N_e or Q_L
<E≠{k}>,<E={k}>,<X rec>,<r>	if R is $Q_{L,k}$
<F_k≠{h}>,<Y rec>,<r>	if R is $Q_{L,k,h}$.

We describe the actions of an R–strategy σ in form of PASCAL–like programs. Whenever σ is accessible, it carries out one step of its program, thereby changing the values of its parameters and determining its outcome. The step to be carried out is given by an instruction received from a node $\subset\sigma$, or has been determined at the last σ–stage if there is no such instruction. If no outcome is specified, we assume the default value <r>. We let s_{init} be the last stage where σ was initialized, and let s be the current stage. A number x is *unused for the strategy* σ if $x \geq s_{init}$ and x is not currently appointed by any other strategy which is not $>_L \sigma$.

Construction.

Stage 0. Let $\delta_0=\lambda$. Initialize all strategies.

Stage s, s>0. Carry out substage 0.

Substage p: Let $\alpha=\delta_s|p$. Carry out one step of the program of node α. Let $\delta_s(p)$=outcome(α). If p<s carry out substage p+1. Else carry out the terminating substage.

Terminating substage. For each set S among A_i,B_j and $D_{i,j}$ do the following. Let σ be the minimal node such that some $\langle x,\sigma\rangle$ has been enumerated into $\widetilde{S}$ at stage s. If σ is defined, enumerate each corresponding x into S. Initialize all strategies $>\sigma$.

Program for a $P_e^{i,j}$–strategy σ. Parameter: x

APPOINT	Let x be an unused number
REALIZE	IF $\{e\}_s(x)=0$ BEGIN enumerate $\langle x,\sigma\rangle$ into $\widetilde{D}_{i,j}$, $\widetilde{A}_i$ and $\widetilde{B}_j$; goto WIN END ELSE goto REALIZE
WIN	goto WIN.

Program for a $Q_{L,k,h}$-strategy γ. Parameters: m,y

(y is the maximal candidate for $F_k \neq \{h\}$ such that $\{h\}(y)=0$).

START	Let m be an unused number
WAIT Φ	IF m<$\text{length}_{L,1}(s)$ goto APPOINT (in this case $\Phi(X\oplus Y)(m)=0$ as m was not enumerated so far) ELSE goto WAIT□Φ
APPOINT	Appoint an unused number y'>use$\Phi(X\oplus Y)(m)$
REALIZE	IF $\{h\}(y')=0 \wedge y'<\text{length}_{L,2}(s)$ for some y'>y which was appointed at a stage $s'>s_{init}$ BEGIN y:=y'; Outcome:=<Yrec> END; Goto APPOINT
WIN	Outcome:=<$F_k \neq \{h\}$>; Goto WIN

Program for a $Q_{L,k}$-strategy β. Parameters: γ,m,y,x

(γ is the strategy currently run on node β, m,y are the parameters of γ, and x is the maximal candidate for $E\neq\{k\}$ such that $\{k\}(x)=0$)

Let $t<s$ be the preceding β–stage.

APPOINT 1. (Update run). A strategy γ' *wants to activate* β if γ' is a $Q_{L,k,h'}$–strategy, and at t, y(γ') was defined and $Y_t|y(\gamma')\neq Y_s|y(\gamma')$. Let γ_0 be the <-minimal such node. IF $\gamma_0<\gamma$ OR (γ is undefined AND γ_0 is defined)

BEGIN initialize γ (thereby redefining $F_k\leq_{wtt}Y(y(\gamma))$; $\gamma:=\gamma_0$; $m:=m(\gamma_0)$; $y:=y(\gamma_0)$; Declare $F_k\leq_{wtt}Y(y)$ to be undefined END;

2. (Appointing). Appoint an unused number $x'>\phi(m)$

REALIZE 1. (Update run). As 1. in APPOINT. IF the value of γ changed goto APPOINT;

2. IF$\{k\}(x')=0 \wedge x'<\mathrm{length}_{L,2}(s)$ for some $x'>x$ which was appointed at a stage $s'>s_{init}$

BEGIN x:=x'; Outcome:=<Xrec>, Goto APPOINT END;

ELSE Goto REALIZE

FORWARD Outcome:=$\langle E=\{k\}\rangle$; Goto APPOINT (this state is only reached by instruction)

WIN Outcome:=$\langle E\neq\{k\}\rangle$; Goto WIN

Program for a Q_L-strategy α ($L=\langle C,\Phi,\Psi,X,Y\rangle$).

Parameters: oldlength1,oldlength2,β,γ,m,x,y

Let $t<s$ the preceding $\alpha\hat{}\langle\infty\rangle$–stage.

