Jumps of nontrivial splittings of r.e. sets

MICHAEL A. INGRASSIA, STEFFEN LEMPP

Abstract. In an infinite injury construction, we construct a nonrecursive recursively enumerable (r.e.) set A such that whenever A is split into nonrecursive r.e. sets A_0 and A_1 then $A_0', A_1' <_T A'$.

1. The theorem. A pair of recursively enumerable (r.e.) sets A_0 and A_1 is said to *split* an r.e. set A if $A = A_0 \sqcup A_1$ (i.e., $A = A_0 \cup A_1$ and $\emptyset = A_0 \cap A_1$). Friedberg [5] was the first to prove that any nonrecursive r.e. set can be split into two nonrecursive r.e. sets. Sacks [15] improved this result by showing that the two halves can be made of low incomparable degrees. Other well-known splitting results were obtained by Owings [13], R. W. Robinson [14], Morley and Soare [12], and Lachlan [8].

Lerman and Remmel introduced the *universal splitting property (USP)* of an r.e. set A, namely, that any r.e. degree $\mathbf{d} \leq \deg(A)$ is realized as the degree of a splitting half of A. They showed [10, 11] that both the degrees containing USP sets and the degrees not containing any USP set are downward dense in the partial order $\mathbf{R}$ of the r.e. degrees. Downey [3] exhibited a non-USP set in every nonrecursive r.e. degree. The so-called *strong universal splitting property* (in which the degrees of both splitting halves can be prescribed) was introduced and studied by Ambos-Spies and Fejer [2].

In a different direction, call an r.e. set A *mitotic* if A can be split into r.e. sets of the same degree. Lachlan [7] proved the existence of nonmitotic r.e. sets, and Ingrassia [6] improved this result by showing that their degrees are dense in $\mathbf{R}$. Ambos-Spies [1], and independently Downey and L. Welch [4], constructed *antimitotic* sets (r.e. sets such that the degrees of any splitting into nonrecursive r.e. sets form a minimal pair).

Ambos-Spies [1] also initiated the study of jumps of splittings of r.e. sets by building an r.e. set A such that for any splitting into r.e. sets A_0 and A_1, not both A_0 and A_1 have the same jump as A (a property he called *strong nonmitoticity*). We strengthen this result and answer a question of Remmel (see Downey and L. Welch [4]) as follows:

THEOREM. *There is a nonrecursive r.e. set A such that whenever A is split into two nonrecursive r.e. sets A_0 and A_1 then $A_0', A_1' <_T A'$.*

The proof uses a new technique for handling jumps of r.e. sets, developed by Lempp and Slaman [9] in their solution to the deep degree problem.

Our notation follows Soare [16].

2. The requirements and the strategies. We will build an r.e. set A satisfying the following requirements (for all e, i, j):

$$\mathcal{R}_e : A \neq \{e\},$$
$$\mathcal{S}_{i,j} : A = W_i \sqcup W_j \implies A' \not\leq_T W_i' \text{ or } W_j \leq_T \emptyset.$$

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For each $\mathcal{S}_{i,j}$ we construct a partial recursive functional $\Gamma_{i,j}$. Using the Limit Lemma, we will then ensure $\mathcal{S}_{i,j}$ by satisfying for all k the requirements

$$\mathcal{S}_{i,j,k} : A = W_i \sqcup W_j \implies \lim_s \Gamma^A_{i,j}(-, s) \neq \lim_v \Phi^{W_i}_k(-, v) \text{ or } W_j \leq_T \emptyset$$

(There is also a hidden requirement that if $A = W_i \sqcup W_j$ and $W_j >_T \emptyset$ then $\Gamma^A_{i,j}$ is total and $\lim_s \Gamma^A_{i,j}(x, s)$ exists for all x.)

