THE COMPUTATIONAL COMPLEXITY OF TORSION-FREENESS OF FINITELY PRESENTED GROUPS

Steffen Lempp

Department of Mathematics
University of Wisconsin
Madison, WI 53706–1388, USA

Abstract. We determine the complexity of torsion-freeness of finitely presented groups in Kleene's arithmetical hierarchy as Π^0_2-complete. This implies in particular that there is no effective listing of all torsion-free finitely presented groups, or of all non-torsion-free finitely presented groups.

0. Introduction. One way of describing a group G is to give its *presentation*, i.e., to write G as

$$G = \langle x_i \ (i \in I) \mid R \rangle$$

(where $\{x_i \mid i \in I\}$ is a set of "generators" and R (the set of "relators") is a set of words in $\{x_i, x_i^{-1} \mid i \in I\}$ such that $G \cong F/H$ where F is the free group generated by $\{x_i \mid i \in I\}$ and H is the *normal* subgroup of F generated by R. If we can find a free group F of finite rank and a finite set of relators R, then we call G a *finitely presented group*.

Groups arising in applications, such as fundamental groups in topology, often are given naturally via their presentations. Unfortunately, a finite presentation does not yield very good information about the group. Novikov [No55] and Boone [Bo54-57] showed that in some finitely presented groups, one cannot even tell whether a particular word in $x_1, \dots, x_n$ and their inverses is the identity in G. (Such groups are said to have *unsolvable word problem.*) Further work of Baumslag, Boone, and Neumann [BBN59] revealed that many other properties of elements of G also cannot be determined from words denoting the elements.

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On the other hand, the finite presentation of a group G also does not allow us to determine almost any conceivable global property of G, such as whether G is trivial, finite, abelian, torsion-free, simple, etc. This follows immediately by a theorem of Adian [Ad57, Ad57a] and Rabin [Ra58], stating that any Markov property of a finitely presented group cannot be effectively determined from its presentation. (A property of finitely presented groups is called *Markov* if it holds of some finitely presented group G_+ and fails in any finitely presented group containing some finitely presented group G_-.)

Once a problem turns out to be undecidable, it is natural to ask exactly how complicated the problem is. Computability (or recursion) theory provides a tool for measuring complexity of unsolvable problems in the form of *Kleene's arithmetical hierarchy.* For $n > 0$, a property is called Σ^0_n if it can be expressed (in the language of arithmetic) by a formula of the form $\exists\vec{x}_1\forall\vec{x}_2\exists\vec{x}_3 \dots Q\vec{x}_n R(\vec{x}_1, \dots, \vec{x}_n)$ where R contains only bounded quantifiers; it is Π^0_n if it can be expressed by a formula of the form $\forall\vec{x}_1\exists\vec{x}_2\forall\vec{x}_3 \dots Q\vec{x}_n R(\vec{x}_1, \dots, \vec{x}_n)$. By Post's Theorem, a property is *(recursively) enumerable* (i.e., the set of all objects satisfying the property can be effectively listed) iff it is Σ^0_1; and a property is *decidable* (i.e., can be determined by an effective algorithm) iff it is both Σ^0_1 and Π^0_1. By a theorem of Kleene, these classes of properties form a proper hierarchy satisfying for all $n > 0$ that $\Sigma^0_n, \Pi^0_n \subset \Sigma^0_{n+1} \cap \Pi^0_{n+1} \subset \Sigma^0_{n+1}, \Pi^0_{n+1}$. A property P is called Σ^0_n-*hard* if any Σ^0_n-property P' can be *effectively reduced* to it, i.e., there is a computable function f such that, for any (code for a mathematical object) m, P' holds of m iff P holds of $f(m)$; a property is Σ^0_n-*complete* if it is both Σ^0_n and Σ^0_n-complete. Note here that Σ^0_n gives an upper bound on the complexity of a property whereas Σ^0_n-hardness gives a lower bound; thus Σ^0_n-competeness gives a precise classification of a property in terms of computability and definability. (Π^0_n-*hardness* and Π^0_n-*completeness* are defined analogously.)

The above-mentioned Adian-Rabin Theorem actually shows that any Markov property is Σ^0_1-hard. Since many Markov properties, such as being trivial, finite, abelian, etc., are also enumerable and so Σ^0_1, these properties are thus in fact Σ^0_1-complete. Other Markov properties, however, such as being solvable, simple, torsion-free, or having a decidable word problem, are not readily seen to be Σ^0_1. One of these, having a decidable word problem, was shown by Boone and Rogers [BR66] to be Σ^0_3-complete.

1. The theorem. The main result of this paper is to establish the completeness of the only other Markov property known to be complete at a level other than Σ^0_1:

Theorem. *The property of a finitely presented group being torsion-free is* Π^0_2*-complete. (Thus, of course, the property of a finitely presented group not being torsion-free is* Σ^0_2*-complete.)*

This theorem has an immediate consequence about effective enumerations of finitely presented groups:

Corollary. *There are no effective listings of (presentations of) all torsion-free finitely presented groups, or of all non-torsion-free finitely presented groups.*

Proof. If there were such an effective listing then the set of *all* finite presentations of such groups would be enumerable, i.e., Σ^0_1, contradicting the above theorem. ■

2. The proof of the theorem. The proof is based on two important theorems of combinatorial group theory, the Higman Embedding Theorem and Britton's Lemma.