START RUN 1. (Test if s is expansionary). IF $\mathrm{length}_{L,1}(s)\leq$oldlength1 OR $\mathrm{length}_{L,2}(s)\leq$oldlength2 goto START RUN

ELSE BEGIN Outcome:=$\langle\infty\rangle$; oldlength1:=$\mathrm{length}_{L,1}(s)$; oldlength2:=$\mathrm{length}_{L,2}(s)$ END

2. (Check) For each $Q_{L,k,h}$–strategy γ', $|\gamma'|\leq s$, $\alpha\hat{}\langle\infty\rangle\subseteq\gamma'$, $\gamma'\neq\gamma$ if γ is defined, which is not in state WIN see if the computation $\Phi(X\oplus Y)(m(\gamma'))$ has changed at some stage s', $t<s'\leq s$. If so, send the instruction "Continue at WAIT Φ" to node γ'. (Note that still $m(\gamma)\notin C$.)

3. (Start a new run) A $Q_{L,k}$-strategy β *wants to activate* α if at t, x(β) was defined and $X_t|x(\beta)\neq X_s|x(\beta)$. IF no such β exists GOTO START RUN ELSE BEGIN

Let β be the strategy which wants to activate α so that γ(β) is <-minimal; γ:=γ(β), m:=m(β),x:=x(β),y:=y(β); Enumerate m into $\tilde{C}$; Declare $E\leq_{wtt}X(x)$ to be undefined END;

END RUN 1.,2. as in START RUN

3. (Terminate run) Redefine $E \leq_{wtt} X(x)$ and $F_k \leq_{wtt} Y(y)$ to the correct value;
IF $X_t|x \neq X_s|x$ BEGIN enumerate y into F_k; send the instruction "Continue at FORWARD" to node β and "Continue at WIN" to node γ END
ELSE (Y|y has changed) BEGIN enumerate x into E; send the instruction "Continue at WIN" to node β END;
Goto START RUN

Program for an $N_e^{A_i,A_j}$ or $N_e^{B_i,B_j}$-strategy μ. Parameter: oldlength.
Let t<s is the preceding μ^<∞>–stage.

ACT IF $\text{length}_e(s)$>oldlength BEGIN oldlength:=$\text{length}_e(s)$; outcome:=<∞>;
initialize all strategies $>_L \mu$^<∞> END;
goto ACT

Verification.
Note that each primary strategy causes an enumeration of at most one number into an associated set E. Since initialization of a strategy $\sigma \subset f$ is only caused by an N_e–strategy or, at a terminating substage, by a primary strategy ν, a strategy $\sigma \subset f$ is initialized only finitely often.

Lemma 1. $D_{i,j} \leq_{wtt} A_i,B_j$ *and* $D_{i,j}$ *is nonrecursive.*

Proof. $D_{i,j} \leq_{wtt} A_i,B_j$ is immediate by the program for the requirements $P_e^{i,j}$. To show that $D_{i,j} \neq \{e\}$, let $\sigma \subset f$ be a $P_e^{i,j}$–strategy. Let s_0 be a stage such that σ is not initialized after s_0. Then, from the σ–stage following s_0 on the parameter x is defined. If $\neg\{e\}(x)=0$ then x is not enumerated into $D_{i,j}$. If $\{e\}_t(x)=0$ for a minimal σ–stage t where x is appointed, then x is enumerated at t.

Lemma 2. *Fix* L.
(i) *If* γ^<Yrec>⊂f *for some* $Q_{L,k,h}$–strategy γ, *then* Y *is recursive.*
(ii) *If* β^<Xrec>⊂f *for some* $Q_{L,k}$–strategy β, *then* X *is recursive.*

Proof. (i). Suppose that γ is a strategy of minimal length associated with L such that γ^<Yrec>⊂f. Let γ belong to the cooperating triple of Q–strategies α⊂β⊂γ, and let s_0 be a stage such that γ^<Yrec>$\leq \delta_s$ for $s \geq s_0$ and

(6) if σ<γ is a primary strategy then it does not cause an enumeration into any set at a stage $s \geq s_0$

(7) if γ' is a $Q_{L,k,h'}$–strategy such that $\gamma' <_L \gamma$ or γ'<r>⊆γ, and if $y(\gamma')[s_0]$

is defined, then $Y|y[s_0]=Y|y$.

