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NOTATION

Our notation is fairly standard and generally follows Soare's forthcoming book "Recursively Enumerable Sets and Degrees" [**Sota**].

We consider sets and functions on the natural numbers $\omega = \{1, 2, 3, \dots\}$. Usually lower-case Latin letters $a, b, c, \dots$ denote natural numbers; $f, g, h, \dots$ total functions on ω; Greek letters $\Phi, \Psi, \dots, \varphi, \psi, \dots$ partial functions on ω; and upper-case Latin letters $A, B, C, \dots$ subsets of ω. For a partial function φ, $\varphi(x)\downarrow$ denotes that $x \in \operatorname{dom}\varphi$, otherwise we write $\varphi(x)\uparrow$. We identify a set A with its characteristic function χ_A. $f \upharpoonright x$ denotes f restricted to arguments less than x, likewise for sets.

We let $A \subset B$ denote that $A \subseteq B$ but $A \neq B$; $A \subseteq^* B$ that $A - B$ is finite; and $A \subset_\infty B$ that $A \subseteq B$ and $|B - A| = \infty$. $A \sqcup B$ will denote the disjoint union. For each $n \in \omega$, we let $\langle x_1, x_2, \dots, x_n \rangle$ denote the coded n-tuple (where $x_i \leq \langle x_1, x_2, \dots, x_n \rangle$ for each i); and $(x)_i$ the ith projection function, mapping $\langle x_1, x_2, \dots, x_n \rangle$ to x_i. $A^{[k]} = \{y \mid \langle y, k \rangle \in A\}$ denotes the kth "row" of A; and $A^{[<l]} = \bigcup_{k<l} A^{[k]}$.

In a partial order, $x \,|\, y$ denotes that x and y are incomparable. $[k, l)$ denotes the interval $\{n \in \omega \mid k \leq n < l\}$.

The logical connectives "and" and "or" will be denoted by $\wedge$ and $\vee$, respectively. We allow as additional quantifiers (in the meta-language) $(\exists^\infty x)$, $(\exists^{<\infty} x)$, and (a.e. x) to denote that the set of such x is infinite, finite, and cofinite, respectively.

$\{e\}$ (or φ_e) and W_e ($\{e\}^X$ (or Φ_e^X) and W_e^X) denote the eth partial recursive function and its domain (with oracle X) under some fixed standard numbering. $\leq_1$ and $\leq_T$ denote one-one and Turing reducibility, respectively, and $\equiv_1$ and $\equiv_T$ the induced equivalence relations. The *use* of a computation $\Phi_e^X(x)$ (denoted by $u(X; e, x)$) is 1 plus the largest number from oracle X used in the computation if $\Phi_e^X(x)\downarrow$; and 0 otherwise (likewise for $u(X; e, x, s)$, the use at stage s). Sets, functionals, and parameters are often viewed as being in a state of formation, so, when describing a construction, we may write A (instead of the full Lachlan notation A_s, $A[s]$, or $A_t[s]$ for the value at the end of stage s or at the end of substage t of stage s).

In the context of trees, $\rho, \sigma, \tau, \ldots$ denote *finite strings*; $|\sigma|$ the *length* of σ; $\sigma\widehat{\ }\tau$ the *concatenation* of σ and τ; $\langle a \rangle$ the one-element string consisting of a; $\langle a^n b^m \ldots \rangle$ the finite string consisting of n many a's, followed by m many b's, ...; $\sigma \subseteq \tau$ $(\sigma \subset \tau)$ that σ is a (*proper*) *initial segment* of τ; $\sigma <_L \tau$ that for some i, $\sigma \upharpoonright i = \tau \upharpoonright i$ and $\sigma(i) <_\Lambda \tau(i)$ (where $<_\Lambda$ is a given order on Λ and $T \subseteq \Lambda^{<\omega}$); and $\sigma \leq \tau$ $(\sigma < \tau)$ that $\sigma <_L \tau$ or $\sigma \subseteq \tau$ $(\sigma \subset \tau)$.

The set $[T]$ of *infinite paths* through a tree $T \subseteq \Lambda^{<\omega}$ is $\{p \in \Lambda^\omega \mid (\forall n)[p \upharpoonright n \in T]\}$. The *extendible part of a tree* T is $\{\sigma \in T \mid (\exists p \in [T])[\sigma \subset p]\}$. The *part of a tree above* σ is $T(\sigma) = \{\tau \mid \sigma\widehat{\ }\tau \in T\}$.

In $\mathbf{0}'''$-priority arguments, we use the following conventions: Upper-case letters at the beginning of the alphabet are used for sets $A, B, C, \ldots$ and functionals $\Gamma, \Delta, \ldots$ constructed by *us*; those at the end of the alphabet are used for sets $U, V, W, \ldots$ and functionals $\Phi, \Psi, \ldots$ constructed by the *opponent*. A functional Φ $(\Psi, \Theta, \ldots)$ is viewed as an r.e. set of triples $\langle x, y, \sigma \rangle$ (denoting $\Phi^\sigma(x) \downarrow = y$), and the corresponding Greek lower-case letter φ $(\psi, \vartheta, \ldots)$ denotes a modified use function for Φ $(\Psi, \Theta, \ldots)$, namely, $\varphi(x) = |\sigma| - 1$ (so changing X at $\varphi(x)$ will change $\Phi^X(x)$). Parameters, once assigned a value, retain this value until reassigned.

Strategies are identified with strings on the tree corresponding to their guess about the outcomes of higher-priority strategies and are viewed as finite automata described in flow charts. In these flow charts, states are denoted by circles, instructions to be executed by rectangles, and decisions to be made by diamonds. To *initialize* a strategy means to put it into state *init* and to set its restraint to zero. A strategy is initialized at stage 0 and whenever specified later. At a stage when a strategy is allowed to act, it will proceed to the next state along the arrows and according to whether the statements in the diamonds are true (y) or false (n). Along the way, it will execute the instructions. Half-circles denote points in the diagram where a strategy starts from through the action of another strategy. Sometimes, parts of a flow chart are shared, the arrows are then labeled i and ii. The *strategy control* decides which strategy can act when. For some further background on $\mathbf{0}'''$-priority arguments, we refer to Soare ([**Sota**] or [**So85**])

There will be no cross references between chapters, so all references refer to theorems, equations, etc. within the same chapter.

ABSTRACT

In Chapter I, we exhibit a high strongly noncappable degree.

Chapter II answers negatively the question whether a deep degree exists. It also shows a weak converse of this.

Chapter III is devoted to index sets. We define a family of properties on hyperhypersimple sets and show that they yield index sets at each level of the hyperarithmetical hierarchy. We also classify the index set of quasimaximal sets, of coinfinite r.e. sets not having an atomless superset, and of r.e. sets major in a fixed nonrecursive r.e. set.

Chapter IV investigates properties of the partial order of ω-degrees. We show that the ω-degree of $\mathbf{0}'$ splits and that there is a minimal pair in the r.e. ω-degrees. A forcing argument shows that the ω-degrees below $\emptyset^{(\omega)}$ do not form an upper semilattice.

Perseverance is a great element of success.
If you knock long enough and loud enough at the gate,
you are sure to wake up somebody.

Henry Wadsworth Longfellow

INTRODUCTION

The question of whether a construction can be done effectively appears frequently throughout mathematics and particularly in mathematical logic. Classical recursion theory deals with this question on the set ω of nonnegative integers. The most frequently studied constructions yield sets of natural numbers which are computable (*recursive*), can be enumerated in a recursive way (*recursively enumerable*, abbreviated *r.e.*), or are limits of a recursive approximation. This thesis focuses on the structures of the lattice $\mathcal{E}$ of r.e. sets and of the upper semilattice $\mathbf{R}$ of r.e. degrees.

The early study of $\mathbf{R}$ revealed certain "nice" properties. For example, the Sacks Splitting Theorem [**Sa63a**] showed that any nonrecursive r.e. degree is the supremum of two incomparable r.e. degrees. The Sacks Density Theorem [**Sa64**] showed that $\mathbf{R}$ is a dense partial order. These and other results led Shoenfield [**Sh65**] to conjecture that if $\vec{a} \in \mathbf{R}$ satisfies a diagram $D(\vec{x})$ in the language $\mathcal{L} = \{0, 1, \leq, \cup\}$ of upper semilattices and $D_0(\vec{x}, y)$ is a consistent extension of $D(\vec{x})$, then there is $\mathbf{b} \in \mathbf{R}$ such that $\vec{a}$ and $\mathbf{b}$ satisfy $D_0(\vec{x}, y)$. A consequence of this would be that no two incomparable r.e. degrees have an infimum (*cap* to some lower r.e. degree). This was refuted independently by Lachlan [**La66**] and Yates [**Ya66**] through the construction of a *minimal pair* (capping to $\mathbf{0}$). Yates [ibid.] also showed that some r.e. degrees are *noncappable* (not half of a minimal pair). Soare [**So80**] defined the notion of a *strongly noncappable* (*s.n.c.*) degree (an r.e. degree $\neq \mathbf{0}, \mathbf{0}'$ that does not have an infimum with any incomparable r.e. degree). Ambos-Spies [**AS84**] proved the existence of s.n.c. degrees and various stronger results, but all his such degrees were constructed by finite injury arguments and thus are low. A (much more difficult) $\mathbf{0}'''$-priority argument in Chapter I of this thesis establishes the existence of a strongly noncappable degree, which is high and thus is not obtainable by

Ambos-Spies's methods. This is a step in the characterization of the range of the *jump* operator (halting problem) on certain classes of r.e. degrees. Which degrees actually are the jumps of s.n.c. degrees still remains an open question. A recent related result by Cooper [**Cota**] (and independently by Shore [**Shta**]) shows that the range of the jump operator on the set of cappable degrees is not the set of all degrees r.e. in and above (REA in) $\mathbf{0}'$.

Adding the jump operator to the language of the structure of $\mathbf{R}$ complicates the picture even more: The Sacks Jump Theorem [**Sa63b**] asserts that any degree REA in $\mathbf{0}'$ is the jump of an r.e. degree (even uniformly). However, a recent result by Shore [**Shta**] shows that even a slight extension is impossible. He constructed two degrees $\mathbf{a}$ and $\mathbf{b}$ REA in $\mathbf{0}'$ such that $\mathbf{a} \cup \mathbf{b} < \mathbf{0}''$, but for any r.e. (even Δ_2) degree $\mathbf{c} < \mathbf{0}'$, either $\mathbf{a}$ or $\mathbf{b}$ is not r.e. in $\mathbf{c}$. He thus established that it is not always possible to simultaneously invert the jump on two degrees REA in $\mathbf{0}'$ and their join.

In the last few years, two major problems about the existence of certain nonrecursive "extremely low" degrees (i.e., degrees with properties stronger than lowness) were solved negatively: Soare and Stob [**SS82**] refuted the existence of a *superlow* degree (a nonrecursive r.e. degree $\mathbf{a}$ such that any degree REA in $\mathbf{a}$ is actually r.e.). A closely related question is whether there is a *deep* degree (a nonrecursive r.e. degree $\mathbf{w}$ such that for all r.e. degrees $\mathbf{a}$, $(\mathbf{a} \cup \mathbf{w})' = \mathbf{a}'$). This question had been raised by Bickford and Mills and had been worked on since by several mathematicians. A joint result with Slaman in Chapter II of this thesis shows that there is no deep degree. We also show a weak converse: There is a nonrecursive low r.e. degree that does not join to a high degree with any other low r.e. degree, this question first being raised by Jockusch.

Most questions concerning the jump remain open and will prove fertile ground for future research.

Chapter III deals with index sets, i.e., sets of indices of partial recursive (p.r.) functions and r.e. sets that are defined through the p.r. functions or r.e. sets they code. The early results in index sets used geometric arguments in one- or two-dimensional arrays: Rogers showed the Σ_3 and Π_3-completeness of the index sets of recursive and simple sets, respectively, in a finite injury argument. Lachlan, D.A. Martin, R.W. Robinson, and Yates (1968, unpublished, later appearing in Tulloss [**Tu71**]) showed the Π_4-completeness of the index set of maximal sets in an infinite injury argument. Tulloss [ibid.] also mentions for the first time the question whether the index set of quasimaximal sets is Σ_5-complete. However, the geometric method was too complex at higher levels of the arithmetical hierarchy. During the 1970's, progress in index sets was mainly made in other areas by several Russian mathematicians as well as L. Hay.

Schwarz [**Schta**] was the first to introduce induction into index set proofs (in the r.e. degrees) and was able to show that the index sets of low_n and high_n r.e. sets are Σ_{n+3} and Σ_{n+4}-complete, respectively. Solovay [**JLSSta**] then extended Schwarz's methods to show the $\Sigma_{\omega+1}$-completeness of the index sets of $\text{low}_{<\omega}$ (low_n for some n) and of $\text{high}_{<\omega}$ (high_n for some n) r.e. sets as well as the $\Pi_{\omega+1}$-completeness of the index set of intermediate degrees (degrees neither $\text{low}_{<\omega}$ nor $\text{high}_{<\omega}$).

We exhibit a family of algebraically invariant properties $\mathbf{L}_{\omega_1,\omega}$-definable in $\mathcal{E}$, that yields index sets at any level of the *hyper*arithmetical hierarchy. The proof is based on induction and Lachlan's theorem [**La68**] that any Σ_3-Boolean algebra is isomorphic to the lattice of r.e. supersets of some r.e. set (modulo finite sets). It uses tree arguments and the fact that the Cantor-Bendixson rank of a tree corresponds to certain properties of the lattice of r.e. supersets of the set constructed. A corollary shows the Σ_5-completeness of the index set of quasimaximal sets, thereby settling this long-open question. Further results classify the index sets of atomlessless sets and of r.e. sets major in a fixed nonrecursive r.e. set.

Interesting open questions in index sets include whether major subsets (in some r.e. superset) or cuppable degrees yield Σ_5-complete index sets.

Chapter IV returns to questions about the jump. Jockusch, Lerman, Soare, and Solovay [**JLSSta**] defined a reflexive, transitive relation $\leq_\omega$ on r.e. degrees by $\mathbf{a} \leq_\omega \mathbf{b}$ iff $(\exists n)[\mathbf{a}^{(n)} \leq_T \mathbf{b}^{(n)}]$. This relation easily extends to all Turing degrees and induces equivalence classes, called *ω-degrees*. They showed the density of the r.e. ω-degrees (even more strongly, that any interval of r.e. ω-degrees allows an independent set of r.e. ω-degrees of size $\aleph_0$). In Chapter IV, we show that the ω-degree of $\mathbf{0}'$ splits in the r.e. ω-degrees, and we exhibit a minimal pair of r.e. ω-degrees. We also show that the ω-degrees do not form an upper semilattice by constructing two nonarithmetical ω-degrees below $\mathbf{0}^{(\omega)}$ not having a supremum. It is still open whether this is also true in the r.e. ω-degrees.

CHAPTER I

A HIGH STRONGLY NONCAPPABLE DEGREE

1. The Theorem

Soare [So80] defined:

Definition: An r.e. degree $\mathbf{a} \neq \mathbf{0}, \mathbf{0}'$ is *strongly noncappable* (*s.n.c.*) if $\mathbf{a}$ does not have an infimum with any incomparable r.e. degree $\mathbf{v}$, i.e., in the r.e. degrees,

$$(1) \qquad (\forall \mathbf{v})(\forall \mathbf{u})\big[\mathbf{a} \mid \mathbf{v} \wedge \mathbf{u} \leq \mathbf{a}, \mathbf{v} \rightarrow (\exists \mathbf{b})[\mathbf{b} \leq \mathbf{a}, \mathbf{v} \wedge \mathbf{b} \not\leq \mathbf{u}]\big].$$

Ambos-Spies [**AS84**] showed the existence of various low s.n.c. degrees. We prove in this chapter:

Theorem. *There is a high strongly noncappable degree.*

Proof: Actually, we will prove, similarly to Ambos-Spies, a slightly stronger result, namely, we will construct a high r.e. degree $\mathbf{a} \neq \mathbf{0}'$ such that in the r.e. degrees,

$$(2) \qquad (\forall \mathbf{v})(\forall \mathbf{u})\big[\mathbf{u} < \mathbf{a} \wedge \mathbf{v} \not\leq \mathbf{a} \rightarrow (\exists \mathbf{b})[\mathbf{b} \leq \mathbf{a}, \mathbf{v} \wedge \mathbf{b} \not\leq \mathbf{u}]\big].$$

(This implies (1) by letting $\mathbf{u} \leq \mathbf{v}$ also.)

2. The Requirements

We will build a high r.e. set A of s.n.c. degree by satisfying the following three requirements:

To ensure that A is high we let J be an r.e. set which in the limit codes $\mathbf{0}''$ as follows:

$$(3) \qquad (\forall e)\big[(e \in \emptyset'' \rightarrow J^{[2e]} =^* \emptyset) \wedge (e \notin \emptyset'' \rightarrow J^{[2e]} = \omega^{[2e]})\big].$$

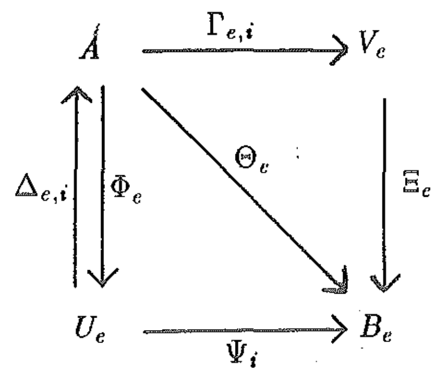

Diagram 1. Sets and functionals used

Then the usual thickness requirements will suffice to make A high:

(4) $$\mathcal{P}_e : A^{[2e]} =^* J^{[2e]}.$$

To make A incomplete we require for all e:

(5) $$\mathcal{N}_e : K \neq \{e\}^A,$$

where $K = \emptyset'$ (although we could in this construction replace K by any nonrecursive r.e. set W). Our basic strategy for $\mathcal{N}_e$ will be the Sacks preservation strategy, using a typical tree argument to deal with infinite injury from the $\mathcal{P}$-strategies but a new coding strategy for such injury from the $\mathcal{S}$-strategies as explained below.

To ensure (2) for strong noncappability, we stipulate that for all e,

(6) $$\tilde{\mathcal{R}}_e : U_e = \Phi_e^A \rightarrow \left[A \leq_T U_e \vee V_e \leq_T A \vee (\exists B_e)[B_e \leq_T A, V_e \wedge B_e \not\leq U_e]\right],$$

where $\{U_e, V_e, \Phi_e\}_{e \in \omega}$ is an enumeration of all triples of r.e. sets U, V and functionals Φ (given by the opponent), and where the B_e are built by us. (See Diagram 1.)

However, the $\tilde{\mathcal{R}}_e$ are still too complicated to be satisfied at one level of the tree, so we split each $\tilde{\mathcal{R}}_e$ up into

(7) $$\hat{\mathcal{R}}_e : U_e = \Phi_e^A \rightarrow B_e \leq_T A, V_e,$$

and for all $i \in \omega$,

$$(8) \qquad \hat{S}_{e,i} : U_e = \Phi_e^A \wedge B_e = \Psi_i^{U_e} \rightarrow A \leq_T U_e \vee V_e \leq_T A,$$

where $\{ \Psi_i \}_{i \in \omega}$ is an enumeration of all functionals Ψ (given by the opponent).

For the sake of $\hat{\mathcal{R}}_e$, we will build functionals Θ_e, Ξ_e such that

$$(9) \qquad \mathcal{R}_e : U_e = \Phi_e^A \rightarrow B_e = \Theta_e^A \wedge B_e = \Xi_e^{V_e}.$$

For $\hat{S}_{e,i}$, we will construct functionals $\Gamma_{e,i}, \Delta_{e,i}$ such that

$$(10) \qquad S_{e,i} : U_e = \Phi_e^A \wedge B_e = \Psi_i^{U_e} \rightarrow A =^* \Delta_{e,i}^{U_e} \vee V_e =^* \Gamma_{e,i}^A.$$

The $\mathcal{R}_e$ and $S_{e,i}$ will correspond to actual strategies.

The strategies for satisfying the requirements will be arranged on nodes of a tree. Each strategy will be responsible for one requirement of type $\mathcal{N}$, $\mathcal{P}$, $\mathcal{R}$, or S and will from now on be called $\mathcal{N}$-, $\mathcal{P}$-, $\mathcal{R}$-, or S-strategy. (We will suppress indices whenever they are clear from the context.)

3. Making A S.N.C.

In order to be able to restrain U through A, we will require that

$$(11) \qquad x \in U_{s+1} - U_s \rightarrow \Phi^A(x)[s] = 1.$$

Then $\Phi^{A \upharpoonright u}[s] \upharpoonright x = U_s \upharpoonright x$ and $A_s \upharpoonright u = A \upharpoonright u$ implies $U_s \upharpoonright x = U \upharpoonright x$. We also tacitly assume that all use functions $\varphi_s(x)$, etc. are increasing in x and nondecreasing in s.

For satisfying $\tilde{\mathcal{R}}_e$, we have to ensure first of all $\mathcal{R}_e$. Each $\mathcal{R}_e$-strategy α will build its version of Ξ_e as direct permitting on α-stages ($V_{e,s} \upharpoonright x = V_e \upharpoonright x \wedge s \in S^\alpha \rightarrow B_{e,s}(x) = B_e(x)$), and we will therefore not mention Ξ_e any more. However, V_e and Ξ_e are used by many strategies on the cone below the $\mathcal{R}_e$-strategy. Therefore, in our infinite injury setting, direct permitting requires that the strategy responsible for building Ξ_e (i.e., the $\mathcal{R}_e$-strategy) allows a strategy below on the tree to act immediately if the latter wants to put a number into B_e and thus needs a V_e-change to correct Ξ_e. A version of the functional Θ_e will be built explicitly by each $\mathcal{R}_e$-strategy as the length of agreement between U and Φ_e^A increases. Notice thus that an $\mathcal{R}$-strategy only builds a functional, but does not enumerate numbers into any set or impose any restraint. Its outcomes are $\Phi^A \neq U$ (called 1, in which case Θ will be finite), and (a guess that) $\Phi^A = U$ (called 0, in which case it has to ensure that Θ^A is total and $\Theta^A = B$).

An $\mathcal{S}_{e,i}$-strategy β, which will only ever act if it is below the outcome 0 of an $\mathcal{R}_e$-strategy on the tree, will mainly try to "code V_e into A" by gradually building $\Gamma_{e,i}$ and putting $\gamma_{e,i}(x)$ into A whenever $\Gamma_{e,i}^A(x) \downarrow \neq V_e(x)$ to ensure the correctness of $\Gamma_{e,i}$. If $V_e = K$ then this would make A complete and thus injure one of the $\mathcal{N}$-strategies below, say, $\gamma \supset \beta$. So the *key to the whole construction* is the feature that the $\mathcal{N}$-strategy γ helps the $\mathcal{S}_{e,i}$-strategy β prove $B_e \neq \Psi_i^{U_e}$ and then *immediately* shuts β off. The outcomes of the $\mathcal{S}_{e,i}$-strategy β are again 0 (infinite action) and 1 (finite action).

Now consider an $\mathcal{N}_e$-strategy γ, and assume it is on the true path and thus has to satisfy its requirement. The strategies to the left of γ only have finite effect; γ will put up restraint against the strategies to the right of and below γ. So the only strategies dangerous to γ lie above it on the tree, and they are either $\mathcal{P}_{e'}$ or $\mathcal{S}$-strategies. The former are no problem: γ knows the outcome (either $A^{[2e']} =^* \omega^{[2e']}$ or $A^{[2e']} =^* \emptyset$). For each $\mathcal{S}$-strategy $\beta \subset \gamma$ for which γ guesses that β puts infinitely many numbers into A, γ will take over β's responsibility and try to put up a candidate x for $B(x) \neq \Psi^U(x)$.

If γ succeeds in finding a suitable candidate, there are two possibilities: Either V will change and allow x into B, while the $\mathcal{N}$-strategy preserves $\Psi^U(x) = 0$; thus $B(x) = 1 \neq 0 = \Psi^U(x)$. Then β's requirement has been satisfied by γ, therefore β can be shut off and has finite outcome. So γ is not on the true path after all, and its restraint will have the same priority as if it were imposed by β (since no $\xi \supseteq \beta^\frown\langle 0\rangle$ will act ever again). The other possibility is that V does not change, which constitutes another step towards showing that $V \leq_T A$.

The strategy γ may have to act even when it is not its turn since it needs to redefine a functional of much higher priority. Thus γ might injure higher-priority strategies which have increased their restraint since γ acted last. Therefore, whenever some $\mathcal{N}$-strategy γ' changes states (while it is its turn), the strategy control will initialize all strategies $\xi > \gamma'$ to prevent them from injuring γ'. This is compatible with the rest of the construction since each $\mathcal{N}$-strategy γ on or to the left of the true path will act only finitely often.

On the other hand, if γ fails to find a suitable candidate, then β has to make Δ total and ensure that $\Delta^U = A$. So again β's requirement will be ensured by γ.

Candidates for showing $B \neq \Psi^U$ must have the property that $\vartheta(x) > \varphi(\psi(x))$ so that we can put x into B, put $\vartheta(x)$ into A to correct $\Theta^A(x)$, and at the same time restrain $A \upharpoonright (\varphi(\psi(x))+1)$ to preserve $U \upharpoonright (\psi(x)+1)$ and thus $\Psi^U(x) = 0$. Now an $\mathcal{R}$-strategy can wait with the definition of $\Theta^A(x)$ until $\Phi^A \upharpoonright (y+1)$ is defined (for some y depending on x), but not for $\Psi^U(x)$ (which may not be defined at all). So we introduce the A-recursive *computation function* of A,

$$c_A(x) = \mu s[A_s \upharpoonright (x+1) = A \upharpoonright (x+1)]$$

for the given enumeration of A, and its recursive approximation

$$c_A(x,s) = (\mu t \leq s)[A_s \upharpoonright (x+1) = A_t \upharpoonright (x+1)].$$

Now if $U <_T A$, ψ is a U-recursive function, and S is an infinite recursive set then

$$(\exists^\infty x \in S)[\psi(x) < c^A(x)],$$

and thus if in addition $\Phi^A = U$ and φ is increasing then

$$(\exists^\infty x \in S)[\varphi(\psi(x)) < \varphi(c^A(x))].$$

If $U <_T A$, this will ensure that an $\mathcal{N}$-strategy below an $\mathcal{S}$-strategy can find enough candidates x for $B(x) \neq \Psi^U(x)$ with $\vartheta(x) > \varphi(\psi(x))$ by having at stage $s+1$ the $\mathcal{R}$-strategy put $\vartheta(x) > \varphi(c_A(x,s))$. (The function c_A is Ambos-Spies's function γ as explained in Lemma 1 of [AS84].) So if an $\mathcal{N}$-strategy γ cannot find a suitable candidate for an $\mathcal{S}$-strategy $\beta \subset \gamma$, we can allow γ to shut off β *eventually.*

The outcome of the $\mathcal{N}$-strategy γ is the lim inf of the restraint that γ imposes on the lower-priority strategies. Note that only the $\mathcal{N}$-strategies want to restrain A.

4. The Full Construction

We will first describe the tree of strategies and then give the full module for each type of strategy (in a flow chart) and explain the strategy control to see how the strategies interact.

Let $\Lambda_{\mathcal{N}}$, $\Lambda_{\mathcal{P}}$, $\Lambda_{\mathcal{R}}$, and $\Lambda_{\mathcal{S}}$ be the *sets of outcomes* of the $\mathcal{N}$-, $\mathcal{P}$-, $\mathcal{R}$-, and $\mathcal{S}$-strategies (where $\Lambda_{\mathcal{N}} = \omega$ and $\Lambda_{\mathcal{P}} = \Lambda_{\mathcal{R}} = \Lambda_{\mathcal{S}} = \{0,1\}$), and let Λ be their union. The *tree of strategies* is

$$(12)\quad T = \{\, \xi \in \Lambda^{<\omega} \mid (\forall k < |\xi|)[\xi(k) \in \Lambda_{\mathcal{N}}, \Lambda_{\mathcal{P}}, \Lambda_{\mathcal{R}}, \Lambda_{\mathcal{S}} \text{ for } k \equiv 0,1,2,3 \bmod 4]\,\}.$$

To each node $\xi \in T$, we assign a type of strategy ($\mathcal{N}$, $\mathcal{P}$, $\mathcal{R}$, $\mathcal{S}$ for $|\xi| \equiv 0,1,2,3 \bmod 4$) and a number $e(\xi)$ (or $\langle\, e(\xi), i(\xi)\,\rangle) = \frac{|\xi|-k}{4}$ (for some $k \in \{0,1,2,3\}$) so that ξ works on requirement $\mathcal{N}_{e(\xi)}$, $\mathcal{P}_{e(\xi)}$, $\mathcal{R}_{e(\xi)}$, or $\mathcal{S}_{e(\xi),i(\xi)}$. Then for each infinite path $h \in [T]$, there is exactly one strategy $\xi \subset h$ working on each requirement. Fixing e and i, notice that if α is the $\mathcal{R}_e$-strategy $\alpha \subset h$ and β is the $\mathcal{S}_{e,i}$-strategy $\beta \subset h$, we have that $\alpha \subset \beta$. (Furthermore, β will not act at all unless $\alpha^\frown\langle 0\rangle \subseteq \beta$, i.e., unless β guesses that $\Phi_e^A = U_e$.)

Each $\mathcal{P}_e$-strategy ξ is assigned to $D_\xi = \omega^{[2e]}$ for its thickness strategy. Each strategy ξ of type $\mathcal{R}$ or $\mathcal{S}$ is effectively assigned to an infinite recursive subset D_ξ of ω so that

$$(13)\qquad \bigsqcup_{\xi \text{ of type } \mathcal{R} \text{ or } \mathcal{S}} D_\xi = \bigcup_{e\in\omega} \omega^{[2e+1]}.$$

All $\mathcal{N}$-strategies $\gamma \supseteq \alpha^\frown\langle 0\rangle$ (where α is a fixed $\mathcal{R}_e$-strategy) also help each $\mathcal{S}_{e,i}$-strategy β with $\alpha^\frown\langle 0\rangle \subseteq \beta \subset \gamma$ build its part of the set B_e, so each γ is effectively assigned an infinite recursive subset E^γ_α such that for fixed α,

$$(14)\qquad \bigsqcup_{\substack{\gamma \supseteq \alpha^\frown\langle 0\rangle \\ \gamma \text{ of type } \mathcal{N}}} E^\gamma_\alpha = \omega.$$

Let also $r(\gamma)$ (or r, for short) denote the *A-restraint imposed by the $\mathcal{N}$-strategy* γ (as defined below), and

$$(15)\qquad r'(\xi) = \max\{\, r(\gamma) \mid \gamma < \xi \,\}$$

(or r', for short) the *A-restraint imposed on* ξ by all stronger strategies. (Recall that only $\mathcal{N}$-strategies impose restraint, so $r(\xi) = -1$ for all other strategies ξ.)

