WEAK DENSITY AND CUPPING IN THE D-R.E. DEGREES

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Abstract. Consider the Turing degrees of differences of recursively enumerable sets (the d-r.e. degrees). We show that there is a properly d-r.e. degree (a d-r.e. degree that is not r.e.) between any two comparable r.e. degrees, and that given a high r.e. degree $\mathbf{h}$, every nonrecursive d-r.e. degree $\leq \mathbf{h}$ cups to $\mathbf{h}$ by a low d-r.e. degree.

A set $A \subseteq \omega$ is called *d-r.e.* if there are recursively enumerable (r.e.) sets A_1, A_2 such that $A = A_1 - A_2$. A Turing degree is called a *d-r.e. degree* if it contains a d-r.e. set; it is called *properly d-r.e.* if it is d-r.e. but not r.e. (contains no r.e. set).

Cooper (unpublished) showed that properly d-r.e. degrees do exist, while Lachlan (unpublished) showed that for any d-r.e. degree $\mathbf{d} > \mathbf{O}$ there is an r.e. degree $\mathbf{a}$ such that $\mathbf{O} < \mathbf{a} \leq \mathbf{d}$. These definitions and results easily generalise to n-r.e. degrees. A Turing degree is called *n-r.e.* if it contains an n-r.e. set where r.e. sets are called 1-r.e. and D is $(n+1)$-r.e. iff $D = D_1 - D_2$ for D_1 r.e., D_2 n-r.e. Hay and Lerman [8] have generalised the above results to show that:

(1) For all $k, n > 0$, given any n-r.e. degree $\mathbf{a} > \mathbf{O}$ there exists a k-r.e. degree $\mathbf{b}$ such that $\mathbf{O} < \mathbf{b} < \mathbf{a}$; and

(2) For all n, p such that $n > p \geq 1$, there exist n-r.e. degrees $\mathbf{a} < \mathbf{b}$ such that there is no p-r.e. degree $\mathbf{c}$ between them.

Further results on the structure of the d-r.e. degrees appear in Arslanov [2, 3]. For example, it is shown there that every d-r.e. degree $\mathbf{d} > \mathbf{O}$ can be *cupped* to $\mathbf{O}'$ in the d-r.e. degrees (i.e. there is a d-r.e. degree $\mathbf{a} < \mathbf{O}'$ such that $\mathbf{d} \cup \mathbf{a} = \mathbf{O}'$). Arslanov also proves that for each r.e. degree $\mathbf{a} < \mathbf{O}'$ there exists a properly d-r.e. degree $\mathbf{b}$ such that $\mathbf{a} < \mathbf{b} < \mathbf{O}'$.

We extend these results as follows:

(1) (Weak Density) Given r.e. degrees $\mathbf{a} < \mathbf{b}$ there exists a properly d-r.e. degree $\mathbf{c}$ such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$; and

(2) (Cupping) Given a d-r.e. degree $\mathbf{b} > \mathbf{O}$ and a high r.e. degree $\mathbf{h} \geq \mathbf{b}$ there exists a low d-r.e. degree $\mathbf{a}$ such that $\mathbf{b} \cup \mathbf{a} = \mathbf{h}$. (Here "d-r.e." can be replaced by "n-r.e.").

These results compare with known results about the r.e. case, e.g. (1) with the Sacks Density Theorem (that the r.e. degrees form a dense partial order). (2) contrasts with the

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result by Yates and Cooper [4] that there exists an r.e. degree $\mathbf{a}$ with $\mathbf{O} < \mathbf{a} < \mathbf{O}'$ such that no r.e. degree $\mathbf{b} < \mathbf{O}'$ cups $\mathbf{a}$ to $\mathbf{O}'$, and also with Ambos-Spies, Jockusch, Shore, Soare [1] who showed that the property of low cuppability to $\mathbf{O}'$ in the r.e. degrees defines a strong filter in the r.e. degrees. Downey [6] has recently shown that the Nondiamond Theorem fails for the d-r.e. degrees; thus there is a d-r.e. degree $\mathbf{d}$ that can be complemented in the d-r.e. degrees (i.e. there is a d-r.e. degree $\mathbf{e}$ such that $\mathbf{d} \cap \mathbf{e} = \mathbf{O}$ and $\mathbf{d} \cup \mathbf{e} = \mathbf{O}'$). It is easy to see that not every nonrecursive incomplete d-r.e. degree can be complemented.