Note that for γ' as in (7), $y(\gamma')$ reaches a maximum value, so s_0 exists. By (7) and the minimality of γ, a run of the strategy γ on node β has highest priority at all stages $s\geq s_0$, and by (6) the strategy γ is not initialized at the terminating substage of s. Then $\gamma^\wedge\langle Yrec\rangle\subset f$ implies that $y(\gamma)[s]$ is increasing for $s\geq s_0$ with limit ∞, and we can compute Y as follows. Given z, compute a β–stage $s\geq s_0$ such that $y(\gamma)[s]>z$. Then $Y(z)$ has the final value: any $Y|z+1$–change after s would lead to an activation of the $Q_{L,k}$ strategy β. Since $\beta^\wedge\langle E=\{k\}\rangle\subset f$, each such run of the strategy γ on node β eventually leads to an activation of the Q_L strategy α, which will send an instruction "Continue at WIN" to the strategy γ. This contradicts $\gamma^\wedge\langle Yrec\rangle\subset f$.

(ii). First, if $\beta^\wedge\langle Xrec\rangle\subset f$ then for some γ, there is eventually a permanent run of the strategy γ on the node β . For let $s(\beta)$ be a stage such that $\beta^\wedge\langle Xrec\rangle\leq\delta_s$ for $s\geq s(\beta)$. Now $\beta^\wedge\langle Xrec\rangle\subseteq\delta_s$ only if a $Q_{L,k,h}$–strategy γ is run on node β. Since $\beta^\wedge\langle E=\{k\}\rangle\subseteq\gamma$ for such a γ, γ was accessible before s_0. Hence, for $s\geq s(\beta)$, after finitely many initializations or changes to a run of a higher priority strategy, γ stabilizes. We denote this strategy by $\gamma*(\beta)$

Let β be the strategy of minimal length associated with L such that $\beta^\wedge\langle Xrec\rangle\subset f$. Then $\gamma*(\beta)$ is< minimal among all $\gamma*(\beta')$, β' a strategy associated with L such that $\beta'^\wedge\langle Xrec\rangle\subset f$, as $\beta'^\wedge\langle E=\{k\}\rangle\subseteq\gamma*(\beta')$. Choose $s_0\geq s(\beta)$ such that (6) and (7) hold (by $\gamma<_L f$) and, if $\gamma'<\gamma*(\beta)$ is a $Q_{L,k',h'}$–strategy then γ' is not run at stage $s\geq s_0$ on any node β' such that $\beta'^\wedge\langle Xrec\rangle\subset f$. Let $\alpha\subset\beta$ be the Q_L–strategy. Again a run of the strategy γ on node β or α has highest priority at a stage $s\geq s_0$ and γ cannot be initialized at the terminating substage of s. To compute X, given z, compute an α–stage $s\geq s_1$ such that $x(\beta)[s]>z$. Then $X(z)$ has the final value: any $X|z+1$–change after s would lead to an activation of the Q_L–strategy α at the next α–stage. By (6) Q_L will succeed in enumerating a number m into C, and hence will send an instruction to the node β at the following α–stage. This causes $\delta_t<_L\beta^\wedge\langle Xrec\rangle$ for some $t>s_0$, contradiction.

Lemma 3. *For each* L,k,h, *the* requirements Q_L, $Q_{L,k}$ *and* $Q_{L,k,h}$ *are met.*

Proof. Suppose that $C=\Phi(X\oplus Y)\wedge X\oplus Y=\Psi(C)$. We drop the subscript L.

1. To show that Q is met, suppose that $\alpha\subset f$ for the Q–strategy α. Then $\alpha^\wedge\langle\infty\rangle\subset f$. It is obvious that $E\leq_{wtt}Y$, since a number x is enumerated into E only if $Y|x$ changed since the last $\alpha^\wedge\langle\infty\rangle$–stage. For $E\leq_{wtt}X$, we need to show that if the

wtt–reduction $E\leq_{wtt}X(x)$ is declared to be undefined at stage s by some $Q_{k'}$–strategy β, it is redefined at a later stage which can be bounded effectively in s. Let $s_1<s_2$ be the α^<∞>–stages following s such that α is in state START RUN at stage s_1. If the $\gamma(\beta)[s]$–strategy has been initialized by the end of stage s_2, the reduction is redefined. Otherwise, at s_1 β has the highest priority for activating α. So the reduction is redefined by the end of stage s_2 through the Q_L–strategy.

2. Suppose that $\beta\subset f$ for the Q_L–strategy β. We go through the possible true outcomes of β. If β^<E≠{k}>⊂f, then at some stage we must have diagonalized against E={k}, so Q_k is satisfied. Now suppose β^<E={k}>⊂f. We show $F_k\leq_{wtt}X,Y$. As above, $F_k\leq_{wtt}X$ is immediate. For $F_k\leq_{wtt}Y$, suppose that $F_k\leq_{wtt}Y(y)$ has been declared undefined at a β–stage s, and let s' be the least β^<E={k}>–stage >s. Then the strategy β activated α at some stage t, s<t<s'. If t'≤s' is the least α^<∞>–stage >t, then at the end of stage t', either the $\gamma(\beta)[s]$–strategy has been initialized, or $F_k\leq_{wtt}Y(y)$ has been redefined by the Q–strategy.