At each stage s, we will build substage by substage the approximation $\delta_s = \max\{\, \xi \mid \xi \text{ acted at stage } s \,\}$ to the true path $f \in [T]$ (where $|\delta_s| \leq s$). We say s is a *ξ-stage* ($\xi \in S^\xi$) iff $\xi \subseteq \delta_s$. In this particular construction, each strategy that acts at substage t of stage s will decide which strategy will act at substage $t+1$ (or whether we should go on to stage $s+1$). $\emptyset$ will always be the strategy to act at substage 0. (When an $\mathcal{R}$- or an $\mathcal{S}$-strategy ξ lets an $\mathcal{N}$-strategy γ below it act first, then the action of γ will not count towards the definition of δ_s or as a separate substage.) Any strategy $\xi >_L \delta_s$ will be initialized as soon as δ_s has been defined far enough (i.e., at the least substage t at which $\delta_t[s] <_L \xi$).

The $\mathcal{P}$-strategies are the easiest to describe. They ensure that A is high. Recall that the r.e. set J codes $\mathbf{0}''$ in the limit on the even rows. Then a $\mathcal{P}_e$-strategy ς acts as described in Diagram 2.

The strategy to play next will be $\varsigma^\frown\langle 0\rangle$ if $A^{[2e]}_s \neq A^{[2e]}_t$ where $t = \max\{\, t' < s \mid t' \in S^\varsigma \,\}$, and $\varsigma^\frown\langle 1\rangle$ otherwise.

Each $\mathcal{R}_e$-strategy α is responsible for building its version of the functional Θ_e, and it is the node where the construction of its version of the r.e. set B_e originates on the tree. Then α proceeds as described in Diagram 3.

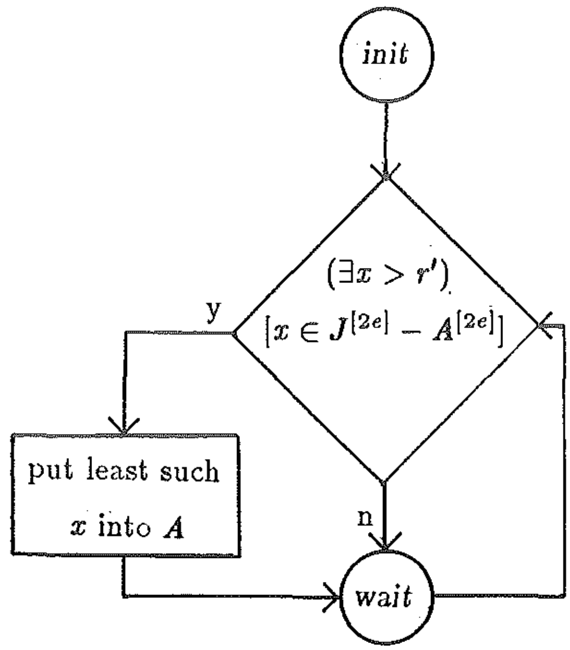

Diagram 2. The $\mathcal{P}$-strategy

Here $m(x)$ (the assigned use for $\Theta^A(x)$) is the least $y \in D_\alpha - A$ such that $y \geq$ previous values of $\vartheta(x)$ and greater than $\vartheta(x-1)$, $\varphi(c_A(x,s))$, and r'.

An $\mathcal{N}$-strategy $\gamma \supseteq \alpha^\frown\langle 0\rangle$ is ready to *special-act* if:

(i) γ has put up a candidate $x_{(k)}$ for an $\mathcal{S}_{e,i}$-strategy $\beta_{(k)} \supseteq \alpha^\frown\langle 0\rangle$ at a previous stage s_0;

(ii) γ has not been initialized since stage s_0;

(iii) no element entered $A \upharpoonright (r_{s_0}(\gamma)+1)$ since stage s_0, but $V_{(k)} \upharpoonright x_{(k)}$ has changed since stage s_0; and

(iv) no candidate for any $\beta_{(j)}$ with $j \leq k$ has been permitted since γ was initialized for the last time.

In this case, γ goes to $spact_k$ and on to the next state and gets a *permitted candidate* $x_{(k)}$ for $\beta_{(k)}$ through its special action (until γ is initialized if ever).

The strategy control will end the current stage if α lets some $\mathcal{N}$-strategy special-act. Otherwise, the next strategy to act will be $\alpha^\frown\langle 0\rangle$ if α just enumerated a new

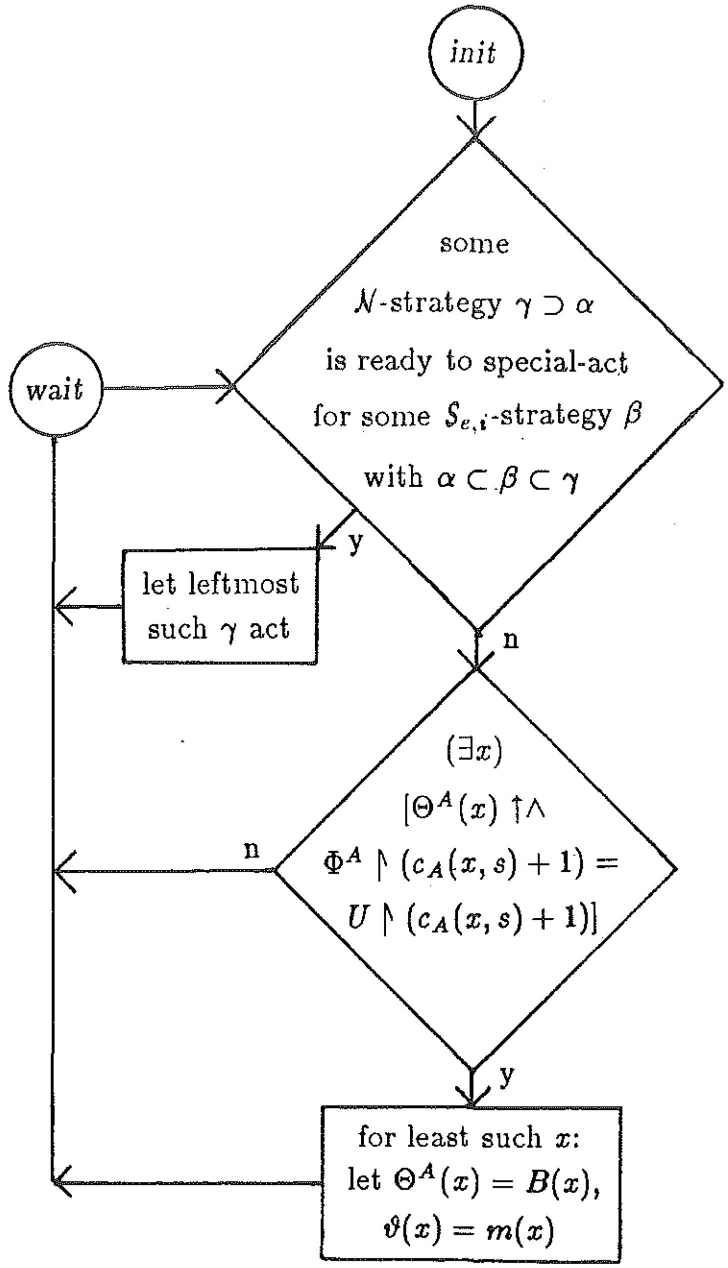

Diagram 3. The $\mathcal{R}$-strategy

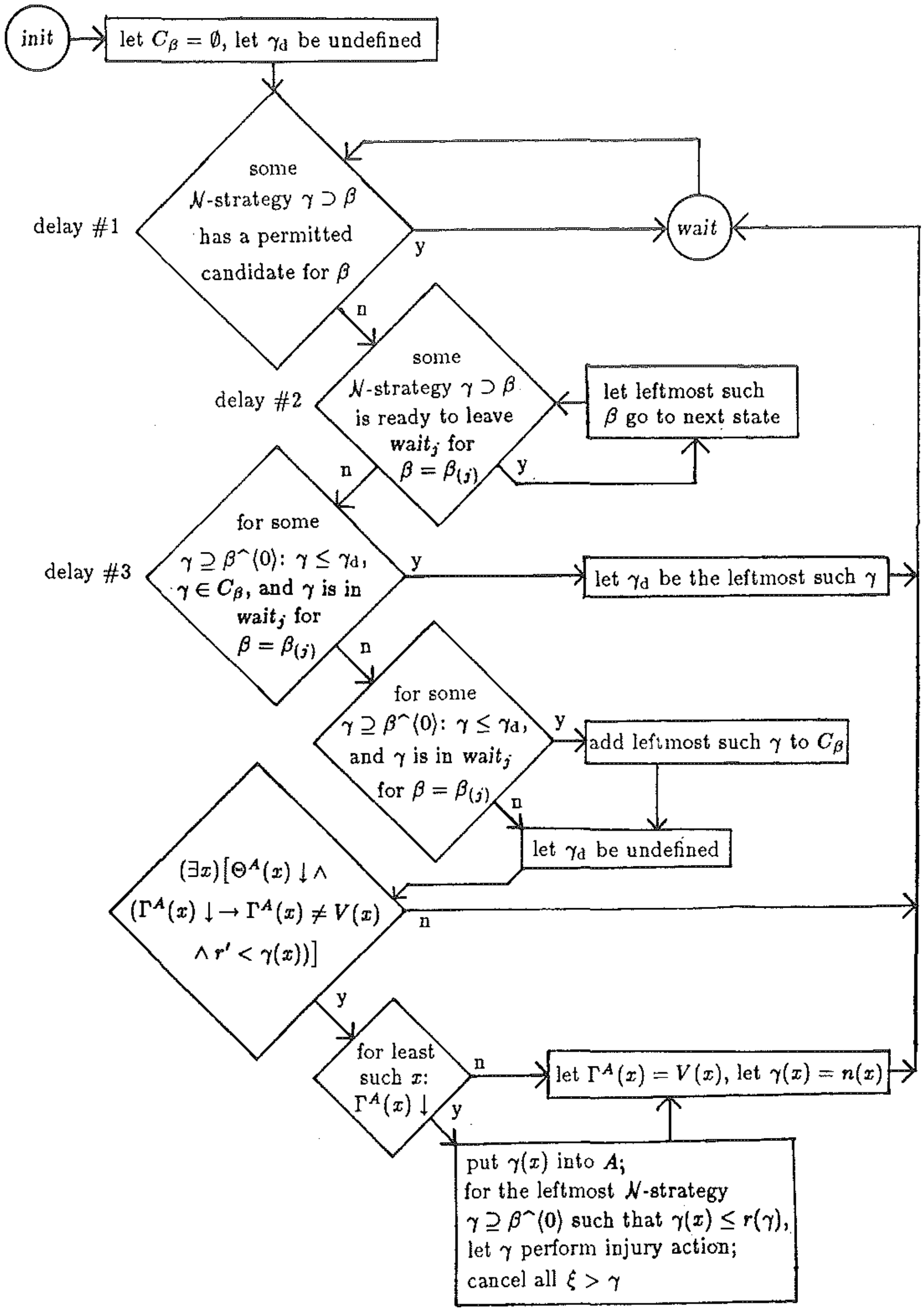

Diagram 4. The $\mathcal{S}$-strategy

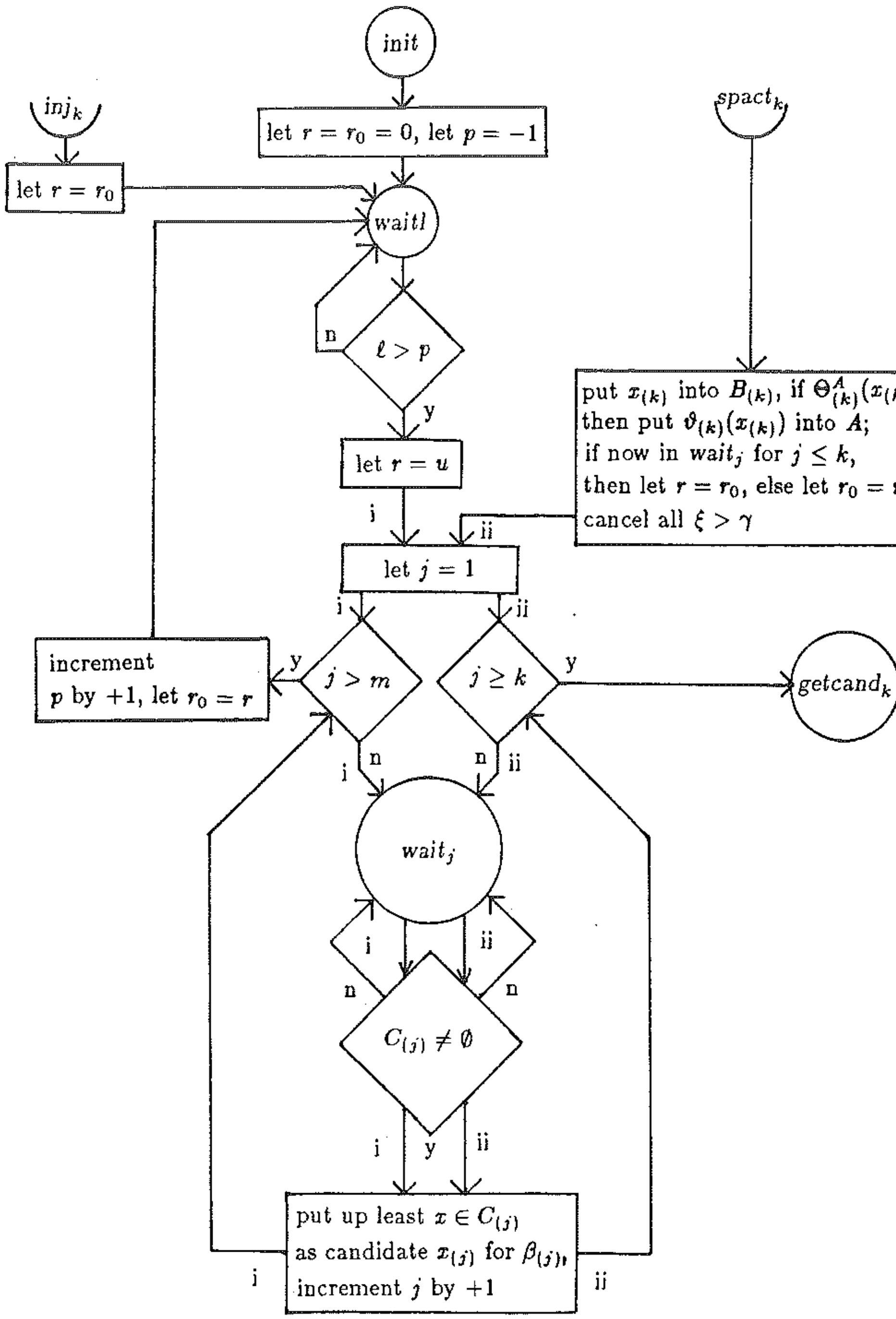

Diagram 5. The $\mathcal{N}$-strategy

axiom for Θ, else it will be $\alpha^\frown\langle 1\rangle$.

An $\mathcal{S}_{e,i}$-strategy β will *only* ever act if $\alpha^\frown\langle 0\rangle \subseteq \beta$ for the $\mathcal{R}_e$-strategy $\alpha \subset \beta$. In this case, it will try to code V_e into A by building its version of $\Gamma_{e,i}$ to show $\Gamma^A_{e,i} = V_e$ unless some $\mathcal{N}$-strategy below it helps it to satisfy $\mathcal{S}_{e,i}$ in some other way. Therefore, β can be delayed in its action in various ways by $\mathcal{N}$-strategies below. An $\mathcal{S}_{e,i}$-strategy will thus act as described in Diagram 4.

Here $n(x)$ is the least $y \in D_\beta - A$ such that $y \geq$ previous values of $\gamma(x)$ and greater than $\gamma(x-1)$, $\vartheta(x)$, and r'.

An $\mathcal{N}$-strategy $\gamma \supseteq \beta^\frown\langle 0\rangle$ performs *injury action* by going to inj_k (where $\beta = \beta_{(k)}$) and on to the next state.

Roughly speaking, γ_{d} is the strategy that caused delay #3 the last time β could act. (We agree that $\gamma \leq \gamma_{\mathrm{d}}$ is satisfied vacuously if γ_{d} is undefined.) Its role is to eventually stop β if some $\mathcal{N}$-strategy below cannot find a candidate for β. Before γ can delay β, however, it has to be injured at least once by definition of C_β. We need C_β in Lemma 2 since for any s, $C_\beta[s]$ is finite and thus well-ordered, whereas $\bigcup_{s\in\omega} C_\beta[s]$ may not be well-ordered.

The next strategy to act will be $\beta^\frown\langle 0\rangle$ if β enumerated a new axiom for Γ, else it will be $\beta^\frown\langle 1\rangle$.

(It is worthwhile to intuitively distinguish the different delays for β here: Delay #1 is immediate and permanent and corresponds to the fact that $B \neq \Psi^U$. Delay #2 is always temporary, the $\mathcal{N}$-strategy below changes states, and then β resumes its action. Delay #3 is permanent again, but will only be activated eventually, corresponding to the outcome that $A \leq_T U$. If β is on the true path f and makes its Γ total, then each $\mathcal{N}$-strategy γ with $\beta \subset \gamma \subset f$ will eventually no longer be injured by β since γ's candidate protects γ against β.)

Finally, we will describe the most complicated of all strategies, the $\mathcal{N}$-strategies. Recall that an $\mathcal{N}$-strategy γ is trying to restrain A in order to ensure $\{e\}^A \neq K$. Towards the strategies $\xi > \gamma$, γ will use the usual Sacks preservation strategy; γ

will have a guess about the $\mathcal{P}$-strategies $\varsigma \subset \gamma$; against the (potentially infinite) injury by the $\mathcal{S}_{e',i'}$-strategies β with $\beta^\frown\langle 0\rangle \subseteq \gamma$, γ will try to put up candidates to show $\Psi_{i'}^{U_{e'}} \neq B_{e'}$. The strategy γ will thus proceed as described in Diagram 5.

Here, p, r, and r_0 are parameters defined in the diagram, roughly denoting the *protected length of agreement* of $K = \{e\}^A$, the *A-restraint imposed by* γ, and the *part of the A-restraint to preserve the protected length of agreement*, respectively.

The other parameters are defined as follows: We call a computation $\{e\}^A(x) \downarrow$ γ*-correct* iff

(16) $$(\forall e' < e)(\forall z \in \omega^{[2e']} = D_{\gamma\restriction(4e'+1)})$$
$$[\gamma(4e'+1) = 0 \wedge r'(\gamma \restriction (4e'+1)) < z < u(A; e, x) \rightarrow z \in A],$$

i.e., if all $\mathcal{P}$-strategies $\varsigma \subset \gamma$ that act infinitely often will not destroy the computation $\{e\}^A(x) \downarrow$. Then the *length of agreement* of $K = \{e\}^A$ is defined by

(17) $$\ell = \max\{\, y \mid (\forall z < y)[K(z) = \{e\}^A(z) \text{ via a } \gamma\text{-correct computation}]\,\}.$$

The *use of the protected length of agreement* is

(18) $$u = \max\{\, u(A; e, y) \mid y \leq p + 1\,\}.$$

For the sake of simplicity, for fixed γ, we denote all $\mathcal{S}$-strategies such that $\beta_{(1)}{}^\frown\langle 0\rangle \subset \beta_{(2)}{}^\frown\langle 0\rangle \subset \cdots \subset \beta_{(m)}{}^\frown\langle 0\rangle \subseteq \gamma$ by $\beta_{(1)}, \ldots, \beta_{(m)}$ (these are the strategies against which γ must put up a candidate), and all of the parameters of $\beta_{(j)}$ are temporarily denoted by $B_{(j)}$, $\Phi_{(j)}$ etc.

Let $\alpha_{(j)}$ be the $\mathcal{R}_{e_{(j)}}$-strategy such that $\alpha_{(j)} \subset \beta_{(j)}$. The *set* $C_{(j)}$ *of possible candidates for* $\beta_{(j)}$ is defined as the set of all $y \in E^{\gamma}_{\alpha_{(j)}}$ such that:

(i) $y > r'$ and $y >$ any previous candidate that γ put up for $\beta_{(j)}$;

(ii) $\Psi_{(j)}^{U_{(j)}}(y) \downarrow = 0$;

(iii) $U_{(j)} \restriction (\psi_{(j)}(y) + 1) = \Phi^A_{(j)} \restriction (\psi_{(j)}(y) + 1) \downarrow$ via a γ-correct computation;

(iv) $\Theta^A_{(j)} \restriction (y + 1) \downarrow$ and $\vartheta_{(j)}(y) > r', r$; and

(v) $c_A(y, s) > \psi_{(j)}(y)$.

If γ changed states then all strategies $\xi > \gamma$ will be initialized. Otherwise, the next strategy to act will be $\gamma^\frown\langle\max\{r, r'\}\rangle$. (Recall that special action or injury action does not count as γ's turn, and that after special action the current stage is ended.)

(Intuitively, an $\mathcal{N}_e$-strategy tries to protect one by one the length of agreement of $K = \{e\}^A$ against $\mathcal{S}$-strategies. Once it is in state $getcand_k$ and thus has a permitted candidate for one of them, it assumes that it is to the left of the true path and will no longer protect longer lengths of agreement.)

5. The Verification

Let δ_s be the string of strategies that act at stage s (except for special action and injury action by the $\mathcal{N}$-strategies). Let $f = \liminf_s \delta_s$ be the true path on the tree T.

The verification consists of several lemmas:

Lemma 1 (Injury Lemma). *No strategy ξ injures a strategy $\xi' < \xi$ by putting into A an element $x \leq r(\xi')$.*

Proof: An $\mathcal{R}$-strategy does not put elements into A at all. The $\mathcal{P}$- and $\mathcal{S}$-strategies observe restraints by stronger strategies explicitly. Moreover, when an $\mathcal{N}$-strategy puts up a candidate, it is greater than stronger restraint so we only have to show that this restraint will not increase until the candidate is cancelled or put into A. But only the $\mathcal{N}$-strategies $\xi' < \xi$ impose stronger restraint. Whenever this restraint increases, some $\mathcal{N}$-strategy $\xi' < \xi$ has changed states, and therefore ξ must have been initialized. ∎

Lemma 2 ($\mathcal{N}$-Strategy Lemma). *Each $\mathcal{N}_e$-strategy $\gamma \subset f$ is injured at most finitely often, is eventually in state waitl (waiting for ℓ to increase), and $\lim_s \ell < \infty$ exists. (Thus $\lim_s r < \infty$ exists, $K \neq \{e\}^A$, and $\mathcal{N}_e$ is satisfied.)*

Proof: First notice that any strategy $\xi <_L f$ acts only finitely often. This is trivial except for $\mathcal{N}$-strategies. But whenever an $\mathcal{N}$-strategy $\gamma <_L f$ performs special action or injury action, it will need $\gamma \subseteq \delta_s$ to act the next time.

We now use induction on $|\gamma|$ and the fact that $\gamma \leq \liminf_s \delta_s$. Let s_0 be minimal such that, after stage s_0, if any $\xi < \gamma$ acts then ξ is not an $\mathcal{N}$-strategy and $\xi \subset \gamma$, and such that every $\mathcal{N}$-strategy $\gamma' \subset \gamma$ is in state *waitl* and is not injured after stage s_0.

Thus, γ is initialized after stage s_0 only if some $\mathcal{S}$-strategy $\beta_{(j)}$ (as defined for γ) with $\beta_{(j)}{}^\frown\langle 0\rangle \subseteq \gamma$ lets γ perform injury action. Since no $\mathcal{N}$-strategies $\gamma' <_L \gamma$ ever act after stage s_0, none of these will *start* delaying any $\mathcal{S}$-strategies $\beta_{(j)}$ (as defined for γ) after stage s_0 more than once (i.e., after they entered $C_{\beta_{(j)}}$); but $\beta_{(j)}{}^\frown\langle 0\rangle \subset f$, and therefore eventually, say, after stage $s_1 \geq s_0$, none of these will ever delay any $\mathcal{S}$-strategy $\beta_{(j)}$. So after stage s_1, for all $j = 1, 2, \ldots, m$, we have that $\gamma_{\mathrm{d}}^{(j)} \geq \gamma$. Thus after stage s_1, once $\gamma \in C_{(j)}$, γ can delay $\beta_{(j)}$ until it has a candidate against it. γ will therefore eventually no longer be injured. (Recall that γ knows which elements will be put into A by $\mathcal{P}$-strategies $\zeta \subset \gamma$ after stage s_0.) But then as in the usual Sacks preservation strategy, $K = \{e\}^A$ would imply that K is recursive, so $\lim_s \ell < \infty$ exists and γ will eventually stop acting and be in state *waitl* forever (waiting for ℓ to increase). So $\lim_s r < \infty$ exists, $K \neq \{e\}^A$, and $\mathcal{N}_e$ is satisfied. ∎

LEMMA 3 ($\mathcal{P}$-STRATEGY LEMMA). *For all e,* $A^{[2e]} =^* J^{[2e]}$. *Thus A is high.*

PROOF: Only the $\mathcal{N}$-strategies impose restraint on A. Lemma 2 shows that this restraint is finite along the true path. ∎

LEMMA 4 ($\mathcal{R}$-STRATEGY LEMMA). *If* $U_e = \Phi_e^A$, *then the* $\mathcal{R}_e$*-strategy* $\alpha \subset f$ *makes* Θ_e *total and* $B_e = \Theta_e^A$. *Thus* $\mathcal{R}_e$ *is satisfied through* α*'s versions of* Θ_e *and* B_e.

PROOF: Suppose by induction that after stage s_0, $\Theta_e^A \upharpoonright x$ has been defined A-correctly; that if strategy $\xi < \alpha$ acts then $\xi \subset \alpha$ and ξ is not an $\mathcal{N}$-strategy; that x is already a candidate for the $\mathcal{N}$-strategy $\gamma \supset \alpha$ (if it ever will be) where $x \in E_\alpha^\gamma$;

and that $\Phi^A \upharpoonright (c_A(x)+1)$ has settled down. But then $m(x)$ changes at most once, namely, when γ puts $\vartheta(x)$ into A, and afterwards x will never again be a candidate. So $m(x)$ will eventually be constant, and thus $\Theta^A(x)$ will eventually be defined A-correctly. Thus Θ^A is total. Furthermore, $B = \Theta^A$ since B only changes on x when $\Theta^A(x)$ is or becomes undefined. ∎

LEMMA 5 (DELAY #3 LEMMA). *For any $\mathcal{S}$-strategy $\beta \subset f$, if β is delayed by delay #3 cofinitely often, then eventually β is always delayed by delay #3 by some fixed $\mathcal{N}$-strategy $\gamma = \lim_s \gamma_{\mathrm{d}}$.*

PROOF: Suppose β is not initialized after stage s_0. If β is delayed cofinitely often by delay #3, then $C_{\beta,\infty} = \bigcup_{s\in\omega} C_\beta[s]$ is finite and thus well-ordered. Let γ_0 be the leftmost $\gamma \in C_{\beta,\infty}$ that causes delay #3 for β infinitely often. Then $\gamma_0 = \lim_s \gamma_{\mathrm{d}}[s]$ since whenever $\gamma_{\mathrm{d}}[s] > \gamma_0$ and later $\gamma_{\mathrm{d}}[s'] = \gamma_0$ then β is not delayed by delay #3 at least once between stages s and s' by the arrangement of delay #3. (This is the reason for having γ_{d} and C_β in this construction.) ∎

LEMMA 6 (TOTAL Γ LEMMA). *If $U_e = \Phi_e^A$ then the $\mathcal{S}_{e,i}$-strategy $\beta \subset f$ makes its version of $\Gamma_{e,i}^A$ total and $V = \Gamma_{e,i}$ unless β is eventually permanently delayed by one fixed $\mathcal{N}$-strategy $\gamma \supseteq \beta^\frown\langle 0\rangle$ through delay #1 or delay #3.*

PROOF: Suppose that if any strategy $\xi < \beta$ acts after stage s_0 then $\xi \subset \beta$ and ξ is not an $\mathcal{N}$-strategy; and that no $\xi \leq \beta$ is initialized after stage s_0. Then β is never initialized after stage s_0, and so either it is permanently delayed by one fixed $\mathcal{N}$-strategy (by Lemma 5 for delay #3 and by the construction for delay #1) and $\beta^\frown\langle 1\rangle \subset f$; or β is not delayed at infinitely many β-stages. (Recall that delay #2 was only temporary.) In the latter case, β can define or redefine $\Gamma_{e,i}$ infinitely often.

Suppose by induction that after stage $s_1 \geq s_0$, $\Gamma_{e,i}^A \upharpoonright x$ has been defined A-correctly; and that $V_e \upharpoonright (x+1) = V_{e,s} \upharpoonright (x+1)$ and $\Theta_e^A \upharpoonright (x+1) \downarrow$ A-correctly. Then $n(x)$ is constant after stage s_1, so $\Gamma_{e,i}^A(x)$ will eventually be defined A-correctly.