Notation. Our notation generally follows Soare [11]; thus sets will be identified with their characteristic functions. One exception is worth mentioning: We shall use $V\lceil x$ in the usual way as V restricted to elements $< x$; however, when we consider the join of two sets $V \oplus W = \{2x \mid x \in V\} \cup \{2x+1 \mid x \in W\}$ then $(V \oplus W)\lceil x$ will denote $(V\lceil x) \oplus (W\lceil x)$. We hope the context will resolve any ambiguity. As a convention, we assume that all functionals Φ given by the "opponent" have use $\varphi_s(x)$ increasing in x and nondecreasing in s.

§1 Weak Density

In this section we show that between any two r.e. degrees there is a properly d-r.e. degree. The proof of this will use an infinite injury argument, involving a tree construction. The reader is referred to Soare [11, Chapter XIV] for the background on these constructions.

Theorem I. *Given r.e. sets $U >_T V$ there is a d-r.e. set C of properly d-r.e. degree such that $U >_T C >_T V$.*

Proof. We construct r.e. sets $A_1, A_2 \leq_T U$. If $A = A_1 - A_2$ then $C = V \oplus A$ will be the desired set.

To ensure that $V \oplus A$ is not of r.e. degree we satisfy for every i the requirement

$$R_i : A \neq \Theta_i^{W_i} \vee W_i \neq \Phi_i^{V \oplus A}$$

Here $\{(W_i, \Theta_i, \Phi_i)\}_{i \in \omega}$ is some enumeration of all possible triples of r.e. sets W and partial recursive functionals Θ and Φ.

In satisfying R_i we shall construct functionals Γ_j ($j \in \omega$) and Δ with the intention that if R_i fails then $U \leq_T V$ via some Γ_j, or Δ, contrary to our hypothesis.

Basic Module. We will choose a sequence of candidates (one for each "cycle" of the strategy), one of which will witness the failure of one or both of the propositions:

(i) $A = \Theta_i^{W_i}$,

(ii) $W_i = \Phi_i^{V \oplus A}$.

This will be sufficient for R_i to succeed.

Let us first consider the requirement without the claim $A \leq_T U$ and in the absence of any V-changes. (This is just the proof that there is a properly d-r.e. degree. There is only one cycle.) The strategy proceeds as follows:

(1) Choose an unused candidate x for R_i.

(2) Wait for a stage s such that for some least u and v

$$0 = A_s(x) = \Theta_{i,s}^{W_{i,s}\lceil u}(x)$$

and

$$W_{i,s}\lceil u = \Phi_{i,s}^{(V \oplus A)_s\lceil v}\lceil u.$$

(If this never happens then x is a witness to the success of R_i.)

(3) Restrain $A\lceil v$ from other strategies from now on.

(4) Put x into A.

(5) Wait for a stage s' such that for some u' and v'

$$1 = A_{s'}(x) = \Theta_{i,s'}^{W_{i,s'}\lceil u'}(x)$$

and

$$W_{i,s'}\lceil u' = \Phi_{i,s'}^{(V \oplus A)_{s'}\lceil v'}\lceil u'.$$

(If this never happens then again x is a witness to the success of R_i. If it does happen then the change in $\Theta_i^{W_i}(x)$ between stages s and s' can only be brought about by a change in $W_i\lceil u$, which is irreversible since W_i is an r.e. set.)

(6) Remove x from A and restrain $A\lceil v'$ from other strategies from now on. (Now x is a permanent witness to the success of R_i because

$$\Phi_i^{V \oplus A}\lceil u = \Phi_{i,s}^{(V \oplus A)_s}\lceil u = W_{i,s}\lceil u \neq W_i\lceil u,$$

i.e. proposition (ii) fails.)

In the above, step 6 assumes, of course, that $V\lceil v$ does not change after stage s. We shall show next how we can impose "indirect" restraint on V by threatening $U \leq_T V$ via a functional Γ. (This is essentially the proof to Arslanov's Theorem, i.e. the case $U \equiv_T \emptyset'$.) We make infinitely many attempts to satisfy R_i as above by an ω-sequence of "cycles", each cycle k proceeding as above but with the following step inserted after step 3:

$3\frac{1}{2}$. Set $\Gamma^V(k) = U_s(k)$ with use $\gamma(k) = v$, start cycle $k+1$ simultaneously, wait for $U(k)$ to change, then stop the cycles $k' > k$ and proceed.

Whenever some cycle sees a $V\lceil v$-change after stage s, it will kill the cycles $k' > k$ and go back to step 2. (Notice that Γ, if defined for $k' \geq k$, becomes undefined through the V-change.)

There are now three possibilities:

A. Eventually each cycle k gets stuck at step $3\frac{1}{2}$, waiting for a U-change. Then $\Gamma^V = U$, contrary to hypothesis.