The case β^<Xrec>⊂f was covered in Lemma 2. Finally assume that β^<r>⊂f. Note that if Y is nonrecursive, infinitely often there is a γ–strategy $Q_{L,k,h}$ which wants to activate β (since there are infinitely many h such that {h} is constant zero). Since β never gives outcome <Xrec> from some stage on, there must be a number x' appointed by β such that ¬{k}(x')=0. Such a number is not enumerated into E, so E≠{k}.

3. Suppose that γ works on $Q_{k,h}$, $\gamma\subset f$. The case γ^<F_k≠{h}>⊂f is treated as the case β^<E≠{k}>⊂f above, and the case γ^<Yrec>⊂f was covered in Lemma 1. Suppose γ^<r>⊂f. Since the strategy γ is initialized only finitely often, from some stage on γ has a stable parameter m. Then, since $C=\Phi(X\oplus Y)$, from some later stage on the strategy is not send to WAIT Φ. Since γ gives outcome <Yrec> only finitely often, from some stage on there must be a number $y'\notin F_k$ appointed by β such that ¬{h}(y')=0. Therefore F_k≠{h}.

Lemma 4. *The requirements* $N_e^{A_i,A_j}$ *are met.*

Proof. Suppose $Z=\{e\}^{A_i}=\{e\}^{A_j}$ and that μ works on $N_e^{A_i,A_j}$. Then μ^<∞>⊂f. Let s_0 be a stage such that, for $s\geq s_0$, μ^<∞>$\leq\delta_s$ and no primary strategy σ<μ^<∞> causes an enumeration at any stage $s\geq s_0$. We verify (5), referring to the discussion there. Suppose that s<t are consecutive μ^<∞> stages, $s\geq s_0$. By the construction, at most one set among A_i,A_j is enumerated into at any stage. Then, for (5), it suffices to show that, if A_j is not enumerated into at stage s, then no secondary enumeration of any number m<s into A_j takes place at a stage t', s<t'<t. Suppose for a

contradiction that $\alpha\subset\beta\subset\gamma$ is a cooperating triple of Q–strategies concerned with A_j and that a run of γ on node α causes an enumeration of the number $m<s$ into A_j at the end of stage t'. By the choice of s_0 and since the γ–strategy is not initialized at stage s, $\mu\wedge<\infty>\subseteq\gamma$. Moreover $\alpha\wedge<\infty>\subseteq\mu$, since $\alpha\wedge<\infty>\subseteq\gamma$, $\alpha\neq\mu$, and t' is an $\alpha\wedge<\infty>$–stage but no $\mu\wedge<\infty>$ stage. Now $\alpha\wedge<\infty>\subseteq\mu$ implies that $X\oplus Y$ cannot change between s and t:

(8) if $z<\text{length}_2(s)$, then $X\oplus Y_s(z)=X\oplus Y_t(z)$.

Else there can be no further $\alpha\wedge<\infty>$–stage after s, since $A_j|s$ does not change at s to correct $\Psi(A_j)(z)[s]$, nor can it (by initialization through μ at s) change during the $\mu\wedge<r>$–stages following s.

Case 1. $\mu\wedge<\infty>\subseteq\beta$ Then the parameters of the strategy β have the same values x,y,m,γ at the end of any stage s', $s\leq s'<t'$, and $x<\text{length}_2(s)$. Since s,t' are $\alpha\wedge<\infty>$–stages and α was activated at t', $X_s|x\neq X_{t'}|x$, contrary to (8).

Case 2. $\beta\wedge<E=\{k\}>\subseteq\mu$. Then the parameters of the strategy γ have the same values y,m at the end of any stage s', $s\leq s'<t'$, and again $y<\text{length}_2(s)$. The activation of the run of γ on β which is terminated at t' takes place at a stage $s''\leq s$, since it cannot be that $Y_{s'}|y\neq Y_{s'+1}|y$ for a stage s', $s\leq s'\leq t'$ by (8). But at stage s, β gives the outcome $<E=\{k\}>$, so actually $s''<s$, and by the end of stage s α already has terminated the run of the strategy run on β from stage s'' on, a contradiction.