Each $\mathcal{R}_e$-strategy acts at most once, so an $\mathcal{S}_{i,j,k}$-strategy need not be concerned about the (finite) injury by higher-priority $\mathcal{R}_e$-strategies. A typical $\mathcal{S}_{i,j,k}$-strategy α will first try to show W_j recursive via some recursive functional Δ, which requires (potentially) infinite A-restraint (to prevent W_j from changing). It will deal with the (necessary) infinite injury by lower-priority $\mathcal{R}_e$-strategies as follows: Whenever a lower-priority $\mathcal{R}_e$-strategy wants to put some number z into A, then α will first start setting $\Gamma^A_{i,j}(x, s) = 1$ with use $\gamma_{i,j}(x, s) = z$ for larger and larger s (where x is the argument at which α is trying to achieve $\lim_s \Gamma^A_{i,j}(x, s) \neq \lim_v \Phi^{W_i}_k(x, v)$) and search for a (new) v such that $\Phi^{W_i}_k(x, v) = 1$. If and when it finds that v, then z is allowed into A, enabling us to reset $\Gamma^A_{i,j}(x, s) = 0$. If only finitely many of these z enter W_j then the injury to Δ is finite, and therefore W_j is recursive. On the other hand, if infinitely many of these z enter W_j then these z will not enter W_i, and if we can protect infinitely many of the corresponding computations $\Phi^{W_i}_k(x, v) = 1$ from later injury to W_i then the limits of $\Gamma^A_{i,j}(x, -)$ and $\Phi^{W_i}_k(x, -)$ will be different (if the latter exists at all).

Protection of these Φ_k-computations of *one* $\mathcal{S}_{i,j,k}$-strategy α from injury by infinitely many $\mathcal{R}_e$-strategies can be ensured by "rearranging the priorities" of the $\mathcal{R}_e$-strategies, using a noneffective function b and letting an $\mathcal{R}_{b(n)}$-strategy β have higher priority than any $\beta' \in \mathcal{C}(n)$, the set of $\mathcal{R}_e$-strategies with $b(n-1) < e < b(n)$. Now when $b(n-1)$ has been determined permanently, then $b(n)$ will be the index of the next $\mathcal{R}_e$-strategy whose z enters W_j, at stage $t(n)$, say, and therefore no $\mathcal{R}_e$-strategy can injure the Φ_k-computation of $\alpha = a(n)$ since they stopped acting for $e \leq b(n-1)$ by hypothesis, or have to respect the restraint for $e > b(n-1)$ by the rearrangement of priorities. (In the construction below, we will actually rearrange the $\mathcal{R}_e$-strategies in the tree priority ordering rather than the linear ordering outlined above. To ensure that every $\mathcal{S}_{i,j,k}$-strategy has infinitely many chances to rearrange the priorities of the $\mathcal{R}_e$-strategies, we will define a function P, rearranging the priorities of the $\mathcal{S}_{i,j,k}$-strategies for this purpose.)

Notice finally that above we have suppressed the two finite outcomes of an $\mathcal{S}_{i,j,k}$-strategy, namely, $A \neq W_i \sqcup W_j$, and that the search for a new v such that $\Phi^{W_i}_k(x, v) = 1$ is unsuccessful in which case $\lim_s \Gamma^A_{i,j}(x, s) = 1$ but not $\lim_v \Phi^{W_i}_k(x, v) = 1$.

3. The construction. The construction is organized on a tree $T = 2^{<\omega}$ of strategies. Strategy $\gamma \in T$ works on requirement $\mathcal{S}_{i,j,k}$ if $|\gamma| = 2\langle i, j, k\rangle$ is even, and on requirement $\mathcal{R}_e$ if $|\gamma| = 2e + 1$ is odd.

Diagrams 1 and 2 show the flow charts for $\mathcal{S}_{i,j,k}$- and $\mathcal{R}_e$-strategies. A strategy, upon initialization, starts in state *init*, picking a witness x or z bigger than any number mentioned in the construction so far, and, whenever eligible to act, proceeds along the arrows to the next state (denoted by a circle). Along the way, it executes the instructions (in rectangular boxes) and makes decisions (in diamonds or hexagons). Through outside action, it may