Thus $\Gamma_{e,i}$ is total. Furthermore, $V_e(x) = \Gamma^A_{e,i}(x)$ at least for all $x > \lim_s r'[s]$ (since $\gamma(x) \geq x$). ∎

LEMMA 7 (CORRECT Ξ_e LEMMA). *Let $\alpha \subset f$ be the $\mathcal{R}_e$-strategy. Then the version of B_e that originates at α is recursive in V_e by direct permitting on α-stages.*

PROOF: Element x can enter B_e only as a candidate through special action of the $\mathcal{N}$-strategies $\gamma \supset \alpha$. This special action can only occur until the first α-stage s at which $V \upharpoonright x = V[s] \upharpoonright x$. ∎

LEMMA 8 ($\mathcal{S}$-STRATEGY LEMMA). *Let $\alpha \subset f$ be the $\mathcal{R}_e$-strategy. If, for α's version of B_e and fixed i, $U_e = \Phi^A_e$, $U_e <_T A$, and $B_e = \Psi^{U_e}_i$ then the $S_{e,i}$-strategy $\beta \subset f$ is not eventually permanently delayed by $\mathcal{N}$-strategies. (Thus, by Lemma 6, $\Gamma_{e,i}$ is total and $V_e = \Gamma^A_{e,i}$, so $S_{e,i}$ is satisfied.)*

PROOF: By Lemma 5, we only have to show that no single $\mathcal{N}$-strategy γ delays β forever. This can only happen if $\gamma \supseteq \beta^\frown\langle 0\rangle$ and $\beta^\frown\langle 1\rangle \subset f$. Suppose that after stage s_0, $\delta_s \geq \beta^\frown\langle 1\rangle$, that no $\mathcal{S}$-strategy injures γ ever again (since otherwise γ cannot delay β at the next β-stage), and that γ does not act ever again. If γ delays β by delay #1 then $B_e \neq \Psi^{U_e}_i$ through the permitted candidate since γ is no longer injured. If γ delays β by delay #3 then we show that $A \leq_T U_e$ as follows (this defines $\Delta_{e,i}$ implicitly): γ delays β because it cannot find a candidate for it. Let $\tilde{C}$ be the set of all $y \in E^{\gamma}_{\alpha} - B_e$ (where α is the $\mathcal{R}_e$-strategy $\alpha \subset \beta$) such that (at some stage $s > s_0$):

(i) $y > r'$ and $y >$ any previous candidate that γ put up for $\beta_{(j)}$;
(ii) $\Psi^U(y)[s] \downarrow = 0$;
(iii) $U \upharpoonright (\psi(y)+1)[s] = \Phi^A \upharpoonright (\psi(y)+1)[s] \downarrow$ via a γ-correct computation; and
(iv) $\Theta^A[s] \upharpoonright (y+1) \downarrow$ and $\vartheta[s](y) > r'[s], r[s]$.

Since $U = \Phi^A$ and $B = \Psi^U$ and r and r' settle down, this is an infinite recursive set, but then $C = \tilde{C} \cap \{\, y \mid c_A(y) > \psi(y) \,\}$ has to be finite, or else the $\mathcal{N}$-strategy

γ would find a candidate eventually. Since ψ is total, we have that $\psi \leq_T U$, and ψ dominates c_A on the set $\tilde{C}$. Therefore, A is recursive in U. ∎

This concludes the proof of the theorem. ∎

CHAPTER II

THERE IS NO DEEP DEGREE

1. The Main Theorem

Bickford and Mills defined the notion of a deep degree:

DEFINITION (Bickford, Mills): An r.e. degree $\mathbf{w} > \mathbf{0}$ is *deep* if for all r.e. degrees $\mathbf{a}$,

$$\mathbf{a}' = (\mathbf{a} \cup \mathbf{w})'. \tag{1}$$

They raised the question of whether a deep degree exists.

MAIN THEOREM (Lempp, Slaman). *There is no deep degree.*

PROOF: For each r.e. set W, we have to construct an r.e. set A such that

$$\hat{\mathcal{R}} : W \leq_T \emptyset \vee A' <_T (A \oplus W)'. \tag{2}$$

Let us first show that A cannot be built uniformly in W. Suppose there is a recursive function f such that for all e,

$$W_e \leq_T \emptyset \vee W'_{f(e)} <_T (W_{f(e)} \oplus W_e)'. \tag{3}$$

We will show that there is a recursive function g such that for all e

$$(W_e \oplus W_{g(e)})' \equiv_T W'_e, \tag{4A}$$

$$(W_e <_T \emptyset' \to W_{g(e)} \not\leq_T W_e) \wedge (W_e \equiv_T \emptyset' \to W_{g(e)} \equiv_T \emptyset'). \tag{4B}$$

Now pick a fixed point e_0 for gf by the Recursion Theorem. Then

$$(W_{e_0} \oplus W_{f(e_0)})' \equiv_T (W_{gf(e_0)} \oplus W_{f(e_0)})' \equiv_T W'_{f(e_0)}. \tag{5}$$

By our assumption (3) on f (which was supposed to pick a counterexample to W_{e_0} deep), W_{e_0} has to be recursive. Therefore, $W_{gf(e_0)}$ is also recursive. This contradicts our claim (4) about g (which is supposed to build nonrecursive sets).

The proof of (4) is a simple infinite injury argument. For given $W = W_e$, we have to uniformly build $A = W_{g(e)}$. To satisfy $(W \oplus A)' \leq_T W'$, we use the Sacks preservation strategy (as in Sacks [**Sa63b**]); it preserves all possible computations to keep $(W \oplus A)'$ down; its restraint drops on W-true stages. In the attempt to satisfy $A \not\leq_T W$, we use the Sacks coding strategy, trying to code K into A (as in Sacks [**Sa64**]). Note that this strategy makes A complete if W is complete.

2. The Requirements and the Basic Module

Fix an r.e. set W. We use the limit lemma for showing (2) by building a functional Γ such that $\lim_s \Gamma^{A\oplus W}(\cdot, s) \neq \lim_v \Psi^A(\cdot, v)$ for all Ψ. A is constructed nonuniformly as in the Lachlan Non-Diamond Theorem [**La66**] in the following way: We will build a pair (A, Γ) consisting of an r.e. set A and a functional Γ, and a sequence $\{ (\hat{A}_\Psi, \hat{\Gamma}_\Psi) \}_{\Psi \text{ functional}}$ of such pairs such that if (A, Γ) fails via Ψ_0, then $(\hat{A}_{\Psi_0}, \hat{\Gamma}_{\Psi_0})$ will work. The requirements will thus be as follows (for all pairs of functionals $(\Psi, \hat{\Psi})$):

(6)
$$\mathcal{R}_{\Psi,\hat{\Psi}} : W \leq_T \emptyset \vee \lim_v \Psi^A(\cdot, v) \neq \lim_s \Gamma^{A\oplus W}(\cdot, s) \\ \vee \lim_v \hat{\Psi}^{\hat{A}_\Psi}(\cdot, v) \neq \lim_s \hat{\Gamma}_\Psi^{\hat{A}_\Psi \oplus W}(\cdot, s).$$

Once we have shown that $W \leq_T \emptyset$ through one strategy, the requirements of lower priority need not be satisfied. (We will from now on suppress the index Ψ on $\hat{A}_\Psi$ and $\hat{\Gamma}_\Psi$ for better legibility.)

The basic idea for the proof is to either force changes in W often enough to make Γ (or $\hat{\Gamma}$) different from Ψ (or $\hat{\Psi}$) in the limit, or else to build an implicit recursive functional $\Delta_{\Psi,\hat{\Psi}}$ (or Δ, for short) to show that W is recursive via Δ.

The highest priority here is to make Γ (and all $\hat{\Gamma}$) total and to ensure that $\lim_s \Gamma^{A\oplus W}(\cdot, s)$ (and all $\lim_s \hat{\Gamma}^{\hat{A}\oplus W}(\cdot, s)$) exist.

To ensure the former for Γ, we will define $\Gamma^{A\oplus W}(x,s)$ at stage s. At stage 0, we set $\Gamma^{A\oplus W}(x,0) = 0$ and its use $\gamma(x,0) = 0$. If at a stage $s' > s$, $\Gamma^{A\oplus W}(x,s)$ becomes undefined because of a change in A or W, we will redefine it by the end of stage s'. This will either be done explicitly by a strategy on the tree, or implicitly at the end of stage s', when the strategy control sets $\Gamma^{A\oplus W}(x,s) = \Gamma^{A\oplus W}(x,s-1)$ and $\gamma(x,s) = \gamma(x,s-1)$. We ensure that $\Gamma^{A\oplus W}(x,s)$ is redefined only finitely often by setting the use $\gamma(x,s)$ only equal to 0 and at most one other number.

To ensure that the limit of Γ exists, we commit ourselves that for all x and s, $\Gamma^{A\oplus W}(x,s) \leq \Gamma^{A\oplus W}(x,s+1) \leq 1$. So actually $\lim_s \Gamma^{A\oplus W}(\cdot,s)$ will be $\Sigma_1^{A\oplus W}$. (There will be one modification later.) We do the same for the $\hat{\Gamma}_{\Psi}$.

The basic module (for one $\mathcal{R}_{\Psi,\hat{\Psi}}$) can now informally be described as follows (call this the A-side of the module):

(i) fix a candidate i (for $\lim_s \Gamma^{A\oplus W}(i,s) \neq \lim_v \Psi^A(i,v)$),

(ii) start setting $\Gamma^{A\oplus W}(i,s) = 0$ (until (iii) holds) at each stage s,

(iii) wait for $\Psi^A(i,v_0) = 0$ for some v_0 (at stage s_1, say),

(iv) impose A-restraint on $A \upharpoonright (\psi(i,v_0)+1)$,

(v) start setting $\Gamma^{A\oplus W}(i,s) = 1$ with $\gamma(i,s) > \psi(i,v_0)$ (until (ix) or (x) holds) at each stage s,

(vi) wait for $\Psi^A(i,v_1) = 1$ for some $v_1 > v_0$ (at stage s_2, say),

(vii) impose A-restraint on $A \upharpoonright (\psi(i,v_1)+1)$,

(Notice that we have now put a *squeeze* on our opponent: either $W \upharpoonright (\gamma(i,s_1)+1)$ changes, and we can reset $\Gamma^{A\oplus W}(i,s') = 0$ (for all $s' \geq s_1$) while $\Psi^A(i,\cdot)$ has a *flip* (a switch from 0 to 1 back to 0), which we preserve; or else $W \upharpoonright (\gamma(i,s_1)+1)$ remains unchanged, which constitutes a step towards showing that W is recursive. In the second case, the effect is to temporarily restrain W until we reset $\Gamma^{A\oplus W}(i,s') = 0$ (for all $s' \geq s$) by changing A below $\gamma(i,s_1)$. The idea is to run a copy of the module (ii)-(vii) (the $\hat{A}$-side) until this copy restrains W in a similar way. Our strategy threatens to compute W

recursively by restraining it by the $\hat{A}$-side while $\Gamma^{A\oplus W}(i,s) = 0$, and by the A-side while $\hat{\Gamma}^{\hat{A}\oplus W}(\hat{\imath},s) = 0$.)

(viii) start the $\hat{A}$-side at (i) or restart the $\hat{A}$-side at (ii) (until (ix) or (x) holds),

(ix) if $W_{s_2} \upharpoonright \gamma(i,s_1) \neq W_s \upharpoonright \gamma(i,s_1)$ at stage s, then immediately reset $\Gamma^{A\oplus W}(i,s') = 0$ for $s_1 \leq s' \leq s$, initialize the $\hat{A}$-side, and go to (ii) (looking for a new v_0 greater than the current v_1),

(x) if the $\hat{A}$-side reaches (vii), then stop it, put $\gamma(i,s_1)$ into A, reset $\Gamma^{A\oplus W}(i,s') = 0$ for $s_1 \leq s' \leq s$, cancel the part of the A-restraint for preserving $A \upharpoonright (\psi(i,v_0)+1)$ and $A \upharpoonright (\psi(i,v_1)+1)$, and restart the A-side at (ii).

We will for this proof tacitly assume that for all x and s, $\psi(x,s) \leq \psi(x,s+1)$ (and likewise for $\hat{\psi}$).

Continuing in this informal way, let us verify that the basic module satisfies the requirement.

The outcomes can be classified as follows:

(a) finitary: One of the sides is waiting forever at (iii) or (vi) for $\Psi^A(i,\cdot)$ (or $\hat{\Psi}^{\hat{A}}(\hat{\imath},\cdot)$) to change. Then, if the limit for Ψ (or $\hat{\Psi}$) exists at all, it must be unequal to the limit of Γ (or $\hat{\Gamma}$).

(b) Ψ-flip: The A-side gets infinitely many W-changes at (ix). Then $\lim_v \Psi^A(i,v)$ cannot exist since we ensured infinitely many flips via A-restraint.

(c) $\hat{\Psi}$-flip: The $\hat{A}$-side gets infinitely many W-changes at (ix), the A-side only finitely many. Then $\lim_v \hat{\Psi}^{\hat{A}}(\hat{\imath},v)$ does not exist. Note that the candidate $\hat{\imath}$ settles down, and that $\lim_s \Gamma^{A\oplus W}(i,s)$ still exists since $\Gamma^{A\oplus W}(i,s)$ is ultimately set to 0 for every s.

(d) (hidden) recursive outcome: Both the A- and the $\hat{A}$-side get only finitely many W-changes at (ix) but both sides change states infinitely often. Then W will turn out to be recursive. Recall that in this case we will not need the other strategies to succeed, so we do not have to put this outcome on the tree.

The full module only requires two minor modifications:

1) If a strategy α has outcome (b) (or (c)) then the A-restraint (or $\hat{A}$-restraint, respectively) that α imposes on a weaker strategy β below this outcome on the tree tends to infinity. So β has to be able to injure α in some controlled way (*explicit injury feature*). But notice that α has some flexibility in preserving Ψ-flips (or $\hat{\Psi}$-flips). It can afford to have the mth flip injured finitely often for each m until it preserves it forever. Then α will still be able to preserve infinitely many flips if it encounters infinitely many. But β may have to put elements into A (or $\hat{A}$) to reset Γ (or $\hat{\Gamma}$), so β has to wait with setting Γ (or $\hat{\Gamma}$) equal to 1 until it would be allowed to injure α if necessary (*delay feature*). Notice that β assumes infinitely many Ψ-flips (or $\hat{\Psi}$-flips) for α, so β can afford to wait.

2) Whenever a strategy α puts an element into A (or $\hat{A}$), a strategy β below it may be injured. However, the set that α puts in can be made strictly increasing, so β (if it is below outcome (b) or (c) of α) will wait until the part of A (or $\hat{A}$) it wants to work on is cleared of possible injury by α (*postponement feature*). Notice that β assumes that the number that α may want to put in increases to infinity, so again β can afford to wait.

3. The Full Construction

We will first describe the tree of strategies, then the full module for each strategy, and finally the strategy control which supervises the interaction between the strategies.

Let $\Lambda = \{\, \mathit{flip} <_\Lambda \widehat{\mathit{flip}} <_\Lambda \mathit{fin} \,\}$ be the set of outcomes. Notice that these outcomes correspond to the outcomes (b), (c), and (a), respectively, of the basic module above, that we collapsed all finitary outcomes into one, and that outcome (d) of the basic module will not be put on the tree since then this one strategy alone will satisfy the overall requirement from (2). Now let $T = \Lambda^{<\omega}$ be the tree of strategies. Fix an effective 1–1 correspondence between all requirements $\mathcal{R}_{\Psi,\hat{\Psi}}$ and the levels of the tree (sets of nodes of equal length). Let each strategy work on the requirement of its level. Also effectively associate each strategy with an infinite recursive set of integers

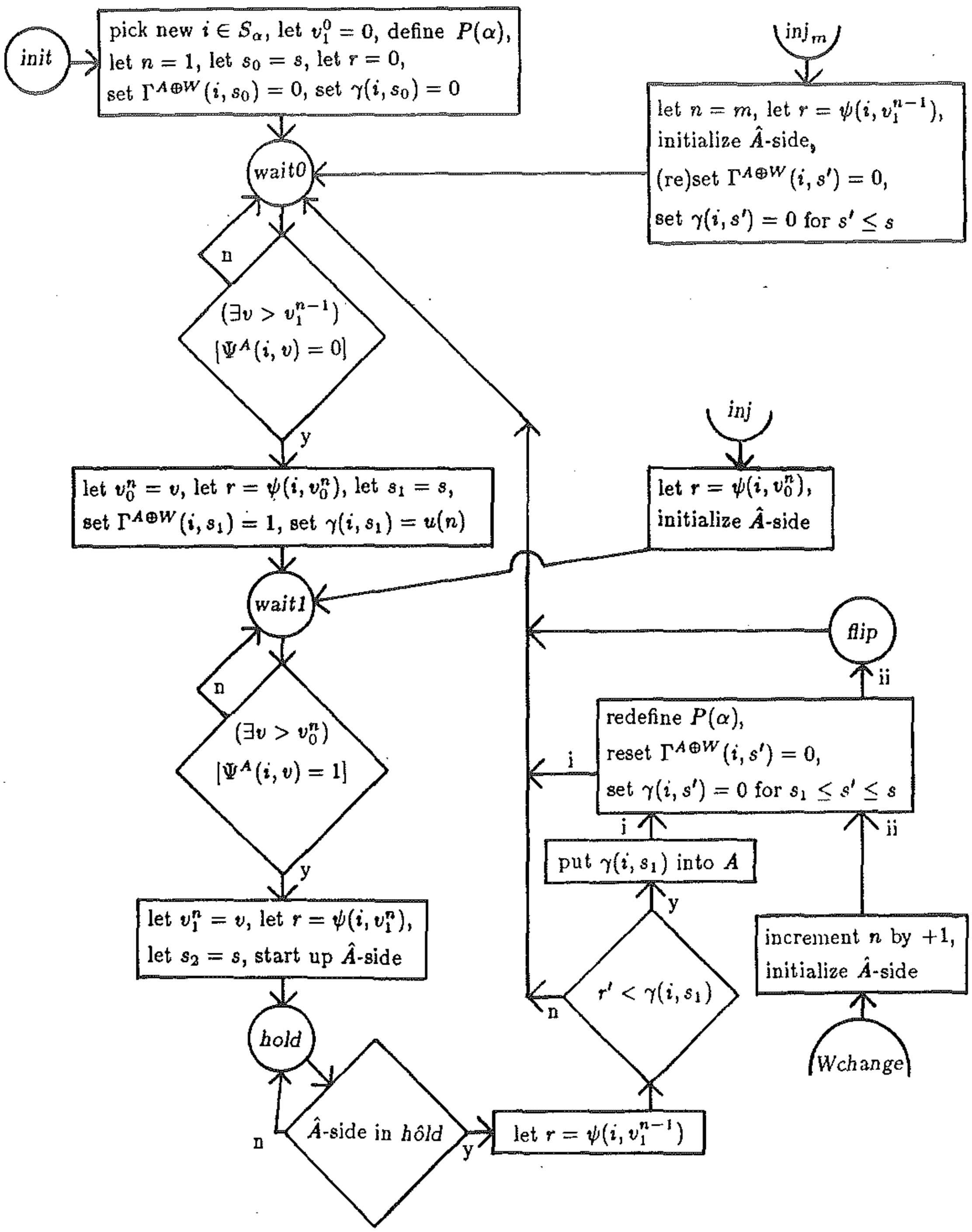

Diagram 6. The A-side

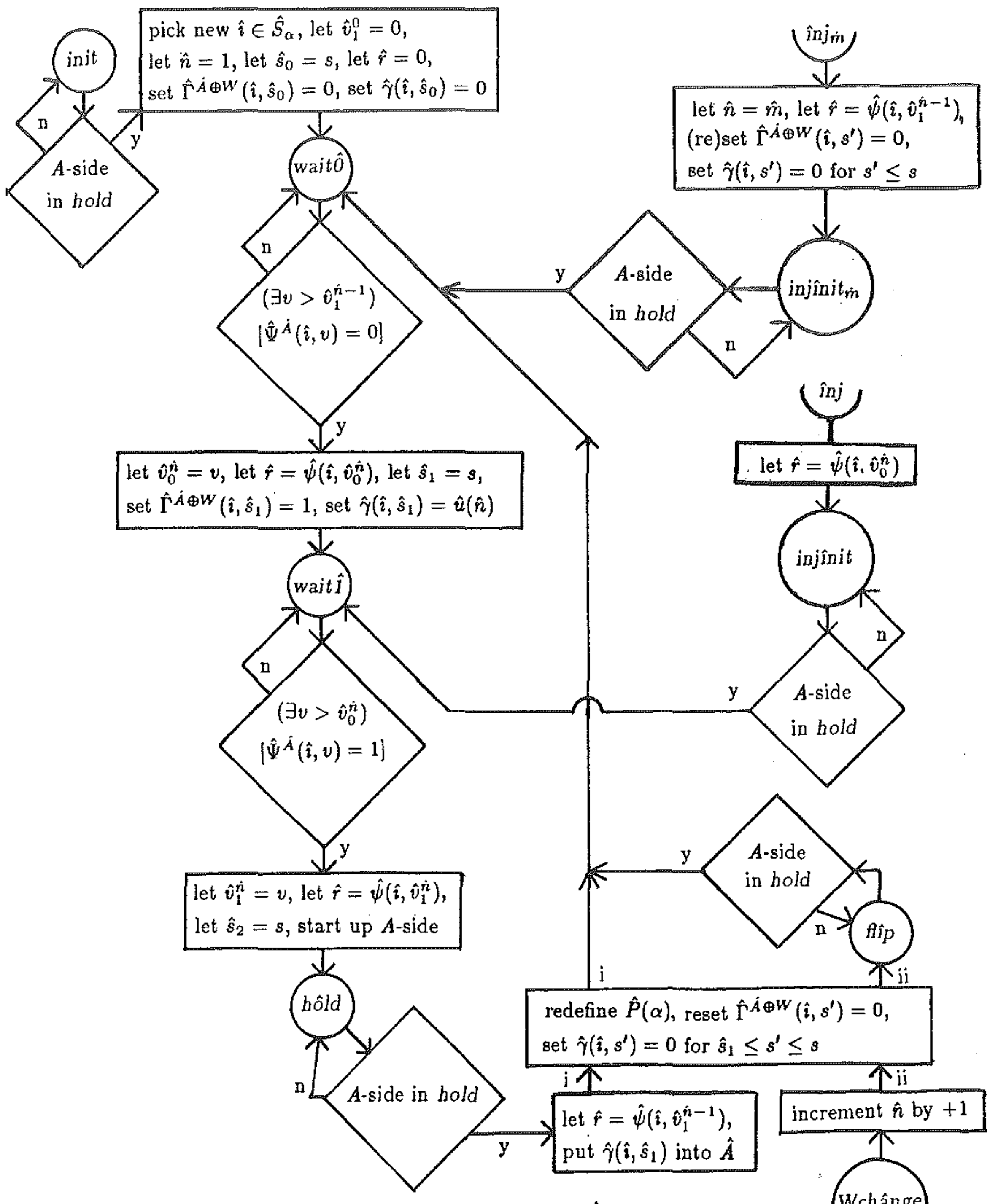

Diagram 7. The $\hat{A}$-side

$S_\alpha = \hat{S}_\alpha$ (such that $\bigsqcup_{\alpha \in T} S_\alpha = \omega$), and let α work with pairs $(i, \hat{\imath}) \in S_\alpha \times \hat{S}_\alpha$.

Now the A-side and the $\hat{A}$-side of the full module of a strategy α proceed as described in Diagrams 1 and 2, respectively.

In general, unhatted parameters refer to the A-side, hatted ones to the $\hat{A}$-side of the module. We assume that γ, the use of Γ, is computed separately on A and W, so $\Gamma^{A \oplus W}(x, s) \downarrow$ implies $\Gamma^{A \restriction (\gamma(x,s)+1) \oplus W \restriction (\gamma(x,s)+1)}(x, s) \downarrow$.

The parameters i, n, r, and v_j^k (for $j = 0, 1$; $k \in \omega$) are defined in the flow chart and roughly denote the *candidate for an inequality* at which α is trying to establish $\lim_s \Gamma^{A \oplus W}(i, s) \neq \lim_v \Psi^A(i, v)$, the *number of the* Ψ*-flip* that α is trying to achieve now, the *A-restraint* α *imposes*, and the *opponent's "stage"* v at which he establishes $\Psi^A(i, v) = j$ for the kth time. The current stage is denoted by s. To *initialize* α means to put both sides into *init* and to set the restraints to zero, to *initialize the* $\hat{A}$*-side* means to do this for the $\hat{A}$-side only.

The following parameters referred to in the diagrams are defined in the text:

The A-side respects A-restraint $r' = \max\{ r(\beta) \mid \beta^\frown\langle \hat{flip} \rangle \subseteq \alpha \}$, the A-restraint imposed by strategies that α assumes will get finitely many Ψ-flips and infinitely many $\hat{\Psi}$-flips. Note that α can afford to do so since it assumes that r' has a finite limit on the set of stages when it acts.

To organize the delay properly, the module defines $P(\alpha)$ and $\hat{P}(\alpha)$ (whenever indicated in the diagram) by setting it to a number greater than all current values of $P(\tilde{\alpha})$ (or $\hat{P}(\tilde{\alpha})$) for any $\tilde{\alpha} \in T$.

We define $u(n)$ (the assigned use for $\gamma(i, s_1)$) to be the least $y \in S_\alpha$ greater than all of the following:

(i) $\psi(i, v_0^n)$;

(ii) all previous values of the parameter $\gamma(i, s_1)$;

(iii) $\max\{ r(\beta) \mid \beta^\frown\langle flip \rangle <_L \alpha \}$;

(iv) $\psi_\beta(i_\beta, v_{1,\beta}^{P(\alpha)})$ for all β with $\beta^\frown\langle flip \rangle \subseteq \alpha$; and

(v) $\gamma(i_\beta, v_{1,\beta}^{n(\beta)})$ for all β with $\beta \supseteq \alpha^\frown\langle flip \rangle$ or $\beta \supseteq \alpha^\frown\langle \hat{flip} \rangle$.

(Here, $r(\beta)$ is the A-restraint imposed by β. Notice that for β with $\beta^\frown\langle \mathit{flip}\rangle \subseteq \alpha$, α observes only the part of the A-restraint imposed by β that it is not allowed to injure.)

Likewise, $\hat{u}(\hat{n})$ (the assigned use for $\hat{\gamma}(\hat{\imath}, \hat{s}_1)$) is the least $y \in \hat{S}_\alpha$ greater than all of the following:

(i) $\hat{\psi}(\hat{\imath}, \hat{v}_0^{\hat{n}})$;

(ii) all previous values of the parameter $\hat{\gamma}(\hat{\imath}, \hat{s}_1)$;

(iii) $\max\{\, \hat{r}(\beta) \mid \beta^\frown\langle \mathit{fl\hat{\imath}p}\rangle <_L \alpha \}$;

(iv) $\hat{\psi}_\beta(\hat{\imath}_\beta, \hat{v}_{1,\beta}^{\hat{P}(\alpha)})$ for all β with $\beta^\frown\langle \mathit{fl\hat{\imath}p}\rangle \subseteq \alpha$ associated with the same Ψ (and thus $\hat{A}$); and

(v) $\hat{\gamma}(\hat{\imath}_\beta, \hat{v}_{1,\beta}^{\hat{n}(\beta)})$ for all β with $\beta \supseteq \alpha^\frown\langle \mathit{flip}\rangle$ or $\beta \supseteq \alpha^\frown\langle \mathit{fl\hat{\imath}p}\rangle$ associated with the same Ψ (and thus $\hat{A}$).

(Notice that we will have $\hat{r}(\beta) = 0$ for β with $\beta^\frown\langle \mathit{flip}\rangle \subseteq \alpha$ since β's $\hat{A}$-side will just have been initialized.)

This ends the description of the full module of an individual strategy. We will now describe the strategy control.

At stage 0, the strategy control will set all parameters to 0 or $\emptyset$ (except $\Gamma^{A\oplus W}(x,s)$ and $\gamma(x,s)$ for $s > 0$ and their hatted counterparts).

At each stage $s > 0$, the strategy control will perform the following three steps:

1) It will let each strategy α whose A-side (or $\hat{A}$-side) is in *hold* (or *hôld*) go to *Wchange* (or *Wchânge*) and on to the next state if $W_s \upharpoonright \gamma(i, s_1) \neq W_{s_2} \upharpoonright \gamma(i, s_1)$ (or $W_s \upharpoonright \hat{\gamma}(\hat{\imath}, \hat{s}_1) \neq W_{\hat{s}_2} \upharpoonright \hat{\gamma}(\hat{\imath}, \hat{s}_1)$, respectively). (Notice that this action does not interfere with any other strategies.)

2) At each substage $t \leq s$ of stage s, some strategy α (with $|\alpha| = t$) will be eligible to act. Strategy $\emptyset$ will be eligible to act at substage 0; if α acted at substage t, then $\alpha^\frown\langle \mathrm{a}\rangle$ will be eligible to act at substage $t+1$ where a is the temporary outcome of α (as defined below).

3) At the end of stage s, the strategy control will define $\Gamma^{A\oplus W}(x, s')$ (and all

$\hat{\Gamma}^{\hat{A}\oplus W}(x,s'))$ for all $x \in \omega$, all $s' \leq s$ as outlined before the description of the basic module.

The rest of this section is devoted to describing in detail the action at substage t under step 2. At each substage t, the strategy control will first check if the strategy α that is eligible to act is delayed or postponed. α is *delayed on the A-side* if there is some β with $\beta^\frown\langle \mathit{flip}\rangle \subseteq \alpha$ such that $n(\beta) \leq P(\alpha)$ where $n(\beta)$ is β's parameter n (the number of the Ψ-flip that β is trying to achieve now). Likewise, α is *delayed on the $\hat{A}$-side* if there is some β with $\beta^\frown\langle \widehat{\mathit{flip}}\rangle \subseteq \alpha$ and associated with the same Ψ (and thus $\hat{A}$) such that $\hat{n}(\beta) \leq \hat{P}(\alpha)$ where $\hat{n}(\beta)$ is defined analogously. α is *postponed on the A-side* if there is some β with $\beta^\frown\langle \mathit{flip}\rangle \subseteq \alpha$ or $\beta^\frown\langle \widehat{\mathit{flip}}\rangle \subseteq \alpha$ such that if α acted now it would measure (in a decision), or restrain, A at or above $\gamma(i(\beta), s_1(\beta))$. Likewise, α is *postponed on the $\hat{A}$-side* if there is some β with $\beta^\frown\langle \mathit{flip}\rangle \subseteq \alpha$ or $\beta^\frown\langle \widehat{\mathit{flip}}\rangle \subseteq \alpha$ associated with the same Ψ (and thus $\hat{A}$) such that if α acted now it would measure or restrain $\hat{A}$ at or above $\hat{\gamma}(\hat{\imath}(\beta), \hat{s}_1(\beta))$.

