STABILITY AND POSETS

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Abstract. Hirschfeldt and Shore have introduced a notion of stability for infinite posets. We define an arguably more natural notion called weak stability, and we study the existence of infinite computable or low chains or antichains, and of infinite Π^0_1 chains and antichains, in infinite computable stable and weakly stable posets. For example, we extend a result of Hirschfeldt and Shore to show that every infinite computable weakly stable poset contains either an infinite low chain or an infinite computable antichain. Our hardest result is that there is an infinite computable weakly stable poset with no infinite Π^0_1 chains or antichains. On the other hand, it is easily seen that every infinite computable stable poset contains an infinite computable chain or an infinite Π^0_1 antichain. In Reverse Mathematics, we show that SCAC, the principle that every infinite stable poset contains an infinite chain or antichain, is equivalent over RCA_0 to WSCAC, the corresponding principle for weakly stable posets.

1. Introduction

If $P \subseteq \omega$ is an infinite set, let $[P]^n$ denote the set of all n-element subsets of P. A k-coloring of $[P]^2$ is called *stable* if for each $a \in P$ there is a color c_a such that the pair $\{a, b\}$ has color c_a for all but finitely many $b \in P$. Stability for 2-colorings of pairs was introduced by Hummel [7, Definition 3.5] and has played a major role in investigations of the effective content and logical strength of Ramsey's Theorem for pairs (see [8], [1], and [5] for example). In [6], Hirschfeldt and Shore introduced corresponding notions of stability for several other combinatorial principles and used them to investigate the effective content and logical strength of those principles. One of those principles was CAC, which is the statement that every infinite partially ordered set (or

2000 *Mathematics Subject Classification.* Primary: 03C57; Secondary: 03D45, 06A06.

The first author thanks all the other authors for their hospitality on his visits to their respective institutions. The third author's research was partially supported by NSF grants DMS-0140120 and DMS-0555381. All the authors would like to thank the anonymous referee for a very careful reading of their paper.

poset) contains either an infinite chain (set of pairwise comparable elements) or an infinite antichain (set of pairwise incomparable elements; they are allowed to be compatible). Because our goal is to study the effective content and logical strength of principles similar to CAC, we restrict our attention to posets with domains $\subseteq \omega$. Note that CAC is a direct consequence of Ramsey's theorem for 2-colorings of pairs RT^2_2. (Given a poset, color a pair of its elements red if its elements are comparable and blue otherwise, and note that the homogeneous sets are the chains and antichains. We call this coloring the *comparability coloring*.) By the argument just given, RT^2_2 implies CAC in RCA_0 (the standard base system for Reverse Mathematics [11]). Hirschfeldt and Shore [6, Corollary 3.12] answered Question 13.8 from [1] by showing that CAC is strictly weaker than RT^2_2 over RCA_0. To prove this result, they introduced a concept of stability for posets and made crucial use of it in their proof. We introduce a notion of weak stability for posets which corresponds more closely to stability for colorings and study the complexity of infinite chains and antichains in infinite computable stable and weakly stable posets. We now give the definitions of stability and weak stability for posets.

Definition 1.1. Fix an infinite poset $\mathcal{P} = (P, <_{\mathcal{P}})$.

(1) We define an element $a \in P$ to be
  - *small* if $a <_{\mathcal{P}} b$ for all but finitely many $b \in P$;
  - *large* if $b <_{\mathcal{P}} a$ for all but finitely many $b \in P$; and
  - *isolated* if a is $\mathcal{P}$-incomparable with all but finitely many $b \in P$.

(2) A poset $\mathcal{P}$ is
  - *weakly stable* if all elements of P are small, large, or isolated; and
  - (Hirschfeldt-Shore [6, Definition 3.2]) *stable* if all elements of P are small or isolated; or all elements of P are large or isolated.

For any infinite poset $\mathcal{P} = (P, <_{\mathcal{P}})$, let $S_{\mathcal{P}}, L_{\mathcal{P}}, I_{\mathcal{P}}$ denote, respectively, the set of small, large, and isolated elements of $\mathcal{P}$. Thus, $\mathcal{P}$ is weakly stable iff $S_{\mathcal{P}} \cup L_{\mathcal{P}} \cup I_{\mathcal{P}} = P$. Also, $\mathcal{P}$ is stable iff $\mathcal{P}$ is weakly stable and either $S_{\mathcal{P}}$ or $L_{\mathcal{P}}$ is empty. When no confusion is possible, we may write S for $S_{\mathcal{P}}$ and similarly for L and I.

A major advantage of computable stable colorings of pairs is that they are closely related to Δ^0_2 sets. Specifically, by [1, Lemma 3.5], for every computable stable coloring f of $[\omega]^2$, there is an infinite Δ^0_2 set A such that every infinite homogeneous set for f is contained in A or $\overline{A}$, and for every infinite set B contained in A or $\overline{A}$ there is an infinite homogeneous set $H \subseteq B$ such that $H \leq_T B$. The following proposition expresses this in the context of weakly stable posets.

Proposition 1.2. *Let $\mathcal{P}$ be an infinite computable weakly stable poset. Then the sets S, L, and I are all Δ^0_2. Also, every infinite chain for $\mathcal{P}$ is contained in $S \cup L$, and every infinite antichain for $\mathcal{P}$ is contained in I. Finally, every infinite subset X of $S \cup L$ contains an infinite subset $C \leq_T X$ which is a chain, and every infinite subset Y of I contains an infinite subset $A \leq_T Y$ which is an antichain.*

Proof. This follows from the proof of [1, Lemma 3.5] by considering the comparability coloring for $\mathcal{P}$, which is easily seen to be a stable coloring. To illustrate this proof, let X be an infinite subset of $S \cup L$. Define the sequence $\{c_n\}$ recursively by letting c_n be the least number $c \in X$ such that $c_i < c$ for all $i < n$ and c is $\mathcal{P}$-comparable with c_i for all $i < n$. Let $C = \{c_n : n \in \omega\}$. Then, as desired, $C \leq_T X$, $C \subseteq X$, and C is an infinite chain. □

In particular, it follows from this that every infinite computable weakly stable poset contains an infinite Δ^0_2 chain or antichain. We will consider refinements of this result involving infinite computable, low, and Π^0_1 chains and antichains.

Obviously, every stable ordering is weakly stable, and easy examples in the next paragraph show that the converse fails, even for linear orderings.

Note that an infinite linear ordering is stable iff it has order type ω or ω^*, and it is weakly stable iff it has order type $\omega + \omega^*$, $\omega + n$, or $n + \omega^*$ for some $n \in \omega$.

The preceding paragraph shows that weak stability for a poset is not equivalent to stability of the comparability coloring for the poset. (Take the order $\omega^* + \omega$ for a very simple counterexample.) On the other hand, the following result shows that these properties are equivalent for a wide class of infinite computable posets.

Proposition 1.3. *Let $\mathcal{P}$ be an infinite computable poset with no infinite computable chain. Then $\mathcal{P}$ is weakly stable iff the comparability coloring for $\mathcal{P}$ is a stable coloring.*

Proof. It is obvious that if $\mathcal{P}$ is weakly stable, then the comparability coloring for $\mathcal{P}$ is stable. To prove the converse, assume for the sake of a contradiction that for every $a \in P$, a is comparable with all but finitely many $b \in P$, or incomparable with all but finitely many $b \in P$ but that $\mathcal{P}$ is not weakly stable. Fix an element $a \in P$ which is neither small nor large nor isolated. Thus there are infinitely many elements of P above a and also infinitely many elements of P below a. Let C be the set of elements comparable with a and note that C is an infinite computable set in which (by the transitivity of $<_{\mathcal{P}}$) every element has the "comparable" limit color in the comparability coloring. By [1, proof of Lemma 3.5], there is an infinite chain $C' \subseteq C$ such that $C' \leq_T C$. Thus $\mathcal{P}$ contains an infinite computable chain. □

This proposition fails for stable posets. Let $\mathcal{P}$ be an infinite computable stable poset with no infinite computable chains. (Such a poset exists by Theorem 4.2.) Let $\mathcal{P}'$ be the poset $\mathcal{P}$ with an added greatest element $1_{\mathcal{P}'}$ and least element $0_{\mathcal{P}'}$. $\mathcal{P}'$ contains no infinite computable chains and is weakly stable, so the comparability coloring for $\mathcal{P}'$ is stable. However, $\mathcal{P}'$ is not a stable poset since $0_{\mathcal{P}'} \in S_{\mathcal{P}'}$ and $1_{\mathcal{P}'} \in L_{\mathcal{P}'}$.

Our goals in this paper are to study the strength of the assertion WSCAC that every infinite weakly stable poset contains an infinite chain or antichain and to study the complexity of infinite chains and antichains in infinite computable stable and weakly stable posets.

In Section 2 we make progress towards the first goal by showing that WSCAC is equivalent over RCA_0 to SCAC, the corresponding statement for stable posets. The statement SCAC was introduced by Hirschfeldt and Shore [6], and the numerous results they obtained on its strength now carry over immediately to WSCAC.