B. Some (least) cycle k_0 gets stuck forever at some other step. Then we were successful in restraining V and satisfy R_i through cycle k_0 as before.

C. Some (least) cycle k_0 gets infinitely many V-changes after step 2. Then $\Phi_i^{V \oplus A}$ or $\Theta_i^{W_i}$ is partial, and R_i is again satisfied by cycle k_0.

Finally, we have to ensure $A \leq_T U$ through a permitting argument. So x has to be permitted by U into A at step 4 and out of A at step 6. The former is already given by the $U(k)$-change, the latter has to be built into the strategy (by asking for permission j many times for larger and larger j).

We thus arrive at a basic module for the R_i-strategy consisting of cycles (j,k) (for $j,k \in \omega$). Cycle $(0,0)$ starts first, and each cycle (j,k) can start cycles $(j,k+1)$ or $(j+1,0)$ and stop, or cancel, cycles (j',k') for $(j,k) < (j',k')$ (in the lexicographical ordering). Each cycle (j,k) can define $\Gamma_j^V(k)$ and $\Delta^V(j)$. At each stage, the least cycle that can act will do so.

A cycle (j,k) now proceeds as follows:

(1) Choose an unused candidate x greater than any number mentioned thus far in the construction.
(2) Wait for a stage s_1 such that for some least u and v

$$\Theta_{i,s_1}^{W_{i,s_1}\lceil u}(x) = 0$$

and

$$\Phi_{i,s_1}^{(V\oplus A)_{s_1}\lceil v}\lceil u = W_{i,s_1}\lceil u.$$

(3) Restrain $A\lceil v$ from other strategies from now on.
(4) Set $\Gamma_j^V(k) = U_{s_1}(k)$ with use $\gamma_j(k) = v$ and start cycle $(j,k+1)$ simultaneously.
(5) Wait for $V\lceil v$ or $U(k)$ to change.
If $V\lceil v$ changes first then cancel cycles $(j',k') > (j,k)$, drop the A-restraint of cycle (j,k) to 0, and go back to step 2.
If $U(k)$ changes first then stop cycles $(j',k') > (j,k)$ and proceed to step 6.
(6) Put x into A.
(7) Wait for a stage s_2 such that for some u' and v'

$$\Theta_{i,s_2}^{W_{i,s_2}\lceil u'}(x) = 1$$

and

$$\Phi_{i,s_2}^{(V\oplus A)_{s_2}\lceil v'}\lceil u' = W_{i,s_2}\lceil u'.$$

(8) Restrain $A\lceil v'$ from other strategies from now on.
(9) Set $\Delta^V(j) = U_{s_2}(j)$ with use $\delta(j) = v'$ and start cycle $(j+1,0)$ simultaneously.
(10) Wait for $V\lceil v'$ or $U(j)$ to change.
If $V\lceil v'$ changes first then cancel cycles $(j',k') \geq (j+1,0)$, drop the A-restraint of cycle (j,k) to v, and go back to step 7.
If $U(j)$ changes first then stop cycles $(j',k') \geq (j+1,0)$ and proceed to step 11.
(11) Remove x from A.
(12) Wait for $V\lceil v \neq V_{s_1}\lceil v$ or $V\lceil v' \neq V_{s_2}\lceil v'$. Proceed to step 13 or 14, respectively.
(13) Reset $\Gamma_j^V(k) = U(k)$, cancel cycles $(j',k') > (j,k)$, start cycle $(j,k+1)$, and halt.
(14) Reset $\Delta^V(j) = U(j)$, cancel cycles $(j',k') \geq (j+1,0)$, start cycle $(j+1,0)$, and halt.

Notice again that whenever a cycle (j,k) is started, any previous version of it has been cancelled and its functionals have become undefined through V-changes. Therefore, Γ_j and Δ are defined consistently.

The basic module now has four possible outcomes:

A. There is a stage s after which no cycle acts. Then some cycle (j_0,k_0) eventually waits at step 2, 7, or 12 forever. Thus we win requirement R_i as before.

B. Some cycle (j_0,k_0) acts infinitely often but no cycle $< (j_0,k_0)$ does so. Then it goes from step 5 to step 2, or from step 10 to step 7, infinitely often. Thus Φ_i or Θ_i is partial. Notice that the overall restraint of all cycles has finite liminf.

C. There is a j_0 and there are stages $s_j (j < j_0)$ and $t_k (k \in \omega)$ such that no cycle (j,k) acts after stage s_j and such that cycle (j_0,k) does not act after stage t_k; but there is no stage s such that no cycle (j_0,k) acts after stage s. ("Row j_0 acts infinitely but rows $j < j_0$ act finitely.") Then every cycle (j_0,k) eventually waits at step 5 or 13 forever, and together these show $U \leq_T V$ via Γ_{j_0} contrary to hypothesis.