If α is delayed or postponed then the strategy control will initialize all $\beta >_L \alpha$ and start the next substage with $\alpha^\frown\langle \mathit{fin}\rangle$. Otherwise, we let α act according to the flow chart on the A-side if that side is not in *hold*; and on the $\hat{A}$-side otherwise. (Notice that only one side of α will act unless the flow chart explicitly starts up the action on the other side in which case both sides will act.)

If there is some β with $\beta^\frown\langle \mathit{flip}\rangle \subseteq \alpha$ and α put some $x \leq r(\beta)$ into A, then β has been *injured explicitly* by α on the A-side as x's entering A changes an A-computation that β was preserving. In this case, each such β will perform *injury action on the A-side* as follows:

(i) if $x \leq \gamma(i,s)$ or $\Gamma^{A\oplus W} = 0$ then β goes to inj_{m_β} where $m_\beta = \min\{\, m \mid x \leq \psi_\beta(i_\beta, v^m_{1,\beta})\,\}$ (the number of the least injured Ψ-flip) and on to the next state;

(ii) otherwise, β goes to *inj* and on to the next state.

Likewise, if there is some β with $\beta^\frown\langle \widehat{\mathit{flip}}\rangle \subseteq \alpha$ associated with the same Ψ and (thus $\hat{A}$) such that α put some $x \leq \hat{r}(\beta)$ into $\hat{A}$, then β has been *injured explicitly* *on the $\hat{A}$-side*, and we let β perform the corresponding *injury action on the $\hat{A}$-side* (using the hatted counterparts of the above).

Furthermore, the strategy control determines the temporary outcome a of α. It will be:

(i) *flip*, if the A-side of α went from *flip* to *wait0*;

(ii) *fl$\hat{\imath}$p*, if the A-side of α went from *hold* to *wait0* and, since the last time the A-side was in *hold*, the $\hat{A}$-side went from *fl$\hat{\imath}$p* to *wait$\hat{0}$* and has not been initialized or injured since (this is the time when α's A-restraint is low); and

(iii) *fin*, otherwise.

Finally, the strategy control will initialize all $\gamma >_L \alpha^\frown\langle \mathrm{a}\rangle$; if either side of α changed states, it will also initialize all $\gamma \supseteq \alpha^\frown\langle \mathit{fin}\rangle$.

4. The Verification

Let δ_s, the *recursive approximation to the true path*, be the string of strategies that act at stage s (excluding special action for W-change under step 1 of the construction, but including strategies that are delayed or postponed at stage s). Let $f = \liminf_s \delta_s$ be the *true path*, and let $f_0 = \bigcup\{\alpha \in f \mid \alpha \text{ initialized at most finitely often}\}$ be the *correct part of the true path* (which is possibly only a finite initial segment of f). Intuitively, f_0 will be finite if we discover at that finite level of the tree that W is recursive. Otherwise, $f = f_0$.

Lemma 1 (Injury from Below Lemma). *If $\alpha < \beta$ then at any stage s, β injures α only* explicitly *(i.e., β does not injure $r_s(\alpha)$ or $\hat{r}_s(\alpha)$ where the restraints are measured at the end of stage s), and β does not injure α's first $P(\alpha)$ (or $\hat{P}(\alpha)$) many Ψ-flips (or $\hat{\Psi}$-flips).*

Proof: β can only injure α on the A-side at stage s if $\beta \subseteq \delta_s$, i.e., if β acts at stage s. At that stage, β will put its $\gamma(i, s_1)$ into A. This $\gamma(i, s_1)$ was defined at stage $s_1 < s$, and at that time $\gamma(i, s_1) > r_{s_1}(\alpha)$ if $\alpha^\frown\langle \mathit{flip}\rangle <_L \beta$, and $\gamma(i, s_1) > \psi_\alpha(i_\alpha, v_{1,\alpha}^{P(\beta)})[s_1]$ if $\alpha^\frown\langle \mathit{flip}\rangle \subseteq \beta$. So, α has increased its restraint or $\psi_\alpha(i_\alpha, v_{1,\alpha}^{P(\beta)})$

since stage s_1, say, at stage s'. Now, if $\alpha <_L \beta$ or $\alpha^\frown\langle fin\rangle \subseteq \beta$ then β was initialized and $\gamma(i, s_1)$ was initialized at stage s'. So assume that β is not initialized between stage s_1 and s. If $\alpha^\frown\langle fl\hat{i}p\rangle \subseteq \beta$ then β explicitly respects α's restraint. If $\alpha^\frown\langle flip\rangle \subseteq \beta$ then α will perform injury action if β injures α.

Furthermore, at stage s_1, $\gamma_\beta(i_\beta, s_1)[s_1] > \psi_\alpha(i_\alpha, v_{1,\alpha}^{P(\beta)})[s_1]$. Now, no strategy $\tilde{\beta}$ with $\tilde{\beta} <_L \beta$ or $\tilde{\beta}^\frown\langle fin\rangle \subseteq \beta$ can injure α's first $P(\beta)$ many Ψ- flips without β being initialized. If some $\tilde{\beta}$ with $\tilde{\beta}^\frown\langle flip\rangle \subseteq \beta$ or $\tilde{\beta}^\frown\langle fl\hat{i}p\rangle \subseteq \beta$ injures α's first $P(\beta)$ many Ψ-flips then $\tilde{\beta}$ puts its $\gamma_{\tilde{\beta}}(i_{\tilde{\beta}}, s_{1,\tilde{\beta}})$ into A. But then β would have been postponed with defining its $\gamma(i, s_1)$ until after $\tilde{\beta}$'s injury to α. Any $\tilde{\beta}$ with $\tilde{\beta} >_L \beta$ or $\tilde{\beta} \supseteq \beta^\frown\langle fin\rangle$ is initialized after stage s_1 and therefore $P(\tilde{\beta}) > P(\beta)$ and $\tilde{\beta}$ cannot injure α's first $P(\beta)$ many Ψ-flips. No $\tilde{\beta}$ with $\tilde{\beta} \supseteq \beta^\frown\langle flip\rangle$ or $\tilde{\beta} \supseteq \beta^\frown\langle fl\hat{i}p\rangle$ can injure α's first $P(\beta)$ many Ψ-flips between stage s_1 and stage s, or else it would also injure β, and $\gamma_\beta(i_\beta, s_{1,\beta})$ would be redefined. Therefore, $\psi_\alpha(i_\alpha, v_{1,\alpha}^{P(\beta)})[s_1] = \psi_\alpha(i_\alpha, v_{1,\alpha}^{P(\beta)})[s]$, and so β will not injure α's first $P(\beta)$ many flips.

The proof for the $\hat{A}$-side is the same except that we note that β cannot injure α on the $\hat{A}$-side if $\alpha^\frown\langle flip\rangle \subseteq \beta$ since the $\hat{A}$-side of α has just been initialized and α's $\hat{A}$-restraint is zero whenever β acts. ∎

LEMMA 2 (NUMBER OF FLIPS LEMMA). *If* $\alpha \subseteq f_0$ *and* $\alpha^\frown\langle flip\rangle \subset f$ *then* $\lim_s n(\alpha) = \infty$. *If* $\alpha \subseteq f_0$ *and* $\alpha^\frown\langle fl\hat{i}p\rangle \subset f$ *then* $\lim_s \hat{n}(\alpha) = \infty$, *and* $\lim_s n(\alpha) < \infty$ *exists.*

PROOF: Assume that α is never initialized after stage s'. Then $n(\alpha)$ is incremented each time $\alpha^\frown\langle flip\rangle \subseteq \delta_s$. Furthermore, for each n, $n(\alpha)$ can be decreased to n through explicit injury only a finite number of times by Lemma 1 and the fact that the $P(\beta)$ increase. Therefore, $\lim_s n_s(\alpha) = \infty$.

The analogous proof shows that $\lim_s \hat{n}(\alpha) = \infty$ if we also assume that $\alpha^\frown\langle fl\hat{i}p\rangle \leq \delta_s$ for all $s > s'$ since the $\hat{A}$-side of α goes from *flîp* to *waitÔ* infinitely often and is initialized only a finite number of times.

On the other hand, $n(\alpha)$ can only decrease after stage s' (or else we would have

$\alpha^\frown\langle flip\rangle \subseteq \delta_s$ for some $s > s'$), so $\lim_s n(\alpha) < \infty$ exists. ∎

The fact that strategies are allowed to injure higher-priority strategies infinitely often seems to prevent A from being low.

LEMMA 3 (INJURY FROM ABOVE LEMMA). *If $\alpha \subset \beta$ then at any stage s, α will not injure β by putting $x \leq r(\beta)$ into A or $x \leq \hat{r}(\beta)$ into $\hat{A}$.*

PROOF: Note that any $\beta \supseteq \alpha^\frown\langle fin\rangle$ will be initialized if α puts any number into A or $\hat{A}$. If $\beta \supseteq \alpha^\frown\langle flip\rangle$ or $\beta \supseteq \alpha^\frown\langle fl\hat{i}p\rangle$ then β will be postponed until α cannot injure it. ∎

Notice the unusual feature that for $\alpha \subset \beta \subseteq f_0$, the weaker β may injure the stronger α infinitely often (in a controlled way), but that β is too smart to be injured infinitely often by α.

LEMMA 4 (DELAY LEMMA). *If $\alpha \subset f$ and both Ψ^A and $\hat{\Psi}^{\hat{A}}$ are total, then α is not delayed at cofinitely many α-stages (stages such that $\alpha \subseteq \delta_s$).*

PROOF: Suppose for the sake of contradiction that α is always delayed or postponed at α-stages after some stage s', say. Now any delay is finite since $\lim_s n(\beta) = \infty$ ($\lim_s \hat{n}(\beta) = \infty$) for each β with $\beta^\frown\langle flip\rangle \subseteq \alpha$ ($\beta^\frown\langle fl\hat{i}p\rangle \subseteq \alpha$, respectively) by Lemma 2, but $P(\alpha)$ (or $\hat{P}(\alpha)$) is constant after stage s'. ∎

LEMMA 5 (CONVERGENCE LEMMA). *(i) $\Gamma^{A\oplus W}$ is total, and for all x, $\lim_s \Gamma^{A\oplus W}(x,s)$ exists.*

(ii) For all Ψ, $\hat{\Gamma}_{\Psi}^{\hat{A}_\Psi\oplus W}$ is total, and for all x, $\lim_s \hat{\Gamma}_{\Psi}^{\hat{A}_\Psi\oplus W}(x,s)$ exists.

PROOF: It follows immediately from the construction (step 3) that $\Gamma^{A\oplus W}$ and all $\hat{\Gamma}_{\Psi}^{\hat{A}_\Psi\oplus W}$ are total. All $\hat{\Gamma}_{\Psi}^{\hat{A}_\Psi\oplus W}$ have limits since we ensure $\hat{\Gamma}_{\Psi}^{\hat{A}_\Psi\oplus W}(x,s) \leq \hat{\Gamma}_{\Psi}^{\hat{A}_\Psi\oplus W}(x,s+1) \leq 1$. The same is almost true for $\Gamma^{A\oplus W}$ as well, except that some strategy α may not be able to reset a computation $\Gamma^{A\oplus W}(i,s) = 1$ on going from *hold* to *wait0* if $\gamma(i,s_1) \leq r'$. But for all β with $\beta^\frown\langle fl\hat{i}p\rangle \subseteq \alpha$, $\lim_s n(\alpha) < \infty$

exists (by Lemma 3), and thus so does $\lim_{s \in S^\beta} r(\alpha) < \infty$ where $S^\beta = \{ t \mid \beta \subseteq \delta_t \}$. So $\lim \Gamma^{A \oplus W}(i,s)$ also exists for those i. ∎

We now analyze the outcomes:

LEMMA 6 (FINITE OUTCOME LEMMA). *Suppose $\alpha \subseteq f_0$ and eventually neither the A-side nor the $\hat{A}$-side changes states. Then:*

(i) the A-side of α is eventually in wait0 or wait1, or the $\hat{A}$-side is in wait$\hat{0}$ or wait$\hat{1}$; and

(ii) either not $\lim_v \Psi^A(i,v) = \lim_s \Gamma^{A \oplus W}(i,s)$ *or not* $\lim_v \hat{\Psi}^{\hat{A}}(\hat{\imath},v) = \lim_s \hat{\Gamma}^{\hat{A} \oplus W}(\hat{\imath},s)$ *for the eventual candidates i and $\hat{\imath}$ of α.*

PROOF: (i) By the construction and Lemma 4, that A-side can get stuck only in *wait0*, *wait1*, or *hold*. If the A-side is stuck in *hold* then the $\hat{A}$-side must be stuck in *wait$\hat{0}$* or *wait$\hat{1}$*.

(ii) By Lemma 4, the delay for α is finite. Suppose α is always postponed after stage s', and that $\lim_v \Psi^A(i,v) = \lim_s \Gamma^{A \oplus W}(i,s)$ and $\lim_v \hat{\Psi}^{\hat{A}}(\hat{\imath},v) = \lim_s \hat{\Gamma}^{\hat{A} \oplus W}(\hat{\imath},s)$. Since Ψ^A and $\hat{\Psi}^{\hat{A}}$ are total, their uses settle down. Furthermore, the restraints used in the computation of $u(n)$ and $\hat{u}(\hat{n})$ settle down. Finally, β's parameters $\gamma(i,s_1)$ and $\hat{\gamma}(\hat{\imath},\hat{s}_1)$ for $\beta^\frown\langle flip\rangle \subseteq \alpha$ or $\beta^\frown\langle fl\hat{\imath}p\rangle \subseteq \alpha$ tend to infinity. Therefore, α will eventually not be postponed. But then α will change states to make the limit of Γ or $\hat{\Gamma}$ different. ∎

LEMMA 7 (FLIP OUTCOMES LEMMA). *(i) If $\alpha \subseteq f_0$ and $\alpha^\frown\langle flip\rangle \subset f$ then* $\lim_v \Psi^A(i,v)$ *does not exist for the eventual candidate i of α.*

(ii) If $\alpha \subseteq f_0$ and $\alpha^\frown\langle fl\hat{\imath}p\rangle \subset f$ then $\lim_v \hat{\Psi}^{\hat{A}}(\hat{\imath},v)$ *does not exist for the eventual candidate $\hat{\imath}$ of α.*

PROOF: By the construction, the candidate i ($\hat{\imath}$) settles down in case (i) (case (ii), respectively), and by Lemma 2, $n(\alpha)$ ($\hat{n}(\alpha)$) tends to infinity. But $n(\alpha)-1$ ($\hat{n}(\alpha)-1$) is the number of protected flips from 0 to 1 back to 0 of $\Psi^A(i,\cdot)$ ($\hat{\Psi}^{\hat{A}}(\hat{\imath},\cdot)$), so the limit of Ψ ($\hat{\Psi}$) cannot exist. ∎

LEMMA 8 (RECURSIVE OUTCOME LEMMA). *If $\alpha = f_0$ is of finite length, then W is recursive.*

PROOF: First of all, $\alpha^\frown\langle \mathit{flip}\rangle \subset f$ or $\alpha^\frown\langle \mathit{fl\hat{i}p}\rangle \subset f$ is impossible by the way the initialization is arranged, thus $\alpha^\frown\langle \mathit{fin}\rangle \subset f$. So suppose that $\alpha^\frown\langle \mathit{fin}\rangle \leq \delta_s$ for all $s > s'$, say. Thus $n(\alpha)$ and $\hat{n}(\alpha)$ eventually come to a finite limit, and by Lemma 3, α is never injured after stage s'. Since $\alpha^\frown\langle \mathit{fin}\rangle$ is initialized infinitely often, α keeps changing states. Both sides settle down on candidates i and $\hat{\imath}$ after stage s'', and $\lim_s \gamma(i, s) = \lim_s \hat{\gamma}(\hat{\imath}, s) = \infty$ and both parameters are nondecreasing in s. Also, after stage s'', both sides always destroy their Γ- and $\hat{\Gamma}$-computations, and thus $W \restriction \gamma(i, s_1)$ does not change while the A-side is in *hold*, and $W \restriction \hat{\gamma}(\hat{\imath}, \hat{s}_1)$ does not change while the $\hat{A}$-side is in *hôld*. Thus W is recursive. ∎

Lemma 8 immediately yields Lemma 9:

LEMMA 9 (INFINITE TRUE PATH LEMMA). *If W is not recursive then f_0 is infinite.* ∎

Thus, if W is not recursive, then $\alpha \subset f_0$ of each level will satisfy its requirement by Lemmas 6 and 7. This concludes the proof of the Main Theorem. ∎

5. A WEAK CONVERSE

The above construction is so difficult that there does not seem to be an obvious way to make A low whenever W is nonrecursive. In fact, it seems quite conceivable to the authors that for some nonrecursive *low* r.e. degree $\mathbf{w}$, $\mathbf{a} \cup \mathbf{w}$ is low for any low r.e. degree $\mathbf{a}$. In the following, we will prove a weaker version of this.

Jockusch (private communication) raised the question whether there is a nonrecursive low r.e. degree that does not join with any other low r.e. degree to a high degree. We answer this question positively (reversing the roles of $\mathbf{a}$ and $\mathbf{w}$ conforming with our convention on names of objects built by us or built by the opponent):

THEOREM (Lempp, Slaman). *There is a low r.e. degree* $\mathbf{a} \neq \mathbf{0}$ *such that for all low r.e. degrees* $\mathbf{w}$, $\mathbf{a} \cup \mathbf{w}$ *is not high.*

PROOF: We will drop the restriction that $\mathbf{a}$ be low, since if $\mathbf{a}$ is not low, choose $\mathbf{a}_0 < \mathbf{a}$ low which satisfies the theorem. (However, a closer analysis shows that our $\mathbf{a}$ is already low.)

We now have, for all r.e. sets V, the usual positive requirements for nonrecursiveness:

(7) $$\mathcal{P}_V : \overline{A} \neq V,$$

and, for all r.e. sets W, the requirements:

(8) $$\hat{\mathcal{R}}_W : W \text{ nonlow or } W \oplus A \text{ nonhigh.}$$

6. THE STRATEGY FOR NONLOW/NONHIGH

We have to construct an r.e. set A satisfying all requirements.

The opponent will try to put up an r.e. set W and a functional Φ claiming that W is low and $\Phi^{W \oplus A}$ is total and dominates all total recursive functions, and thus, by a theorem of Martin [**Ma66**], $W \oplus A$ is high.

We will respond by building a functional $\Gamma_{(W,\Phi)}$ witnessing the nonlowness of W via $\lim_s \Gamma^W_{(W,\Phi)}(\cdot, s) \not\leq_T \emptyset'$.

If the opponent succeeds in refuting this by furnishing some total recursive function Ψ such that $\lim_s \Gamma^W_{(W,\Phi)}(\cdot, s) = \lim_v \Psi(\cdot, v)$ then we will defeat him by constructing a total recursive function $\Delta_{(W,\Phi,\Psi)}$ that is not dominated by $\Phi^{W \oplus A}$. (We will use $\Delta_{(W,\Phi,\Psi)}$ to try to force changes in W to redefine $\Gamma^W_{(W,\Phi)}$.)

The nonlow/nonhigh requirements are thus of the form

(9) $$\mathcal{R}_{W,\Phi,\Psi} : \Phi^{W \oplus A} \text{ total} \rightarrow \Big[\lim_s \Gamma^W_{(W,\Phi)}(\cdot, s) \neq \lim_v \Psi(\cdot, v) \vee \big[\Delta_{(W,\Phi,\Psi)} \text{ total} \wedge (\exists^\infty j)[\Delta_{(W,\Phi,\Psi)}(j) > \Phi^{W \oplus A}(j)]\big]\Big].$$

Now for fixed W and Φ, either $\mathcal{R}_{W,\Phi,\Psi}$ is satisfied for all Ψ by the first disjunct and thus W is nonlow; or one $\mathcal{R}_{W,\Phi,\Psi}$ is satisfied by the second disjunct and therefore

$W \oplus A$ is not high via Φ. (We will suppress the subscripts on Γ and Δ if they are clear from the context.) We assume that φ, the use of Φ, is computed separately on W and A, so $\Phi^{W \oplus A}(x)\downarrow$ implies $\Phi^{W \upharpoonright (\varphi(x)+1) \oplus A \upharpoonright (\varphi(x)+1)}(x)\downarrow$.

The basic module for $\mathcal{R}_{W,\Phi,\Psi}$ consists of a stack of ω copies, each denoted by C_n, of a simple *submodule*. Copy C_0 acts first, each copy C_{n+1} is started by copy C_n, and a copy C_n can be initialized by a copy C_m with $m < n$.

Copy C_n now proceeds as follows:

(i) pick a new candidate i (for $\lim_s \Gamma^W(i,s) \neq \lim_v \Psi(i,v)$),

(ii) pick the least j for which Δ is undefined,

(iii) start setting $\Gamma^W(i,s) = 0$ (until (iv) holds) at each stage s,

(iv) wait for $\Psi(i,v_0)\downarrow = 0$ for some v_0 and $\Phi^{W \oplus A}(j)\downarrow$ (at some stage s_1, say),

(v) impose A-restraint on $A \upharpoonright (\varphi(j)+1)$,

(vi) start setting $\Gamma^W(i,s) = 1$ with $\gamma(i,s) = \varphi(j)$ (until (vii) or (viii) holds) at each stage s,

(vii) if $W_s \upharpoonright (\varphi(j)+1) \neq W_{s_1} \upharpoonright (\varphi(j)+1)$ then immediately reset $\Gamma^W(i,s') = 0$ for $s_1 \leq s' \leq s$, cancel the A-restraint, and go to (iii),

(viii) wait for $\Psi(i,v_1) = 1$ for some $v_1 > v_0$ (at some stage s_2, say),

(ix) set $\Delta(j) > \Phi^{W \oplus A}(j)$ and start copy C_{n+1} (with different i and j),

(Notice that we now have a squeeze on W. If W changes we can reset our Γ while his Ψ has a flip; if W does not change we have another witness j towards showing that $\Phi^{W \oplus A}$ does not dominate Δ.)

(x) if $W_s \upharpoonright (\varphi(j)+1) \neq W_{s_1} \upharpoonright (\varphi(j)+1)$ then initialize copies C_m (for $m > n$), reset $\Gamma^W(i,s') = 0$ for $s_1 \leq s' \leq s$, cancel the A-restraint, and go to (ii) (looking for a new v_0 greater than the current v_1).

Here, all copies work on the same A, Γ, and Δ.

To ensure that Γ is total and that the limits exist, we use the same convention as for the Main Theorem (as described just before the basic module of the Main Theorem). We always pick the least j for which Δ is undefined in order to ensure that Δ is total if we pick infinitely many j.

Let $n_0 = \liminf_s\{n \mid \text{copy } C_n \text{ waiting for (iv) or (viii) at stage } s\}$ (possibly $n_0 = \infty$). Then the possible outcomes of the basic module are as follows:

(a) $n_0 = \infty$: Then each time a copy acts for the last time, it finds some j such that $\Delta(j) > \Phi^{W\oplus A}(j)$, and therefore there are infinitely many such j witnessing that $W \oplus A$ is not high via Φ.

(b) $n_0 < \infty$: We distinguish the following cases:

(b_1) copy C_{n_0} acts finitely often (and therefore so does the whole module): Then C_{n_0} gets stuck at (iv) or (viii), and it is not the case that $\lim_s \Gamma^W(i,s) = \lim_v \Psi(i,v)$.

(b_2) copy C_{n_0} goes infinitely often through (x): Then $\lim_v \Psi(i,v)$ does not exist where i is the eventual candidate of C_{n_0} since we force infinitely many Ψ-flips.

(b_3) copy C_{n_0} goes finitely often through (x), but infinitely often through (vii): Then $\Phi^{W\oplus A}(j)$ is not defined for the eventual candidate j of C_{n_0}, and therefore $\Phi^{W\oplus A}$ is not total.

There are two problems with putting this module on a tree. Firstly, the restraint tends to infinity under outcome (a). But most of all, the natural ordering for the outcomes would be of order type $\omega+2$ (namely, (b_2) for $n_0 = 0 <$ (b_3) for $n_0 = 0 <$ (b_2) for $n_0 = 1 <$ (b_3) for $n_0 = 1 < \cdots <$ (a) $<$ (b_1)), which would be hard to organize on a tree.

On the other hand, the positive strategies for $\mathcal{P}_V$ act at most once, and each copy of the above module can live with finite injury. So we will spread out the copies as separate strategies without giving up their coordination described above. It would be possible to put these on a tree, but ensuring that Δ be total would be rather cumbersome. (The fact that is seems hard to let the positive strategies put more elements into A seems to make strengthening this theorem hard.)

Instead, we will use a linear ordering of the strategies combined with the method of W-true stages and the "hat trick". We observe that in the above module the

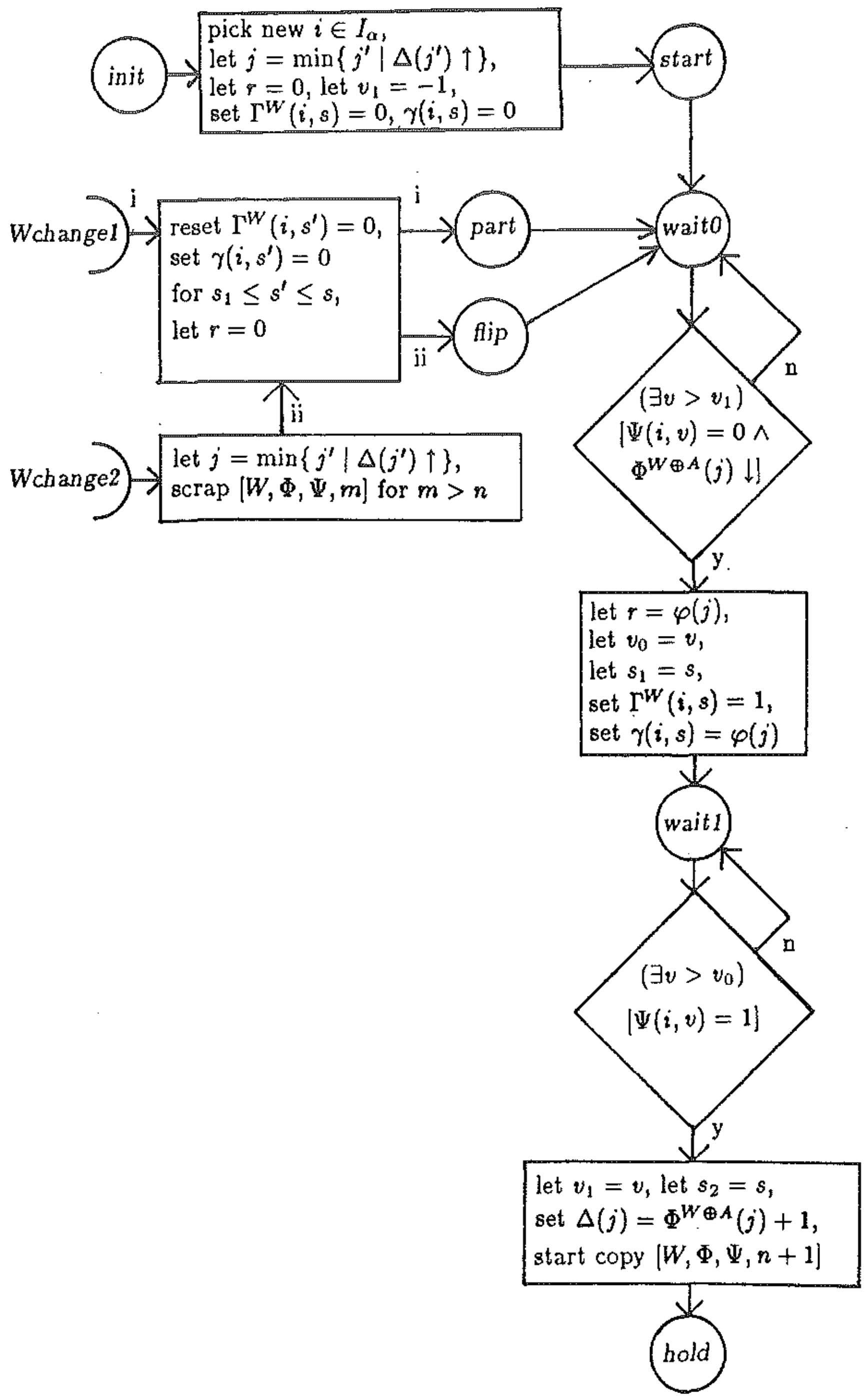

Diagram 8. Copy C_n

restraint of each copy has a finite limit on the set of W-true stages (i.e., the stages at which some element is enumerated into W that is less than any element enumerated later) where we assume that $\Phi^{W\oplus A}(x)[s+1]\uparrow$ if $W_s \upharpoonright (\varphi_s(x)+1) \neq W_{s+1} \upharpoonright (\varphi_s(x)+1)$ and $\Phi^{W\oplus A}(x)[s]\downarrow$ (the *hat trick*, so called because of its original notation).

The construction will thus "look like" a finite injury argument. However, to figure how each requirement $\mathcal{R}_{W,\Phi,\Psi}$ became satisfied will require a $\mathbf{0}'''$-oracle; in fact, it has to since the question whether $W \oplus A$ is high via dominating functional Φ is Π_4-complete and thus the way in which each $\mathcal{R}_{W,\Phi,\Psi}$ becomes satisfied constitutes a Σ_3-complete statement.

7. The Construction for Nonlow/Nonhigh

Fix an effective 1–1 correspondence $\langle \cdot,\cdot,\cdot,\cdot \rangle$ between ω and all quadruples (W, Φ, Ψ, n) where W is an r.e. set, Φ and Ψ are functionals, and n is an integer. (Assume here that always $\langle W,\Phi,\Psi,n\rangle < \langle W,\Phi,\Psi,n+1\rangle$.) This correspondence will yield our priority ranking between strategies. We will denote a strategy α by $[W,\Phi,\Psi,n]$ to specify that α works as the nth copy in the basic module for $\mathcal{R}_{W,\Phi,\Psi}$.