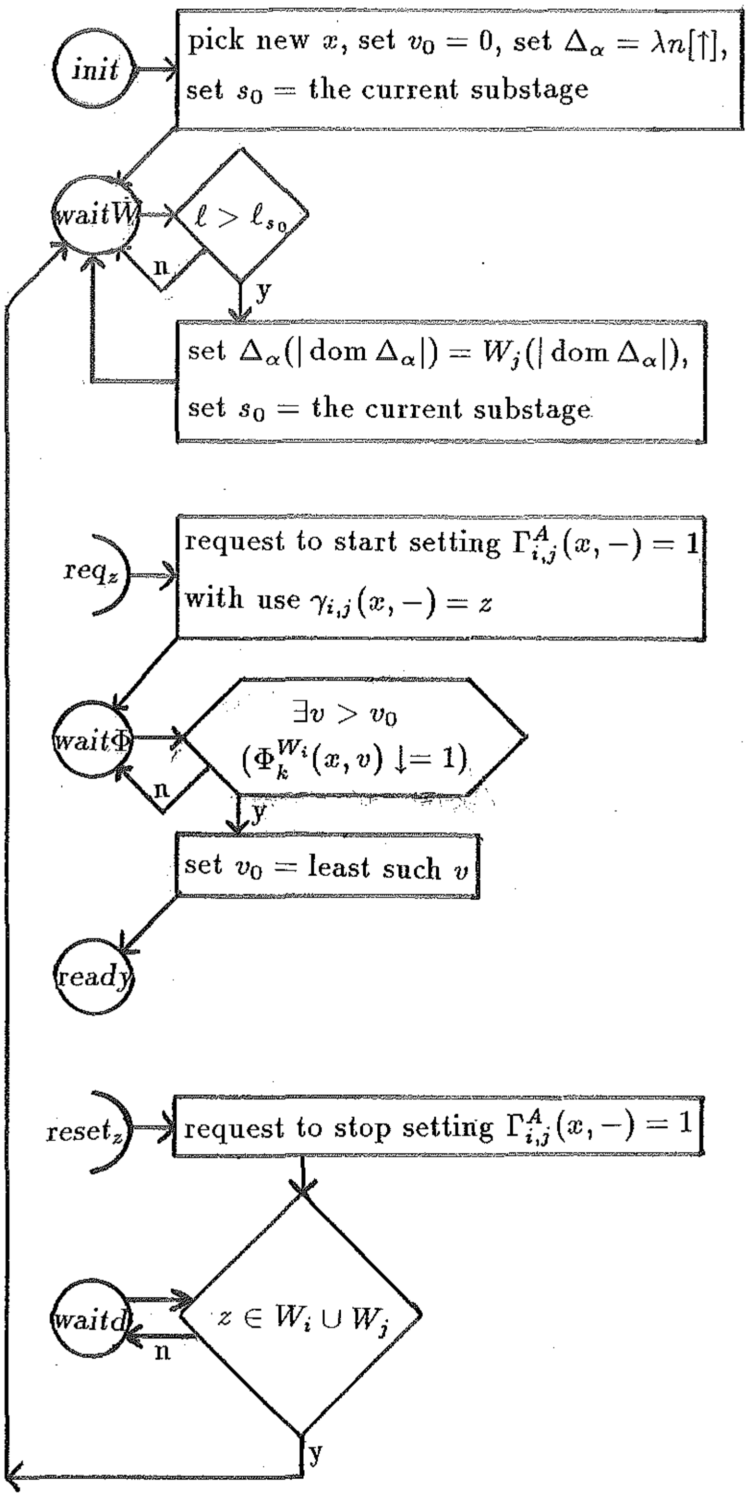

Diagram 1: $S_{i,j,k}$-strategy α

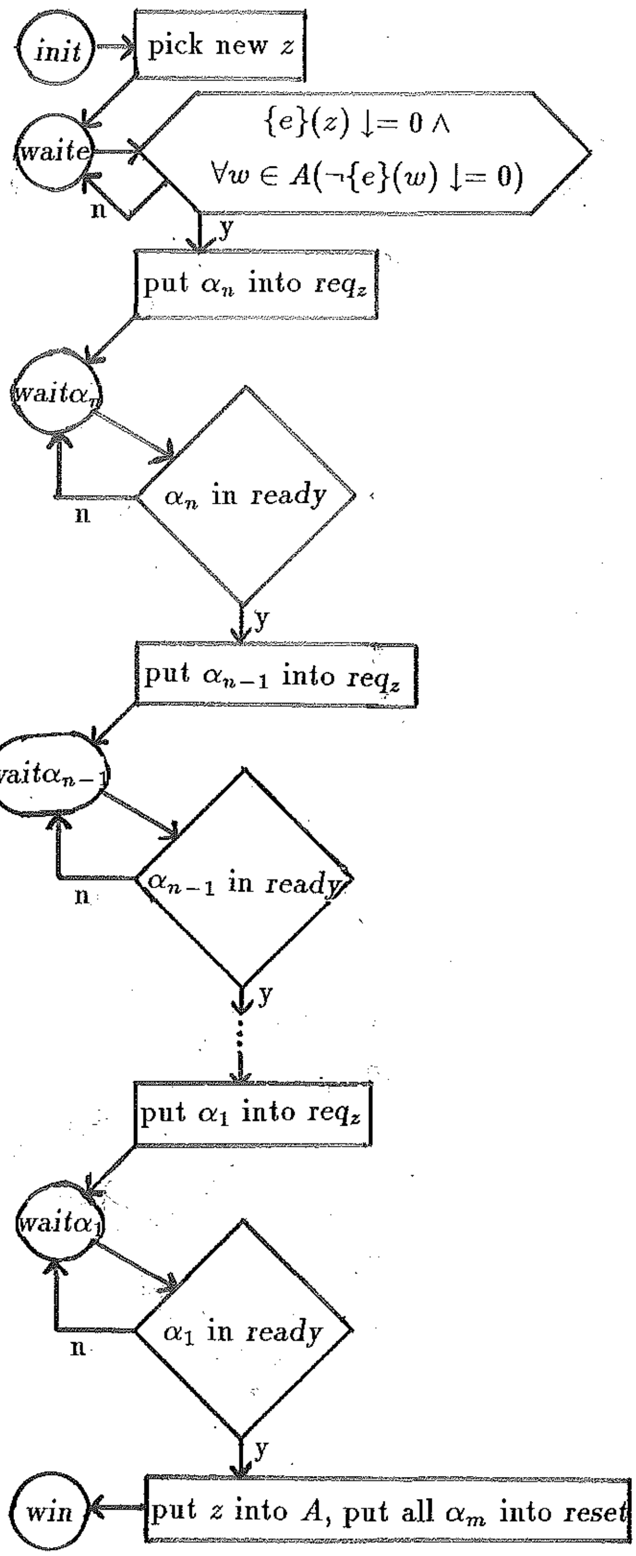

Diagram 2: $\mathcal{R}_e$-strategy β

be put into special states (in half-circles) from which it proceeds immediately to the next state. All parameters are taken at the current substage unless sub-indexed by a previous substage. (We will assume from now on that a substage also codes the corresponding stage.)

For diagram 1, the parameters x, s_0, v_0, and Δ_α are defined in the diagram and roughly denote the witness at which α tries to achieve $\lim_s \Gamma^A_{i,j}(x,s) \neq \lim_v \Phi^{W_i}_k(x,v)$, the last ($A = W_i \sqcup W_j$)-expansionary stage, the last "opponent's stage" at which $\Phi^{W_i}_k(x, v_0) = 1$, and the partial recursive function trying to witness the recursiveness of W_j, respectively. The parameter ℓ is the length of agreement $\ell = \max\{y \mid \forall u < y(A(u) = W_i(u) + W_j(u))\}$. The partial recursive functional $\Gamma_{i,j}$ is global to the construction and shared by all $\mathcal{S}_{i,j,k}$-strategies for this pair $\langle i,j \rangle$. An $\mathcal{S}_{i,j,k}$-strategy can only issue requests for $\Gamma_{i,j}$, which will be observed at the end of a stage as described below.

For Diagram 2, the parameter z is defined in the diagram and denotes the witness at which β is trying to achieve $A \neq \{e\}$. The strategies $\alpha_1, \ldots, \alpha_n$ mentioned in the diagram are exactly the $\mathcal{S}_{i,j,k}$-strategies α_m with $\alpha_m^\wedge \langle 0 \rangle \subseteq \beta$ in increasing order of length.

We are now ready to describe the full construction.