D. There are stages s_j $(j \in \omega)$ such that no cycle (j,k) acts after stage s_j; but there are infinitely many stages at which some cycle acts. ("Every row acts finitely.") Then for every j there is a cycle (j,k_j) that eventually waits at step 10 or 14 forever, and together these show $U \leq_T V$ via Δ contrary to hypothesis.

Only outcomes A. and B. need to be put on the priority tree since the other two contradict the hypothesis of the theorem. We will order the former in order type ω^2+1, with (j,k) (in lexicographical ordering) denoting outcome B. with that cycle, and with the rightmost outcome *fin* denoting outcome A.

We visualize the action of a cycle in Diagram 1. A cycle starts in state *init* and, following the arrows, proceeds to the next state (denoted by a circle) every time it is allowed to do so. Along the way, it will execute instructions (in boxes) and make decisions (in diamonds). At times when U changes, a cycle may directly proceed to *Uchange1* or *Uchange2* and on to *setup2* or *Vchange3*, respectively, to make the direct permitting of A below U work.

The instructions in the flow chart are to be interpreted as follows: After state *init*, x is picked bigger than any number mentioned previously in the construction. (This automatically ensures that restraints are respected by strategies of lower priority.) The parameters x, r, s_1, s_2, u, v, u', v' are different for each cycle and roughly denote the *candidate* for R_i, the *A-restraint* imposed, the *stages* at which the setups are first found, and the *uses* for the setups, respectively. A cycle is *started* by letting it go from *init* to *setup1*. A cycle is *stopped* by putting it into *init* and by setting its restraint is 0. A cycle being *cancelled* denotes that furthermore its part of the functionals has become undefined and that it could be started again.

A strategy is *initialised* by cancelling all its cycles and starting cycle $(0,0)$. A strategy *acts* by letting its least cycle *act* (go to a different state) that can do so and that is not in state *init* (if there is such a cycle). The *restraint* of a strategy is the maximum of the A-restraints of all its cycles.

Construction. Let $\Lambda = \{(j,k) \mid (j,k) \in \omega \times \omega\} \cup \{\textit{ fin }\}$ be the set of outcomes, ordered lexicographically with *fin* rightmost. Let $T = \Lambda^{<\omega}$ be the tree of strategies. A strategy $\alpha \in T$ of length i works on requirement R_i, assuming outcome $\alpha(j)$ of strategies $\alpha\lceil j$

$(j < i)$. Any parameter, once defined, retains that value until redefined; a functional remains defined on an argument until the oracle changes on the use.

At stage 0, all parameters are set to 0 or $\emptyset$; all functionals are completely undefined; all strategies are initialised.

At each stage $s > 0$, we first find the leftmost strategy $\alpha \in T$ that has a (least) cycle (j,k) in state *Vchange1* or *Vchange2* for which $U(k) \neq U_{s_1}(k)$ or $U(j) \neq U_{s_2}(j)$, respectively, is satisfied.

If there is such α then we let cycle (j,k) go from *Uchange1* or *Uchange2*, respectively, to the next state, and we initialise all strategies $\beta \geq \alpha^\wedge(j,k)$.

In either case, we now proceed in substages $t < s$. At each substage t, a strategy α of length t is eligible to act. Once α has acted we determine its outcome o to be (j,k) if this cycle of α has acted, and *fin* if none has. We initialise all strategies $\beta >_L \alpha^\wedge o$ and let $\alpha^\wedge o$ be *eligible to act* at the next substage (if cycle (j,k) went from state *Vchange1* or *Vchange2* to *setup1* or *setup2*, respectively) and $\alpha^\wedge$*fin* otherwise. (This is because the restraint of cycle (j,k) is lowest only at that point.)

Verification. Let the *true path* f be the path through T defined inductively by $f(i) = o$ where $(f\lceil i)^\wedge o$ is the leftmost successor of $f\lceil i$ eligible to act infinitely often. Let the *correct part of the true path* $f_0 = \bigcup\{\alpha \subset f \mid \alpha$ initialised finitely often $\}$. We shall prove that this is only a finite initial segment of f iff $U \leq_T V$.

Injury Lemma. *The restraint of every strategy $\alpha \subseteq f_0$ is injured at most finitely often.*

Proof. Only a strategy $\beta < \alpha$ can have a candidate small enough to injure α. But every time β injures α (in the first part of a stage s), α is initialised. Therefore the lemma holds by the definition of f_0.