4. The Π_4–theory of the r.e. T–degrees

In [Ld,Sa75], the transfer method to carry over results from $\mathbf{R}_{wtt}$ to $\mathbf{R}_T$, using contiguous degrees, was introduced. We give another application of this method: we ensure that all sets involved in the Main Lemma 3.1. have contiguous degree, thereby giving an alternative proof that Π_4–Th($\mathbf{R}_T$,≤) is undecidable. Note that contiguous degrees can be simultaneously viewed as r.e. wtt– and T–degrees, and observe the following two facts:

(9) If $\mathbf{y}_i$ $(1\leq i\leq k)$ and $\mathbf{s}$ are contiguous, then
$$\mathbf{s}=\sup_{wtt}\{\mathbf{y}_i:1\leq i\leq k\} \Leftrightarrow \mathbf{s}=\sup_T\{\mathbf{y}_i:1\leq i\leq k\}.$$

(10) If $\mathbf{y}_i$ $(1\leq i\leq k)$ are contiguous then
$$\inf_{wtt}\{\mathbf{y}_i:1\leq i\leq k\}=\mathbf{0} \Leftrightarrow \inf_T\{\mathbf{y}_i:1\leq i\leq k\}=\mathbf{0}.$$

4.1 Main Lemma. *There exist disjoint r.e. sets* A_i, *disjoint r.e. sets* B_j *and* *sets* $D_{i,j} \leq_{wtt} A_i, B_j$ *satisfying the conclusions of the Main Lemma 3.1 such that in addition, the sets* $A_F = \mathbf{U}_{i \in F} A_i$, $B_G = \mathbf{U}_{j \in G} B_j$ $(\emptyset \neq F, G \subseteq \{1, \ldots, n\})$ *and*

$$D_E = \mathbf{U}\{D_{i,j}: <i,j> E\} \quad (\emptyset \neq E \subseteq \{1, \ldots, n\} \times \{1, \ldots, m\})$$

have contiguous T–degrees.

4.2 Theorem (cf. [A,S93]). Π_4–Th$(\mathbf{R}_T, \leq)$ *is undecidable.*

Proof. Let $\mathbf{x} = \deg_T(X)$ for each set X mentioned in the Main Lemma 4.1. Note that $\deg_{wtt}(A_F) = \sup_{i \in F} \deg_{wtt}(A_i)$ by disjointness. We show that the contiguous degrees $\mathbf{a}_i, \mathbf{b}_i$ and $\mathbf{d}_{i,j}$ satisfy the hypotheses of Theorem 2.1. First, by (9), $\widehat{\mathbf{d}}_{i,j}$ is the same in $\mathbf{R}_T$ and $\mathbf{R}_{wtt}$ and, by (10), $\inf(\mathbf{a}_i, \mathbf{b}_j, \widehat{\mathbf{d}}_{i,j}) = \mathbf{0}$, since this holds in $\mathbf{R}_{wtt}$. To show the definability of $\mathbf{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ via a fixed Σ_2–formula, again let $\mathbf{s}_A = \sup_i \mathbf{a}_i$ and $\mathbf{s}_B = \sup_i \mathbf{b}_i$. We only need to verify that

$$\mathrm{Compl}(\mathbf{x}, \mathbf{s}_A) \Leftrightarrow \mathbf{x} = \mathbf{a}_F \text{ for some nonempty } F \subseteq \{1, \ldots, n\}.$$

also holds in $\mathbf{R}_T$; then we can argue as in the proof of Theorem 3.2. If $\mathbf{x} = \mathbf{a}_F$, then Compl$(\mathbf{x}, \mathbf{s}_A)$ holds via $\mathbf{a}_{\overline{F}}$, since this is the case in $\mathbf{R}_{wtt}$. Now suppose $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x} \wedge \mathbf{y} = \mathbf{0}$ and $\mathbf{x} \vee \mathbf{y} = \mathbf{s}_A$ for $\mathbf{x} = \deg_T(X)$ and $\mathbf{y} = \deg_T(Y)$ Then the same is true for the wtt–degrees of X,Y. Hence $X \equiv_{wtt} A_F$ for some $F \neq \emptyset$ by (ii) of Lemma 2.2, whence $\mathbf{x} = \mathbf{a}_F$.

Proof of the Main Lemma 4.1. We first give a general procedure for building contiguous degrees, which works in an environment with finitary primary strategies as in the proof of the Main Lemma 3.1. To make the T–degrees of a set C contiguous, the requirements

$$\mathrm{Cont}_{C,\Phi,\Psi,i}: \quad C = \Phi(W_i) \wedge W_i = \Psi(C) \Rightarrow W_i \equiv_{wtt} C$$

are satisfied, where Φ, Ψ are T–functionals. If a computation $\Phi(X)(y)$ is defined, $\phi(y)$ denotes the use, i.e. 1+the maximal oracle question asked. Similarly, $\phi(y)[s]$ (or $\phi_s(y)$) denotes the use of $\Phi(X)(y)[s]$, if the latter is defined. We assume that $\phi(y)[s]$ is nondecreasing in y and s. Let $L = <C, \Phi, \Psi, i>$ and define the $\Psi(C)$–correct length of agreement between C and $\Phi(W_i)$ by

$$\mathrm{Clength}_L(s) = \max\{x : (\forall y < x)[C(y) = \Phi(W_i)(y)[s] \wedge (\forall z < \phi(y)[s])[W_i(y) = \Psi(C)(y)[s]]]\}.$$