Section 3 is devoted to the study of infinite computable and low chains and antichains in infinite computable posets. Hirschfeldt and Shore [6, proof of Theorem 3.4] showed that every infinite computable stable poset contains an infinite low chain or an infinite computable antichain. We extend this result from stable to weakly stable posets by a double application of the construction used in Section 2 to show that SCAC implies WSCAC in RCA_0. We also show that every infinite computable stable poset contains an infinite computable chain or an infinite low antichain, but we leave open whether this result extends to weakly stable posets.

In Section 4 we study infinite Π^0_1 chains and antichains. We start by observing that every infinite computable stable poset contains an infinite computable chain or an infinite Π^0_1 antichain. We then show that the "dual" of this result fails, i.e., there is an infinite computable poset with no infinite Π^0_1 chain or infinite computable antichain. This lack of duality is apparently new. This result is proved by a priority argument in which the requirements dealing with chains may act infinitely often, and yet all requirements are injured only finitely often. Finally, as our main result, we show that there is an infinite computable weakly stable poset with no infinite Π^0_1 chains or antichains. This result contrasts with the stable case and also with our results on infinite low chains and antichains. It is proved with a priority argument in which all requirements can act infinitely often and yet are injured only finitely often.

We complete this introduction by surveying some known results on the complexity of chains and antichains in computable posets which are not necessarily stable. First, every infinite computable poset contains an infinite Π^0_2 chain or antichain. This follows from the corresponding result in effective Ramsey theory [9, Theorem 4.2] via the comparability coloring. This is best possible for the arithmetical hierarchy, since Herrmann [4] showed that there is an infinite computable poset with no infinite Σ^0_2 chains or antichains. (This result of Herrmann's was far more difficult than the corresponding negative result in effective Ramsey theory [9, Corollary 3.2].) For the high-low hierarchy, it is known that every infinite computable poset contains an infinite low_2 chain or antichain. This follows from the corresponding result in effective Ramsey theory, due to Cholak, Jockusch, and Slaman, [1, Theorem 3.1], via the comparability coloring. Again, Herrmann's result shows that this is best possible since there are infinite computable posets with no infinite low chains or antichains.

The complexity bounds become much higher if one considers only chains, or only antichains. It was shown by Harizanov, Jockusch, and Knight [3, Theorem 1.1] that there is an infinite computable poset which contains an infinite chain but none which is Σ^1_1 or Π^1_1, and the corresponding result for antichains was proved in Theorem 1.4 of the same paper. In Theorem 1.2 of that paper it was shown that every infinite computable poset which contains an infinite chain contains one which is a difference of Π^1_1 sets. The corresponding result for antichains is open, though by [3, Remark 1.3] every infinite computable poset which contains an infinite antichain contains one which is truth-table reducible to a Π^1_1 set. These bounds can be greatly improved for weakly stable posets. In fact, it follows easily from Proposition 1.2 that every computable weakly stable poset which has an infinite chain has one which is Δ^0_2, and the analogous result for antichains follows likewise.

2. An equivalence result

Hirschfeldt and Shore [6] analyzed the principle CAC defined below. In particular they showed in [6, Corollary 3.12] that CAC is strictly weaker than RT^2_2 (Ramsey's Theorem for pairs) over RCA_0. Stable posets played a major role in their proof, and they also analyzed the strength of the statement that every infinite stable poset contains an infinite chain or antichain. In this section we show that this principle is equivalent over RCA_0 to the corresponding statement for weakly stable posets.

Definition 2.1.
- CAC is the principle "Every infinite poset contains an infinite chain or antichain."
- WSCAC is the principle "Every infinite weakly stable poset contains an infinite chain or antichain."
- SCAC is the principle "Every infinite stable poset contains an infinite chain or antichain."

Clearly, over RCA_0, CAC implies WSCAC, which in turn implies SCAC. Hirschfeldt and Shore [6, Proposition 3.1 and Corollary 3.6] have shown that SCAC does not imply CAC over RCA_0 (since SCAC has an ω-model containing only sets of low degree, whereas CAC does not by Herrmann [4]). We resolve the question of how WSCAC fits in by showing that WSCAC is equivalent to SCAC over RCA_0. The trick used to prove this theorem will also be useful in Section 3.

Theorem 2.2 (Jockusch, Kastermans, and Lempp)**.** *Over* RCA_0*, the principles* WSCAC *and* SCAC *are equivalent.*

Proof. We need only show that SCAC implies WSCAC. We reason in RCA_0. Consider a weakly stable poset $\mathcal{P} = (P, <_{\mathcal{P}})$. Define a new partial ordering $\mathcal{Q} = (P, <_{\mathcal{Q}})$ by $a <_{\mathcal{Q}} b$ iff $a <_{\mathcal{P}} b$ and $a < b$. It is easy to check that $a \in \omega$ is $\mathcal{Q}$-small iff a is $\mathcal{P}$-small; and a is $\mathcal{Q}$-isolated iff a is $\mathcal{P}$-isolated or $\mathcal{P}$-large. Thus, $\mathcal{Q}$ is a stable poset. By SCAC, $\mathcal{Q}$ contains either an infinite chain C or an infinite antichain A. In the first case, C is also a $\mathcal{P}$-chain. In the other case, A consists only of $\mathcal{Q}$-isolated elements, i.e., only of $\mathcal{P}$-isolated or $\mathcal{P}$-large elements. Thus A with the ordering induced by $<_{\mathcal{P}}$ forms a stable poset, to which we can apply SCAC again. □

Let DNR be the assertion that for every set X there is a function f which is X-DNR, i.e., $(\forall e)[f(e) \neq \Phi_e^X(e)]$. Let SRT^2_2 be the assertion that every stable 2-coloring of pairs has an infinite homogeneous set. Let COH be the assertion that for every sequence of sets $R_0, R_1, \ldots$, there is an infinite set C such that, for all i, C has finite intersection with R_i or the complement of R_i.

Corollary 2.3 (Jockusch, Kastermans, and Lempp)**.** *In* RCA_0, WSCAC *does not imply any of the following principles:* DNR, RT^2_2, SRT^2_2, COH, *and* CAC.

Proof. The corresponding results for SCAC in place of WSCAC follow from [6, Corollaries 3.6, 3.8, and 3.12] (or see [6, Diagram 3 on page 195]). □

3. Infinite low chains and antichains

It was shown by Hirschfeldt and Shore [6, Theorem 3.4] that every infinite computable stable poset $\mathcal{P}$ contains an infinite low chain or antichain, and in fact their proof shows that every such poset contains an infinite low chain or an infinite computable antichain. We use the latter result to show that every infinite computable *weakly* stable poset contains an infinite low chain or an infinite computable antichain. We also show by a different method that every infinite computable stable poset contains an infinite computable chain or an infinite low antichain, and we use the method of proof of this theorem to give simplified proofs of some results about infinite computable *linear* orderings in [6].

Theorem 3.1 (Hirschfeldt and Shore [6, proof of Theorem 3.4]). *Every infinite computable stable poset contains either an infinite low chain or an infinite computable antichain.*

We extend this theorem to the weakly stable case using the trick used to prove Theorem 2.2.

Theorem 3.2 (Jockusch, Kastermans, and Lempp). *Every infinite computable weakly stable poset contains either an infinite low chain or an infinite computable antichain.*

Proof. Let $\mathcal{P}$ be an infinite computable weakly stable poset and let the infinite computable stable poset $\mathcal{Q}$ be defined as in the proof of Theorem 2.2. Recall that every $\mathcal{Q}$-chain is a $\mathcal{P}$-chain, and $S_{\mathcal{Q}} = S_{\mathcal{P}}$ and $I_{\mathcal{Q}} = L_{\mathcal{P}} \cup I_{\mathcal{P}}$. By Theorem 3.1 applied to $\mathcal{Q}$, the poset $\mathcal{Q}$ contains either an infinite low chain C or an infinite computable antichain A. If C exists, then it is the desired infinite low $\mathcal{P}$-chain. Otherwise, note that the restriction of $\mathcal{P}$ to A is stable because $A \subseteq I_{\mathcal{Q}} = L_{\mathcal{P}} \cup I_{\mathcal{P}}$. Apply Theorem 3.1 to this restricted ordering. □

It is unknown whether the dual of Theorem 3.2 holds:

Question 3.3. Does every infinite computable weakly stable poset have either an infinite computable chain or an infinite low antichain?

The following theorem will be used to solve the stable case of this problem and to provide simplified proofs of some results from [6].

Theorem 3.4 (Jockusch, Kastermans, and Lempp).

- *Let $\mathcal{P} = \langle P, <_{\mathcal{P}} \rangle$ be an infinite computable poset, and let S be the set of all small elements of $\mathcal{P}$. Then S is either c.e. or hyperimmune. The same holds for the set L of large elements of $\mathcal{P}$.*
- *Let $\mathcal{P}$ be an infinite computable stable poset. Then either $\mathcal{P}$ contains an infinite computable chain, or $\mathcal{P}$ contains an infinite antichain which is Turing reducible to some 1-generic Δ^0_2 set G.*

Proof. To prove the first part, we may assume that S is infinite and not hyperimmune. Fix a computable function f such that the array $\{D_{f(n)}\}_{n\in\omega}$ witnesses that S is not hyperimmune, i.e., the sets $D_{f(n)}$ are pairwise disjoint and have nonempty intersection with S. Then, for all $a \in P$,

$$a \in S \iff (\exists n)(\forall b \in D_{f(n)})[a <_{\mathcal{P}} b]$$

(The implication from left to right holds by definition of smallness: For any small element a and any infinite subset S' of P, there must be an element in S' bounding a. The implication from right to left holds because S is an initial segment of $\mathcal{P}$ and every $D_{f(n)}$ intersects S.) It follows that S is c.e. The proof for L is analogous.