The Γ will be common to all strategies with the same W and Φ, so fix effectively for each α an infinite recursive set of integers I_α such that

$$\bigsqcup_{\substack{\alpha=[W,\Phi,\Psi,n]\\ \text{for some } \Psi,n}} I_\alpha = \omega \tag{10}$$

for fixed W and Φ. Δ will be common to all strategies working on the same $\mathcal{R}_{W,\Phi,\Psi}$. The Γ and the Δ are never discarded even when individual strategies are scrapped.

The module for a strategy $\alpha = [W,\Phi,\Psi,n]$ now acts as described in Diagram 3.

Here, v_0 and v_1 are the "stages" at which the opponent establishes $\Psi(i,v_j) = j$ (for $j = 0,1$). The current stage is denoted by s. The *A-restraint* imposed by α is denoted by r. To *start* α means to let it go from *init* to *start*. To *scrap* it means to put it into *init*. To *initialize* $\alpha = [W,\Phi,\Psi,n]$ is to scrap all $[W,\Phi,\Psi,m]$ for $m \geq n$ and, if $n = 0$ or $[W,\Phi,\Psi,n-1]$ is in *hold*, to start α.

At stage 0, the strategy control starts all strategies $[W, \Phi, \Psi, 0]$ and defines $\Gamma^W(x,0) = 0$ with $\gamma(x,0) = 0$ for all x and all W and Γ.

At a stage $s > 0$, the strategy control proceeds in three steps:

1) If there is some e such that

$$(11) \qquad A_s \cap W_{e,s} = \emptyset \wedge (\exists x \in \omega^{[e]})[x > \max\{\, r(\alpha) \mid \#\alpha \leq e \,\} \wedge x \in W_{e,s}],$$

(where $r(\alpha)$ is the restraint imposed by α and $\#\alpha = \langle\, W, \Phi, \Psi, n \,\rangle$ is the code number of α) then for the least such e, put the least such x into A and initialize all α with $\#\alpha > e$.

2) For each triple $(W,\ \Phi,\ \Psi)$, do the following: First check whether there is a strategy $\alpha = [W, \Phi, \Psi, n]$ in *wait1* or *hold* such that $W_s \restriction \varphi(j) \neq W_{s_1} \restriction \varphi(j)$. If so let the least such α go from *Wchange1* or *Wchange2* to *wait0* (depending on whether α was in *wait1* or *hold*, respectively). Otherwise let the unique $[W, \Phi, \Psi, n]$ that is not in *init* or *hold* act according to the flow chart.

3) The strategy control (re)defines $\Gamma^W(x, s') = \Gamma^W(x, s'-1)$ for all W and Γ with same use for all x and all $s' \leq s$ for which Γ is now undefined.

8. The Verification for Nonlow/Nonhigh

Lemma 1 (Convergence Lemma). *For each pair (W, Φ), $\Gamma^W_{(W,\Phi)}$ is total, and for all x, $\lim_s \Gamma^W_{(W,\Phi)}(x,s)$ exists.*

Proof: $\Gamma^W(x,s)$ is defined at the end of each stage $s' \geq s$ by step 3 of the construction. $\gamma(x,s)$ increases at most once, so W-changes can make $\Gamma^W(x,s)$ undefined at most finitely often. As for the limit, note that for all x, s, $\Gamma^W(x,s) \leq \Gamma^W(x,s+1) \leq 1$. (So Γ^W is actually Σ^W_1.) ∎

Lemma 2 (Finite Injury Lemma). *Action is taken for each $\mathcal{P}_V$ at most once, and thus each α is scrapped at most finitely often under step 1 of the construction.*
∎

Define a stage $s > 0$ to be *W-true* if $W \upharpoonright x = W_s \upharpoonright x$ for some $x \in W_s - W_{s-1}$. Let T be the (infinite) set of W-true stages. Note that, by the hat trick,

$$(12)\quad \Phi^{W \oplus A}(x)[s] \downarrow \wedge\, s \in T \wedge A_s \upharpoonright (\varphi_s(x)+1) = A \upharpoonright (\varphi_s(x)+1) \rightarrow \Phi^{W \oplus A}(x) \downarrow .$$

LEMMA 3 (FINITE RESTRAINT/$\mathcal{P}_V$-STRATEGY LEMMA). *For any strategy α, $\lim_{s \in T} r[s] < \infty$ exists. (Thus each $\mathcal{P}_V$ is satisfied.)*

PROOF: By Lemma 2, let $\alpha = [W, \Phi, \Psi, n]$ not be scrapped under step 1 of the construction after stage s', say. Suppose for some $s \in T$ with $s > s'$, $r[s] > 0$. Then $\Phi^{W \oplus A}(j(\beta)) \downarrow$ for all $\beta = [W, \Phi, \Psi, m]$ with $m \leq n$ via W-correct computations, which are also A-correct by the assumption on s'. Thus in this case $\lim_s r[s] < \infty$ exists. Otherwise $\liminf_s r[s] = \lim_{s \in T} r[s] = 0$. ∎

Now we fix W, Φ, and Ψ and distinguish four cases for the outcome of the strategies $[W, \Phi, \Psi, n]$.

LEMMA 4 (FINITE OUTCOME LEMMA). *Suppose there are only finitely many stages at which any of the strategies $[W, \Phi, \Psi, n]$ (for fixed W, Φ, and Ψ) changes states. Then it is not the case that* $\lim_s \Gamma^W(\cdot, s) = \lim_v \Psi(\cdot, v)$.

PROOF: Let n_0 be the unique n such that $\alpha = [W, \Phi, \Psi, n]$ is eventually not in *init* or *hold*. Then α must be stuck in *wait0* or *wait1*. Therefore not $\lim_s \Gamma^W(i, s) = \lim_v \Psi(i, v)$ for the eventual candidate i of α. ∎

LEMMA 5 (FLIP LEMMA). *Suppose that for some n, $\alpha = [W, \Phi, \Psi, n]$ is scrapped finitely often and goes through Wchange2 infinitely often. Then $\lim_v \Psi(i, v)$ does not exist for the eventual candidate i of α.*

PROOF: Let α not be scrapped after stage s', say. Then the parameters v_0 and v_1 increase to infinity, and each time they increase, $\Psi(i, v_0) = 0$ or $\Psi(i, v_1) = 1$ is established. ∎

LEMMA 6 (PARTIAL Φ LEMMA). *Suppose that for some n,* $\alpha = [W, \Phi, \Psi, n]$ *is scrapped finitely often and changes states infinitely often, but goes through Wchange2 only finitely often. Then* $\Phi^{W \oplus A}$ *is partial.*

PROOF: Suppose that α is not scrapped or goes through *Wchange2* after stage s', say. Then α from now on always goes through *Wchange1* with the same j, so $\Phi^{W \oplus A}(j) \uparrow$. ∎

LEMMA 7 (NONDOMINANCE LEMMA). *Suppose that, for fixed* W, Φ, *and* Ψ, *no* $[W, \Phi, \Psi, n]$ *changes states infinitely often, but that there are infinitely many stages at which some* $[W, \Phi, \Psi, n]$ *changes states. Then* Δ *is total and not dominated by* $\Phi^{W \oplus A}$. *(Thus* $W \oplus A$ *is not high via* Φ.*)*

PROOF: First of all, Δ is total since we always pick the least j for which Δ is currently undefined and since each $[W, \Phi, \Psi, n]$ is eventually in *hold*. But each time some $[W, \Phi, \Psi, n]$ reaches *hold* for the last time, $\Delta(j) > \Phi^{W \oplus A}(j)$ is established for its current j, and this is preserved by the A-restraint. ∎

This concludes the proof of the theorem. ∎

CHAPTER III

Σ_α- AND Π_α-COMPLETE INDEX SETS

We will first prove an easy warm-up theorem to demonstrate our technique for index set classifications in a simple setting. Recall that Lachlan, Martin, Robinson, and Yates classified the index set of maximal sets as Π_4-complete. The classification of the index set of quasimaximal sets had been open since then. Our warm-up theorem reproves the above and another previously known result and classifies for the first time the index sets of quasimaximal sets and of coinfinite r.e. sets not having atomless supersets (the so-called *atomlessless* sets) as Σ_5- and Π_6-complete, respectively.

Next, we will generalize the definitions of maximal and quasimaximal sets by alternation of generating filters and taking coatoms, using a correspondence with the Cantor-Bendixson derivative of certain trees. The main theorem generalizes the classification of the index sets of cofinite, maximal, and quasimaximal sets in a transfinite recursion argument through Kleene's hyperarithmetical hierarchy. We prove this by establishing a correspondence between r.e. sets and binary trees through the Cantor-Bendixson derivative. Finally, we will classify the index set of r.e. sets major in some fixed nonrecursive r.e. set, using a different technique.

First of all, however, we will explain the tree machinery needed to prove the main results of this chapter. All trees using this machinery will from now on be binary.

1. The Machinery

Lachlan [**La68**] showed that any Σ_3-Boolean algebra can be represented as the lattice of r.e. supersets (modulo finite sets) of some hyperhypersimple set A. The proof uses an argument that can be generalized substantially. From an arbitrary

Σ_2-tree $T \in 2^{<\omega}$ (i.e., $\sigma \in T$ iff $R(\sigma)$, for some Σ_2-predicate R), Lachlan constructs a (hyperhypersimple) r.e. set A_T with a 1–1 correspondence between nodes $\sigma \in T$ and elements $a_\sigma \in \overline{A}$ satisfying the following two properties:

(1) $(\forall \sigma \in T)[A \cup C_\sigma$ is r.e.$]$, and

(2) $(\forall W \supseteq A$ r.e.$)(\exists S \subseteq T$ finite$)[W =^* A \cup \bigcup_{\sigma \in S} C_\sigma]$,

where $C_\sigma = \{ a_\tau \mid \tau \in T \wedge \tau \supseteq \sigma \}$ is the "cone" of elements of $\overline{A}$ "above" a_σ.

The idea is now to reduce index set proofs to proofs about trees by the above correspondence between trees T and r.e. sets A_T.

Using Lachlan's construction as a starting point, we can break up an index set classification into easier parts. Suppose we are trying to show that $(\Sigma_n, \Pi_n) \leq_1 (\mathrm{A}, \mathrm{B})$ for certain disjoint index sets A and B which are closed modulo finite sets, i.e., which satisfy

$$e \in \mathrm{A} \wedge W_e =^* W_i \to i \in \mathrm{A}, \tag{1}$$

and likewise for B. (The technique works just as well if we replace the integer n by a recursive ordinal α.) Then it suffices to establish the following two lemmas:

(I) Correspondence Lemma: The mapping index of $T \mapsto$ index of A_T maps the Σ_2-trees of $\mathcal{S}$ into A, and the Σ_2-trees of $\mathcal{T}$ into B, for certain disjoint classes of index sets of Δ_3-binary trees $\mathcal{S}$, $\mathcal{T}$.

(II) Reduction Lemma: A recursive function f maps C into the set of recursive trees of $\mathcal{S}$ and $\overline{C}$ into the set of recursive trees of $\mathcal{T}$.

Here C is a Σ_m-complete set (where $2+m=n$), and we require that membership of T in $\mathcal{S}$ and $\mathcal{T}$ only depends on $[T]$, namely, for Δ_3-trees T and $\tilde{T}$,

$$T \in \mathcal{S} \wedge [\tilde{T}] = [T] \to \tilde{T} \in \mathcal{S}, \tag{2}$$

and likewise for $\mathcal{T}$.

Once we have established (I) and (II), we can complete the proof of the index set classification as follows:

LEMMA.

(i) *We can relativize the construction of f to $\emptyset''$ to obtain a recursive function $\tilde{f}$ mapping a $\Sigma_m^{\emptyset''}$-complete (i.e., Σ_n-complete) set $\tilde{C}$ to the $\Delta_1^{\emptyset''}$-trees (i.e., Δ_3-trees) of $\mathcal{S}$, and the complement of $\tilde{C}$ to the Δ_3-trees of $\mathcal{T}$.*

(ii) *We can approximate the Δ_3-trees $\tilde{T}$ obtained in (i) by Σ_2-trees $\hat{T}$ with $[\tilde{T}] = [\hat{T}]$, and denote this approximation of $\tilde{f}$ by $\hat{f}$.*

PROOF: (i) Straightforward relativization of the construction of f first yields a function $g \leq_T \emptyset''$. Now it is easy to find the desired partial recursive function $\tilde{f}$ such that $W_{\tilde{f}(e)}^{\emptyset''} = W_{g(e)}^{\emptyset''}$ (where these sets code the trees) by "pushing the oracle of the index function into the main oracle". Since g is total, so is $\tilde{f}$.

(ii) Notice that for a Δ_3-tree (i.e., $\Delta_2^{\emptyset'}$-tree) $\tilde{T}$, there is a function $h \leq_T \emptyset'$ such that $\sigma \in \tilde{T}$ iff $\lim_s h(\sigma, s) = 1$, and $\sigma \notin \tilde{T}$ iff $\lim_s h(\sigma, s) = 0$. Now enumerate $\hat{T}$ (relative to $\emptyset'$) by putting σ into $\hat{T}$ at stage s if

$$|\sigma| \leq s \wedge (\forall n \leq |\sigma|)[h(\sigma \restriction n, s) = 1]. \blacksquare$$

Now the composition of $\hat{f}$ with the mapping index of $T \mapsto$ index of A_T yields the desired reduction $(\Sigma_n, \Pi_n) \leq_1 (\mathrm{A}, \mathrm{B})$.

Three typical examples of a correspondence as in (I) are the following: A finite tree T (i.e., $[T] = \emptyset$) corresponds to a cofinite set A_T. A Σ_2-tree with exactly one infinite path corresponds to a maximal set A_T. A *perfect* tree T is a tree such that for all $\sigma \in T$, there are $\tau_1, \tau_2 \in T$ such that $\sigma \subset \tau_1, \tau_2$ and $\tau_1 \mid \tau_2$. A perfect Σ_2-tree corresponds to an atomless hyperhypersimple set A_T. (We will give a proof below for the latter two correspondences.)

In the Reduction Lemmas below, since the construction is recursive we will ensure that the tree T constructed is recursive by letting $T_s = T \cap 2^{\leq s}$, where T_s is the part of T constructed by the end of stage s.

2. A WARM-UP THEOREM

DEFINITION: Let A be a coinfinite r.e. set.

(i) A is *maximal* if for all r.e. sets $W \supseteq A$, either $W =^* A$ or $W =^* \omega$.

(ii) A is *quasimaximal* if it is a finite intersection of maximal sets.

(iii) A is *atomless* if it has no maximal superset.

(iv) A is *atomlessless* if it has no atomless superset.

(v) A is *hyperhypersimple* if $\mathcal{L}(A)$, the *lattice of r.e. supersets of A*, forms a Boolean algebra. (By Lachlan [**La68**], this is equivalent to the original definition.)

Notice that a coinfinite r.e. set having no atomlessless superset is the same as an atomless set, so the hierarchy collapses at that level. Atomlessless sets are usually called atomic sets. However, this would conflict with our notation below.

PROPOSITION. *The index sets of maximal, quasimaximal, atomless, and atomlessless sets are* Π_4, Σ_5, Π_5, *and* Π_6, *respectively.*

PROOF: By the fact that Max is Π_4 and the usual Tarski-Kuratowski algorithm. ∎

We denote these index sets by Max, QMax, Atomless, and Atomlessless, respectively. Our machinery now allows an easy classification of these four index sets:

THEOREM A. *The following reductions hold:*

(i) $(\Pi_4, \Sigma_4) \leq_1 (\mathrm{Max}, \mathrm{QMax} - \mathrm{Max})$;

(ii) $(\Sigma_5, \Pi_5) \leq_1 (\mathrm{QMax}, \mathrm{Atomless})$; *and*

(iii) $\Pi_6 \leq_1$ Atomlessless.

COROLLARY.

(i) (Lachlan, D.A. Martin, R.W. Robinson, Yates (unpublished); later appearing in Tulloss [**Tu71**]) *The index set of maximal sets is* Π_4*-complete.*

(ii) *The index set of quasimaximal sets is* Σ_5*-complete.*

(iii) (Jockusch) *The index set of atomless sets is* Π_5*-complete.*

(iv) *The index set of atomlessless sets is* Π_6*-complete.* ∎

PROOF OF THEOREM A: We have to establish (I) and (II) above for our machinery to apply. Call T *essentially perfect* if $\mathrm{Ext}(T)$ is a perfect tree, i.e., if there is a 1–1 map e from $2^{<\omega}$ into the *extendible part* $\mathrm{Ext}(T)$ *of* T such that

(a) $(\forall \sigma, \tau \in 2^{<\omega})[\sigma \subset \tau \leftrightarrow e(\sigma) \subset e(\tau)]$, and

(b) $(\forall \rho \in \mathrm{Ext}(T))(\exists \sigma \in 2^{<\omega})[\rho \subseteq e(\sigma)]$.

We define four classes of trees:

$$
(3)\qquad
\begin{aligned}
\mathcal{T}_1 &= \{\, T \subseteq 2^{<\omega} \text{ tree} \mid \|[T]\| = 1 \,\},\\
\mathcal{T}_2 &= \{\, T \subseteq 2^{<\omega} \text{ tree} \mid [T] \neq \emptyset,\ \text{finite} \,\},\\
\mathcal{T}_3 &= \{\, T \subseteq 2^{<\omega} \text{ tree} \mid T \text{ is essentially perfect} \,\},\\
\mathcal{T}_4 &= \{\, T \subseteq 2^{<\omega} \text{ tree} \mid [T] \neq \emptyset \wedge (\forall \sigma \in T)[T(\sigma) \text{ is not essentially perfect}] \,\},
\end{aligned}
$$

CORRESPONDENCE LEMMA. *Let $T \subseteq 2^{<\omega}$ be a Σ_2-tree. Then:*

(i) *If $T \in \mathcal{T}_1$ then A_T is maximal, and conversely.*

(ii) *If $T \in \mathcal{T}_2$ then A_T is quasimaximal.*

(iii) *If $T \in \mathcal{T}_3$ then A_T is atomless.*

(iv) *If $T \in \mathcal{T}_4$ then A_T is atomlessless.*

PROOF: (i) Let $W \supseteq A_T$ be an r.e. superset. Then $W =^* A_T \cup \bigcup_{\sigma \in S} C_\sigma$ for some finite set $S \subseteq T$. If $S \cap \mathrm{Ext}(T) = \emptyset$ then $W =^* A_T$, and, since $\|[T]\| = 1$, if $S \cap \mathrm{Ext}(T) \neq \emptyset$ then $W =^* \omega$. So A_T is maximal. The converse is shown analogously.

(ii) Similar to (i).

(iii) Suppose $W \supseteq A_T$ is a maximal superset. Then $W =^* A_T \cup \bigcup_{\sigma \in S} C_\sigma$ for some finite set $S \subseteq T$. Since W is coinfinite there is some $\sigma_0 \in \mathrm{Ext}(T)$ such that $C_{\sigma_0} \cap W = \emptyset$. Let $\tau_0 \in 2^{<\omega}$ be such that $\sigma_0 \subseteq e(\tau_0)$. Then $W \subset_\infty W \cup C_{e(\tau_0{}^\frown\langle 0 \rangle)} \subset_\infty W \cup C_{e(\tau_0)}$, contradicting W's maximality.

(iv) Suppose $W \supseteq A_T$ is an atomless superset. Then $W =^* A_T \cup \bigcup_{\sigma \in S} C_\sigma$ for some finite set $S \subseteq T$. Since W is coinfinite there is some $\sigma_0 \in \mathrm{Ext}(T)$ such that

$C_{\sigma_0} \cap W = \emptyset$. Let

$$W_0 = A_T \cup \bigcup_{\substack{|\sigma|=|\sigma_0| \\ \sigma \in T - \{\sigma_0\}}} C_\sigma.$$

Then W_0 is coinfinite and $W_0 \supseteq^* W$, so W_0 is also atomless. We will show that $T(\sigma_0)$ is essentially perfect to reach a contradiction. Let $T_0 = \mathrm{Ext}(T(\sigma_0))$. It suffices to show that, for all $\tau \in T_0$, there exist $\tau_1, \tau_2 \in T_0$ such that $\tau \subset \tau_1, \tau_2$ and $\tau_1 \mid \tau_2$. Suppose $\tau_0 \in T_0$ does not admit such a splitting. Then

$$W_1 = A_T \cup \bigcup_{\substack{|\tau|=|\tau_0| \\ \tau \in T_0 - \{\tau_0\}}} C_{\sigma_0 {}^\frown \tau}$$

is maximal by an argument similar to (i). ∎

REDUCTION LEMMA. *We have the following reductions (where all images of the reducing maps are* recursive *trees):*

(i) $(\Pi_2, \Sigma_2) \leq_1 (\mathcal{T}_1, \mathcal{T}_2 - \mathcal{T}_1)$,

(ii) $(\Sigma_3, \Pi_3) \leq_1 (\mathcal{T}_2, \mathcal{T}_3)$, *and*

(iii) $\Pi_4 \leq_1 \mathcal{T}_4$.

PROOF: (i) We choose Inf and Fin, the index sets of infinite and finite r.e. sets, respectively, as Π_2- and Σ_2-complete index sets. We will build a reduction $k \mapsto T_k$ such that $k \in \mathrm{Inf}$ implies $T_k \in \mathcal{T}_1$, and $k \in \mathrm{Fin}$ implies $T_k \in \mathcal{T}_2 - \mathcal{T}_1$. Fix k. At stage 0, let $T_{k,0} = \{\emptyset\}$; at stage 1, we put $\langle 0 \rangle$ and $\langle 1 \rangle$ into $T_{k,1}$. At a stage $s \geq 2$, if $W_{k,s} \neq W_{k,s-1}$, we put $\langle 0^s \rangle$ and $\langle 0^{s-1} 1 \rangle$ into $T_{k,s}$; otherwise, we put $\tau^\frown\langle 0 \rangle$ into $T_{k,s}$ for the two $\tau \in T_{k,s-1}$ with $|\tau| = s - 1$. Then

$$k \in \mathrm{Inf} \to (\exists^\infty s)[W_{k,s} \neq W_{k,s-1}] \to [T_k] = \{\langle 0^\omega \rangle\} \to T_k \in \mathcal{T}_1,$$

$$(4) \qquad k \in \mathrm{Fin} \to (\exists^{<\infty} s)[W_{k,s} \neq W_{k,s-1}] \to [T_k] = \{\langle 0^\omega \rangle, \langle 0^{s_0 - 1} 1 0^\omega \rangle\} \to T_k \in \mathcal{T}_2 - \mathcal{T}_1,$$

where $s_0 = \max\{ s \mid W_{k,s} \neq W_{k,s-1} \}$.

(ii) We choose Cof and Coinf, the index sets of cofinite and coinfinite r.e. sets, respectively, as Σ_3- and Π_3-complete index sets. We will again build a reduction $k \mapsto T_k$ such that $k \in$ Cof implies $T_k \in \mathcal{T}_2$, and $k \in$ Coinf implies $T_k \in \mathcal{T}_3$. Fix k and let $\overline{W_{k,s}} = \{ w^0_{k,s} < w^1_{k,s} < w^2_{k,s} < \ldots \}$. Let $\{ \mu_\sigma \}_{\sigma \in 2^{<\omega}}$ be a sequence of markers. At stage 0, let $n_0 = 0$, let $\mu_{\emptyset,0} = \emptyset$, let all other markers be undefined, and put $\emptyset$ into T_0. At a stage $s > 0$, let $n_s = \min(\{ n_{s-1} + 1 \} \cup \{ n \mid w^n_{k,s-1} \neq w^n_{k,s} \})$. For $|\sigma| < n_s$, let $\mu_{\sigma,s} = \mu_{\sigma,s-1}$. For $|\sigma| = n_s$, let $\mu_{\sigma,s}$ be equal to some string τ with $|\tau| = s$ and $\tau \supset \mu_{\sigma^-,s}$ where $\sigma^- = \sigma \upharpoonright (n_s - 1)$, and put all these τ into $T_{k,s}$. For $|\sigma| > n_s$, let $\mu_{\sigma,s}$ be undefined.

Now assume that W_k is cofinite. Then there is some (least) $\tilde{n}$ such that $\lim_s w^{\tilde{n}}_{k,s} = \infty$, so $\lim_s |\mu_{\sigma,s}| = \infty$ for all σ with $|\sigma| \geq \tilde{n}$. But then $\liminf_s |T_k \cap 2^s| = 2^{\tilde{n}}$, so $[T_k]$ is finite. $[T_k]$ is nonempty since for all s, $T_k \cap 2^s \neq \emptyset$. Thus $T_k \in \mathcal{T}_2$.

On the other hand, if W_k is coinfinite, then $\lim_s w^n_{k,s} < \infty$ exists for all n, so $\lim_s n_s = \infty$. We can thus define, for all n, a stage $s_n = \max\{ s \mid n_s = n \}$. Therefore, $\lim_s \mu_{\sigma,s} = \mu_\sigma$ exists for all $\sigma \in 2^{<\omega}$. The mapping $\sigma \mapsto \mu_\sigma$ now shows that T_k is essentially perfect.

(iii) The final part of the proof allows us a first glimpse at how the uniformity of the construction can be used to yield more and more complicated index set results.

There is a recursive function g such that

$$k \in \emptyset^{(4)} \leftrightarrow (\exists i)[W_{g(k,i)} \text{ coinfinite}], \text{ and}$$
(5)
$$k \notin \emptyset^{(4)} \leftrightarrow (\forall i)[W_{g(k,i)} \text{ cofinite}].$$

Fix k. At stage 0, we let $T_{k,0} = \{ \emptyset \}$. At a stage $s > 0$, put $\langle 0^s \rangle$ and $\langle 0^{s-1}1 \rangle$ into $T_{k,s}$ and start the construction described in part (ii) but above $\langle 0^{s-1}1 \rangle$ in place of $\emptyset$ and using $W_{g(k,s-1)}$ in place of W_k.

Now, if $k \notin \emptyset^{(4)}$, then for all i, $W_{g(k,i)}$ is cofinite, so $[T_k(\langle 0^i1 \rangle)]$ is finite for all i by (ii), and therefore $T_k(\sigma)$ is not essentially perfect for any $\sigma \in T_k$. Thus $T_k \in \mathcal{T}_4$.

On the other hand, if $k \in \emptyset^{(4)}$, then $W_{g(k,i)}$ is coinfinite for some i, so, again by (ii), $[T_k(\langle 0^i1 \rangle)]$ is essentially perfect. Thus $T_k \notin \mathcal{T}_4$. ∎

This establishes Theorem A by our machinery. ∎

3. The Main Theorem

Call a set $A \subseteq \omega$ 0-atomic iff $|\overline{A}| \leq 1$. Then a set B is cofinite iff B is in the filter generated by the 0-atomic sets. A set C is maximal iff its equivalence class is a coatom of the lattice of r.e. sets modulo the cofinite filter. A coinfinite set D is quasimaximal iff D is in the filter in $\mathcal{E}$ generated by the maximal sets, etc. This alternation of generating a filter and considering the coatoms leads to the following definition:

DEFINITION: Let A be a hyperhypersimple or cofinite set, α a recursive ordinal, and λ a recursive limit ordinal. Then:

(i) A is 0-*atomic* if $|\overline{A}| \leq 1$;

(ii) A is α-*quasiatomic* if A is a finite intersection of α-atomic sets, i.e., if A is in the filter generated by the α-atomic sets;

(iii) A is $(\alpha+1)$-*atomic* if for all r.e. sets $W \supseteq A$, W or $A \cup \overline{W}$ is α-quasiatomic, i.e., if A is α-quasiatomic or its equivalence class is a coatom of the lattice of r.e. sets modulo the α-quasiatomic filter (notice here and in (v) that $A \cup \overline{W}$ is r.e. if A is hyperhypersimple);

(iv) A is $<\lambda$-*atomic* if A is α-atomic for some $\alpha < \lambda$, i.e., if A is in the filter generated by the α-atomic sets for $\alpha < \lambda$;

(v) A is λ-*atomic* if for all r.e. sets $W \supseteq A$, W or $A \cup \overline{W}$ is $<\lambda$-quasiatomic, i.e., if A is $<\lambda$-quasiatomic or its equivalence class is a coatom of the lattice of r.e. sets modulo the $<\lambda$-quasiatomic filter.

The notions of α-atomic, α-quasiatomic, and $<\lambda$-atomic are natural generalizations of the notions of cofinite sets, maximal sets, and quasimaximal sets. Namely, A is cofinite iff A is 0-quasiatomic; A is maximal (or cofinite) iff A is 1-atomic; and A is quasimaximal (or cofinite) iff A is 1-quasiatomic.

Let At_α, QAt_α, and $\mathrm{At}_{<\lambda}$ denote the index sets of α-atomic, α-quasiatomic, and $<\lambda$-atomic sets, respectively.

(We chose not to call these sets coatoms (as common in the literature) since, e.g., a 0-atomic set is 1-atomic but its equivalence class is not a coatom modulo the 0-atomic filter, etc.)

The importance of the above definition lies in the correspondence of these properties with the Cantor-Bendixson rank of binary trees, as explained below. This correspondence allows the classification of their index sets, yielding a family of index sets of properties $\mathcal{L}_{\omega_1,\omega}$-definable over $\mathcal{E}$, which goes all the way through the *hyper*arithmetical hierarchy.