A Cont_L–strategy ν works at stages where $\mathrm{Clength}_L$ has increased, in which case it gives outcome $<\infty>$ and initializes the strategies $>_L \nu\hat{}<\infty>$. Moreover, it defines a *stream* (R. Downey) of numbers $x^{\nu}_{0,s} < x^{\nu}_{1,s} < \ldots$ which are $< \mathrm{Clength}_L(s)$ and have the property that if $x^{\nu}_{i+1,s}$ is defined then

(11) $\psi(\phi(x^{\nu}_{i,s}))[s]<x^{\nu}_{i+1,s}$.

The strategy relies on the following:

(12) a primary strategy σ, $\nu\hat{}<\infty>\subseteq\sigma$, only causes numbers $x^{\nu}_{i,s}$ to be enumerated into C at stage s.

(13) for almost every $\nu\hat{}<\infty>$–stage t, if x is enumerated into C at t, then $C_r\cap[x,r)=C\cap[x,r)$ for the $\nu\hat{}<\infty>$–stage r following t.

Also suppose that a primary strategy σ, $\nu\hat{}<\infty>\subseteq\sigma$, only enumerates $x^{\nu}_{i,s}$ if $i\geq|\sigma|$. Then $\lim_s x^{\nu}_{n,s}$ exists for each n. Now $C\leq_{wtt}W_i$, since it can be shown that $C(y)=C_t(y)$ for the first stage t such that, for some s<t $Clength_L(s)>y$, some $x^{\nu}_{i,s}\geq y$ is defined, and the oracle $W_{i,t}$ has stabilized on $[0,\phi(y))[s]$. To show $W_i\leq_{wtt}C$, given input y, compute If k and s≥s* such that $y<\phi_s(x^{\nu}_{k,s})$ (by convention, we assume $\phi_s(z)\geq z$). Ask the oracle if $x^{\nu}_{k',s}\in C$ for some k'≤k If not, $W_i(y)$ has the final value already at s. If so, compute a minimal–$\nu\hat{}<\infty>$stage t>s such that $x^{\nu}_{k',s}\in C_t$. Then, by (13), $C|r=C_r|r$ for the $\nu\hat{}<\infty>$–stage r following t. Since at stage r again $W_i(y)=\Psi(C)(y)$, $W_{i,r}(y)$ has the final value.

We now give the program and the verification in detail. Since (12) must be met for all contiguity strategies, each contiguity strategy ν refines the stream put out by $\hat{\nu}$, where $\hat{\nu}$ is the string of maximal length such that $\hat{\nu}=\lambda$ or $\hat{\nu}$ is a contiguity strategy and $\hat{\nu}\hat{}<\infty>\subseteq\nu$. The root λ just puts out an increasing sequence of unused numbers x^{λ}_k.

Program for a $Cont_L$*–strategy* ν. Parameter: oldClength, i (the maximal index of a number appointed so far), numbers x^{ν}_k $(-1\leq k\leq i)$.

The strategy is initialized by setting i to –1 and cancelling all $x^{\nu}_{k,s}$. Formally we let $x^{\nu}_{-1,s}=0$. Let t<s be the preceding $\nu\hat{}<\infty>$–stage.

ACT IF $Clength_L(s)$>oldClength BEGIN

oldClength:=$Clength_L(s)$; outcome:=<∞>;

initialize all strategies $>_L\nu\hat{}<\infty>$;

(Adjust i) IF $C_t|x^{\nu}_{k,t}+1\neq C_s|x^{\nu}_{k,t}+1$ for some minimal k, let i:=k–1 (we say that the $x^{\nu}_{k'}$ are *cancelled* for k'≥k);

(Appoint) IF there is a number $x=x^{\hat{\nu}}_{j,s}$ such that

$x\geq s_{init}$, $\psi(\phi(x^{\nu}_{i,s}))[s]<x$ and

(14) x≥the $\nu\hat{}<\infty>$–stage following the last stage when x^{ν}_{i+1} was cancelled, if there is such a stage (to meet (13))

BEGIN i:=i+1; appoint $x^{\nu}_{i,s}$:=x END

goto ACT

Verification.
Suppose that $\nu\hat{\ }\langle\infty\rangle$ is on the true path for the Cont_L–strategy ν. Let s_0 be a stage such that $\nu\hat{\ }\langle\infty\rangle\leq\delta_s$ for all $s\geq s_0$ and no primary strategy $<\nu$ causes an enumeration at s. Note that

(15) if $s<s'$ are consecutive $\nu\hat{\ }\langle\infty\rangle$ stages, $s_0\leq s$, $x<\text{Clength}_L(s)$ and $C_s|\psi_s(x)=C_{s+1}|\psi_s(x)$ then $W_{i,s+1}(x)=W_{i,s'}(x)$.