To prove the second part, we may assume without loss of generality that every element of $\mathcal{P}$ is small or isolated. Let S be the set of small elements. If S is finite, then the set I of isolated elements is cofinite and hence computable. In this case, $\mathcal{P}$ contains an infinite computable antichain by Proposition 1.2. If S is infinite and c.e., then S contains an infinite computable subset, and hence $\mathcal{P}$ contains an infinite computable chain. Otherwise, S is hyperimmune, and by an old result attributed to Jockusch (cf. Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp, and Slaman [5, Proposition 4.7]), S is contained in a 1-generic Δ^0_2 set G. Therefore, $\overline{G} \subseteq I$, so by Proposition 1.2 there is an infinite antichain A such that $A \leq_T \overline{G} \leq_T G \leq_T 0'$.

□

The following corollary is the dual of Theorem 3.1.

Corollary 3.5 (Jockusch, Kastermans, and Lempp)**.** *Every infinite computable stable poset contains either an infinite computable chain or an infinite low antichain.*

Proof. Apply Theorem 3.4 and the fact that every 1-generic Δ^0_2 set is low. □

It is natural to attempt to answer Question 3.3 by using Corollary 3.5 and the trick used to derive Theorem 3.2 from Theorem 3.1. However, this method seems to show only that every infinite computable weakly stable poset contains either an infinite low chain or an infinite low antichain, and of course this result follows already from Theorem 3.2.

The following corollary is a variation of [6, Proposition 3.9]. In fact, it follows from the proof of the Hirschfeldt-Shore result that if $\mathcal{P}$ is an infinite computable stable poset, then $\mathcal{P}$ contains either an infinite computable antichain or an infinite chain C such that there is no DNR function $f \leq_T C$. The following corollary is the dual of that result.

Corollary 3.6 (Jockusch, Kastermans, and Lempp)**.** *Every infinite computable stable poset contains either an infinite computable chain or an infinite antichain A such that there is no DNR function $f \leq_T A$.*

Proof. Apply Theorem 3.4 and the fact (due to Demuth and Kučera [2, Corollary 9]) that no DNR function is Turing reducible to a 1-generic set. □

It is not known whether the above corollary holds for weakly stable posets.

The following result is similar to Theorem 3.4.

Theorem 3.7 (Jockusch, Kastermans, and Lempp)**.** *Let $\mathcal{L}$ be an infinite computable linear ordering of order type $\omega + \omega^*$. Then either $\mathcal{L}$ contains an infinite computable subset of order type ω or a 1-generic Δ^0_2-subset of order type ω^*.*

Proof. Let S be the set of small elements of $\mathcal{L}$, so S is the ω-part of $\mathcal{L}$. The proof of Theorem 3.4 shows that either S contains an infinite computable subset (which must have order type ω), or there is a 1-generic Δ^0_2 set G which is disjoint from S. In the latter case, the order type of G is obviously ω^*. □

Note that Theorem 3.7 has the following results of Hirschfeldt and Shore as corollaries, using, as before, that every Δ^0_2 1-generic set is low, and that no 1-generic set computes a DNR function.

Corollary 3.8 (Hirschfeldt and Shore [6, Theorem 2.11])**.** *Every infinite computable linear order of order type $\omega+\omega^*$ contains a low subset G which is of order type ω or ω^*.*

Corollary 3.9 (Hirschfeldt and Shore [6, Theorem 2.26])**.** *Every infinite computable linear order of order type $\omega + \omega^*$ contains an infinite subset G which is of order type ω or ω^* such that no DNR function is G-computable.*

4. Infinite Π^0_1 chains and antichains

We begin with an easy observation about infinite computable stable posets. This result is the best possible with respect to the arithmetical hierarchy since by [6, proofs of Corollary 2.5 and Proposition 3.1] there exists an infinite computable stable poset with no infinite computable chain or antichain. This latter result also follows from our Theorem 4.2, or can be proved directly by the method of proof of that result but with a simpler proof in which each requirement acts only finitely often.

Proposition 4.1 (Jockusch)**.** *Every infinite computable stable poset contains an infinite computable chain or an infinite Π^0_1 antichain.*

Proof. By symmetry, we assume that every element of the infinite computable poset $\mathcal{P}$ is small or isolated. We now distinguish two cases: If $\mathcal{P}$ contains infinitely many maximal elements, then these form an infinite Π^0_1 antichain. Otherwise, since every element of P is small or isolated and hence bounds only finitely many elements, there must be some element $a \in P$ which is does not lie below any maximal element. It is now easy to generate an infinite computable chain in $\mathcal{P}$ of order type ω starting with a. □

We contrast the above observation with the following result, which is the first known result exhibiting a difference between the complexity of infinite chains and antichains in infinite computable posets.

Theorem 4.2 (Jockusch, Kastermans, and Lempp)**.** *There is an infinite computable stable poset with no infinite Π^0_1 chain and no infinite computable antichain.*

Proof. We effectively construct an infinite stable poset $\mathcal{P} = (\omega, <_{\mathcal{P}})$ containing only small or isolated elements. Let S be the set of small elements and let I be the set of isolated elements (so $S = \overline{I}$). The sets S and I are Δ^0_2 by Proposition 1.2, and we will give computable approximations to them (denoted S^s and I^s, respectively) during the construction. Also, S must be closed downward in $\mathcal{P}$, and hence we require at every stage s that S^s must also be closed downward in the part of $\mathcal{P}$ already constructed. At the end of stage $s+1$, we will add s to the field of $\mathcal{P}$ and extend $<_{\mathcal{P}}$ by putting s above all elements of S^{s+1} and making s incomparable with all elements of I^{s+1}. This preserves transitivity since S^s is downward closed for all s. This procedure produces a stable ordering provided that every number is either in S^s for all sufficiently large s or is in I^s for all sufficiently large s. The following special property of $\mathcal{P}$ will be important:

$$(*) \quad (\forall a)(\forall b)[a <_{\mathcal{P}} b \implies a < b]$$

This property holds because elements are added to the field of $\mathcal{P}$ in the natural order of ω and a new element is never put below any existing element.

We ensure that $\mathcal{P}$ contains no infinite c.e. antichains or infinite co-c.e. chains by meeting the following requirements:

$$\mathcal{A}_e : \text{If } W_e \text{ is infinite, then } W_e \cap S \neq \emptyset$$
$$\mathcal{C}_i : \text{If } \overline{W_i} \text{ is infinite, then } \overline{W_i} \cap I \neq \emptyset$$

Meeting these requirements suffices because, by Proposition 1.2, every infinite chain is contained in S and every infinite antichain is contained in I.

The strategy for $\mathcal{A}_e$ alone is to search for a witness $w \in W_e$ and then to put all $z \leq_{\mathcal{P}} w$ into S.

The strategy for $\mathcal{C}_i$ alone is to search for a witness w not yet in W_i and then to put all $z \geq_{\mathcal{P}} w$ into I. If w later appears in W_i, we cancel this witness and start over.

Obviously, these requirements conflict and may also threaten stability. We assign priorities as follows: $\mathcal{C}_0 > \mathcal{A}_0 > \mathcal{C}_1 > \mathcal{A}_1 > \dots$. We then require that no action for any of these requirements can change the assignment (to S or I) of a witness for a higher priority requirement. Thus, if an $\mathcal{A}$-requirement acts, it will be satisfied forever (and never act again), provided no higher priority requirement acts later. Similarly, if $\mathcal{C}_i$ acts on a witness w which is not in (the final version of) W_i, then it will be satisfied forever and never act again, provided no higher priority requirement acts later. However, it is possible that a requirement $\mathcal{C}_i$ may act infinitely often because all of its witnesses chosen after the higher-priority $\mathcal{A}$-requirements stop acting are in W_i. We will show that $\overline{W_i}$ is finite in this case.

Also, for the sake of stability, we require that the assignment of z can be changed from I to S only for the sake of some requirement $\mathcal{A}_e$ for $e < z$. Since each such requirement acts only finitely often, it follows that the assignment of z can be changed from I to S only finitely often, and hence z has a limiting assignment.

We say that a number $w < s$ is *eligible* for the requirement $\mathcal{C}_e$ at stage $s+1$ if $w \notin W_e^{s+1}$ and there is no $z < s$ such that $w \leq_{\mathcal{P}} z$, $z \in S^s$, and z is a witness at the end of stage s for some requirement $\mathcal{A}_i$ with $i < e$. This means that w is an appropriate choice to serve as a witness for $\mathcal{C}_e$ according to the above restrictions. We say that $\mathcal{C}_e$ *requires attention* at stage $s+1$ if either $\mathcal{C}_e$ has no witness at the end of stage s and there is a number z which is eligible for $\mathcal{C}_e$ at stage $s+1$, or else $\mathcal{C}_e$ has a witness z at the end of stage s and $z \in W_e^{s+1}$.