We first recall that a group $G = \langle X \mid R\rangle$ is *recursively presented* if X is a finite set of generators and R is a (recursively) enumerable set of relators; and that G is *infinitely recursively presented* if X is a countable set of generators and R is a (recursively) enumerable set of relators.

Our proof proceeds by a sequence of reductions, each given by a proposition:

Proposition 1. *The property of a recursively presented group being torsion-free can be effectively reduced to the property of a finitely presented group being torsion-free.*

Proposition 2. *The property of an infinitely recursively presented group being torsion-free can be effectively reduced to the property of a recursively presented group being torsion-free.*

Proposition 3. *The property of an infinitely recursively presented group being torsion-free is Π^0_2-hard (in fact, Π^0_2-complete).*

Proof of the theorem. The property of a finitely presented group being torsion-free is readily seen to be Π^0_2:

$$G \text{ is torsion-free iff } \forall w \forall n > 0 (w^n \neq_G 1 \text{ or } w =_G 1)$$

where w ranges over words on X and their inverses and n ranges over integers. (Note that equality in G, denoted by $=_G$, is enumerable and thus Σ^0_1.)

But, by Propositions 1, 2, and 3, the property of a finitely presented group being torsion-free is also Π^0_2-hard, establishing the theorem. ■

It now remains to verify the propositions. We first need another definition.

A group G is called an *HNN-extension* (or *Britton extension*) of a group $H = \langle X \mid S\rangle$ (for a set of generators X and a set of relators S) if $G = \langle X, t \mid S, R\rangle$ where t is a generator (called a *stable letter*) not occurring in X and R is a set of relators (i.e., words) of the form $a_i^t b_i^{-1}$ (where i ranges over some (possibly infinite, or possibly empty) index set I) such that the map sending each a_i to b_i induces an isomorphism of the subgroups of H generated by the a_i's and the b_i's, respectively. (This latter condition is often called the *isomorphism condition.*)

Proof of Proposition 1. Given (a presentation of) a recursively presented group $H = \langle X \mid S\rangle$, we must effectively produce (a presentation of) a finitely presented group G such that G is torsion-free iff H is. We note that the Higman Embedding Theorem embeds a recursively presented group H into a finitely presented group G. A careful analysis of the proof (e.g., in [Ro95]) shows that G is obtained from H by a finite sequence of HNN-extensions. But, by Britton's Lemma, HNN-extensions preserve torsion-freeness, i.e., G is torsion-free iff H is, as desired. ■

Proof of Proposition 2. Given (a presentation of) an infinitely recursively presented group $H = \langle X \mid S\rangle$, we must effectively produce (a presentation of) a recursively presented group G such that G is torsion-free iff H is.

Let $X = \{x_1, x_2, \dots\}$. Let H' be the free product $H \star \langle a, b\rangle$ where $\langle a, b\rangle$ is a free group of rank 2. We now form an HNN-extension of H' by a stable letter t:

$$G = \langle a, b, x_1, x_2, \dots, t \mid R, a^t b^{-1}, a^{b^n t}(x_n b^{a^n})^{-1} (n > 0)\rangle$$

Note here that the isomorphism condition holds since both $\{a^{b^n} \mid n \geq 0\}$ and $\{b\} \cup \{x_n b^{a^n} \mid n > 0\}$ freely generate subgroups in H'.

But now G is an HNN-extension of H', which in turn trivially is an HNN-extension of H; so G is torsion-free iff H is. Finally, G is finitely generated, namely, by a and t. ■

Proof of Proposition 3. We use the well-known fact (see, e.g., [So87, p. 66]) that the (index) set of all infinite enumerable sets is Π^0_2-complete. Given (an index for) an enumerable set W, we must thus effectively produce (a presentation of) an infinitely recursively presented group G such that W is infinite iff G is torsion-free.

We let $G = \langle x_1, x_2, \cdots \mid R\rangle$ where the set of relators is enumerated as follows: Start with $R = \{x_1^2\}$ and enumerate the set W stage by stage, enumerating at most one element at any stage. At the stage at which the nth element (in the order of enumeration) enters W, enumerate into R the set $\{x_n, x_{n+1}^2\}$.

Now, if W is finite, say, it contains n elements, then we have

$$R = \{x_1^2, x_1, x_2^2, x_2, \dots, x_n^2, x_n, x_{n+1}^2\},$$

so $G = \langle x_1, x_2, \cdots \mid x_1, x_2, \dots, x_n, x_{n+1}^2\rangle \cong \langle x_{n+1}, x_{n+2}, \cdots \mid x_{n+1}^2\rangle$, which has a torsion element x_{n+1} of order 2. On the other hand, if W is infinite, then $R = \{x_1^2, x_1, x_2^2, x_2, \dots\}$, so $G = \langle x_1, x_2, \cdots \mid x_1, x_2, \dots\rangle = 1$, which is torsion-free. ■

3. Open Questions. As mentioned in the introduction, some Markov properties are not known to be (and in fact conjectured not to be) Σ^0_1: Solvability is only known to be Σ^0_3, residual finiteness and simplicity are only known to be Π^0_2, etc. Our technique above, however, does not seem to work for these properties since these are not properties localized at some element and thus not preserved under HNN-extensions.

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