Otherwise, by initialization at s, a $W_i(x)$ change at t', $s<t'\leq s'$ would cause Clength_L to drop back permanently and $\nu\hat{\ }\langle\infty\rangle$ would not be on the true path.
We first prove that $x^{\nu}_n=\lim_s x^{\nu}_{n,s}$ exists for each n. By induction suppose this holds for $\hat{\nu}$. Choose a $\nu\hat{\ }\langle\infty\rangle$–stage $s_1\geq s_0$ such that $x^{\nu}_{m,s}$ (m<n) have reached their limits and $\text{Clength}(s_1)>x^{\nu}_{n-1}$. Then $\psi(\phi(x^{\nu}_{n-1}))$ has reached a final value at s_1 by (11) and (15). By inductive hypothesis for $\hat{\nu}$, arbitrarily big numbers $x=x^{\hat{\nu}}_{j,t}$ appear as possible choices for $x^{\nu}_{n,t}$ for stages $t\geq s_1$. Thus $x^{\nu}_{n,t}$ is defined infinitely often. Now by initialization at $\nu\hat{\ }\langle\infty\rangle$–stages, $x^{\nu}_{n,t}$ can only be enumerated by primary strategies σ, $|\sigma|<i$, $\nu\hat{\ }\langle\infty\rangle\subseteq\sigma$. So after finitely many such enumerations, $x^{\nu}_{n,t}$ reaches its limit. Note that, for $s\geq s_0$, $\psi(\phi(x^{\nu}_{i,s}))[s]<x^{\nu}_{i+1,s}$ remains valid unless $x^{\nu}_{i,s}$ is cancelled. We verify (13) for $t\geq s_0$. Suppose $x=x^{\nu}_{i,s}$ is enumerated into C at stage t. By initialization at stage r, we only need to consider numbers enumerated into C at $\nu\hat{\ }\langle\infty\rangle$–stages>r. The next possible value for x^{ν}_i is $\geq r$ by (14). Since the next possible value for x^{ν}_j (j<i) (if it is cancelled at a stage $\geq r$) is also $\geq r$, this shows (13).

Suppose that $C=\Phi(W_i)$ and $W_i=\Psi(C)$. We drop the superscript ν. To show $W_i\leq_{wtt}C$, given input y, compute a $\nu\hat{\ }\langle\infty\rangle$–stage $s\geq s_0$ such that for some minimal k, $y<\phi(x_{k,s},s)$.
Case 1. If $C_s|x_{k,s}+1=C|x_{k,s}+1$, then $\psi(\phi(x_{k,s}))[s]$ has reached a final value at stage s by (11). Then $W_{i,s}(y)=W_i(y)$.
Case 2. Else let $k'\leq k$ be minimal such that $x_{k',s}\in C$. Compute a minimal stage t>s such that $x_{k',s}\in C_t$ and let r be the $\nu\hat{\ }\langle\infty\rangle$–stage following t. Then $\Psi_r(C)(y)$ is defined. Since $\psi_r(y)<r$ and $C|r=C_r|r$ by minimality of k' and (12), $W_{i,r}(y)=\Psi_r(C)(y)$ has the final value.
To show $C\leq_{wtt}W_i$, given input y, compute a $\nu\hat{\ }\langle\infty\rangle$–stage $s\geq s_0$ such that some $x_{k,s}\geq y$ is defined. If not $y=x_{k,s}$, then $y\notin C$ by (12) and the choice of s_0. If $y=x_{k,s}$ determine a $\nu\hat{\ }\langle\infty\rangle$–stage $t\geq s$ such that $W_{i,t}|\phi_s(y)=W_i|\phi_s(y)$. We claim that

$C_t(y)=C(y)$. Since $\phi_s(y)$ was computed effectively, this gives a wtt–reduction of C to W_i.
If $C_s|y+1\neq C_t|y+1$, then the claim follows from (12). Else actually $C_s|x_{n+1,s}=C_t|x_{n+1,s}$ by (12) and initialization at s, and so by (14) and since $\psi(\phi(y))[s]<x_{k+1,s}$

$$W_{i,s}|\phi_s(y)=W_{i,t}|\phi_s(y)=W_i|\phi_s(y),$$

i.e. the computation $\Phi(W_i)(y)[s]$ was already final. Since $C(y)=\Phi(W_i)(y)[s]$, this proves the claim.