Permitting Lemma. *$A \leq_T U$ by direct permitting.*

Proof. Suppose some strategy α changes A at x in the first part of some stage $s+1$. Since α did not do so at stage s, either $U_s\lceil x \neq U_{s+1}\lceil x$, or else α was not ready to change A at x. But in the latter case, by the construction, we also conclude $U_s\lceil x \neq U_{s+1}\lceil x$. Thus $A \leq_T U$.

Outcome Lemma. *i) If $\alpha \subset f_0$ then α satisfies requirement $R_{|\alpha|}$.*
ii) If $\alpha = f_0$ then α shows $U \leq_T V$, contrary to hypothesis.

Proof. i) Let s' be the least stage such that $f_0\lceil(|\alpha|+1)$ is not initialised after stage s'. (This stage exists by the definition of f_0.) Then α is no longer injured by the way the construction is set up.

First assume $\alpha^\wedge$*fin* $\subseteq f_0$. Then some cycle (j,k) is eventually waiting in *setup1*, *setup2*, or *Vchange3* forever (and every cycle $> (j,k)$ is in *init*). Thus requirement R_i is satisfied.

On the other hand, assume $\alpha^\wedge(j,k) \subseteq f_0$ for some (j,k). Let $s'' \geq s'$ be the least stage such that no cycle $< (j,k)$ acts after stage s''. Now cycle (j,k) is no longer stopped or cancelled, and it will work on a fixed candidate x from now on. Since $\alpha^\wedge(j,k)$ is not initialised after stage s' we conclude that $A_s(x) = A(x)$ for all $s > s''$. Then after stage s'', cycle (j,k) keeps going between *Vchange1*, or *Vchange2*, and *setup1*, or *setup2*, respectively. Thus Φ_i or Θ_i is partial, and again R_i is satisfied.

ii) Let s' be the least stage such that α is not initialised after stage s'.

First assume that infinitely often some cycle $(j,k) < (j_0,k_0)$ acts for (j_0,k_0) some fixed (least) cycle. For the sake of a contradiction, assume that $k_0 > 0$. Fix a stage $s'' \geq s'$ such that no cycle (j_0,k) (for $k < k_0$) acts after stage s'. Then infinitely often some cycle $(j,k) < (j_0,0)$ acts, contradicting the choice of s''.

For all k, let $v_k \geq s'$ be the last stage at which cycle (j_0-1,k) acts. Since cycle $(j_0,0)$ is cancelled infinitely often, each cycle (j_0-1,k) waits at *Vchange1* or *halt1* forever after stage v_k. But then $\Gamma^V_{j_0-1}(k) = U(k)$ for all k. (Notice that the definitions of $\Gamma^V_{j_0-1}(k)$ by all previous actions of cycle (j_0-1,k) have become undefined through V-changes.)

On the other hand, assume no cycle (j,k) acts infinitely often. For all k, let $w_j \geq s'$ be the last stage at which any cycle (j,k) acts. By the definition of w_j, some cycle (j,k) waits at *Vchange2* or *halt2* forever after stage w_j. But then $\Delta^V(j) = U(j)$ for all j. (Notice again that previous definitions of $\Delta^V(j)$ have become undefined.)

The above lemmas establish Theorem I.

§2 Cupping

It is known that there are r.e. degrees $\mathbf{a} > \mathbf{b} > \mathbf{O}$ such that $\mathbf{b}$ does not *cup* to $\mathbf{a}$ in the Δ^0_2-degrees, i.e. there is no Δ^0_2-degree $\mathbf{c} < \mathbf{a}$ such that $\mathbf{a} = \mathbf{b} \cup \mathbf{c}$ (Cooper [5], Slaman and Steel [10]). In this part we show that $\mathbf{a}$ cannot be high, and we extend a result of Arslanov [2, 3].

Theorem II. *Given r.e. sets V and H such that H is high and $\emptyset <_T V \leq_T H$, there exists a low d-r.e. set $A <_T H$ such that $H \equiv_T V \oplus A$.*

This implies a cupping theorem for all n-r.e. degrees ($n \geq 2$):

Corollary. *Let $\mathbf{h}$ be a high r.e. degree and $n \geq 2$. Then any n-r.e. degree $\mathbf{w}$ with $\mathbf{O} < \mathbf{w} \leq \mathbf{h}$ cups to $\mathbf{h}$ by a low d-r.e. degree.*

Proof. There is an r.e. degree $\mathbf{v}$ with $\mathbf{O} < \mathbf{v} \leq \mathbf{w}$ (see e.g. [7]). Now the degree from Theorem II cupping $\mathbf{v}$ to $\mathbf{h}$ will also do so for $\mathbf{w}$.