Similarly, we say that a number $w < s$ is *eligible* for the requirement $\mathcal{A}_e$ at stage $s+1$ of the construction if $w \in W_e^{s+1}$ and there is no $z < s$ such that $z \leq_{\mathcal{P}} w$, $z \in I^s$, and either $z \leq e$ or z is a witness at the end of stage s for some requirement $\mathcal{C}_i$ with $i \leq e$. Say that the requirement $\mathcal{A}_e$ *requires attention* at stage $s+1$ if $\mathcal{A}_e$ has no witness at the end of stage s and there is a number z which is eligible for $\mathcal{A}_e$ at stage $s+1$.

We now describe the construction. Effectively assign each stage to a requirement in such a way that each requirement has infinitely many stages assigned to it. At the end of every stage s, the domain of the part of $\mathcal{P}$ defined so far is $\{i : i < s\}$, and S^s and I^s partition this set.

Stage 0. Let $S^0 = I^0 = \emptyset$. No requirement has a witness assigned to it.

Stage $s+1$. Suppose first that stage $s+1$ is assigned to the requirement $\mathcal{C}_e$. If $\mathcal{C}_e$ does not require attention, let $S^{s+1} = S^s$ and $I^{s+1} = I^s \cup \{s\}$. If $\mathcal{C}_e$ requires attention and has no witness, let w be the least number (in the standard ordering) eligible for $\mathcal{C}_e$. Appoint w as the witness for $\mathcal{C}_e$ and define $I^{s+1} = I^s \cup \{z < s : w \leq_{\mathcal{P}} z\} \cup \{s\}$ and $S^{s+1} = \{z < s : z \notin I^{s+1}\}$. If $\mathcal{C}_e$ requires attention and has a witness w, then cancel w as a witness for $\mathcal{C}_e$ and define $S^{s+1} = S^s$ and $I^{s+1} = I^s \cup \{s\}$. (In this case, $w \in W_e^{s+1}$.)

Now suppose that stage $s+1$ is assigned to the requirement $\mathcal{A}_e$. If $\mathcal{A}_e$ does not require attention, let $S^{s+1} = S^s$ and $I^{s+1} = I^s \cup \{s\}$. If $\mathcal{A}_e$ requires attention, let w be the least number (in the standard ordering) eligible for $\mathcal{A}_e$. Appoint w as the witness for $\mathcal{A}_e$ and define $S^{s+1} = S^s \cup \{z < s : z \leq_{\mathcal{P}} w\}$ and $I^{s+1} = \{z < s : z \notin S^{s+1}\} \cup \{s\}$.

In both cases, for $i < s$, put $i <_{\mathcal{P}} s$ iff $i \in S^{s+1}$. (Thus s is $\mathcal{P}$-incomparable with all $i < s$ such that $i \in I^{s+1}$.) Also, cancel the witness of any $\mathcal{A}$-requirement with a witness in I^{s+1} and the witness of any $\mathcal{C}$-requirement with a witness in S^{s+1}. (It is easily seen that this action causes the witness of a requirement to be cancelled only when an opposing requirement of higher priority acts. However, it does not seem safe to cancel a witness whenever an opposing requirement of higher priority acts because a $\mathcal{C}$-requirement might act infinitely often.)

This completes the description of the construction.

It is easy to verify by induction on s that S^s is an initial segment of the restriction of $<_{\mathcal{P}}$ to $\{j : j < s\}$. It then follows by induction on s that this restricted ordering is transitive for each s. Therefore $\mathcal{P}$ is a poset. Also, it is clearly computable.

Lemma 4.3.

- *If w is cancelled as a witness for $\mathcal{C}_e$ at stage $s+1$, then either $w \in W_e^{s+1}$, or, at stage $s+1$, some requirement $\mathcal{A}_i$ for $i < e$ appoints a witness.*
- *If w is cancelled as a witness for $\mathcal{A}_e$ at stage $s+1$, then at stage $s+1$, some requirement $\mathcal{C}_i$ for $i \leq e$ appoints a witness $z \leq_{\mathcal{P}} w$ (and hence $z \leq w$ by (*)).*

Proof. To prove the first part, assume that w is cancelled as a witness for $\mathcal{C}_e$ at stage $s+1$. If $s+1$ is assigned to $\mathcal{C}_e$, then w is cancelled because $w \in W_e^{s+1}$. If $s+1$ is not assigned to $\mathcal{C}_e$, then w is cancelled because $w \in S^{s+1} - S^s$. It follows that stage $s+1$ is assigned to $\mathcal{A}_i$ for some i, as otherwise $S^s \supseteq S^{s+1}$. Furthermore, $\mathcal{A}_i$ appoints a number $z \geq_{\mathcal{P}} w$ as its witness. Since z is eligible for $\mathcal{A}_i$ at stage $s+1$ and $z \geq_{\mathcal{P}} w$, it follows that $i < e$.

The proof of the second part is similar. □

The remainder of the verification that the construction works is included in the following lemma.

Lemma 4.4. *For all e, we have:*

(1) *Either $e \in S^s$ for all sufficiently large s or $e \in I^s$ for all sufficiently large s. Hence $\mathcal{P}$ is stable.*
(2) *$\mathcal{C}_e$ is met. Also, for every number w, there are only finitely many stages at which w is appointed or cancelled as a witness for $\mathcal{C}_e$.*
(3) *$\mathcal{A}_e$ is met and either has a permanent witness w or eventually has no witness.*

Proof. Assume that (1)-(3) hold for all $i < e$.

To prove (1) for e, we claim that e can be switched from I into S only when some $\mathcal{A}_i$ for $i < e$ puts its witness into S. This suffices because, by inductive assumption, this happens only finitely often for each $i < e$. The proof of this claim is similar to the proof of Lemma 4.3.

We now prove (2) for e. We show first that every number w is cancelled as a witness for $\mathcal{C}_e$ only finitely often (and hence is also appointed only finitely often). Let s_0 be a stage so large that no requirement $\mathcal{A}_i$ for $i < e$ appoints or cancels a witness after s_0. After stage s_0 any witness w in existence for $\mathcal{C}_e$ is either permanent or is cancelled at a stage $s+1$ with $w \in W_e^{s+1}$. In the latter case, w is never appointed as a witness for $\mathcal{C}_e$ after stage $s+1$ because it is not eligible. Since a witness can be cancelled at most finitely many times before s_0, it follows that every number is cancelled or appointed as a witness for $\mathcal{C}_e$ only finitely often.

In order to show that $\mathcal{C}_e$ is met, we first show that if W_e is coinfinite then there is a fixed number w_0 which is eligible for $\mathcal{C}_e$ from some stage on. Let A be the set of all numbers which are $<_\mathcal{P}$-above all permanent witnesses for requirements $\mathcal{A}_i$ for $i < e$. Since each such permanent witness is in S^s for all sufficiently large s, A is cofinite. Assume that W_e is coinfinite. Choose $w_0 \in \overline{W_e} \cap A$. Then w_0 is eligible for $\mathcal{C}_e$ at every stage after s_0. (If z is a witness for $\mathcal{A}_i$ with $i < e$ at stage $s > s_0$, then z is a permanent witness for $\mathcal{A}_i$ and hence $z <_\mathcal{P} w_0$, so it is not the case that $w_0 \leq_\mathcal{P} z$.) Let $s_1 > s_0$ be a stage such that no witness $z \leq w_0$ for $\mathcal{C}_e$ is cancelled after stage s_1. If $\mathcal{C}_e$ has a permanent witness, it is obviously met. Otherwise, there is a stage $s+1 > s_1$ which is assigned to $\mathcal{C}_e$ at the beginning of which $\mathcal{C}_e$ has no witness. Then w_0 is eligible for $\mathcal{C}_e$ at stage $s+1$, so $\mathcal{C}_e$ requires attention and some witness $z \leq w_0$ is appointed for $\mathcal{C}_e$ at stage $s+1$. The witness z is never cancelled after $s+1$ since $s+1 \geq s_1$, and hence z is a permanent witness for $\mathcal{C}_e$, and $\mathcal{C}_e$ is met. This completes the proof of (2) for e.

We now prove (3) for e. We show first that every number w is cancelled as a witness for $\mathcal{A}_e$ only finitely many times. By Lemma 4.3, it suffices to prove that there are only finitely many stages at which, for some $i \leq e$, $\mathcal{C}_i$ appoints a witness $z \leq_\mathcal{P} w$. Recall that, by (*), if $z \leq_\mathcal{P} w$, then $z \leq w$. Thus it suffices to show that for each $i \leq e$ and each z, there are only finitely many stages at which z is cancelled as a witness for $\mathcal{C}_i$. This follows from the fact that (2) holds for all $i \leq e$.