At stage 0 of the construction, all strategies are initialized in order of increasing length, the functions $a, b, \mathcal{C}$, and t are completely undefined, and we set $P(\gamma) = |\gamma|$ for all $\gamma \in T$.

A stage $s+1$ consists of three steps:

First, pick the highest-priority $\mathcal{R}_e$-strategy β that is in some state *wait*α_m and that can proceed to state *wait*α_{m-1} or *win*. If β exists let it act. If β also reaches *win* and $\beta \in \mathcal{C}(n_0)$ for some n_0, then initialize all $\gamma > \beta$, make the functions $a, b, \mathcal{C}$, and t undefined for arguments $n > n_0$, and set $P(\gamma) = P_{t(n_0)}(\gamma)$ for all $\gamma \in T$.

Secondly, we proceed in substages $t \leq s$. At a substage $t \leq s$, a strategy γ of length t is eligible to act according to its flow chart.

If γ is an $\mathcal{S}_{i,j,k}$-strategy and has changed states from *waitd* to *waitW* at this substage while its $z \in W_j$ then let n_0 be the greatest n such that $a(n)$ is defined and $P_{t(n)}(a(n)) \leq P(\gamma)$. (Allow $n_0 = -1$ here.) Then (re)define

$$
\begin{aligned}
a(n_0+1) &= \gamma, \\
b(n_0+1) &= \text{the } \mathcal{R}_e\text{-strategy that put } z \text{ into } A, \\
\mathcal{C}(n_0+1) &= \{\beta \in T - \bigcup_{n \leq n_0} \mathcal{C}(n) \mid |\beta| \leq s \text{ odd}\}, \\
t(n_0+1) &= \text{ the current substage.}
\end{aligned}
$$

Make the functions $a, b, \mathcal{C}$, and t undefined for arguments $n > n_0 + 1$.

Increment $P(\gamma)$ by $+1$ and set $P(\alpha) = P_{t(n_0)}(\alpha)$ for all $\mathcal{S}_{i,j,k}$-strategies $\alpha \neq \gamma$. Initialize all $\beta \in \mathcal{C}(n_0+1)$.

The strategy eligible to act at the next substage is $\gamma^\wedge\langle 0 \rangle$ if γ is an $\mathcal{S}_{i,j,k}$-strategy and has extended the definition of Δ_γ at the current substage, otherwise $\gamma^\wedge\langle 1 \rangle$.

At the end of the second step of stage $s+1$, we initialize all $\gamma' > \gamma$ where γ is the strategy that acted at substage s.

In the third and final step of stage $s+1$, we (re)define $\Gamma^A_{i,j}(x,u)$ for all i, j, x and all $u \leq s$ if it is now undefined. If some $\mathcal{S}_{i,j,k}$-strategy α currently works with witness

x and requests to start setting $\Gamma^A_{i,j}(x,-) = 1$, i.e. is currently in state *wait*Φ or *ready*, then set $\Gamma^A_{i,j}(x,u) = 1$ with requested use $\gamma_{i,j}(x,u) = z$; otherwise set $\Gamma^A_{i,j}(x,u) = 0$ with $\gamma_{i,j}(x,u) = 0$.

4. The verification. We first need to show that the rearrangement of priorities works properly:

LEMMA 1 (REARRANGEMENT OF PRIORITIES LEMMA).

(i) *The limit functions* $a = \lim_s a_s$, $b = \lim_s b_s$, $\mathcal{C} = \lim_s \mathcal{C}_s$, *and* $t = \lim_s t_s$ *are well-defined and total.*

(ii) *If for an* $\mathcal{S}_{i,j,k}$*-strategy* α *there are infinitely many* n *and* s *such that* $a_s(n) = \alpha$ *then there are infinitely many* n *such that* $a(n) = \alpha$. *(Thus* $P(\alpha) = \lim_s P_s(\alpha) \leq \infty$ *exists for all* α.*)*

PROOF: i) Since $W_i = \emptyset$, $W_j = A$ for some i, j, and since A is infinite, we will set $a_s(n) = \alpha$ infinitely often (for some α). Observe that all strategies working on a fixed requirement $\mathcal{R}_e$ combined put at most finitely many numbers into A. Since $P_s(a_s(n))$ is nondecreasing in n (at all s), and since $P_{s-1}(a_{s-1}(n)) \uparrow$ or $> P_s(a_s(n))$ when we define $a_s(n)$, part (i) follows by induction on $p = P_s(a_s(n))$ and on n.