Proof of Theorem II. We shall construct Γ to satisfy $H = \Gamma^{V\oplus A}$. We ensure $A \leq_T H$ by permitting; and finally $A <_T H$ since A will be low and H is high.

We ensure A low by the usual lowness requirements:

$$R_i : (\exists^\infty)[\Phi^{A_s}_{i,s}(i)\downarrow] \rightarrow \Phi^A_i(i)\downarrow$$

Basic Module. Let us first see how to satisfy R_i in the absence of high permitting but just ensuring $H = \Gamma^{V\oplus A}$. (This is just the proof for H complete.) The requirement $H = \Gamma^{V\oplus A}$ has absolute priority over the R_i's; whenever a number x enters H and $\Gamma^{V\oplus A}(x)$ is defined, then Γ *must* be corrected at x by a change in V or A. A change in A, of course, may injure R_i, so we shall first change A, force V to change (threatening to prove it recursive via a functional Δ) and then change A back to satisfy R_i. We may need several attempts before we can force a V-change.

The strategy thus consists of cycles k (for $k \in \omega$) and works with a fixed parameter b_i. Cycle 0 will start first. Each cycle k proceeds as follows:

(1) Wait for a stage s such that $\Phi_{i,s}^{A_s}(i)\downarrow$.
(2) Restrain $A\lceil\varphi(i)$ from other strategies from now on (to preserve $\Phi_i^A(i)\downarrow$).
(3) If $\Gamma^{V\oplus A}(b_i)\uparrow$ or $\gamma(b_i)$ (the use of $\Gamma^{V\oplus A}(b_i)$) is greater than $\varphi(i)$ (the use of $\Phi_i^A(i)$) then halt (since numbers $z < b_i$ entering H will only cause finite injury).
Otherwise put $\gamma(b_i) - 1$ into A (to destroy $\Gamma^{V\oplus A}(b_i)$ and force it to be redefined with a use $> \varphi_{i,s}(i)$).
(4) Set $\Delta(y) = V(y)$ for $y < \gamma_s(b_i)$ unless already defined (to try to force $V\lceil\gamma_s(b_i)$ to change).
(5) Start cycle $k+1$ (starting the next step to show V recursive via Δ).
(6) Wait for $V\lceil\gamma_s(b_i)$ to change.
(7) Cancel all other cycles, remove $\gamma_s(b_i) - 1$ from A, and halt. (Now we are in the same situation as in the first case of step 3.)

There are now two possible outcomes:

A. Some cycle k eventually halts or waits forever at step 2. Then either $\Phi_i^A(i)\downarrow$ (if cycle k halts) or $\Phi_{i,t}^A(i)\uparrow$ for all stages t at which cycle k waits at step 2.

B. Every cycle eventually waits forever at step 6. Then $V\lceil\gamma_s(b_i)$ (for this $\gamma_s(b_i)$) never changes once a cycle has reached step 6. Then $\Gamma^{V\oplus A}$ will be undefined but $\gamma_s(b_i)$ tends to infinity, and thus V is recursive via Δ, contrary to hypothesis.

We now need to include the permitting of A below H. First, when some y is removed from A then $V\lceil(y+1)$ has changed. Since $V \leq_T H$, H can compute which elements are removed from A in the construction above. So we only have to permit numbers *into* A. We use a version of Martin permitting below high r.e. degrees as in Robinson [9]:

Without loss of generality, we may assume that H is *e-dominant*, namely, the *computation function* c_H defined by

$$c_H(y) = (\mu s)[H_s\lceil y = H\lceil y]$$

dominates every total recursive function f, i.e.

$$(\text{a.e. } y)[c_H(y) > f(y)].$$

The final version of the basic module for R_i thus consists of cycles (j,k) (for $j \geq b_i; k \in \omega$; where b_i is a fixed parameter). Cycle $(b_i, 0)$ starts first, and each cycle (j,k) can start cycles $(j+1, 0)$ or $(j, k+1)$ and cancel other cycles.

Recursive functions Δ_j (for $j \geq b_i$) and f will be constructed by the cycles (j,k) (for fixed j, and all j jointly, respectively), showing V recursive via some Δ_j or f not dominated by c_H, if R_i cannot be satisfied.

A cycle (j,k) then proceeds as follows:

(1) Wait for a stage s such that $\Phi_{i,s}^{A_s}(i)\downarrow$.