To show that $\mathcal{A}_e$ is met, assume that W_e is infinite. We first show that there is a fixed number w_0 which is eligible for $\mathcal{A}_e$ from some stage on. Let H be the set of w such that $w \leq e$ or w is a permanent witness for $\mathcal{C}_i$ for some $i \leq e$. We claim that all elements of H stabilize, where a number *stabilizes* if it is in S^s for all sufficiently large s or in I^s for all sufficiently large s. All $i \leq e$ stabilize since (1) holds for $i \leq e$. All permanent witnesses for $\mathcal{C}$-requirements stabilize because they are in I^s for all sufficiently large s. Let A be the set of all numbers which are incomparable with all $i \in H$ which are in I^s for all sufficiently large s and are $\mathcal{P}$-above all $i \in H$ which are in S^s for all sufficiently large s. By construction, A is cofinite. Choose $w_0 \in W_e \cap A$. Let s_0 be a stage so large that $w_0 \in W_{e,s_0}$ and no number $z \leq w_0$ is cancelled or appointed as a witness by any requirement $\mathcal{C}_i$ for $i \leq e$ after stage s_0. Such a number exists because, as remarked in the previous paragraph, no number is cancelled infinitely often as a witness for any $\mathcal{C}_i$ for $i \leq e$. Choose $s_1 > s_0$ such that for all $i \leq e$, if $\mathcal{C}_i$ has a permanent witness z_i, then z_i is the witness for $\mathcal{C}_i$ at stage s_1 and is never cancelled after stage s_1. We claim that w_0 is eligible for $\mathcal{A}_e$ at every stage $s+1 \geq s_1$. If not, choose $z \leq_\mathcal{P} w_0$ such that z is a witness at stage $s+1$ for some requirement $\mathcal{C}_i$ for $i \leq e$. Then $\mathcal{C}_i$ cannot have a permanent witness because in this case z would be the permanent witness (as $s+1 > s_1$) and then z and w_0 would be incomparable. Thus z must be cancelled as a witness for $\mathcal{C}_i$ at some stage after $s+1$. Since $s+1 > s_0$, it follows that $z > w_0$. But $z \leq w_0$ by (*). This contradiction shows that w_0 is eligible for $\mathcal{A}_e$ at every stage after s_1. The proof that $\mathcal{A}_e$ is met is now virtually the same as the proof that $\mathcal{C}_e$ is met. In fact, the argument shows that $\mathcal{A}_e$ has a permanent witness if W_e is infinite.

It remains to show in general that $\mathcal{A}_e$ acts only finitely often. This follows from the previous paragraph if W_e is infinite. If W_e is finite, it follows from the fact that only elements of W_e can be appointed as witnesses for $\mathcal{A}_e$, and each number is cancelled as a witness for $\mathcal{A}_e$ only finitely often. This completes the proof of (3). □

This completes the proof of Theorem 4.2. □

The next result contrasts with Proposition 4.1 and thus establishes a difference between the effective properties of stable posets and those of weakly stable posets. It is best possible with respect to the arithmetic hierarchy since, by Proposition 1.2, every infinite computable weakly stable poset contains an infinite Δ^0_2 chain or antichain. The proof is a priority argument and, as in Theorem 4.2, every requirement is injured only finitely often. However, in the next result, both the requirements for chains and those for antichains concern Π^0_1 sets and thus have the potential to act infinitely often. This makes the argument considerably more delicate.

Theorem 4.5 (Jockusch, Lerman, and Solomon)**.** *There is an infinite computable weakly stable poset which contains no infinite Π^0_1 chains or antichains.*

In a weakly stable partial ordering $\mathcal{P}$, every infinite chain is a subset of $S_\mathcal{P} \cup L_\mathcal{P}$, and every infinite antichain is a subset of $I_\mathcal{P}$, by Proposition 1.2. Thus to prove the theorem it suffices to construct an infinite computable weakly stable partial ordering $\mathcal{P} = \langle P, \leq_\mathcal{P} \rangle$ with domain ω such that neither $S_\mathcal{P} \cup L_\mathcal{P}$ nor $I_\mathcal{P}$ contains an infinite co-c.e. subset. In order to achieve weak stability, in addition to defining $\mathcal{P}$, we define a computable partial function $t : P \times \omega \to \{S, L, I\}$ with computable domain such that $\lim_s t(x,s) \downarrow$ for all x, $S_\mathcal{P} = \{x : \lim_s t(x,s) = S\}$, $L_\mathcal{P} = \{x : \lim_s t(x,s) = L\}$, and $I_\mathcal{P} = \{x : \lim_s t(x,s) = I\}$. The following requirements imply that neither $S_\mathcal{P} \cup L_\mathcal{P}$ nor $I_\mathcal{P}$ contains an infinite co-c.e. subset:

$$C_e : |\overline{W_e}| = \infty \to \overline{W_e} \cap (S_{\mathcal{P}} \cup L_{\mathcal{P}}) \neq \emptyset$$
$$I_e : |\overline{W_e}| = \infty \to \overline{W_e} \cap I_{\mathcal{P}} \neq \emptyset.$$

We form an effective list $\{R_i : i < \omega\}$ of all requirements.

Our original approach was the standard one; place the requirements C_e, I_e and the convergence requirements for t on a tree of strategies, write down the additional properties needed for the approximation t and the approximations to $\mathcal{P}$, and devise a $\mathbf{0}''$-priority argument to satisfy the requirements and the approximation properties. The proof we found, using approximations that worked with finite blocks of numbers instead of single numbers, seemed unnatural. We then realized that essentially the same construction, when viewed not from the point of view of the manner in which requirements are satisfied, but rather from the dual point of view of satisfying the requirements specifying how the blocks are defined and their labels designated, is a natural finite injury priority construction, and this is the way we will present the proof.

The notion of satisfying a dual set of requirements is not a new one. One example is the construction of a maximal c.e. set. Instead of casting the construction in terms of how to satisfy the requirements generated by the definition of maximality, the construction is described in terms of an attempt to define a Π^0_1 set while satisfying certain e-state properties for its members. Even though each e-state requirement has a potentially infinite effect on the construction, the effect of all requirements on any given marker whose position approximates an element of the Π^0_1 set is finite. Similarly, Rogers' [10] movable markers describe a construction from the point of view of marker movement to satisfy certain properties, producing a linear description, rather than from the viewpoint of how the underlying requirements are satisfied, which is most naturally done through a tree description. In some cases, the dual requirements are implicit, but can be formalized. Our proof, obtained by satisfying an implicit dual set of requirements, is very similar to the Rogers approach. However, instead of movable markers, we have movable finite blocks of numbers.

The poset $\mathcal{P} = \langle P, \leq_{\mathcal{P}} \rangle$ needs to be computable, so we will define it as the union of an increasing sequence of finite posets $\mathcal{P}^s = \langle P^s, \leq^s \rangle$. The *labeling function* $t(x, s)$ will identify the predicted weak stability type of the number $x \in P^s$, and we will need the limit of the predicted types to exist and to be the true type. Thus the range of t will be the set $\{S, L, I\}$, which make the obvious predictions. In order for the limit process to work correctly, we will need the *labelings* $\lambda x t(x,s)$ to be *viable* for each s, as defined below.

Definition 4.6. Let $\mathcal{P} = \langle P, \leq_{\mathcal{P}} \rangle$ be a poset, and let $t : P \to \{S, L, I\}$ be given. We say that the labeling function t is *viable* if it satisfies the following properties for all $x, y \in P$:

(V1) $x <_{\mathcal{P}} y \;\&\; t(y) = S \implies t(x) = S$
(V2) $x <_{\mathcal{P}} y \;\&\; t(x) = L \implies t(y) = L$
(V3) $t(x) = S \;\&\; t(y) = L \implies x <_{\mathcal{P}} y$

If t is constant on a nonempty set $S \subseteq P$, we will use $t(S)$ for the value of $t(x)$ for $x \in S$.

It is easily seen that if P is infinite and t is the natural labeling function corresponding to the sets $S_{\mathcal{P}}$, $L_{\mathcal{P}}$ and $I_{\mathcal{P}}$, then t is viable. Hence it is natural to require viability for our finite approximations to $\mathcal{P}$, $\leq_{\mathcal{P}}$ and t. However, in order to build a weakly stable poset $\langle P, \leq_{\mathcal{P}} \rangle$, we will also need to have conditions that will allow us to extend a finite poset with a viable labeling to another finite poset with a possibly revised viable labeling. Conditions that need to be imposed in order to accomplish this are introduced in the next several definitions.

Definition 4.7. Let $\mathcal{P} = \langle P, \leq_{\mathcal{P}} \rangle$ be a poset, and let A and B be disjoint subsets of P. We say that A *upwardly restricts* B if for all $a \in A$ and $c \in P$, if $c >_{\mathcal{P}} a$ then $c \notin B$, and that A *downwardly restricts* B if for all $a \in A$ and $c \in P$, if $c <_{\mathcal{P}} a$ then $c \notin B$. If $A = \{a\}$, then we say that *a upwardly (downwardly) restricts* B if A upwardly (downwardly) restricts B.