In the following, we will use ordinal arithmetic to compute expressions like $2\alpha+2$, etc. A set of integers is $\Sigma_{\lambda+n}$ ($\Pi_{\lambda+n}$) (for λ a recursive limit ordinal, $n \in \omega - \{0\}$) iff it is $\Sigma_n^{\emptyset^{(\lambda)}}$ ($\Pi_n^{\emptyset^{(\lambda)}}$). We use Rogers's book **[Ro67]** for the background on recursive ordinals. He defines a system of ordinal notations $|\cdot| : \mathcal{O} \to \omega_1^{CK}$ from Kleene's $\mathcal{O} \subseteq \omega$ into the set of recursive ordinals as well as a partial order $<_{\mathcal{O}}$ on $\mathcal{O}$ by

$$
\begin{aligned}
&|1| = 0 \\
&|x| = \alpha \to |2^x| = \alpha + 1, \text{ and } z \leq_{\mathcal{O}} x \to z <_{\mathcal{O}} 2^x \\
&\{\varphi_y(n)\}_{n\in\omega} \text{ a } <_{\mathcal{O}}\text{-increasing sequence and } \sup_n |\varphi_y(n)| = \alpha \to \\
&\qquad |3 \cdot 5^y| = \alpha, \text{ and } (\exists n)[z <_{\mathcal{O}} \varphi_y(n)] \to z <_{\mathcal{O}} 3 \cdot 5^y]
\end{aligned} \tag{6}
$$

The hyperarithmetical hierarchy $H : \mathcal{O} \to 2^\omega$ is then defined by

$$
\begin{aligned}
H(1) &= \emptyset \\
H(2^x) &= (H(x))' \\
H(3 \cdot 5^y) &= \{\langle u, v\rangle \mid u \in H(v) \wedge v <_{\mathcal{O}} 3 \cdot 5^y\}
\end{aligned} \tag{7}
$$

Now $|x| \leq |y|$ implies $H(x) \leq_T H(y)$. In particular, the Turing degree of $H(3 \cdot 5^y)$ does not depend upon the specific notation for a limit ordinal $\lambda = 3 \cdot 5^y$. Thus the definition of $\Sigma_{\lambda+n}$ and $\Pi_{\lambda+n}$ does not depend upon which $H(3\cdot 5^y)$ with $|3\cdot 5^y| = \lambda$ we use for $\emptyset^{(\lambda)}$. (Recall also that for any $y \in \mathcal{O}$, $\{x \mid x <_{\mathcal{O}} y\}$ is r.e. uniformly in y.)

The following theorem generalizes Theorem A (i) and (ii) to the hyperarithmetical hierarchy. We can do so by bounding the Cantor-Bendixson rank of the associated trees more carefully.

THEOREM B. *Let α be a recursive ordinal and λ a recursive limit ordinal. Then:*

(i) $(\Pi_{2\alpha+2}, \Sigma_{2\alpha+2}) \leq_1 (\mathrm{At}_\alpha, \mathrm{QAt}_\alpha - \mathrm{At}_\alpha)$;

(ii) $(\Sigma_{2\alpha+3}, \Pi_{2\alpha+3}) \leq_1 (\mathrm{QAt}_\alpha, \mathrm{At}_{\alpha+1} - \mathrm{QAt}_\alpha)$; *and*

(iii) $(\Sigma_{\lambda+1}, \Pi_{\lambda+1}) \leq_1 (\mathrm{At}_{<\lambda}, \mathrm{At}_\lambda - \mathrm{At}_{<\lambda})$.

COROLLARY 1.

(a) At_α *is* $\Pi_{2\alpha+2}$*-complete;*

(b) QAt_α *is* $\Sigma_{2\alpha+3}$*-complete; and*

(c) $\mathrm{At}_{<\lambda}$ *is* $\Sigma_{\lambda+1}$*-complete.*

PROOF: By Theorem B and the fact that At_α, QAt_α, and $\mathrm{At}_{<\lambda}$ are $\Pi_{2\alpha+2}$, $\Sigma_{2\alpha+3}$, and $\Sigma_{\lambda+1}$, respectively, by the Tarski-Kuratowski algorithm. ∎

COROLLARY 2.

(a) (Lachlan, D.A. Martin, R.W. Robinson, Yates (unpublished); later appearing in Tulloss [**Tu71**]) *The index set of maximal sets is* Π_4*-complete.*

(b) *The index set of quasimaximal sets is* Σ_5*-complete.*

PROOF: Set $\alpha = 1$ in Corollary 1. ∎

PROOF OF THEOREM B: The proof for the 0-atomic case is trivial and will be omitted here since it does not fit into our machinery. Using this machinery, we again have to prove a Correspondence Lemma and a Reduction Lemma.

Recall the definitions of Cantor-Bendixson derivative and Cantor-Bendixson rank. The *Cantor-Bendixson derivative* of a tree $T \subseteq 2^{<\omega}$ is T minus its isolated paths, i.e.,

(8) $$D(T) = \{\, \sigma \in \mathrm{Ext}(T) \mid (\exists \tau_1, \tau_2 \in \mathrm{Ext}(T))[\sigma \subset \tau_1, \tau_2 \wedge \tau_1 \mid \tau_2] \,\}.$$

We also define its iterates:

$$(9)\qquad \begin{aligned} D^0(T) &= T, \\ D^{\alpha+1}(T) &= D(D^\alpha(T)), \\ D^\lambda(T) &= \bigcap_{\alpha<\lambda} D^\alpha(T), \end{aligned}$$

where α is an ordinal, λ is a limit ordinal. Then the *Cantor-Bendixson rank* of T is

$$(10)\quad \rho(T) = \begin{cases} -1 & \text{if } T \text{ is finite,} \\ \min\{\,\alpha \mid D^{\alpha+1}(T) \text{ finite}\,\} & \text{if } T \text{ is infinite} \\ \qquad = \min\{\,\alpha \mid \|[D^\alpha(T)]\| \text{ finite}\,\} & \qquad \text{and this ordinal exists,} \\ \infty & \text{otherwise.} \end{cases}$$

It is a well-known fact that $D^\alpha(T) = D^\beta(T)$ for any uncountable ordinals α and β; and that $D^\lambda(T)$ finite for some limit ordinal λ implies $D^\alpha(T)$ finite for some $\alpha < \lambda$ by compactness.

These definitions lead to the

CORRESPONDENCE LEMMA. *Let α be a recursive ordinal, $T \subseteq 2^{<\omega}$ a Σ_2-tree. Then:*

(i) $\rho(T) = -1$ *iff A_T is 0-quasiatomic;*

(ii) $\|[D^\alpha(T)]\| \leq 1$ *iff A_T is $(1+\alpha)$-atomic; and*

(iii) $\rho(T) \leq \alpha$ *iff A_T is $(1+\alpha)$-quasiatomic.*

PROOF: By induction on α:

(i). $\rho(T) = -1$ iff T is finite iff A_T is cofinite iff A_T is 0-quasiatomic.

(ii)$_{\alpha=0}$. By (i) and the Correspondence Lemma for Theorem A.

(ii)$_\alpha \rightarrow$(iii)$_\alpha$. Assume (ii) for an ordinal α.

Suppose first that $\rho(T) \leq \alpha$. Then $[D^\alpha(T)]$ is finite, say, $[D^\alpha(T)] \subseteq \{\,p_1, p_2, \ldots, p_n\,\}$. Let k be large enough such that $i \neq j$ implies $p_i \restriction k \neq p_j \restriction k$. Then $\|[D^\alpha(\sigma^\frown T(\sigma))]\| \leq 1$ for all $\sigma \in T \cap 2^k$. By induction,

$$A_\sigma =_{\text{def}} A_T \cup \bigcup_{\substack{|\tau|=|\sigma|,\tau\neq\sigma \\ \tau\in T}} C_\tau$$

is $(1+\alpha)$-atomic, thus $A_T =^* \bigcap_{\sigma \in T \cap 2^k} A_\sigma$ is $(1+\alpha)$-quasiatomic.

On the other hand, if A_T is $(1+\alpha)$-quasiatomic then $A_T = \bigcap_{i=1}^n A_i$ for a finite set of $(1+\alpha)$-atomic sets $A_1, A_2, \ldots, A_n$. For each i, let $A_i =^* A_T \cup \bigcup_{\sigma \in S_i} C_\sigma$ for some finite set $S_i \subseteq T$, and let $T_i = T - \{\, \sigma\hat{\ }T(\sigma) \mid \sigma \in S_i \,\}$. Then $\bigcup_{i=1}^n T_i =^* T$, and, by induction, $[D^\alpha(T_i)] \subseteq \{\, p_i \,\}$ for some $p_i \in 2^\omega$. Thus $[D^\alpha(T)] \subseteq \{\, p_1, p_2, \ldots, p_n \,\}$ is finite, and $\rho(T) \leq \alpha$.

(iii)$_{<\alpha}$$\rightarrow$(ii)$_\alpha$. Assume $\alpha > 0$, and that (iii) holds for all ordinals less than α. Without loss of generality, let α be a successor ordinal and put $\beta + 1 = \alpha$ (if α is a limit ordinal, replace β by $<\alpha$ throughout this part of the proof).

Suppose first that $|[D^\alpha(T)]| \leq 1$, say, $[D^\alpha(T)] \subseteq \{\, p \,\}$. If $W \supseteq A_T$ is r.e. then $W =^* A_T \cup \bigcup_{\sigma \in S} C_\sigma$ for some finite set $S \subseteq T$ (assume that all $\sigma \in S$ are of the same length, say, k). Let $S_0 = (2^k - S) \cap T$, and put $W_0 = A_T \cup \bigcup_{\sigma \in S_0} C_\sigma$. Then W_0 is the relative complement (w.r.t. A_T) of W (modulo a finite set). Without loss of generality, suppose that $p \upharpoonright k \in S_0$ (the other case is symmetric). Then $T_0 = T - \bigcup_{\sigma \in S_0} C_\sigma$, the tree associated with W_0, satisfies $[D^\alpha(T_0)] =^* \emptyset$, and so W_0 is $(1+\beta)$-quasiatomic. Thus A_T is $(1+\alpha)$-atomic.

On the other hand, let A_T be $(1+\alpha)$-atomic. Suppose for the sake of contradiction that $[D^\alpha(T)]$ contains two distinct infinite paths, say, p_1 and p_2. Let k be large enough that $p_1 \upharpoonright k \neq p_2 \upharpoonright k$; let S_1 and S_2 be such that $S_1 \sqcup S_2 = 2^k \cap T$, $p_1 \upharpoonright k \in S_1$, and $p_2 \upharpoonright k \in S_2$; and let $W_1 = A_T \cup \bigcup_{\sigma \in S_1} C_\sigma$ and $W_2 = A_T \cup \bigcup_{\sigma \in S_2} C_\sigma$. Thus W_1 and W_2 are relative complements (w.r.t. A) to each other (modulo a finite set). Then for both $T_1 = T - \bigcup_{\sigma \in S_1} C_\sigma$ and $T_2 = T - \bigcup_{\sigma \in S_2} C_\sigma$, $[D^\alpha(T_1)]$ and $[D^\alpha(T_2)]$ are nonempty (namely, $p_1 \in [D^\alpha(T_2)]$ and $p_2 \in [D^\alpha(T_1)]$), and thus, by induction, neither of their associated r.e. sets W_1 and W_2 is $(1+\beta)$-quasiatomic, a contradiction. ∎

4. The Reduction Lemma for the Main Theorem

Let α be a recursive ordinal. We define

$$\mathcal{S}_\alpha = \{\, T \in 2^{<\omega} \text{ tree} \mid |[D^\alpha(T)]| \leq 1 \,\},$$

$$\mathcal{T}_\alpha = \{\, T \in 2^{<\omega} \text{ tree} \mid \rho(T) \leq \alpha \,\} \text{ (allow } \alpha = -1\text{)},$$

$$\mathcal{T}_{<\alpha} = \bigcup_{\beta<\alpha} \mathcal{T}_\beta.$$

It remains to prove the

Reduction Lemma. *Let α be a recursive ordinal and λ a recursive limit ordinal. Then:*

(i) $(\Pi_{2\alpha+2}, \Sigma_{2\alpha+2}) \leq_1 (\mathcal{S}_\alpha, \mathcal{T}_\alpha - \mathcal{S}_\alpha)$;

(ii) $(\Sigma_{2\alpha+3}, \Pi_{2\alpha+3}) \leq_1 (\mathcal{T}_\alpha, \mathcal{S}_{\alpha+1} - \mathcal{T}_\alpha)$ *(also allow $\alpha = -1$); and*

(iii) $(\Sigma_{\lambda+1}, \Pi_{\lambda+1}) \leq_1 (\mathcal{T}_{<\lambda}, \mathcal{S}_\lambda - \mathcal{T}_{<\lambda})$.

Notice that this lemma is an extension of the Reduction Lemma for Theorem A. Let LOR be the class of limit ordinals.

Proof: All constructions will be uniform in an ordinal notation for α (or λ), so we can use transfinite induction and the following four statements for $\alpha, \lambda \geq 0$:

(A) $(\Sigma_1, \Pi_1) \leq_1 (\mathcal{T}_{-1}, \mathcal{S}_0 - \mathcal{T}_{-1})$;

(B) $(\Sigma_{2\alpha+1}, \Pi_{2\alpha+1}) \leq_1 (\mathcal{T}_{<\alpha}, \mathcal{S}_\alpha - \mathcal{T}_{<\alpha}) \to (\Sigma_{2\alpha+3}, \Pi_{2\alpha+3}) \leq_1 (\mathcal{T}_\alpha, \mathcal{S}_{\alpha+1} - \mathcal{T}_\alpha)$;

(C) $(\Sigma_{2\alpha+1}, \Pi_{2\alpha+1}) \leq_1 (\mathcal{T}_{<\alpha}, \mathcal{S}_\alpha - \mathcal{T}_{<\alpha}) \to (\Pi_{2\alpha+2}, \Sigma_{2\alpha+2}) \leq_1 (\mathcal{S}_\alpha, \mathcal{T}_\alpha - \mathcal{S}_\alpha)$; and

(D) $(\Sigma_1, \Pi_1) \leq_1 (\mathcal{T}_{-1}, \mathcal{S}_0 - \mathcal{T}_{-1}) \wedge (\forall \gamma \in \text{LOR} \cap \lambda)[(\Sigma_{\gamma+1}, \Pi_{\gamma+1}) \leq_1 (\mathcal{T}_{<\gamma}, \mathcal{S}_\gamma - \mathcal{T}_{<\gamma})] \to (\Sigma_{\lambda+1}, \Pi_{\lambda+1}) \leq_1 (\mathcal{T}_{<\lambda}, \mathcal{S}_\lambda - \mathcal{T}_{<\lambda})$.

Then (ii) for $\alpha = -1$ follows from (A); (ii) for $\alpha \geq 0$ and (i) follow from (ii) for $\alpha - 1$ (if $\alpha \notin \text{LOR}$) or from (iii) (if $\alpha \in \text{LOR}$) by (B) and (C), respectively; and (iii) for λ follows from (ii) for $\alpha = -1$ and (iii) for $\gamma \in \text{LOR} \cap \lambda$ by (D). (Notice that the proof of (D) will require an induction argument separate from the successor ordinal case (B)-(C), as explained later.)

We will now prove (A)–(D):

(A) Given k, we will construct a recursive tree T_k such that

$$k \in \emptyset' \to T_k \text{ finite},$$
(11)
$$k \notin \emptyset' \to |[T_k]| = 1.$$

At any stage s, put $\langle 0^s \rangle$ into $T_{k,s}$ iff $\{k\}_s(k)\uparrow$. This construction obviously satisfies the claim.

(B) By (A) (for $\alpha = 0$), (B) (for $\alpha \notin \mathrm{LOR} \cup \{0\}$), or (D) (for $\alpha \in \mathrm{LOR}$), we have a uniformly recursive sequence of trees $\{\tilde{T}_l\}_{l\in\omega}$ satisfying

$$l \in \emptyset^{(2\alpha+1)} \to [D^\alpha(\tilde{T}_l)] = \emptyset,$$
(12)
$$l \notin \emptyset^{(2\alpha+1)} \to |[D^\alpha(\tilde{T}_l)]| = 1.$$

Now $\emptyset^{(2\alpha+3)} \equiv_1 \mathrm{Cof}^{\emptyset^{(2\alpha)}}$, so, given k, it suffices to uniformly build a recursive tree T_k such that

$$k \in \mathrm{Cof}^{\emptyset^{(2\alpha)}} \to |[D^\alpha(T_k)]| < \aleph_0,$$
(13)
$$k \notin \mathrm{Cof}^{\emptyset^{(2\alpha)}} \to |[D^{\alpha+1}(T_k)]| = 1.$$

Define a recursive function f such that $f(k,l) \in \emptyset^{(2\alpha+1)}$ iff $l \in W_k^{\emptyset^{(2\alpha)}}$. Fix k. At stage 0, put $\emptyset$ into $T_{k,0}$. At any stage $s > 0$, put $\langle 0^s \rangle$ and $\langle 0^{s-1}1 \rangle$ into $T_{k,s}$ and start the construction of $\tilde{T}_{f(k,s-1)}$ on top of $\langle 0^{s-1}1 \rangle$.

If $k \in \mathrm{Cof}^{\emptyset^{(2\alpha)}}$ then $f(k,l) \notin \emptyset^{(2\alpha+1)}$ for only finitely many l, say, l_0 is greater than all such l. Then $[D^\alpha(T_k(\langle 0^l 1 \rangle))] = \emptyset$ for all $l \geq l_0$, so $[D^\alpha(T_k(\langle 0^{l_0} \rangle))] \subseteq \{\langle 0^\omega \rangle\}$. Also $[D^\alpha(T_k(\langle 0^l 1 \rangle))]$ is finite for all $l < l_0$, so $[D^\alpha(T_k)]$ is finite.

On the other hand, if $k \notin \mathrm{Cof}^{\emptyset^{(2\alpha)}}$ then $f(k,l) \notin \emptyset^{(2\alpha+1)}$ for infinitely many l, so $|[D^\alpha(T_k(\langle 0^l 1 \rangle))]| = 1$ for infinitely many l. Thus $[D^{\alpha+1}(T_k)] = \{\langle 0^\omega \rangle\}$.

(C) The proof is similar to the proof for (B). We use that $(\mathrm{Tot}^{\emptyset^{(2\alpha)}}, \mathrm{Cotwo}^{\emptyset^{(2\alpha)}})$ is $(\Pi_{2\alpha+2}, \Sigma_{2\alpha+2})$-complete, where Tot^X and Cotwo^X are the index sets of total functions recursive in X and functions recursive in X undefined for exactly two integers, respectively.

Given k and $\{\tilde{T}_l\}_{l\in\omega}$ as in the proof of (B), we have to uniformly build a recursive tree T_k such that

$$(14)\qquad \begin{aligned} k \in \mathrm{Tot}^{\emptyset^{(2\alpha)}} &\to \|[D^\alpha(T_k)]\| \leq 1, \\ k \in \mathrm{Cotwo}^{\emptyset^{(2\alpha)}} &\to 1 < \|[D^\alpha(T_k)]\| < \aleph_0. \end{aligned}$$

The construction is the same as in (B).

If $k \in \mathrm{Tot}^{\emptyset^{(2\alpha)}}$ then $f(k,l) \in \emptyset^{(2\alpha+1)}$ for all l, so $[D^\alpha(T_k(\langle 0^l 1\rangle))] = \emptyset$ for all l. Thus $[D^\alpha(T_k)] \subseteq \{\langle 0^\omega\rangle\}$.

On the other hand, if $k \in \mathrm{Cotwo}^{\emptyset^{(2\alpha)}}$ then $f(k,l) \notin \emptyset^{(2\alpha+1)}$ for exactly two distinct l, say, l_1 and l_2, and so $D^\alpha(T_k(\langle 0^l 1\rangle))$ has exactly one infinite path for $l = l_1$ or l_2, and none for all other l. Thus $2 \leq \|[D^\alpha(T_k)]\| \leq 3$ (since possibly $\langle 0^\omega\rangle \in [D^\alpha(T_k)]$).

Part (D) is much harder to prove and requires some preparation.

5. The Reduction Lemma: The Limit Ordinal Case

The first lemma generalizes a lemma by Solovay for $\lambda = \omega$ **[JLSSta]** to arbitrary recursive limit ordinals:

Lemma 1 (Approximation Lemma). *Let λ be a recursive limit ordinal and $\{\alpha_n\}_{n\in\omega}$ the increasing sequence with $\sup_n \alpha_n = \lambda$ given by our ordinal notation for λ (i.e., $\lambda = |3\cdot 5^x|$, $|\varphi_x(n)| = \alpha_n$). Then there is a recursive function d (uniformly in a notation for λ) such that*

$$(15)\qquad (\forall y)\big[y \in \emptyset^{(\lambda+1)} \leftrightarrow (\exists n)[d(y,n) \in \emptyset^{(\alpha_n+1)}]\big].$$

Here $\emptyset^{(\lambda+1)} = (H(3\cdot 5^x))'$, and $\emptyset^{(\alpha_n+1)} = (H(\varphi_x(n)))'$.

Proof: Recall that there are recursive functions $h_{a,b}$ (uniformly in a, b) and r.e. sets P_a (uniformly in a) such that

$$(16)\qquad \begin{aligned} &H(a) \leq_1 H(b) \text{ via } h_{a,b} \text{ (for } a \leq_O b\text{), and,} \\ &P_a = \{\, b \mid b <_O a \,\} \text{ for } a \in O. \end{aligned}$$

(See Rogers [**Ro67**] for details.)

Now

(17)

$$\begin{aligned}
&y \in \emptyset^{(\lambda+1)} \\
&\leftrightarrow \{y\}^{H(3\cdot 5^x)}(y)\downarrow \\
&\leftrightarrow (\exists u,v,s)\big[\{y\}_s^{(D_u,D_v)}(y)\downarrow \wedge D_u \subseteq H(3\cdot 5^x) \wedge \\
&\qquad D_v \cap H(3\cdot 5^x) = \emptyset\big] \\
&\leftrightarrow (\exists u,v,s)\big[\{y\}_s^{(D_u,D_v)}(y)\downarrow \wedge (\forall\langle z_1,z_2\rangle \in D_u)[z_1 \in H(z_2) \wedge z_2 <_0 3\cdot 5^x] \wedge \\
&\qquad (\forall\langle z_1,z_2\rangle \in D_v)[z_1 \notin H(z_2) \vee z_2 \not<_0 3\cdot 5^x]\big] \\
&\leftrightarrow (\exists u,v,s,n)\big[\{y\}_s^{(D_u,D_v)}(y)\downarrow \wedge \\
&\qquad (\forall\langle z_1,z_2\rangle \in D_u)[h_{z_2,\varphi_x(n)}(z_1) \in H(\varphi_x(n)) \wedge z_2 \in P_{\varphi_x(n),s} \wedge z_2 \in P_{3\cdot 5^x}] \wedge \\
&\qquad (\forall\langle z_1,z_2\rangle \in D_v)[(h_{z_2,\varphi_x(n)}(z_1) \notin H(\varphi_x(n)) \wedge z_2 \in P_{\varphi_x(n),s}) \vee z_2 \notin P_{3\cdot 5^x}]\big] \\
&\leftrightarrow (\exists u,v,s,n)\big[\Delta_1 \wedge (\mathrm{Q})[\Delta_1^{H(\varphi_x(n))} \wedge \Delta_1 \wedge \Sigma_1] \wedge (\mathrm{Q})[(\Delta_1^{H(\varphi_x(n))} \wedge \Delta_1) \vee \Pi_1]\big]
\end{aligned}$$

where (Q) denotes a bounded quantifier, and $\{y\}^{(D_u,D_v)}$ that the computation uses from the oracle set X at most that $z \in X$ for $z \in D_u$ and that $z \notin X$ for $z \in D_v$.

Now the matrix of the last expression is recursive in $H(\varphi_x(n)) \oplus \emptyset'$, and thus certainly in $(H(\varphi_x(n+1)))' = \emptyset^{(\alpha_{n+1}+1)}$. This establishes the claim of the lemma. ∎

The first try at the construction of T_k at a limit ordinal level λ satisfying (D) would be to build $T^{\alpha_n}_{d(k,n)}$ on top of $\langle 0^n 1\rangle$. However, we only know $\rho(T^{\alpha_n}_{d(k,n)}) = \alpha_n$ or $< \alpha_n$, so $\sup_n \rho(T^{\alpha_n}_{d(k,n)}) = \lambda$ is possible independent of whether $k \in \emptyset^{(\lambda+1)}$.

Our second try is to let level α_n, say, at which we "discover" that $k \in \emptyset^{(\lambda+1)}$ by Lemma 1, stop the higher levels by some kind of "permission" for extending branches above $\langle 0^m 1\rangle$ for $m > n$. However, this is hard since $T^{\alpha_m}_{d(k,n)}$ looks very different from $T^{\alpha_n}_{d(k,n)}$, so we have to introduce a very strong kind of permission at all branchings of the much bigger tree $T^{\alpha_m}_{d(k,m)}$. Keeping this in mind should make the following construction seem less mysterious. This requires also a new induction argument at the successor ordinal level.

For the sake of convenience, let $\sigma(k_1, k_2, \ldots, k_n) = \langle 0^{k_1} 1 0^{k_2} 1 \ldots 0^{k_n} 1 \rangle \in 2^{<\omega}$. For α a recursive ordinal, the *field of the α-strategy F_α* (i.e., the largest possible tree that T_k^α could be) is defined by

$$
\begin{aligned}
F_0 &= \{ \langle 0^n \rangle \mid n \in \omega \}, \\
F_{\alpha+1} &= \{ \sigma(n)^\frown\langle\sigma\rangle \mid \sigma \in F_\alpha, n \in \omega \} \cup F_0, \\
F_\lambda &= \{ \sigma(n)^\frown\langle\sigma\rangle \mid \sigma \in F_{\alpha_n}, n \in \omega \} \cup F_0 \\
&\qquad \text{for } \lambda \in \text{LOR},\ \lambda = |3 \cdot 5^y|,\ \alpha_n = |\varphi_y(n)|.
\end{aligned} \tag{18}
$$

(Notice that the F_α's are all recursive sets, and that they *do* depend upon the particular ordinal notation chosen. However, since we will always fix an ordinal notation in advance this will not matter in the following.)

The *ordinal β_σ^α associated with a branching node σ on F_α* is defined by

$$
\begin{aligned}
\beta_\emptyset^\alpha &= \alpha, \\
\beta^\alpha_{\sigma^\frown\langle\sigma(k)\rangle} &= \begin{cases} \beta_\sigma^\alpha - 1 & \text{for } \beta_\sigma^\alpha \notin \text{LOR} \cup \{0\}, \\ \gamma_k & \text{for } \beta_\sigma^\alpha = \gamma \in \text{LOR},\ \gamma = |3 \cdot 5^z|,\ \gamma_n = |\varphi_z(n)|, \\ \text{undefined} & \text{for } \beta_\sigma^\alpha = 0. \end{cases}
\end{aligned} \tag{19}
$$

(Thus β_σ^α is defined exactly for all nodes $\sigma \in F_\alpha$ of the form $\sigma = \sigma(k_1, k_2, \ldots, k_n)$. The ordinals β_σ^α will determine the strategy above the node σ.)

The following lemma will be essential later:

LEMMA 2 (FINITE EXCEPTIONS LEMMA). *For any subtree $S \subseteq F_\alpha$ and any infinite path $p \in [S]$, $\{ i \mid p(i) = 1 \}$ is finite.*

PROOF: Otherwise there are $n_1, n_2, n_3, \cdots \in \omega$ such that $\emptyset \subset \sigma(n_1) \subset \sigma(n_1, n_2) \subset (n_1, n_2, n_3) \subset \cdots \subset p$, so that all these nodes are in S and thus in F_α, but then $\beta_\emptyset^\alpha$, $\beta^\alpha_{\sigma(n_1)}$, $\beta^\alpha_{\sigma(n_1,n_2)}$, $\beta^\alpha_{\sigma(n_1,n_2,n_3)}$, … is an infinite descending sequence of ordinals. ∎

We call a tree $T \subseteq F_\alpha$ *α-dense* (for α a recursive ordinal) iff

$$
\begin{aligned}
&(\forall n \in \omega \cap (\alpha + 1))(\text{a.e. } k_1)(\text{a.e. } k_2) \ldots (\text{a.e. } k_n) \\
&\qquad [\rho(T(\sigma(k_1, k_2, \ldots, k_n))) = \beta^\alpha_{\sigma(k_1,k_2,\ldots,k_n)}].
\end{aligned} \tag{20}
$$

I.e., in an α-dense tree, all appropriate subtrees of T have maximal rank possible. For example, the only 0-dense tree is F_0 itself; a tree $T \subseteq F_1$ is 1-dense iff $T(\sigma(n)) = F_0$ for almost all n, etc.

LEMMA 3 (DENSITY LEMMA). *Let $\alpha > 0$ be a recursive ordinal, $T \subseteq F_\alpha$ a tree. Then T is α-dense iff* (a.e. m)[$T(\sigma(m))$ is $\beta^\alpha_{\sigma(m)}$-dense].

PROOF: ($\rightarrow$) Trivial by definition.

($\leftarrow$) We only need to show (20) for $n = 0$. Suppose that for all $m > m_0$, $\rho(T(\sigma(m))) = \beta^\alpha_{\sigma(m)}$. Since $\beta^\alpha_{\sigma(m)} = \alpha - 1$ (for $\alpha \notin \text{LOR}$) or $\alpha = \sup_m \beta^\alpha_{\sigma(m)}$ (for $\alpha \in \text{LOR}$), we obtain $\rho(T) = \alpha$. ∎

LEMMA 4 (INTERSECTION LEMMA). *Let α be a recursive ordinal. If T and $\tilde{T}$ are α-dense, then so is $T \cap \tilde{T}$.*

PROOF: By induction on α: For $\alpha = 0$, note that $T = \tilde{T} = \{\langle 0^m \rangle \mid m \in \omega\}$. For $\alpha > 0$, use Lemma 3 and the fact that $\beta^\alpha_{\sigma(m)} < \alpha$. ∎

Notice that this would be false, for example, if we had defined α-dense just as having rank α. For example, then the intersection of $T, \tilde{T} \subseteq F_1$, both of rank 1, could have rank 0.

The following lemma will be essential later for showing that the nesting of trees works properly. (It is the first example of the property of trees that the subtree above a certain node $\sigma(k_1, k_2, \ldots, k_n)$ looks exactly as if it were constructed by itself.)