(2) Restrain $A\lceil\varphi_i(i)$ from other strategies from now on.
(3) If $\Gamma^{V\oplus A}(j)\uparrow$ or $\gamma(j)>\varphi_i(i)$ then cancel all other cycles and halt. Otherwise set $f(x)=s$ for the least x at which f is undefined (to force $H\lceil\gamma(j)$ to change, threatening f is not dominated by c_H), start cycle $(j+1,0)$, and wait for $H\lceil\gamma(j)$ to change.
(4) Cancel all cycles (j',k') for $j'>j$, put $\gamma_s(j)-1$ into A, set $\Delta_j(y)=V(y)$ for $y<\gamma_s(j)$ (unless already defined), start cycle $(j,k+1)$, and wait for $V\lceil\gamma_s(j)\neq V_s\lceil\gamma_s(j)$ to change.
(5) Cancel all other cycles, remove $\gamma_s(j)-1$ from A, and halt.

(Here, in step 4, cancelling cycles (j',k') for $j'>j$ includes discarding the functions $\Delta_{j'}$ for $j'>j$ and starting from scratch.)

Since the requirement $H=\Gamma^{V\oplus A}$ has highest priority there may be injury to any cycles (j,k) (for fixed j) whenever $H\lceil(j+1)$ changes. In this case we cancel all these cycles and restart the least one of them if it had been active. The crucial point here is that we shall not discard f, and we shall prove below that this will cause only finite injury if c_H dominates f.

The basic module now has three possible outcomes:

A. After some stage s_0, no cycle acts. Then some cycle (j,k) has halted or is waiting at step 1 forever. Thus $\Phi_i^A(i)\downarrow$, or $\Phi_{i,s}^{A_s}(i)\uparrow$ for all $s>s_0$.

B. There is a (least) fixed j_0 such that infinitely often some cycle (j_0,k) acts. Then every cycle (j_0,k) eventually is waiting at step 4. Thus $\Gamma^{V\oplus A}(j_0)\uparrow$ but $\gamma_s(j_0)$ tends to infinity and therefore $V=\Delta_{j_0}$, contrary to hypothesis.

C. Infinitely often some cycle acts but for each $j\geq b_i$ there is a stage s_j such that no cycle (j,k) acts after stage s_j. Then infinitely many cycles wait at step 3, and each has found some y such that $c_H(y)\leq f(y)$ (since the $\gamma(j)$'s dominate the y's), contrary to hypothesis.

Notice that only outcome A. does not contradict the hypotheses of Theorem II, and that this outcome is finitary. We shall therefore be able to use a finite injury argument.

We visualize the action of a cycle in Diagram 2. A cycle starts in state *init* and, following the arrows, proceeds to the next state (denoted by a circle) every time it is allowed to do so. Along the way, it will execute instructions (in boxes) and make decisions (in diamonds).

The *A-restraint* that each cycle imposes, is denoted by r (and is enforced automatically). A cycle (j,k) is *started* by letting it go from *init* to *wait*Φ; it is *cancelled* by putting it into *init*, setting its restraint to 0, and (if $k=0$) discarding its function Δ_j.

A strategy is *initialised* at j by cancelling all its cycles (j',k') (for $j'\geq j$) and starting cycle $(\hat{\jmath},0)$ for $\hat{\jmath}=\max(j,b_i)$ (if cycle $(\hat{\jmath},0)$ was not in state *init)*. A strategy is *initialised* by initialising it at 0, setting b_i bigger than any number mentioned before in the construction, and starting cycle $(b_i,0)$. A strategy *acts* by letting its least cycle *act* (go to a different state) that can do so and that is not in state *init*.

The *restraint* of a strategy is the maximum of the restraints of its cycles.

Construction. Any parameter, once defined, retains that value until redefined. We have one strategy for each R_i acting as described above.

At stage 0, all parameters are set to 0 or $\emptyset$, and all R_i-strategies are initialised in sequence.

Each stage $s > 0$ consists of three steps:

First determine if there is a (least) j such that $H(j) \neq \Gamma^{V \oplus A}(j) \downarrow$. If so then put $\gamma(j) - 1$ into A (so that $\Gamma^{V \oplus A}(j)$ can be corrected) and initialise all R_i-strategies at j.

Next find the least j such that $\Gamma^{V \oplus A}(j) \uparrow$. Set $\Gamma^{V \oplus A}(j) = H(j)$. Define its use to be the same as last time if since that time only $V \lceil \gamma(j)$ has changed but not $A \lceil \gamma(j)$. Otherwise define the use to be more than twice as big as any number mentioned thus far in the construction.

Finally, find the least i such that the R_i-strategy can act. For this i (if it exists), let the R_i-strategy act and initialise all $R_{i'}$-strategies (for $i' > i$).

Verification.