Definition 4.8. Let $\mathcal{P} = \langle P, \leq_{\mathcal{P}} \rangle$ be a poset, let t be a viable labeling of $\mathcal{P}$, and let A and B be disjoint subsets of P. We say that B *respects* A if the following conditions hold for all $a \in A$ and $b \in B$:

(R1) $t(a) = S \;\to\; b >_{\mathcal{P}} a$.
(R2) $t(a) = L \;\to\; b <_{\mathcal{P}} a$.
(R3) $t(a) = I \;\to\; a \,|_{\mathcal{P}}\, b$.

Let $\langle B_i : i \leq n \rangle$ be a finite sequence of sets ("blocks"). For $i \leq n$, we define $B_{\geq i} = \bigcup \{B_j : n \geq j \geq i\}$. $B_{>i}$, $B_{\leq i}$ and $B_{<i}$ are defined in a similar fashion.

At each stage of our construction, we will have defined a finite poset $\mathcal{P} = \langle P, \leq_{\mathcal{P}} \rangle$ and a viable labeling t of $\mathcal{P}$. We will also have defined a partition $\langle B_i : i \leq n \rangle$ of P such that t is constant on each block B_i, and a *target function* $g : [0, n] \to \{0, 1\}$ telling us which element of $\{S, L\}$ is a safe label for the block, with 0 representing S and 1 representing L.

Using this information, we will want to revise both t and the block structure in a way that enables us to carry out the next step of the construction. In order to do this, we require the block structure to have certain properties that are listed in the next definition.

Definition 4.9. Let $\mathcal{P} = \langle P, \leq_{\mathcal{P}} \rangle$ be a poset and let t be a viable labeling of $\mathcal{P}$. Let $\langle B_i : i \leq n \rangle$ be a partition of P and let $g : [0, n] \to \{0, 1\}$ be a target function. Then $\langle B_i : i \leq n \rangle$ is a $\langle t, g \rangle$-*respectful block sequence* if the following conditions hold for all $i \leq n$ and $x, y \in P$:

(G1) t is constant on B_i. (We write $t(i)$ for $t(B_i)$.)
(G2) $(g(i) = 0 \implies t(i) \in \{S, I\})$ & $(g(i) = 1 \implies t(i) \in \{L, I\})$.
(G3) $B_{\geq i+2}$ respects $B_{\leq i}$.
(G4) If $g(i) = 0$ then B_i downwardly restricts $B_{>i}$, and if $g(i) = 1$ then B_i upwardly restricts $B_{>i}$.
(G5) $(i < n$ & $g(i) = 0$ & $t(i) = S) \implies g(i + 1) = 0$
(G6) $(i < n$ & $g(i) = 1$ & $t(i) = L) \implies g(i + 1) = 1$
(G7) $(i < n$ & $t(i) = I) \implies g(i + 1) = 1 - g(i)$

We now outline the ideas behind the proof. We will use blocks to satisfy requirements, and each block will be used for at most a predetermined finite set of requirements. At each stage, any given block is trying to satisfy at most one requirement. Each time we change the requirement that a block B_i is trying to satisfy, we will collapse all blocks B_j for $j > i$ into a single block B_{i+1} and may change the label of B_i and change both the label and target of B_{i+1}. This will happen only finitely often for each i, so each B_i will have a limiting value, label, and target. Since every number will belong to some B_i, it follows that every number will have a limiting label. Because of the way the labels are allowed to change and the requirements are assigned after collapsing a block, all sufficiently large numbers will be considered for all requirements that are not permanently assigned to a block. This fact is exactly what ensures that those requirements are satisfied. The label change may prevent the new block from respecting B_i, but the upward and downward restriction conditions will still be in force. The rules on the way g is revised will allow us to show that the new labeling is viable and the new block sequence retains the properties of the old one. The next lemma covers the way that this will be done.

Lemma 4.10. *Fix a poset $\langle P, \leq_{\mathcal{P}} \rangle$, a viable labeling t of $\mathcal{P}$, a target function g with domain $[0, n]$, and a $\langle t, g \rangle$-respectful block sequence $\langle B_i : i \leq n \rangle$ partitioning P. Fix $k < n$ and an element $X \in \{S, L, I\}$ such that $X \in \{S, I\}$ if $g(k) = 0$ and $X \in \{L, I\}$ if $g(k) = 1$. Define $\widetilde{t}$, $\widetilde{g}$ and $\langle \widetilde{B}_i : i \leq k + 1 \rangle$ as follows: $\widetilde{B}_i = B_i$ for all $i \leq k$ and $\widetilde{B}_{k+1} = B_{>k}$;*

$\widetilde{t} \restriction B_{<k} = t \restriction B_{<k}$, $\widetilde{t}(B_k) = X$ and $\widetilde{t}(\widetilde{B}_{k+1}) = I$; and $\widetilde{g}(i) = g(i)$ for all $i \leq k$ and $\widetilde{g}(k+1)$ is uniquely determined by (G5)–(G7). Then $\widetilde{t}$ is a viable labeling of $\mathcal{P}$ and $\langle \widetilde{B}_i : i \leq k+1 \rangle$ is a $\langle \widetilde{t}, \widetilde{g} \rangle$-respectful block sequence.

Proof. Recall that we will write $t(i)$ for $t(B_i)$. The first step in the proof is to verify the three viability conditions. Fix $x, y \in P$. If $x, y \in B_{<k}$, then (V1)–(V3) for $\widetilde{t}$ follow from (V1)–(V3) for t.

We first consider (V1). Suppose that $x <_{\mathcal{P}} y$ and $\widetilde{t}(y) = S$. We must show that $\widetilde{t}(x) = S$. Because (V1) holds when $x, y \in B_{<k}$ and because $\widetilde{t}(k+1) = I$ (so $y \notin B_{>k} = \widetilde{B}_{k+1}$), it suffices to consider the two remaining cases: when $y \in B_{<k}$ and $x \in B_{\geq k}$, and when $y \in B_k = \widetilde{B}_k$.

First, suppose that $y \in B_i$ for some $i < k$ and $x \in B_{\geq k}$. By definition, $\widetilde{t}(y) = t(y) = t(i)$, so $t(i) = S$. By (G2), $t(i) = S$ implies $g(i) = 0$. Applying (G3) and (G4), we have that $B_{\geq k+1}$ respects B_i and that B_k is downwardly restricted by B_i. Therefore, $x <_{\mathcal{P}} y$ implies that $x \notin B_{\geq k}$, so this case cannot arise.

Second, suppose that $y \in B_k$. Since $\widetilde{t}(y) = \widetilde{t}(B_k) = S$, we have $X = S$ and hence $g(k) = 0$. By (G4), B_k downwardly restricts $B_{>k}$. Therefore, no number $z \in \widetilde{B}_{k+1} = B_{>k}$ can satisfy $z <_{\mathcal{P}} y$. In particular, $x \notin B_{>k}$, so we split into three cases depending on whether $x \in B_k$, $x \in B_{k-1}$ or $x \in B_{<k-1}$.

If $x \in B_k$ then $\widetilde{t}(x) = S$ (as desired). If $x \in B_i$ for some $i < k-1$, then as B_k respects B_i by (G3), it follows from (R1)–(R3) and the definition of $\widetilde{t}$ that $\widetilde{t}(x) = t(x) = S$. Finally, suppose that $x \in B_{k-1}$. If $g(k-1) = 1$, then by (G4), B_{k-1} upwardly restricts B_k so we cannot have $y \in B_k$ and $y >_{\mathcal{P}} x$. If $g(k-1) = 0$, then $t(x) \in \{S, I\}$ by (G2). If $t(x) = S$, then $\widetilde{t}(x) = S$ by the definition of $\widetilde{t}$, and we are done. Otherwise, $t(x) = I$, so $g(k) = 1$ by (G7), and hence we cannot define $\widetilde{t}(y) = S$. We conclude that $\widetilde{t}(x) = t(x) = S$.

(V2) is proved using a symmetric argument to that given in the preceding paragraphs for (V1).

We now consider (V3). Suppose that $\widetilde{t}(x) = S$ and $\widetilde{t}(y) = L$. We have already treated the case in which $x, y \in B_{<k}$. If $x, y \in \widetilde{B}_k$, then $\widetilde{t}(x) = \widetilde{t}(y)$, and any $z \in \widetilde{B}_{k+1}$ satisfies $\widetilde{t}(z) = I$. Hence we need only treat the case in which one of x and y lies in $B_{<k}$ and the other lies in B_k. If one of these elements lies in $\widetilde{B}_i$ for some $i < k-1$, then by (G3), B_k respects B_i and so (V3) follows. Otherwise, one of these elements lies in B_{k-1}, so by (G5) and (G6), we must have $g(k-1) = g(k)$. By (G2) and the choice of X, it follows that if $g(k-1) = g(k) = 0$ then $X \neq L$ and $t(k-1) \neq L$, and if $g(k-1) = g(k) = 1$ then $X \neq S$ and

$t(k-1) \neq S$. Furthermore, by the definition of $\widetilde{t}$, $t(k-1) = \widetilde{t}(k-1)$ and $\widetilde{t}(k) = X$. Thus if $\widetilde{t}(k-1) = S$ then $\widetilde{t}(k) \neq L$, and if $\widetilde{t}(k-1) = L$ then $\widetilde{t}(k) \neq S$, so this case cannot occur, completing the proof of (V3).