LEMMA 5 (NESTING LEMMA). *Let $\beta < \alpha$ be two recursive ordinals, and let $T \subseteq F_\beta$ be a β-dense tree. Then $\tilde{T} = \{\sigma \in F_\alpha \mid (\forall \tau \subseteq \sigma)[\tau \in F_\beta \rightarrow \tau \in T]\}$ is α-dense.*

PROOF: By induction on β: If $\beta = 0$ then $T = \{\langle 0^m \rangle \mid m \in \omega\}$, and $\tilde{T} = F_\alpha$. If $\beta > 0$ then for almost every m, $\beta^\beta_{\sigma(m)} < \beta^\alpha_{\sigma(m)}$, and, by Lemma 3, for almost every m, $T(\sigma(m))$ is $\beta^\beta_{\sigma(m)}$-dense. Therefore, by induction, for almost every m, $\tilde{T}(\sigma(m))$ is $\beta^\alpha_{\sigma(m)}$-dense. Thus, again by Lemma 3, $\tilde{T}$ is α-dense. ∎

The following lemma is the key to the construction. We build trees, again by induction, but with much stronger properties. (However, in the successor ordinal case, we lose a finite number of levels, so we can use this construction only for the proof in the limit ordinal case.)

For the sake of convenience, for an arbitrary $\beta < \omega_1^{CK}$ with fixed ordinal notation, define a sequence of predicates $\{ P_\alpha \}_{\alpha \leq \beta}$

$$P_\alpha(k) \leftrightarrow \begin{cases} k \in \emptyset^{(\alpha+1)} & \text{if } \alpha \text{ is an even ordinal,} \\ k \notin \emptyset^{(\alpha+1)} & \text{otherwise,} \end{cases} \tag{21}$$

where α is an *even ordinal* if $\alpha = \lambda + 2n$ for $\lambda \in \mathrm{LOR} \cup \{ 0 \}$ and $n \in \omega$.

LEMMA 6 (STRONG REDUCTION LEMMA). *For any recursive ordinal α, there exists (uniformly in an ordinal notation for α) a uniformly recursive sequence $\{ T_k^\alpha \}_{k \in \omega}$ of trees $T_k^\alpha \subseteq F_\alpha$ such that*

$$\begin{aligned} &P_\alpha(k) \to (\text{a.e.}\, k_1)(\text{a.e.}\, k_2) \ldots (\text{a.e.}\, k_m)[\rho(T_k^\alpha(\sigma(k_1, k_2, \ldots, k_m))) < \lambda], \text{ and} \\ &\neg P_\alpha(k) \to T_k^\alpha \text{ is } \alpha\text{-dense,} \end{aligned} \tag{22}$$

where $\alpha = \lambda + m$, $\lambda \in \mathrm{LOR} \cup \{ 0 \}$, $m \in \omega$.

PROOF: For $\alpha = 0$, use the construction from (A) above.

For α a successor ordinal, say, $\alpha = \beta + 1$, assume without loss of generality that α is even (the odd case is similar). Using $(\emptyset^{(\beta+2)}, \overline{\emptyset^{(\beta+2)}}) \leq_1 (\mathrm{Fin}^{\emptyset^{(\beta)}}, \mathrm{Cof}^{\emptyset^{(\beta)}})$, there are recursive functions h and h_0 such that

$$\begin{aligned} P_\alpha(k) &\to k \in \emptyset^{(\beta+2)} \to W_{h_0(k)}^{\emptyset^{(\beta)}} \text{ finite} \to \{ l \mid l \in W_{h_0(k)}^{\emptyset^{(\beta)}} \} \text{ finite} \\ &\to \{ l \mid h(k,l) \in \emptyset^{(\beta+1)} \} \text{ finite} \to (\text{a.e.}\, l)[P_\beta(h(k,l))], \\ \neg P_\alpha(k) &\to k \notin \emptyset^{(\beta+2)} \to W_{h_0(k)}^{\emptyset^{(\beta)}} \text{ cofinite} \to \{ l \mid l \in W_{h_0(k)}^{\emptyset^{(\beta)}} \} \text{ cofinite} \\ &\to \{ l \mid h(k,l) \in \emptyset^{(\beta+1)} \} \text{ cofinite} \to (\text{a.e.}\, l)[\neg P_\beta(h(k,l))]. \end{aligned}$$

Fix k. At stage 0, put $\emptyset$ into $T_{k,0}^\alpha$. At a stage $s > 0$, put $\langle 0^s \rangle$ and $\langle 0^{s-1}1 \rangle$ into $T_{k,s}^\alpha$ and start the construction of $T_{h(k,s-1)}^\beta$ on top of $\langle 0^{s-1}1 \rangle$. The claim that this works is immediate by (23) and Lemma 3.

For α a limit ordinal, let $\alpha = |3 \cdot 5^x|$, $\alpha_n = |\varphi_x(n)|$, so $\{ \alpha_n \}_{n \in \omega}$ is an increasing sequence of ordinals with $\alpha = \sup_n \alpha_n$. Slightly modify the function d from Lemma 1 so that

$$(\forall y)\left[y \in \emptyset^{(\alpha+1)} \leftrightarrow (\exists n)[P_{\alpha_n}(d(y,n))]\right],$$

and, for simplicity,

$$(\forall n)[P_{\alpha_n}(d(y,n)) \rightarrow P_{\alpha_{n+1}}(d(y,n+1))].$$

Given $\sigma \in 2^{<\omega}$, we define the *branch number* $b(\sigma) = \min\{ n \mid \langle 0^n \rangle \subseteq \sigma \}$, and the *decision set* $D(\sigma) = \{ \tau \subseteq \sigma \mid (\exists \tilde{\tau})[\tilde{\tau}^\frown\langle 1 \rangle = \tau] \}$. ($b(\sigma)$ will determine the main strategy at σ, the nodes of $D(\sigma)$ the secondary strategies from lower levels.)

The construction for α a recursive limit ordinal now proceeds as follows: Fix k. At stage 0, put $\emptyset$ into $T^\alpha_{k,0}$. At a stage $s > 0$, put $\langle 0^s \rangle$ and $\langle 0^{s-1}1 \rangle$ into $T^\alpha_{k,s}$; also put any $\sigma \in 2^{<\omega}$ into $T^\alpha_{k,s}$ for which the following conditions are satisfied:

(i) $|\sigma| = s$, $\sigma \upharpoonright (s-1) \in T^\alpha_{k,s-1}$,

(ii) $\sigma \in F_\alpha$, and

(iii) $(\forall \tau \in D(\sigma))(\forall m \leq b(\sigma))[\alpha_m \leq \beta^\alpha_\tau \wedge \sigma \in \tau^\frown F_{\alpha_m} \rightarrow \sigma \in \tau^\frown T^{\alpha_m}_{d(k,m)}]$.

(Notice here that the construction is arranged in such a way that to any $\sigma(k_1, k_2, \ldots, k_m)$, the construction above it looks the same as to a $\sigma(n)$ above it. This will be an essential feature for the verification.)

Now suppose first that $k \in \emptyset^{(\alpha+1)}$, i.e., by the modification of Lemma 1, $P_{\alpha_n}(d(k,n))$ holds for all $n \geq$ some fixed n_0. We then claim that $\rho(T^\alpha_k(\sigma(n))) \leq \alpha_{n_0}$ for all n, thus $\rho(T^\alpha_k) \leq \alpha_{n_0} + 1 < \alpha$ as desired. The proof requires induction on α_{n_0}. (Of course, there is nothing to prove for $\alpha_n \leq \alpha_{n_0}$.)

$\alpha_{n_0} = 0$: Let $\tilde{\tau} = \sigma(n)$. Then $\tilde{\tau}^\frown F_{\alpha_{n_0}} = \{ \sigma(n)^\frown\langle 0^m \rangle \mid m \in \omega \}$, so $\langle 0^{m_0} \rangle \notin T^{\alpha_{n_0}}_{d(k,n_0)}$ for some m_0, and thus $T^\alpha_k(\tilde{\tau}^\frown\langle 0^{m_0} \rangle)$ is finite. As for $T^\alpha_k(\sigma(n,m))$ for $m < m_0$, apply the same proof to $\tilde{\tau} = \sigma(n,m)$, etc. By Lemma 2, there is no infinite sequence $\sigma(n)$, $\sigma(n,m)$, $\sigma(n,m,l)$, ... of such $\tilde{\tau}$'s, so $T^\alpha_k(\sigma(n))$ is finite and $\rho(T^\alpha_k(\sigma(n))) \leq \alpha_{n_0}$.

$\alpha_{n_0} = \beta + 1$: There is m_0 such that $P_\beta(h(d(k, n_0), m))$ holds for all $m \geq m_0$ where h is the function for α_{n_0} and β mentioned above in the proof for the successor ordinal case. Now the α_{n_0}-construction works at $\sigma(n)$, and thus the β-construction at $\sigma(n, m)$ for all m, through condition (iii) of the construction (putting $\tau = \sigma(n_0)$). Thus by induction (replacing α_{n_0} and α_n by β and $\beta^{\alpha_n}_{\sigma(m)}$), there is some m_0 such that $\rho(T^\alpha_k(\sigma(n, m))) \leq \beta$ for all $m \geq m_0$, so $\rho(T^\alpha_k(\sigma(n)^\frown\langle 0^{m_0}\rangle)) \leq \alpha_{n_0}$. As for $T^\alpha_k(\sigma(n, m))$ for $m < m_0$, apply the same proof with $\tau = \sigma(n, m)$, etc. By Lemma 2, there is no infinite sequence $\sigma(n)$, $\sigma(n, m)$, $\sigma(n, m, l)$, $\ldots$ of such τ's, so $T^\alpha_k(\sigma(n))$ consists of finitely many subtrees, each of rank $\leq \alpha_{n_0}$, and thus $\rho(T^\alpha_k(\sigma(n))) \leq \alpha_{n_0}$.

The above establishes $\rho(T^\alpha_k(\sigma(n))) \leq \alpha_{n_0} < \lambda$ for all n, so $\rho(T^\alpha_k) \leq \alpha_{n_0} + 1 < \lambda$ in the successor ordinal case of α_{n_0}.

$\alpha_{n_0} \in \mathrm{LOR}$: Then $\{\, \beta^{\alpha_{n_0}}_{\sigma(m)} \,\}_{m \in \omega}$ is an increasing sequence with limit α_{n_0}. There is m_0 such that $P_{\beta^{\alpha_{n_0}}_{\sigma(m)}}(\tilde{d}(d(k, n_0), m))$ holds for all $m \geq m_0$ where $\tilde{d}$ is the counterpart of f for α_{n_0} as a limit ordinal. Now the α_{n_0}-construction works at $\sigma(n)$, and thus the $\beta^{\alpha_{n_0}}_{\sigma(m)}$-construction at $\sigma(n, m)$ for all m, through condition (iii) of the construction (putting $\tau = \sigma(n_0)$). Thus by induction (replacing α_{n_0} and α_n by $\beta^{\alpha_{n_0}}_{\sigma(m)}$ and $\beta^{\alpha_n}_{\sigma(m)}$), we have that $\rho(T^\alpha_k(\sigma(n, m))) \leq \beta^{\alpha_{n_0}}_{\sigma(m_0)}$ for all $m \geq m_0$ (this part does not follow by induction for m with $\beta^{\alpha_n}_{\sigma(m)} \leq \beta^{\alpha_{n_0}}_{\sigma(m)}$ but in that case it is trivial anyway). Therefore, $\rho(T^\alpha_k(\sigma(n)^\frown\langle 0^{m_0}\rangle)) \leq \alpha_{n_0}$. As for $T^\alpha_k(\sigma(n, m))$ for $m < m_0$, apply the same proof with $\tau = \sigma(n, m)$, etc. By Lemma 2, there is no infinite sequence $\sigma(n)$, $\sigma(n, m)$, $\sigma(n, m, l)$, $\ldots$ of such τ's, so $T^\alpha_k(\sigma(n))$ consists of finitely many subtrees, each of rank $\leq \alpha_{n_0}$, so $\rho(T^\alpha_k(\sigma(n))) \leq \alpha_{n_0}$.

The above establishes $\rho(T^\alpha_k(\sigma(n))) \leq \alpha_{n_0} < \lambda$ for all n, so $\rho(T^\alpha_k) \leq \alpha_{n_0} + 1 < \lambda$ in the limit ordinal case of α_{n_0}.

On the other hand, assume that $k \notin \emptyset^{(\alpha+1)}$. Then $P_{\alpha_n}(d(k, n))$ does not hold for any n. We claim that T^α_k is α-dense (and thus $[D^\alpha(T^\alpha_k)] = \{\, \langle 0^\omega \rangle \,\}$). We proceed by induction on $\beta = \alpha_n$, using Lemma 3:

$\alpha_n = 0$: We have $T^\alpha_k(\sigma(n)) = T^0_{d(k,n)} = \{\, \langle 0^m \rangle \mid m \in \omega \,\}$, so $\rho(T(\sigma(n))) = \alpha_n$.

$\alpha_n > 0$: We have

(23) $T^\alpha_k(\sigma(n)) = \{\, \sigma \in F_{\alpha_n} \mid (\forall \tilde\sigma \subseteq \sigma)(\forall \tau \in D(\tilde\sigma) \cup \{\emptyset\})(\forall m \leq n)$

$$[\alpha_m \leq \beta^{\alpha_n}_\tau \wedge \tilde\sigma \in \tau^\frown F_{\alpha_m} \rightarrow \tilde\sigma \in \tau^\frown T^{\alpha_m}_{d(k,m)}]\,\}.$$

Among these restrictions, we can distinguish three types:

(a) $\tau \neq \emptyset$ (and thus $m < n$);

(b) $\tau = \emptyset$ and $m = n$; and

(c) $\tau = \emptyset$ and $m < n$.

Thus $T(\sigma(n))$ is the intersection of the following three trees:

(a) $T_1 = \{\, \sigma \in F_{\alpha_n} \mid (\forall \tilde\sigma \subseteq \sigma)(\forall \tau \in D(\tilde\sigma))(\forall m < n)[\alpha_m \leq \beta^{\alpha_n}_\tau \wedge \tilde\sigma \in \tau^\frown F_{\alpha_m} \rightarrow \tilde\sigma \in \tau^\frown T^{\alpha_m}_{d(k,m)}]\,\}$;

(b) $T_2 = \{\, \sigma \in F_{\alpha_n} \mid (\forall \tilde\sigma \subseteq \sigma)[\tilde\sigma \in T^{\alpha_n}_{d(k,n)}]\,\} = T^{\alpha_n}_{d(k,n)}$; and

(c) $T_3 = \{\, \sigma \in F_{\alpha_n} \mid (\forall \tilde\sigma \subseteq \sigma)(\forall m < n)[\tilde\sigma \in F_{\alpha_m} \rightarrow \tilde\sigma \in T^{\alpha_m}_{d(k,m)}]\,\} = \bigcap_{m<n}\{\, \sigma \in F_{\alpha_n} \mid (\forall \tau \subseteq \sigma)[\tau \in F_{\alpha_m} \rightarrow \tau \in T^{\alpha_m}_{d(k,m)}]\,\}$. (Call these trees $T_{3,m}$ for $m < n$.)

By Lemma 4, it suffices to show that each of T_1, T_2, and the $T_{3,m}$ is α_n-dense.

(a) Recall again the remark that the construction above $\sigma(n)$ looks to α_n just as it does to α above $\emptyset$. For all l,

$$T_1(\sigma(l)) = \{\, \sigma \in F_{\beta^{\alpha_n}_{\sigma(l)}} \mid (\forall \tilde\sigma \subseteq \sigma)(\forall \tau \in D(\tilde\sigma) \cup \{\emptyset\})(\forall m < n)$$
$$[\alpha_m \leq \beta^{\alpha_n}_{\sigma(l)^\frown\tau} \wedge \tilde\sigma \in \tau^\frown F_{\alpha_m} \rightarrow \tilde\sigma \in \tau^\frown T^{\alpha_m}_{d(k,m)}]\,\}.$$

Therefore, by induction on $\beta = \alpha_n$ in (23) (with $\beta^{\alpha_n}_{\sigma(l)}$ in place of α_n, and $\beta^{\alpha_n}_{\sigma(l)^\frown\tau}$ in place of $\beta^{\alpha_n}_\tau$), $T_1(\sigma(l))$ is $\beta^{\alpha_n}_{\sigma(l)}$-dense for almost every l. Thus, by Lemma 3, T_1 is α_n-dense.);

(b) T_2 is α_n-dense by induction on the overall construction; and

(c) each $T_{3,m}$ is α_n-dense by induction and Lemma 5.

This concludes the proof of Lemma 6. ∎

Lemma 6 now implies part (D) of the proof of the Reduction Lemma, and thus Theorem B has been established. ∎

6. An Index Set in Major Subsets

Lachlan [**La68**] defined the following notion of two r.e. sets $A \subset_\infty B$ being "close" to each other:

DEFINITION: Let $A \subset_\infty B$ be r.e. sets. Then A is *major in* B ($A \subset_{\mathrm{m}} B$) iff

$$(\forall W \text{ r.e.})[\overline{B} \subseteq^* W \to \overline{A} \subseteq^* W]. \tag{24}$$

(24) is equivalent to either of the following two conditions:

$$(\forall W \text{ r.e.})[\overline{B} \subseteq W \to \overline{A} \subseteq^* W], \tag{24'}$$

$$\mathcal{L}^*(A) = \mathcal{L}^*(B), \tag{24"}$$

where $\mathcal{L}^*(X)$ is the lattice of r.e. supersets of X (modulo finite sets).

The classification of the index set $\{\langle e, i\rangle \mid W_e \subset_{\mathrm{m}} W_i\}$ has been one of the open questions in index sets for a while. The major obstacle here is that $A \subset_{\mathrm{m}} B$ implies that B is nonrecursive. This makes the uniformity required for the classification hard. We present below a partial result towards the classification of this index set:

THEOREM C. *Let V be a nonrecursive r.e. set. Then the index set* $\mathrm{Maj}_V = \{k \mid W_k \subset_{\mathrm{m}} V\}$ *is* Π_4*-complete.*

PROOF: It is easy to see that Maj_V is Π_4:

$$\begin{aligned} W_k \subset_{\mathrm{m}} V &\leftrightarrow W_k \subset_\infty V \wedge (\forall e)[V \cup W_e \neq \omega \vee W_k \cup W_e =^* \omega] \\ &\leftrightarrow \Pi_3 \wedge (\forall e)[\Sigma_2 \vee \Sigma_3] \\ &\leftrightarrow \Pi_4. \end{aligned} \tag{25}$$

We will build (uniformly in k) an r.e. set $A_k \subset_\infty V$ such that $A_k \subset_{\mathrm{m}} V$ iff $k \notin \emptyset^{(4)}$. (We will usually suppress the index k on A from now on.)

We use the fact that there is a recursive function h such that

$$\begin{aligned} k \notin \emptyset^{(4)} &\to (\forall i)[W_{h(k,i)} \text{ cofinite}], \\ k \in \emptyset^{(4)} &\to (\exists i)[W_{h(k,i)} \text{ coinfinite}]. \end{aligned} \tag{26}$$

Fix k from now on, and let $\overline{W_{h(k,i),s}} = \{ h^s_{i,0} < h^s_{i,1} < h^s_{i,2} < \dots \}$.

The idea of the proof is now to have for each i two conflicting strategies, a positive strategy trying to establish (24′) for W_i, and a negative strategy trying to build a counterexample B to $A \subset_{\mathrm{m}} V$. Which strategy succeeds will depend on whether $W_{h(k,i)}$ is cofinite or not. (If $W_{h(k,i)}$ is coinfinite then the strategies working on $i' > i$ will not matter.)

For the basic module of the positive $\mathcal{P}_e$-strategy, we use a variant of Lachlan's strategy [La68] to construct a major subset. Let $\tilde{W}_{e,s} = \{ x \in W_{e,s} \mid (\forall y < x)[y \in W_{e,s} \cup V_s] \}$, and let $\tilde{W}_e = \bigcup_s \tilde{W}_{e,s}$. Then $W_e = \tilde{W}_e$ if $W_e \supseteq \overline{V}$, and $\tilde{W}_e$ is finite if $W_e \not\supseteq \overline{V}$. In the former case, we have to take action for the sake of W_e; in the latter case, the strategy will only have a finite effect on the rest of the construction. Furthermore, let f be a 1–1 enumeration of V (recall that V has to be infinite). Finally, let $V_s - A_s = \{ d^s_0, d^s_1, d^s_2, \dots, d^s_{n_s} \}$ where the markers d^s_n need not be in order. (The markers d^s_n will be undefined for $n > n_s$.)

At stage 0, let $A_0 = \emptyset$, let $d^0_0 = f(0)$, and let d^0_n be undefined for $n > 0$. At a stage $s+1$, first determine if $f(s+1) \in \tilde{W}_{e,s}$ and $d^s_{\tilde{n}} \notin \tilde{W}_{e,s}$ for some $\tilde{n} \le n_s$. If so, for the least such $\tilde{n}$, put $d^s_{\tilde{n}}$ into A_{s+1}, let $d^{s+1}_{\tilde{n}} = f(s+1)$, and let $d^{s+1}_n = d^s_n$ for all $n \ne \tilde{n}$ (for the sake of $\overline{A} \subseteq^* W_e$). Otherwise, let $d^{s+1}_{n_s+1} = f(s+1)$, and let $d^{s+1}_n = d^s_n$ for $n \ne n_s + 1$ (for the sake of $A \subset_\infty V$).

Since V is nonrecursive, $\overline{V}$ is not r.e. Suppose $\overline{V} \subseteq W_e$ (and thus $W_e = \tilde{W}_e$). Then we have that

$$(\exists^\infty s)(\exists x)[x \in V_{s+1} - V_s \wedge x \in \tilde{W}_{e,s}]. \tag{27}$$

Therefore, $f(s+1) \in \tilde{W}_{e,s}$ for infinitely many s, so any marker d^s_n will be moved until it is in $\tilde{W}_e$, and so $\overline{A} \subseteq \tilde{W}_e$. (These strategies will later be combined using e-states as first introduced by Friedberg in his maximal set construction [**Fr58**].)

The basic module for the negative $\mathcal{N}$-strategy tries to build a set B refuting $A \subset_{\mathrm{m}} V$, i.e., such that $\overline{V} \subseteq B$ and that $V - (A \cup B)$ is infinite. At the nth time the strategy acts, it will wait for $|V - (A \cup B)| > n$, then put $\min(V)$ into B (for the sake of $\overline{V} \subseteq B$) and restrain another element of $V - (A \cup B)$ from entering A (to make $V - (A \cup B)$ infinite).

Suppose that $A \subset_\infty V$. Then the strategy will act infinitely often (else B and thus $V - A$ would be finite). So $\overline{V} \subseteq B$ and $V - (A \cup B)$ is infinite. (Notice that we really only have to restrain forever from A an infinite subset of the restrained elements of $V - (A \cup B)$.

We have to let the success (or failure) of the $\mathcal{N}$-strategy depend on whether $W_{h(k,i)}$ is coinfinite (or cofinite). Recall that $\overline{W_{h(k,i),s}} = \{ h^s_{i,0} < h^s_{i,1} < h^s_{i,2} < \dots \}$. Let the $\mathcal{N}$-strategy only restrain at stage $s+1$ at most $m_s = \min\{ n \mid h^n_{i,s+1} \neq h^n_{i,s} \}$ many elements. If $W_{h(k,i)}$ is coinfinite then $\lim_s m_s = \infty$, so the $\mathcal{N}$-strategy can eventually restrain more and more elements from A permanently. If $W_{h(k,i)}$ is cofinite then $m = \liminf_s m_s < \infty$, so the $\mathcal{N}$-strategy can restrain at most m elements permanently from A. (Notice that if one $\mathcal{N}$-strategy is allowed to succeed the lower-priority $\mathcal{P}$-strategies will not matter since this $\mathcal{N}$-strategy will satisfy the overall requirement $A \not\subset_{\mathrm{m}} V$.)

Combining all strategies requires two minor changes:

First of all, a stronger $\mathcal{P}$-strategy may injure a weaker $\mathcal{N}$-strategy by putting infinitely many elements into A that are restrained by the $\mathcal{N}$-strategy. So the latter has to be able to predict which elements the $\mathcal{P}$-strategy will put into A. This is done in a straightforward tree argument fashion.

Secondly, if a $\mathcal{P}$-strategy is forced to always observe the *current* restraint of the stronger $\mathcal{N}$-strategies then a synchronization problem may arise. Good elements (i.e., numbers $f(s+1) \in \tilde{W}_{e,s}$) may come up only when the restraint is high, so the $\mathcal{P}$-strategy may not achieve its objective even if the lim inf of the restraint is finite. To resolve this conflict, we will, roughly speaking, make the $\mathcal{P}$-strategy only observe (for d^s_n) the lowest restraint since some d^s_m with $m \leq n$ moved. (This will be done through the control function Q. An alternative way to resolve this conflict would be to delay putting the elements into A.)

Before describing the full construction, we will define all the parameters. Let $\Lambda_1 = \omega$ and $\Lambda_2 = 2$ be the sets of outcomes of the $\mathcal{N}$- and $\mathcal{P}$-strategies, respectively. Let $T_1 = (\Lambda_1 \times \Lambda_2)^{<\omega}$, $T_2 = (\Lambda_1 \times \Lambda_2)^{<\omega} \times \Lambda_1$, and let $T = T_1 \cup T_2$ be the tree of strategies. (T_1 and T_2 are the sets of even nodes ($\mathcal{N}$-strategies) and odd nodes ($\mathcal{P}$-strategies) of the tree T, respectively.) For each k, let $\{ W_{h(k,i)} \}_{i\in\omega}$ be a uniformly r.e. sequence of sets such that $k \in \emptyset^{(4)}$ iff $(\exists i)[W_{h(k,i)}$ coinfinite]. Without loss of generality, assume that $W_{h(k,i),s} \neq W_{h(k,i),s+1}$ for all k, i, s. The construction of $A = A_k$ will be controlled by markers $h^n_{i,s}$ where $\overline{W_{h(k,i),s}} = \{ h^0_{i,s} < h^1_{i,s} < h^2_{i,s} < \dots \}$.

Fix a recursive 1–1 enumeration f of V, and let $V_s = \{ f(0), f(1), f(2), \dots, f(s) \}$. Let $\tilde{W}_{e,s} = \{ x \in W_{e,s} \mid (\forall y < x)[y \in W_{e,s} \cup V_s] \}$, and let $\tilde{W}_e = \bigcup_s \tilde{W}_{e,s}$. Define the *e-states* $\sigma(e,x,s) = \{ e' \leq e \mid x \in \tilde{W}_{e',s} \}$, and $\sigma(e,x) = \lim_s \sigma(e,x,s)$. Denote the elements of the difference set $V - A$ by markers d^s_n so that $V_s - A_s = \{ d^s_0, d^s_1, d^s_2, d \dots, d^s_{n_s} \}$. The order of these markers will be determined by the construction, and markers d^s_n will be undefined for $n > n_s$.

Each $\mathcal{N}$-strategy $\alpha \in T_1$ builds its own set B_α, trying to disprove $A \subset_{\mathrm{m}} V$ by B_α. It has to take into account the action of stronger $\mathcal{P}$-strategies in building B_α and imposing restraint of A. So it will use

$$U_{\alpha,s} =_{\mathrm{def}} \Big(\Big(\bigcap_{\substack{2e' < |\alpha| \\ \alpha(2e'+1)=0}} \tilde{W}_{e,s}\Big) \cap V_s\Big) - (A_s \cup B_{\alpha,s}) \tag{28}$$

(instead of $V_s - (A_s \cup B_{\alpha,s})$ as in the basic module). Notice that $U_\alpha =^* V - (A \cup B_\alpha)$ if the $\mathcal{P}$-strategies above α succeed.

We define δ_s (with $|\delta_s| = 2s$), the *recursive approximation to the true path*, by induction:

(29)

$$\delta_s(2e) = \min\{\, n \mid h^n_{e,s} \neq h^n_{e,s'} \,\} \text{ where } s' = \max(\{\,0\,\} \cup \{\, t < s \mid \delta_s \upharpoonright 2e \subseteq \delta_t \,\}),$$

$$\delta_s(2e+1) = \begin{cases} 0 & \text{if } \tilde{W}_{e,s} \neq \tilde{W}_{e,s'} \text{ where} \\ & \quad s' = \max(\{\,0\,\} \cup \{\, t < s \mid \delta_s \upharpoonright (2e+1) \subseteq \delta_t \,\}), \\ 1 & \text{otherwise.} \end{cases}$$

For $\mathcal{P}$-strategies $\alpha = \beta^\frown\langle m\rangle \in T_2$, define the *restraint function* by:

$$r_s(\beta^\frown\langle m\rangle) = \begin{cases} \min\{\, r \mid |U_{\alpha,s} \cap [0,r)| = m & \text{if } \beta \subseteq \delta_s \text{ or } s = 0, \\ \qquad \vee\ r = 1 + \max(U_{\alpha,s}) \,\} & \\ r_{s-1}(\beta^\frown\langle m\rangle) & \text{otherwise.} \end{cases}$$

(Recall that restraint is imposed by $\mathcal{N}$-strategies $\beta \in T_1$, but the restraint that β imposes depends on $W_{h(k,i)}$ and thus differs below distinct outcomes m (the current guess for $|\overline{W_{h(k,i)}}|$) of β.)

For $\mathcal{P}$-strategies $\alpha \in T_2$, define the *control function* by:

(30)

$$Q_s(\alpha) = \begin{cases} \infty & \text{if } \alpha \subseteq \delta_s \text{ or } \alpha >_L \delta_s \text{ or } s = 0, \\ n & \text{if } \alpha <_L \delta_s \text{ and } \alpha \text{ moved } \Gamma_n \text{ at stage } s \text{ (as defined below)}, \\ Q_{s-1}(\alpha) & \text{otherwise.} \end{cases}$$

The construction of the r.e. set A and the r.e. sets B_α (for all $\alpha \in T_1$) now proceeds as follows:

At stage 0, let $A_0 = B_{\alpha,0} = \emptyset$ (for $\alpha \in T_1$), let $d^0_0 = f(0)$, and let d^0_n be undefined for all $n > 0$.

At a stage $s+1$, perform the following two steps:

For all $\mathcal{N}$-strategies $\alpha \in T_1$ with $\alpha \subseteq \delta_s$, put $\min(\overline{V_s \cup B_{\alpha,s}})$ into $B_{\alpha,s+1}$ if $|U_{\alpha,s}| > |B_{\alpha,s}|$.