We now apply this method to prove the Main Lemma 4.1, by modifying the construction in the proof of the Main Lemma 3.1. Let $A=\bigcup A_i$, $B=\bigcup B_j$ and $D=\bigcup D_{i,j}$. There are three types of contiguity requirements: for $C=A_F$, $C=B_G$ and $C=D_E$. We call these A–type, B–type and D–type contiguity requirements. The primary strategies choose their candidates from the appropriate streams: if a number is targeted for A_i, say, it is chosen from the stream of the contiguity requirement σA, where, for a string σ and $X\in\{A,B,D\}$, σX is the string ξ of maximal length such that $\xi=\lambda$ or ξ is an X–type contiguity strategy and $\xi\hat{}\langle\infty\rangle\subseteq\sigma$.
Although the A–type requirements (say) are concerned with different sets A_F, they all refine each others stream. An X–type contiguity strategy ν now refines the stream of νX (which plays the role of $\hat{\nu}$ above) and works with X instead of C when i is adjusted. Thus any change of X below x^ν_j+1 leads to the cancellation of x^ν_j. However, it measures the length of agreement with respect to C.
The contiguity requirements (with the possible outcomes $\langle\infty\rangle$ and $\langle r\rangle$) are included into the priority list of requirements R, and the tree of strategies is modified accordingly. The programs for the primary strategies are modified as follows.

If γ is a $Q_{L,k,h}$-strategy, $L=\langle A_i,\Phi,\Psi,X,Y\rangle$:

START IF there is an unused number $m=x^{\gamma A}_{k,s}$, $k\geq|\gamma|$ BEGIN appoint m;goto WAIT Φ END (here it is essential that "unused for σ" means "not used by any strategy which is not $>_L\sigma$", as the possible choices for m are now more restricted)
ELSE goto START
...(as before)

If $L=\langle B_j,\Phi,\Psi,X,Y\rangle$, appoint $m=x^{\gamma B}_{k,s}$ instead.

A $P^{i,j}_e$–strategy σ has to appoint three parameters x and $a,b \le x$ from streams of the appropriate contiguity requirements σD, σA and σB. The numbers a,b are needed to ensure $D_{i,j} \le_{wtt} A_i$ and $D_{i,j} \le_{wtt} B_j$.

APPOINT 1. Initialize all primary strategies $\sigma' >_L \sigma$

2.IF there are unused numbers x,a,b such that $a,b \le x$ and

$x = x^{\sigma D}_{k,s}$, $a = x^{\sigma A}_{k',s}$, $b = x^{\sigma B}_{k'',s}$ for some $k,k',k'' \ge |\sigma|$

appoint x,a,b.

REALIZE IF $\{e\}_s(x)=0$ BEGIN enumerate $\langle x,\sigma \rangle$ into $\tilde{D}_{i,j}$; enumerate $\langle a,\sigma \rangle$ into $\tilde{A}_i$ and $\langle b,\sigma \rangle$ into $\tilde{B}_j$; goto WIN END

ELSE goto REALIZE

WIN goto WIN.

Now (13) holds for each set X among A,B,D and therefore also for the sets A_F,B_G and D_E. The verification for the requirements from the Main Lemma 3.1 can be carried out mostly as before. We use the fact that the candidates in a stream reach a limit to prove that the primary strategies finally appoint fixed candidates. The verification for the contiguity requirements is as above, with the exception that in the proof of $W_i \le_{wtt} C$ we have to include one more case due to the fact that a number $x_{i,s}$ may be enumerated , but not into C. Suppose for instance that $C=A_F$.

Case 1. $C|x_{k,s}+1 = C_s|x_{k,s}+1$..

Case 1a. $A|x_{k,s}+1 = A_s|x_{k,s}+1$. Then $x_0,\dots,x_k$ have reached a limit, and we argue as before in Case 1.

Case 1b. Else. Then, for some minimal $k' \le k$, there is a first $v\hat{}\langle\infty\rangle$–stage $p \ge s$ such that at p $x_{k',s}$ has been enumerated into A (but not into A_F). By (13), this implies $C_p|p = C|p$, so $\Psi(C)(y)[s]$ already has the final value.□

4.3. Open problems.

(i) *Is* Π_3–Th($\mathbf{R}_{wtt}$) *undecidable ?*

(ii) *Does every nontrivial initial segment* $[\mathbf{0},\mathbf{a}]$ *of* $\mathbf{R}_{wtt}$ *have an undecidable theory ?*

Our coding methods cannot be applied in every nontrivial initial segment, for if **a** is the degree of an antimitotic set [A85], then each closed subinterval of $[\mathbf{0},\mathbf{a}]$ embeds the 4–element Boolean algebra preserving the least and the greatest element.

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