We now verify (G1)–(G7). (G1) is immediate from the definitions of $\widetilde{B}$, $\widetilde{g}$ and $\widetilde{t}$, as is (G2). For all $i \leq k$, $\widetilde{B}_i = B_i$, $\widetilde{B}_{\geq i} = B_{\geq i}$ and $\widetilde{t}(i) = t(i)$, so (G3) for $\langle \widetilde{B}_i : i \leq k+1 \rangle$ follows from (G3) for $\langle B_i : i \leq n \rangle$, and the same holds for (G4). (G5)–(G7) are immediate from the definitions of $\widetilde{g}$ and $\widetilde{t}$ and since these properties hold in the starting situation. □

We will also need a lemma to apply when extending $\mathcal{P}^s$.

Lemma 4.11. *Fix a poset $\mathcal{P} = \langle P, \leq_{\mathcal{P}} \rangle$, a partition $\langle B_i : i \leq n \rangle$ of P, a viable labeling function t for $\mathcal{P}$ and a target function g with domain $[0, n]$ such that $\langle B_i : i \leq n \rangle$ is a $\langle t, g \rangle$-respectful block sequence, and fix m such that $m \notin P$. For all $i \leq n$, define $\widetilde{B}_i = B_i$, $\widetilde{t}(i) = t(i)$ and $\widetilde{g}(i) = g(i)$. Define $\widetilde{B}_{n+1} = \{m\}$ and $\widetilde{t}(n+1) = t(n)$; and if $t(n) = I$ then define $\widetilde{g}(n+1) = 1 - g(n)$, and define $\widetilde{g}(n+1) = g(n)$ otherwise. Let $\widetilde{P} = P \cup \{m\}$, and all for $x \in P$, specify that $x <_{\widetilde{\mathcal{P}}} m$ if $t(x) = S$, $x >_{\widetilde{\mathcal{P}}} m$ if $t(x) = L$, and $x \mid_{\widetilde{\mathcal{P}}} m$ if $t(x) = I$. Then $\langle \widetilde{B}_i : i \leq n \rangle$ is a $\langle \widetilde{t}, \widetilde{g} \rangle$-respectful block sequence, $\widetilde{t}$ is a viable labeling of $\widetilde{\mathcal{P}}$, and $\langle \widetilde{P}, \leq_{\widetilde{\mathcal{P}}} \rangle$ is a poset.*

Proof. (G1)–(G7) for the new block sequence follow easily from (G1)–(G7) for the original block sequence. The extension of the ordering to $\widetilde{P}$, the definition of $\widetilde{t}(n+1)$ and the viability of t for $\mathcal{P}$ are easily seen to imply the viability of $\widetilde{t}$ for $\widetilde{\mathcal{P}}$. Finally, the fact that $\langle \widetilde{P}, \leq_{\mathcal{P}} \rangle$ is a partial ordering follows easily from (V1)-(V3). We leave the formal verifications to the reader. □

Proof of Theorem 4.5. We will present a movable marker construction in the sense of Rogers [10], using the blocks as markers and without necessarily preserving the order of the markers. When a block B_i receives attention, all blocks B_j for $j > i$ are combined into a single block B_{i+1}. Now consider the behavior of a fixed block B_i after all blocks B_j for $j < i$ have stopped receiving attention. The block B_i starts in state O at this point. If it is empty, some element will be put into it. Once it becomes nonempty, its content may grow finitely often, and its state may change finitely often. Specifically, it may be assigned to some requirement R_{n_1} with $n_1 \leq i$. Here n_1 is chosen as small as possible so that no block B_j for $j < i$ is assigned to n_1. Then B_i remains assigned to R_{n_1} until, if ever, B_i is a subset of W_e, where W_e is the c.e. set associated with R_{n_1}. At this point, B_i is known to be permanently useless for meeting R_{n_1}, and B_i may be reassigned to some requirement R_{n_2} with $n_1 < n_2 \leq i$ and such that no B_j for $j < i$ is assigned to R_{n_2}. The process continues in this way until either B_i is permanently assigned to a fixed requirement, or no requirement is available to assign it to, in which case it permanently enters the state F. It is clear by induction that B_i will have a final content and state.

We will show that, at the end of the construction, each requirement either has a block assigned to it witnessing its satisfaction, or it is satisfied by default. The current requirement or state to which the block is assigned will be tracked by an assignment function $f : \omega^2 \to \omega \cup \{O, F\}$; $f(i, s)$ will denote the requirement or state assigned to the block B_i^s at stage s. At each stage we also define a target function g^s.

We say that B_i^s *requires attention* at stage $s+1$ if one of the following conditions holds:

$$B_i^s = \emptyset. \tag{4.1}$$

$$B_i^s \neq \emptyset \ \& \ f(i, s) = O. \tag{4.2}$$

$$f(i, s) = m \ \& \ R_m \in \{C_e, I_e\} \ \& \ \overline{W_e^s} \cap B_i^s = \emptyset. \tag{4.3}$$

The Construction: We proceed by stages. Blocks will be empty at a given stage unless the construction states otherwise. The target and labeling functions will not be defined on empty blocks. Functions and blocks will be defined identically at stages s and $s+1$ unless specifically redefined at stage $s + 1$ of the construction. If $x \in B_i^s$, we set $t(x) = t(B_i^s) = t(i)$. We often write $f^s(i)$ for $f(i, s)$. To handle the case where $i = 0$, we make the convention that $g^s(-1) = 0$ for all s. Because the ordering and incomparability relations between elements do not change once they are specified, we do not attach stage numbers when we specify relations such as $x \leq_{\mathcal{P}} y$ or $x \mid_{\mathcal{P}} y$.

Stage 0: We define $B_0^0 = \{0\}$, $f^0(0) = O$, $g^0(0) = 0$, and $t^0(B_0^0) = I$.

Stage s+1: Fix the smallest i such that B_i^s requires attention. (Such an i will exist, as only finitely many blocks will be nonempty at stage s.) We say that B_i^s *receives attention* at stage $s + 1$ *through* the first of (4.1)–(4.3) that holds for B_i^s.

Case 1: B_i^s receives attention through (4.1). Let x be the smallest number that does not lie in any block B_j^s, and set $B_i^{s+1} = \{x\}$, $t^{s+1}(B_i^{s+1}) = I$, $f^{s+1}(i) = O$, $g^{s+1}(i) = 1 - g^s(i-1)$ if $t^s(i-1) = I$, and $g^{s+1}(i) = g^s(i-1)$ otherwise. For $y \in P^s = \bigcup\{B_j : j < i\}$, specify that $y <_{\mathcal{P}} x$ if $t^s(y) = S$, $y >_{\mathcal{P}} x$ if $t^s(y) = L$, and $y \mid_{\mathcal{P}} x$ if $t^s(y) = I$.

Case 2: B_i^s receives attention through (4.2). Fix the smallest $n \leq i$ such that $f(j,s) \neq n$ for all $j < i$, and define $f^{s+1}(i) = n$; if no such n exists, define $f^{s+1}(i) = F$. Set $t^{s+1}(B_i^{s+1}) = t^s(B_i^s)$ if $f^{s+1}(i) = F$, $t^{s+1}(B_i^{s+1}) = S$ if $R_n = C_e$ and $g^s(i) = 0$, $t^{s+1}(B_i^{s+1}) = L$ if $R_n = C_e$ and $g^s(i) = 1$, and $t^{s+1}(B_i^{s+1}) = I$ if $R_n = I_e$. We set $B_{i+1}^{s+1} = \bigcup\{B_j^s : j > i \ \&\ B_j^s \neq \emptyset\}$ and define $f^{s+1}(i+1) = O$ and $t^{s+1}(B_{i+1}^{s+1}) = I$. We define $g^{s+1}(i+1) = 1 - g^s(i)$ if $t^{s+1}(i) = I$, and $g^{s+1}(i+1) = g^s(i)$ otherwise. For $j > i+1$, we set $B_j^{s+1} = \emptyset$, and $f^{s+1}(j)$, $g^{s+1}(j)$ and $t^{s+1}(B_j^{s+1})$ are undefined.

Case 3: B_i^s receives attention through (4.3). We proceed as in Case 2, except that the search for n is restricted to $(m, i]$.

The following six lemmas show that the above construction succeeds.

Lemma 4.12. *Fix $i < \omega$. Then:*

(i) $B_i = \lim_s B_i^s$ *exists, and* B_i *is finite and nonempty.*
(ii) $\lim_s f^s(i)$ *exists.*
(iii) $\lim_s g^s(i)$ *exists.*
(iv) $\lim_s t^s(B_i^s)$ *exists.*

Proof. We proceed by induction on i.