Secondly, for the sake of the $\mathcal{P}$-strategies, choose n_0 to be the least $n \leq n_s$ such that

$$(31)\quad (\exists e \leq n)[\sigma(e-1, f(s+1), s) = \sigma(e-1, d_n^s, s) \wedge$$
$$f(s+1) \in \tilde{W}_{e,s} \wedge d_n^s \notin \tilde{W}_{e,s} \wedge$$
$$d_n^s > \max\{ r_s(\alpha) \mid \alpha \leq \gamma \wedge \alpha \in T_2 \}$$
$$(\text{where } \gamma \leq \delta_s \text{ is leftmost with } |\gamma| = 2e+1 \text{ and } Q_s(\gamma) > n)].$$

If n_0 exists then put $d_{n_0}^s$ into A_{s+1}, let $d_{n_0}^{s+1} = f(s+1)$, and let $d_n^{s+1} = d_n^s$ for $n \neq n_0$. (We say γ *moved* Γ_{n_0} *at stage* $s+1$.) Otherwise, let $d_{n_s+1}^{s+1} = f(s+1)$, and let $d_n^{s+1} = d_n^s$ for $n \neq n_s + 1$.

This concludes the construction.

LEMMA 1 (MARKER CONVERGENCE LEMMA). *For all n, $d_n = \lim_s d_n^s$ is defined. (Thus $A \subset_\infty V$.)*

PROOF: By induction on n: Suppose d_m is defined for all $m < n$, and $d_m^s = d_m$ for all $s \geq s_0$, say. Then d_n^s is defined for all $s > s_0$ and changes only finitely often since it increases its n-state each time (and the n-state is nondecreasing between these changes). ∎

LEMMA 2 (TRUE PATH EXISTENCE LEMMA). *If $W_{h(k,i)}$ is cofinite for all $i < i_0$, then $\alpha_0 = \liminf_s \delta_s \upharpoonright 2i_0$ exists.*

PROOF: By the definition of δ_s, we have for $i < i_0$:

$$(32)\qquad \begin{aligned} \alpha_0(2i) &= |\overline{W_{h(k,i)}}|, \\ \alpha_0(2i+1) &= \begin{cases} 0 & \text{if } \tilde{W}_i \text{ is infinite,} \\ 1 & \text{otherwise .} \end{cases} \end{aligned} \blacksquare$$

LEMMA 3 (OUTCOME LEMMA). *Fix i_0.*

(i) If $\alpha_0 = \liminf_s \delta_s \upharpoonright 2i_0$ exists, then $\overline{V} \subseteq B_{\alpha_0}$, and

$$(33)\quad \beta_0 = \alpha_0{}^\frown\langle m\rangle \wedge (\exists^{<\infty} s)[\delta_s <_L \beta_0] \rightarrow$$
$$(\forall \beta \in T_2)[\beta \leq \beta_0 \rightarrow r(\beta) = \liminf r_s(\beta) < \infty \text{ exists}] \wedge |U_{\alpha_0} \cap [0, r(\beta_0))| = m.$$

(ii) If $\gamma_0 = \liminf_s \delta_s \upharpoonright (2i_0+1)$ *exists, then either* $\tilde{W}_{i_0}$ *is finite (if* $\gamma_0{}^\frown\langle 1\rangle = \liminf_s \delta_s \upharpoonright (2i_0+1)$*) or* $\overline{A} \subseteq^* W_{i_0}$ *(if* $\gamma_0{}^\frown\langle 0\rangle = \liminf_s \delta_s \upharpoonright (2i_0+1)$*).*

PROOF: By simultaneous induction on i_0:

(i) We first establish $\overline{V} \subseteq B_{\alpha_0}$. By the construction, it suffices to show that B_{α_0} is infinite (since we always put $\min(\overline{V_s \cup B_{\alpha_0,s}})$ into B_{α_0}). Suppose for the sake of a contradiction that B_{α_0} is finite. Then for all s with $\alpha_0 \subseteq \delta_s$, $|U_{\alpha_0,s}| \leq |B_{\alpha_0,s}|$. But U_{α_0} is a difference of r.e. sets, so $|U_{\alpha_0}| \leq |B_{\alpha_0}|$. By (ii), $\overline{A} \subseteq^* \tilde{W}_i$ for $i < i_0$ with $\alpha_0(2i) = 0$, and therefore $U_{\alpha_0} =^* V - (A \cup B_{\alpha_0})$. But then $U_{\alpha_0} =^* V - A$ is finite, contradicting Lemma 1.

Let us now show (33). By induction on (i), choose s_0 such that

$$(\forall s \geq s_0)(\forall \alpha \in T_2)[\alpha \leq \alpha_0 \upharpoonright (2i_0 - 1) \rightarrow r_s(\alpha) = r(\alpha)].$$

(This assumption is vacuous for $i_0 = 0$.) Next, by our assumption on β_0 and the definition of $r_s(\beta)$, pick $s_1 \geq s_0$ such that

$$(\forall s \geq s_1)(\forall \beta \in T_2)[\beta < \beta_0 \wedge \beta \upharpoonright (|\beta| - 1) \neq \alpha_0 \rightarrow r_s(\beta) = r(\beta)].$$

Furthermore, since by the construction $Q_s(\beta)$ cannot increase while $\beta <_L \delta_s$, pick $s_2 \geq s_1$ such that

$$(\forall s \geq s_2)(\forall \beta \in T_2)[\beta <_L \beta_0 \rightarrow Q_s(\beta) = \lim_t Q_t(\beta)].$$

Finally, let $\sigma = \{\, i < i_0 \mid \tilde{W}_i \text{ infinite} \,\}$. Then by (ii),

$$(\exists n_0)(\forall n \geq n_0)[\sigma(i_0 - 1, d_n) = \sigma].$$

Pick $s_3 \geq s_2$ such that

$$(\forall s \geq s_3)(\forall n < n_0)[d_n^s = d_n].$$

We will now show (33) by induction on m (for fixed α_0). For $m = 0$, trivially $r(\beta_0) = 0$. Let $m > 0$. Let $r = 1 + \max(\{ r(\alpha_0{}^\frown\langle m-1\rangle) \} \cup \{ d_n \mid n < n_0 \})$. Pick $s_4 \geq s_3$ such that

$$(\forall s \geq s_4)[r_s(\alpha_0{}^\frown\langle m-1\rangle) = r(\alpha_0{}^\frown\langle m-1\rangle) \wedge$$
$$X_{s_4} \upharpoonright (r+1) = X \upharpoonright (r+1) \text{ for all } X = W_i \text{ (for } i < i_0\text{)}, V, A, \text{ and } B_{\alpha_0}].$$

By the first part of (i), we have $\limsup\{ |U_{\alpha_0,s}| \mid \alpha_0 \leq \delta_s \} = \infty$, so pick $s_5 \geq s_4$ such that $\alpha_0 \subseteq \delta_{s_5}$ and $|U_{\alpha_0,s_5}| \geq m$.

We claim that

$$(34) \qquad (\forall s \geq s_5)[r_s(\beta_0) \leq r_{s+1}(\beta_0) \wedge |U_{\alpha_0,s} \cap [0, r_s(\beta_0))| \geq m].$$

Suppose for the sake of a contradiction that for some $s \geq s_5$, $U_{\alpha_0,s} \cap [0, r_s(\beta_0)) \not\subseteq U_{\alpha_0,s+1} \cap [0, r_s(\beta_0))$. Then some $x \in U_{\alpha_0,s}$ entered B_{α_0} or A. The former is impossible by the construction of B_{α_0} (since $x \in V_s$). But x cannot enter A since:

(a) no $\gamma \geq \beta_0$ can move x by the restraint imposed;

(b) no $\gamma <_L \beta_0$ can move x, or else $Q_s(\gamma) > Q_{s+1}(\gamma)$, contradicting the assumption on s_2; and

(c) no $\gamma \subset \beta_0$ will move x since either $x \notin \tilde{W}_{i,s}$ (if $|\gamma| = 2i+1$ and $\beta_0(2i+1) = 0$), or γ no longer moves any element (if $|\gamma| = 2i+1$ and $\beta_0(2i+1) = 1$).

(Notice that $r_s(\beta_0)$ may still drop a finite number of times as U_{α_0} gets new small elements.)

Now (34) establishes (33).

(ii) By (i), pick s_0 such that

$$(\forall s \geq s_0)(\forall \gamma \in T_2)[\gamma \leq \gamma_0 \rightarrow r(\gamma) = r_s(\gamma)].$$

Let $R(\gamma_0) = \max\{ r(\gamma) \mid \gamma \leq \gamma_0 \wedge \gamma \in T_2 \}$. Since $\gamma \subseteq \delta_{s_0}$ for infinitely many s, we also have $\lim_s Q_s(\gamma_0) = \infty$. Let $\sigma = \{ i \leq i_0 \mid \tilde{W}_i \text{ infinite} \}$, and assume that $\tilde{W}_{i_0}$ is infinite. Then $\tilde{W}_\sigma = \bigcap_{i \in \sigma} \tilde{W}_i \supseteq \overline{V}$. By induction on (ii), pick $n_0 > i_0$ such that

$$(\forall n \geq n_0)[\sigma(i_0 - 1, d_n) = \sigma - \{ i_0 \}].$$

Since V is not recursive,

(35) $$(\exists^\infty s)[f(s+1) \in \tilde{W}_{\sigma,s}].$$

Suppose that $\sigma(i_0, d_n) = \sigma - \{ i_0 \}$ for some $n \geq n_0$ with $d_n > R(\gamma_0)$. Pick $s_1 \geq s_0$ such that

$$(\forall s \geq s_1)\big[Q_s(\gamma_0) > n \wedge (\forall n' \leq n)[d^s_{n'} = d_{n'}]\big].$$

Then d_n will be moved by (35), contradicting our assumption. Thus $W_{i_0} = \tilde{W}_{i_0} \supseteq^* \overline{A}$. ∎

It is now easy to see that the lemmas imply Theorem C.

First suppose that $k \in \emptyset^{(4)}$. Then $W_{h(k,i_0)}$ is coinfinite for some (least) i_0. By Lemma 2, $\alpha_0 = \liminf_s \delta_s \restriction 2i_0$ exists, and

$$(\forall m)(\exists^{<\infty} s)[\delta_s <_L \alpha_0{}^\frown\langle m \rangle].$$

Therefore, by Lemma 3 (i), $\overline{V} \subseteq B_{\alpha_0}$, and U_{α_0} is infinite. But then $V - (A \cup B_{\alpha_0})$ is infinite, so B_{α_0} witnesses that $A \not\subset_{\mathrm{m}} V$.

On the other hand, assume that $k \notin \emptyset^{(4)}$. Then $W_{h(k,i)}$ is cofinite for all i. By Lemma 2, $\liminf_s \delta_s \restriction 2i$ exists for all i. Therefore, by Lemma 3 (ii), either $\tilde{W}_i$ is finite or $\overline{A} \subseteq^* \tilde{W}_i = W_i$ for all i. Furthermore, by Lemma 1, $A \subset_\infty V$. Thus $A \subset_{\mathrm{m}} V$.

This concludes the proof of Theorem C. ∎

CHAPTER IV

ω-DEGREES

Jockusch, Lerman, Soare, and Solovay [JLSSta] defined a new partial order $\leq_\omega$ on the r.e. degrees. This partial order can easily be extended to all Turing degrees and induces equivalence classes of r.e. (or Turing) degrees, called *ω-degrees.*

DEFINITION (Jockusch, Lerman, Soare, Solovay [JLSSta]): Let $\mathbf{a}$ and $\mathbf{b}$ be r.e. (or Turing) degrees.

(i) The *partial order* $\leq_\omega$ is defined by

$$\mathbf{a} \leq_\omega \mathbf{b} \leftrightarrow (\exists n)[\mathbf{a}^{(n)} \leq \mathbf{b}^{(n)}]$$

(where $\leq$ is the usual Turing reducibility of degrees).

(ii) The induced *equivalence relation* $\sim_\omega$ is defined by

$$\mathbf{a} \sim_\omega \mathbf{b} \leftrightarrow \mathbf{a} \leq_\omega \mathbf{b} \wedge \mathbf{b} \leq_\omega \mathbf{a} \leftrightarrow (\exists n)[\mathbf{a}^{(n)} = \mathbf{b}^{(n)}].$$

The equivalence classes of $\sim_\omega$ are called *ω-degrees* and are denoted by $[\mathbf{a}], [\mathbf{b}], \ldots$ (or $[A], [B], \ldots$ where $A \in \mathbf{a}$, $B \in \mathbf{b}, \ldots$).

The study of the structure of the ω-degrees is interesting both in its own right and because it leads to the study of the decidability of fragments of the theory of the r.e. (or Turing) degrees with jump.

Jockusch, Lerman, Soare, and Solovay showed in their paper, among other things, that the r.e. ω-degrees are dense and indeed allow an independent set in any interval. We will show below the existence of a splitting of $[\emptyset']$ and of a minimal pair. Furthermore, we will show the surprising fact that the Turing ω-degrees do not form an upper semilattice. It is still open whether this is true also for the r.e. ω-degrees.

1. SPLITTING AND MINIMAL PAIR IN THE R.E. ω-DEGREES

We begin by recalling three important theorems.

SACKS SPLITTING THEOREM (Sacks [Sa63a]). *Let $A >_T \emptyset$ be an r.e. set. Then there are low r.e. sets A_1 and A_2 such that $A_1 \sqcup A_2 = A$ and $A_1 \mid_T A_2$. In particular,* $\deg(A_1) \cup \deg(A_2) = \deg(A)$ *and $A_1' \equiv_T A_2' \equiv_T \emptyset'$. Furthermore, indices for A_1 and A_2 can be found uniformly in the index for A.*

We will use this theorem only for $A = \emptyset'$.

THEOREM (Lachlan [La66]). *There is a minimal pair of high r.e. sets A and B, i.e., such that* $\deg(A) \cap \deg(B) = \mathbf{0}$ *and $A' \equiv_T B' \equiv_T \emptyset''$.*

(Notice here that A and B are constructed by thickness strategies. Therefore the reductions from A' or B' to $\emptyset''$ are the same in all relativizations of this theorem.)

ROBINSON JUMP INTERPOLATION THEOREM (R.W. Robinson [Ro71]). *Let $C <_T D$ be r.e. sets, let $n > 0$, and let S be such that $C^{(n)} \leq_T S \leq_1 D^{(n)}$. Then there is an r.e. set A such that $C <_T A <_T D$ and $A^{(n)} \equiv_T S$. Furthermore, the index for A can be found uniformly in n and indices for C, D, and the reduction $S \leq_1 D^{(n)}$.*

All three theorems relativize uniformly to an arbitrary oracle $X \subseteq \omega$.

Given two r.e. sets A and B, the supremum and infimum of their ω-degrees (if they exist) can be characterized as follows:

LEMMA. *Let $A, B \subseteq \omega$. Then:*

(i) $[C]$ *(for some $C \subseteq \omega$) is the supremum of $[A]$ and $[B]$, written $[A] \cup [B]$, iff*

$$(\forall n)(\exists m)[(A^{(n)} \oplus B^{(n)})^{(m)} \equiv_T C^{(n+m)}].$$

(ii) $[D]$ *(for some $D \subseteq \omega$) is the infimum of $[A]$ and $[B]$, written $[A] \cap [B]$, if*

$$(\forall n)(\exists n_0 \geq n)(\exists m)[\deg(A^{(n_0)}) \cap \deg(B^{(n_0)}) \text{ exists and } (\deg(A^{(n_0)}) \cap \deg(B^{(n_0)}))^{(m)} \equiv_T D^{(n_0+m)}].$$

(Notice that, in (ii), we only claim one direction of the implication.)

PROOF: (i)

$$\begin{aligned}
&[A] \cup [B] = [C] \\
&\leftrightarrow A, B \leq_\omega C \wedge (\forall X)[A, B \leq_\omega X \rightarrow C \leq_\omega X] \\
&\leftrightarrow (\exists n)[A^{(n)}, B^{(n)} \leq_T C^{(n)}] \wedge \\
&\qquad (\forall X)\left[(\exists n)[A^{(n)}, B^{(n)} \leq_T X^{(n)}] \rightarrow (\exists m)[C^{(m)} \leq_T X^{(m)}]\right] \\
&\leftrightarrow (\exists n)[A^{(n)} \oplus B^{(n)} \leq_T C^{(n)}] \wedge \\
&\qquad (\forall X)(\forall n)(\exists m)[A^{(n)} \oplus B^{(n)} \leq_T X^{(n)} \rightarrow C^{(m)} \leq_T X^{(m)}] \\
&\leftrightarrow (\forall n)(\exists m)[(A^{(n)} \oplus B^{(n)})^{(m)} \equiv_T C^{(n+m)}].
\end{aligned}$$

(ii) Similar to (i). ∎

We can now prove the first two theorems.

THEOREM A. *$[\emptyset']$ splits in the r.e. ω-degrees, i.e., there are r.e. sets $A, B <_\omega \emptyset'$ such that $[A] \cup [B] = [\emptyset']$.*

PROOF: Using the Sacks Splitting Theorem, find indices j_0 and k_0 such that, for all $X \subseteq \omega$,

(1A) $$W_{j_0}^X \geq_T X \text{ and } W_{k_0}^X \geq_T X,$$

(1B) $$(W_{j_0}^X)' \equiv_T (W_{k_0}^X)' \equiv_T X' \equiv_T W_{j_0}^X \oplus W_{k_0}^X.$$

Using the Robinson Jump Interpolation Theorem, find recursive functions f and g such that, for all $X \subseteq \omega$ and all indices e and i,

(2A) $$(W_{f(e,i)}^X)' \equiv_T W_e^{X'} \text{ and } (W_{g(e,i)}^X)' \equiv_T W_i^{X'},$$

(2B) $$W_{j_0}^X <_T W_{f(e,i)}^X <_T X' \text{ and } W_{k_0}^X <_T W_{g(e,i)}^X <_T X'.$$

By the Double Recursion Theorem, find indices e_0 and i_0 such that, for all $X \subseteq \omega$,

(3) $$W_{f(e_0,i_0)}^X = W_{e_0}^X \text{ and } W_{g(e_0,i_0)}^X = W_{i_0}^X.$$

Now let $A = W_{e_0}^{\emptyset}$ and $B = W_{i_0}^{\emptyset}$. Then, for all n,

(4A) $$A^{(n)} = (W_{e_0}^{\emptyset})^{(n)} \equiv_T W_{e_0}^{\emptyset^{(n)}} <_T \emptyset^{(n+1)},$$

(4B) $$B^{(n)} = (W_{i_0}^{\emptyset})^{(n)} \equiv_T W_{i_0}^{\emptyset^{(n)}} <_T \emptyset^{(n+1)},$$

(4C) $$A^{(n)} \oplus B^{(n)} \equiv_T W_{e_0}^{\emptyset^{(n)}} \oplus W_{i_0}^{\emptyset^{(n)}} \geq_T W_{j_0}^{\emptyset^{(n)}} \oplus W_{k_0}^{\emptyset^{(n)}} \equiv_T \emptyset^{(n+1)}.$$

This establishes the claim by the lemma. ∎

THEOREM B. *There is a minimal pair in the r.e. ω-degrees, i.e., there are r.e. sets $A, B >_\omega \emptyset$ such that $[A] \cap [B] = [\emptyset]$.*

PROOF: Using Lachlan's Theorem stated above, find indices j_0 and k_0 such that, for all $X \subseteq \omega$,

(5A) $$W_{j_0}^X \geq_T X \text{ and } W_{k_0}^X \geq_T X,$$

(5B) $$(W_{j_0}^X)' \equiv_T (W_{k_0}^X)' \equiv_T X'',$$

(5C) $$\deg(W_{j_0}^X) \cap \deg(W_{k_0}^X) = \deg(X).$$

Notice that (5B) is equivalent to

(6) $$(W_{j_0}^X)'' \equiv_1 (W_{k_0}^X)'' \equiv_1 X'''.$$

Using the uniformity of the reduction in (6) and the Robinson Jump Interpolation Theorem, find recursive functions f and g such that, for all $X \subseteq \omega$ and all indices e and i,

(7A) $$(W_{f(e,i)}^X)'' \equiv_T W_e^{X''} \text{ and } (W_{g(e,i)}^X)'' \equiv_T W_i^{X''},$$

(7B) $$X <_T W_{f(e,i)}^X <_T W_{j_0}^X \text{ and } X <_T W_{g(e,i)}^X <_T W_{k_0}^X.$$

By the Double Recursion Theorem, find indices e_0 and i_0 such that, for all $X \subseteq \omega$,

(8) $$W_{f(e_0,i_0)}^X = W_{e_0}^X \text{ and } W_{g(e_0,i_0)}^X = W_{i_0}^X.$$

Now let $A = W^{\emptyset}_{e_0}$ and $B = W^{\emptyset}_{i_0}$. Then, for all n,

(9A) $$A^{(2n)} = (W^{\emptyset}_{e_0})^{(2n)} \equiv_T W^{\emptyset^{(2n)}}_{e_0} >_T \emptyset^{(2n)},$$

(9B) $$B^{(2n)} = (W^{\emptyset}_{i_0})^{(2n)} \equiv_T W^{\emptyset^{(2n)}}_{i_0} >_T \emptyset^{(2n)},$$

(9C) $$\deg(A^{(2n)}) \cap \deg(B^{(2n)}) = \deg(W^{\emptyset^{(2n)}}_{e_0}) \cap \deg(W^{\emptyset^{(2n)}}_{i_0}) \leq_T \deg(W^{\emptyset^{(2n)}}_{j_0}) \cap \deg(W^{\emptyset^{(2n)}}_{k_0}) \equiv_T \emptyset^{(2n)}.$$

This establishes the claim by the lemma. ∎

2. No Join in the ω-Degrees Below $\emptyset^{(\omega)}$

The following theorem shows that the ω-degrees below $\emptyset^{(\omega)}$ do not form an upper semilattice:

THEOREM C. *There are sets $A, B \leq_T \emptyset^{(\omega)}$ such that, for all n and m,*

$$(A^{(n+1)} \oplus B^{(n+1)})^{(m)} <_T (A^{(n)} \oplus B^{(n)})^{(m+1)}.$$

COROLLARY. *There are sets $A, B \leq_T \emptyset^{(\omega)}$ such that $[A] \cup [B]$ does not exist.*

PROOF: For $X \geq_T \emptyset^{(m)}$, let

$$[X]_m = \{\, Y \subseteq \omega \mid (\exists n)[Y^{(n+m)} \equiv_T X^{(n)}]\,\}.$$

Then for A and B as in the theorem,

$$[A \oplus B] > [A' \oplus B']_1 > [A'' \oplus B'']_2 > \dots$$

This establishes the claim by the lemma. ∎

PROOF OF THEOREM C: We use Cohen forcing and refer to Lerman [**Le83**] for the background.

The idea of the proof is to put more information into the join of A and B than A and B can compute individually. We define the *symmetric difference* of A and B as $A \bigtriangleup B = (A - B) \cup (B - A)$. The *coded difference* of A and B is defined by:

(10) $$(A \bigtriangledown B)(i) = \begin{cases} 1 & \text{if the } i\text{th element of } A \bigtriangleup B \text{ is in } A - B, \\ 0 & \text{if the } i\text{th element of } A \bigtriangleup B \text{ is in } B - A. \end{cases}$$

It is now easy to force, for example, that A or B cannot alone compute $A \bigtriangledown B$. The jump is handled using the limit theorem.

We will build sets $A, B \leq_T \emptyset^{(\omega)}$ and a sequence of sets $\{S_i\}_{i\in\omega-\{0\}}$ uniformly below $\emptyset^{(\omega)}$ which will satisfy the following requirements:

(i) for all n, $A^{(n)} \equiv_T A \oplus \emptyset^{(n)}$ and $B^{(n)} \equiv_T B \oplus \emptyset^{(n)}$;

(ii) for all n and m, $(A^{(n)} \oplus B^{(n)})^{(m)} \equiv_T A \oplus B \oplus S_1 \oplus S_2 \oplus \cdots \oplus S_m \oplus \emptyset^{(n+m)}$;

(iii) for all n and m, $S_m \not\leq_T A \oplus B \oplus S_1 \oplus S_2 \oplus \cdots \oplus S_{m-1} \oplus \emptyset^{(n)}$.

Then A and B will satisfy the claim of the theorem:

$$\begin{aligned}(A^{(n+1)} \oplus B^{(n+1)})^{(m)} &\equiv_T A \oplus B \oplus S_1 \oplus S_2 \oplus \cdots \oplus S_m \oplus \emptyset^{(n+m+1)} <_T \\ A \oplus B \oplus S_1 \oplus S_2 \oplus \cdots \oplus S_{m+1} \oplus \emptyset^{(n+m+1)} &\equiv_T (A^{(n)} \oplus B^{(n)})^{(m+1)}.\end{aligned} \tag{11}$$

We will thus construct through forcing a generic G of the form $A \times B \times (\prod_{i>0} S_i)$ where

$$\begin{aligned} S_i &= T_i \times M_i \text{ for } i > 0, \\ T_0 &= A \bigtriangledown B, \\ T_i(x) &= \lim_s T_{i-1}(\langle x, s\rangle) \text{ for } i > 0, \text{ and} \\ M_i(x) &= \mu s\big[(\forall t \geq s)[T_{i-1}(\langle x, t\rangle) = T_i(x)]\big] \text{ for } i > 0. \end{aligned} \tag{12}$$

I.e., M_i is a modulus function for T_i relative to T_{i-1}. Our set of forcing conditions can only contain finite initial segments of possible generics that observe the modulus function to be correct.

We can thus formally define our set of forcing conditions P to consist of all ordered tuples $(\rho, \sigma, \tau_1, \tau_2, \ldots, \tau_n)$ satisfying the following conditions:

$$\rho, \sigma \in 2^{<\omega}; |\rho| = |\sigma|; \tag{13A}$$

$$n \in \omega; (\forall i < n)[\tau_{i+1} \in (2 \times \omega)^{<\omega}]; \tag{13B}$$

$$(\forall i < n)(\forall x)(\forall s \geq (\tau_{i+1}(x))_2) \left[(\tau_{i+1}(x))_1 = \begin{cases} (\tau_i(\langle x, s\rangle))_1 & \text{if } i > 0, \\ (\rho \bigtriangledown \sigma)(\langle x, s\rangle) & \text{if } i = 0, \end{cases}\right] \tag{13C}$$

(where (13C) is only required to hold for strings for which both sides of the equation are defined).

Now let G be a generic filter through P in the sense of set forcing (i.e., G is a generic filter meeting all dense Σ_n-sets for all $n \in \omega$). Define A, B, and the S_i by

$$G = A \times B \times (\prod_{i>0} S_i).$$

G is a total characteristic function for all these sets by the usual forcing argument. Furthermore, G can be built recursively in $\emptyset^{(\omega)}$.

By the usual forcing machinery, it suffices to verify that the requirements (i)-(iii) correspond to dense subsets of P.

(i) By induction on n: Given $(\rho, \sigma, \tau_1, \tau_2, \ldots, \tau_n) \in P$, we want to force $\{e\}^{A^{(n)}}(e)$, i.e., $\{e\}^{A\oplus\emptyset^{(n)}}(e)$. This can be done by extending ρ to ρ' if $\{e\}^{\rho'\oplus\emptyset^{(n)}}(e)\downarrow$; otherwise, let $\rho' = \rho$; extend σ to σ' by letting $\sigma'(x) = \rho'(x)$ for $x \geq |\sigma|$. This will not affect the S_i's. Thus $A \oplus \emptyset^{(n+1)}$ can compute $A^{(n+1)}$. (B is handled analogously.)

(ii) By induction on m: For $m = 0$, this follows from (i). Fix m and establish (ii) for $m+1$ as follows: Given $(\rho, \sigma, \tau_1, \tau_2, \ldots, \tau_n) \in P$, we want to force $\{e\}^{(A^{(n)}\oplus B^{(n)})^{(m)}}(e)$, i.e., $\{e\}^{A\oplus B\oplus S_1\oplus S_2\oplus\cdots\oplus S_m\oplus\emptyset^{(n+m)}}(e)$. Assume $n > m$ for convenience. This can be done by extending ρ to ρ', σ to σ', and each τ_i to τ_i' such that $(\rho', \sigma', \tau_1', \tau_2', \ldots, \tau_n') \in P$ if this achieves $\{e\}^{\rho\oplus\sigma\oplus\tau_1'\oplus\tau_2'\oplus\cdots\oplus\tau_m'\oplus\emptyset^{(n+m)}}(e)\downarrow$; otherwise, we let the strings be as before. Thus $A \oplus B \oplus S_1 \oplus S_2 \oplus \cdots \oplus S_{m+1} \oplus \emptyset^{(n+m+1)}$ can compute $(A^{(n)} \oplus B^{(n)})^{(m+1)}$. (Notice that S_{m+1} is needed here because τ_m cannot be extended arbitrarily by the definition of P.)

(iii) Suppose we are given $(\rho, \sigma, \tau_1, \tau_2, \ldots, \tau_l) \in P$, and we want to force $S_m \neq \{e\}^{A\oplus B\oplus S_1\oplus S_2\oplus\cdots\oplus S_{m-1}\oplus\emptyset^{(n)}}$. (Assume $l \geq m$ for convenience.) This can be done by extending ρ to ρ', σ to σ', and each τ_i to τ_i' (for $i \leq m$) if this achieves

$$(\exists x)[\tau_m(x)\downarrow \neq \{e\}^{\rho\oplus\sigma\oplus\tau_1\oplus\tau_2\oplus\cdots\oplus\tau_{m-1}\oplus\emptyset^{(n)}}(x)\downarrow].$$

Otherwise, we do not extend any string. In that case, we argue that

$\{e\}^{A \oplus B \oplus S_1 \oplus S_2 \oplus \cdots \oplus S_{m-1} \oplus \emptyset^{(n)}}$, if total, is recursive in $\emptyset^{(n)}$ and thus unequal S_m (since again we force $S_m \neq \{e\}^{\emptyset^{(n)}}$ for all n).

This concludes the proof of properties (i)-(iii) and thus the proof of the theorem. ∎

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