Injury Lemma. *No R_i-strategy injures any $R_{i'}$-strategy (for $i' < i$).*

Proof. When the R_i-strategy changes A at some $\gamma(j)$ then $\gamma(j) \geq j \geq b_i >$ any number the $R_{i'}$-strategy worked on so far.

Permitting Lemma. *$A \leq_T V \oplus H \leq_T H$ by direct permitting.*

Proof. Any A-change at some x in the first step of the construction is permitted by a change in $H \lceil x$ (since γ is increasing). Any A-change at some x by an R_i-strategy has to be permitted by a change in $H \lceil x$ or $V \lceil x$, and H can compute the r.e. set V.

Outcome Lemma. *Fix $i \in \omega$ and assume all $R_{i'}$-strategies act finitely often (for $i' < i$).*

i) If the R_i-strategy acts finitely often then R_i is satisfied.

ii) If the R_i-strategy acts infinitely often then either V is recursive via some Δ_j, or f is total recursive and not dominated by c_H, contrary to the e-dominance of H. (In either case, a hypothesis of Theorem II fails.)

Proof. i) Let t be a stage such that no $R_{i'}$-strategy (for $i' \leq i$) acts after stage t. Let (j, k) be the cycle of the R_i-strategy that was started last. Let $v \geq t$ be a stage such that $H \lceil (j+1)$ has settled down by stage v. Then after stage v, cycle (j, k) waits forever at *wait*Φ or has halted at *win1* or *win2*. In either case, R_i is satisfied.

ii) Let t be a stage such that no $R_{i'}$-strategy (for $i' < i$) acts after stage t. Then after stage t, the parameter b_i is fixed.

We distinguish two cases:

For some (least) fixed j_0, there are infinitely many stages such that some cycle (j_0, k) of the R_i-strategy acts. Let $v \geq t$ be a stage such that $H \lceil (j_0 + 1)$ has settled down by stage v. For each $k \in \omega$, let $s_k \geq v$ be the least stage such that cycle (j_0, k) does not act after stage s_k. Then Δ_{j_0} is never discarded after stage v, and cycle (j_0, k) waits at *waitV* forever after stage s_k. Since the parameter $\gamma_s(j_0)$ tends to infinity by the construction, V is recursive via Γ_{j_0}.

On the other hand, assume that for every $j \geq b_i$ there is a least stage $s_j \geq t$ such that no cycle (j, k) of the R_i-strategy acts after stage s_j and such that $H \lceil (j+1)$ has settled down. Then for each j, some cycle (j, k_j) will wait at *waitH* forever after stage s_j, just having set $f(x_j) = s_j$ and having used parameter $z_j = \gamma_{s_j}(j)$. We shall show below that $z_j \geq x_j$ for all $j \geq b_i$. Thus $H_{s_j}\lceil z_j = H\lceil z_j$ (by the definition of s_j). Therefore $f(x_j) = s_j \geq c_H(z_j) \geq c_H(x_j)$. Since the R_i-strategy acts infinitely often, f is total and, by the above, not dominated by c_H.

It remains to show that $z_j \geq x_j$ for all $j \geq b_i$. Let $t_j \leq s_j$ be the last stage at which $\gamma(j)$ was *increased*. At that stage $z_j > 2 \cdot \max(|\operatorname{dom} f|, j) \geq |\operatorname{dom} f| + j$. Now between stages t_j and s_j, cycles (j', k') (for $j' \leq j$) can only go from *wait*Φ to *wait*H (else Γ becomes undefined by an A-change), and no cycle (j', k') (for $j' > k'$) can act at all. Thus at most $j + 1$ times cycles can act between stages t_j and s_j, extending $\operatorname{dom} f$ by at most $j + 1$ values. Thus at stage s_j, $z_j \geq |\operatorname{dom} f| = x_j$ as desired.

Convergence Lemma. *If no R_i-strategy acts infinitely often then $\Gamma^{V \oplus A}$ is total, and $H = \Gamma^{V \oplus A}$.*

Proof. If $\Gamma^{V \oplus A}$ is total, it will also be correct by the first step of the construction.

Assume $\Gamma^{V \oplus A}\lceil j$ has been defined permanently by stage t, and that after that stage no R_i-strategy acts (for $i \leq j$). Let $v \leq t + 1$ be the last stage at which $\Gamma^{V \oplus A}(j)$ is defined. (Such a stage exists by the second step of the construction). No R_i-strategy (for $i > j$) can change A below $\gamma(j)$ (since $b_i > j$). Thus $\Gamma^{V \oplus A}$ will be defined at the end of stage $t + 1$, and its use will never again increase. Now, again by the second part of the construction, $\Gamma^{V \oplus A}(j)$ will eventually be defined permanently.

The above lemmas establish Theorem II.

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