Suppose that (i) holds for all $j < i$. Fix the smallest stage s such that for all $j < i$, B_j^r does not require attention at stage $r \geq s$. Then Case 1 of the construction will be followed for i at stage s if $B_i^{s-1} = \emptyset$, so that $B_i^s \neq \emptyset$ in any case. Furthermore, $B_i^{r+1} = B_i^r$ for all $r \geq s$, so $B_i = B_i^s$ exists and is finite. (i) now follows. In addition, $f^r(i)$, $g^r(i)$ and $t^r(B_i^r)$ will be defined for all $r \geq s$.

If $r \geq s$ and $f^{r+1}(i) \neq f^r(i)$, then either $f^r(i) = O$, or $f^r(i) \in [0,i]$ and $f^{r+1}(i) \in (f^r(i), i] \cup \{F\}$; thus $\{r > s : f^{r+1}(i) \neq f^r(i)\}$ is finite. (ii) now follows. Furthermore, if $f^{r+1}(i) = f^r(i)$ then $g^{r+1}(i) = g^r(i)$ and $t^{r+1}(B_i^{r+1}) = t^r(B_i^r)$, so (iii) and (iv) follow. □

Let $f(i) = \lim_s f^s(i)$, $g(i) = \lim_s g^s(i)$, and $t(B_i) = \lim_s t^s(B_i^s)$.

Lemma 4.13. *Fix $x < \omega$. Then there is an i such that $x \in B_i$.*

Proof. We always choose the least number not yet in any block when following Case 1 of the construction, and once a number is placed in a block B_i, it will lie in a block B_j for some $j \leq i$ at all subsequent stages. The lemma now follows. □

Lemma 4.14. *For all $s < \omega$, if n_s is the largest number n such that $B_n^s \neq \emptyset$, then $\langle B_i^s : i \leq n_s \rangle$ is a $\langle t^s, g^s \rangle$-respectful block sequence.*

Proof. We proceed by induction on s. The lemma follows easily for $s = 0$, and from Lemmas 4.10 and 4.11 for $s > 0$. $\square$

Let $\mathcal{P} = \bigcup\{\mathcal{P}^s : s < \omega\}$, and let $\leq_{\mathcal{P}}$ be the unique extension of the orderings defined during the construction of $\mathcal{P}$.

Lemma 4.15. *$\langle P, \leq_{\mathcal{P}}\rangle$ is a weakly stable poset.*

Proof. The fact that $\langle P, \leq_{\mathcal{P}}\rangle$ is a poset is immediate from Lemma 4.11. By Lemma 4.12(i), for each i, we can fix a stage s_i such that $f^{r+1}(i) = f^r(i)$ for all $r \geq s_i$, and $B_i^r = B_i$ for all $r \geq s_i$. Then $B_j^r = B_j$ for all $j \leq i$ and $r \geq s_i$. If $i > 0$, then by (G3), $\omega - B_{\leq i}$ will respect $B_{<i}$, so weak stability follows from (R1)–(R3). $\square$

Lemma 4.16. *If there is an i such that B_i is assigned to R_m at all sufficiently large stages, then R_m is satisfied.*

Proof. Fix e such that $R_m \in \{C_e, I_e\}$. By Lemma 4.12, we can fix a stage s such that for all $r \geq s$, $f^r(i) = f(i)$, $B_i^r = B_i$, $g^r(i) = g(i)$ and $t^r(B_i^r) = t(B_i)$. Furthermore, when $f^s(i)$ is permanently set to m at stage s, we define $t(B_i^s) \in \{S, L\}$ if $R_m = C_e$, and $t(B_i^s) = I$ if $R_m = I_e$. R_m cannot require attention at any $r > s$, else it would receive attention at stage r and we would have $f^r(i) \neq f^{r-1}(i)$. It is now easily seen that $\overline{W_e} \cap B_i \neq \emptyset$, so R_m is satisfied. $\square$

Lemma 4.17. *If there is no i such that B_i is assigned to R_m at all sufficiently large stages, then R_m is satisfied.*

Proof. We say that the block B_a has *settled down by stage s* if for every $r \geq s$, $B_a^r = B_a$, $f^r(a) = f(a)$, $g^r(a) = g(a)$ and $t^r(B_a^r) = t(B_a)$. Assume that $R_m \in \{C_e, I_e\}$.

Let $\widehat{m}$ be the maximum of $\{m\} \cup \{j \mid f(j) \in \omega \wedge f(j) \leq m\}$. The number $\widehat{m}$ exists because f is injective (when taking values in ω). By Lemma 4.12, there is a stage s_0 such that for all $a \leq \widehat{m}$, B_a has settled down by stage s_0. Let X be the set of all x which are not in any block at stage s_0. X is cofinite, so to prove R_m is satisfied, it suffices to show that $X \subseteq W_e$.

We fix an arbitrary $x \in X$ and show $x \in W_e$. Let $s_1 > s_0$ be the first stage at which x is placed in a block (by the action of Case 1 of the construction) and let $B_d^{s_1} = \{x\}$. For all $r \geq s_1$, x will be in some block B_b^r such that $\widehat{m} < b \leq d$. Furthermore, if $x \in B_b^r$, $x \in B_i^{r+1}$ and $b \neq i$, then $\widehat{m} < i < b$ and $f^{r+1}(i) = O$.

By Lemma 4.12, we can fix a stage $s_2 \geq s_1$ and an index i such that $\widehat{m} < i$, $x \in B_i^{s_2}$, $B_i^r = B_i$ for all $r \geq s_2$, and $B_i^{s_2} \neq B_i^{s_2-1}$. By the comments above, these conditions imply that $f^{s_2}(i) = O$.

Notice that no block B_a with $a < i$ can act at any stage $r \geq s_2$, since any such action would cause $B_i^{r+1} \neq B_i^r$. In particular, we cannot have an $a < i$ and $r \geq s_2$ such that $f^r(a) = m$. This claim follows because such a B_a cannot be permanently assigned to R_m. Hence, if such an a and r did exist, then at some stage $t \geq r$, we will have $f^{t+1}(a) \neq f^r(a)$ either because B_a^t acts through (4.3) or some B_c^t with $c < a$ acts causing $f^{t+1}(a)$ to become undefined or set to O. Since no B_a with $a < i$ can act at or after s_2, neither of these situations can occur.

Since $f^{s_2}(i) = O$ and no B_a with $a < i$ acts at or after stage s_2, $B_i^{s_2}$ will act through (4.2) at stage s_2. $B_i^{s_2}$ chooses the least $n \leq i$ such that $f^{s_2}(j) \neq n$ for all $j < i$ and sets $f^{s_2+1}(i) = n$. Since $m < i$, m is one of the potential choices for n, and by the comments in the previous paragraph, $n \leq m$. We split into two cases.

Suppose $f^{s_2+1}(i) = l < m$. By our choice of $\widehat{m}$, there is no c such that $f(c) = l$ and $x \in B_c$ (and so, in particular, $f(i) \neq l$). Therefore, there must be a first stage $s_4 > s_2$ such that $f^{s_4+1}(i) \neq l$. Since this change cannot be caused by the action of B_a for $a < i$, it must be caused by $B_i^{s_4}$ acting through (4.3). At stage $s_4 + 1$, $B_i^{s_4}$ looks for the least $n \in (l, i]$ such that $f^{s_4}(j) \neq n$ for all $j < i$. Since $m \in (l, i]$, we must have $l < f^{s_4+1}(i) \leq m$. Repeating the argument for $f^{s_4+1}(i)$ in place of l, it is clear that eventually there is a stage $t > s_2$ at which $f^t(i) = m$ and a stage $u > t$ at which B_i^u acts through (4.3) because $\overline{W_e} \cap B_i^u = \emptyset$, proving that $x \in W_e$ as required.

Finally, suppose $f^{s_2+1}(i) = m$. Since B_i is not assigned permanently to R_m and no B_a with $a < i$ acts after s_2, B_i must eventually act through (4.3) at a stage $s_3 > s_2$. This action implies that $\overline{W_e} \cap B_i^{s_3} = \emptyset$, so $x \in W_e$ as required because $x \in B_i^{s_2} = B_i^{s_3}$.

□

The theorem now follows from Lemmas 4.15–4.17. □

Corollary 4.18 (Jockusch, Lerman, and Solomon)**.** *There is an infinite computable partial ordering $\mathcal{P}$ such that*

1. *$\mathcal{P}$ contains no infinite Π^0_1 chains or antichains*
2. *Every copy $\mathcal{Q}$ of $\mathcal{P}$ contains an infinite chain and also an infinite antichain which are both $\Delta^0_2(\mathcal{Q})$.*

Proof. Let $\mathcal{P}$ be as in the theorem. Then $I_{\mathcal{P}}$ and $S_{\mathcal{P}} \cup L_{\mathcal{P}}$ are both infinite since otherwise $\mathcal{P}$ would have an infinite computable chain or antichain by Proposition 1.2. If $\mathcal{Q}$ is a copy of $\mathcal{P}$ then $\mathcal{Q}$ is also weakly stable, and furthermore $I_{\mathcal{Q}}$ and $S_{\mathcal{Q}} \cup L_{\mathcal{Q}}$ are both infinite. It follows by relativizing Proposition 1.2 that $\mathcal{Q}$ contains an infinite chain and also an infinite antichain which are both $\Delta^0_2(\mathcal{Q})$. □

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