(ii) We will first show that for any p we define $a(n)$ with $P_{t(n)}(a(n)) \leq p$ only finitely often. We proceed by induction on p and assume the statement for $P_{t(n)}(a(n)) < p$. (Allow $p = 0$ here.) Suppose the statement is false for $P_{t(n)}(a(n)) \leq p$. Since $P_s(\alpha) \geq |\alpha|$, it suffices to show that there are not $n_1 < n_2$ such that $a(n_1) = a(n_2) = \alpha$ and $P_{t(n_1)}(\alpha) = P_{t(n_2)}(\alpha) = p$. For the sake of a contradiction, assume there is such an α. It is impossible that some $\beta \in \bigcup_{n<n_1} \mathcal{C}(n)$ decreased $P(\alpha)$ between $t(n_1)$ and $t(n_2)$ since this would have caused a redefinition of $a(n_1)$. So some $\mathcal{S}_{i,j,k}$-strategy α' must have decreased $P(\alpha)$ to $p-1$ between $t(n_1)$ and $t(n_2)$, say, at some (least) substage s'. Then $P_{s'}(\alpha) = P_{t_{s'}(n_0)}(\alpha)$ for some n_0 with $n_1 < n_0 < n_2$, and $t_{s'}(n_0) \geq t(n_1)$, so $P_{t_{s'}(n_0)}(\alpha) \geq P_{t(n_1)}(\alpha) = p$, a contradiction.

For part (ii), we now just observe that, by the above, each α will eventually either satisfy $P_s(\alpha) \geq p$ for all p, or else eventually not want to set $a(n) = \alpha$ for any n.

We now define the *true path* f of the construction as the leftmost path on T on which any strategy is eligible to act infinitely often.

LEMMA 2 (INITIALIZATION LEMMA). *Any* $\gamma \subset f$ *is initialized at most finitely often.*

PROOF: By induction on $|\gamma|$, let s' be the least substage $> |\gamma|$ after which γ^- is no longer initialized. (Set $s' = 0$ for $\gamma = \emptyset$.) If γ is an $\mathcal{R}_e$-strategy, we define $n_s(\gamma)$ to be the unique n such that $\gamma \in \mathcal{C}_s(n)$ at s ($> |\gamma|$), observe that $n_s(\gamma)$ is nonincreasing in s, and set $n(\gamma) = \lim_s n_s(\gamma)$. Then we assume furthermore that $\mathcal{C}(n(\gamma))$ has been defined permanently before s'.

Now, by our assumptions on s', the construction can initialize γ at a substage s after s' only if $\gamma = \gamma^{-\wedge}\langle 1\rangle$ and $\gamma^{-\wedge}\langle 0\rangle$ is eligible to act at s, or if γ puts its z into A. By the definition of the true path or by the construction, respectively, this will happen at most finitely often.

We are now in a position to prove the two main lemmas that establish the theorem:

LEMMA 3 (CONVERGENCE LEMMA). *For all i and j:*

(i) $\Gamma^A_{i,j}$ *is total, and*

(ii) $\lim_s \Gamma^A_{i,j}(x,s)$ *exists for all* x.

PROOF: Since $\gamma_{i,j}(x,s)$ increases at most once for fixed x and s, (i) follows by the third step of each stage of the construction.

Again by the third step, part (ii) is trivial if eventually no $\mathcal{S}_{i,j,k}$-strategy works on x. Otherwise, some fixed $\mathcal{S}_{i,j,k}$-strategy will eventually always work on x. But then $\Gamma^A_{i,j}(x,s)$ is set or reset to 0 eventually for all s unless α is eventually always in state *wait*Φ or *ready* in which case $\Gamma^A_{i,j}(x,s) = 1$ for almost all s.

LEMMA 4 (OUTCOME LEMMA). *Each* $\gamma \subset f$ *satisfies its requirement.*

PROOF: By Lemma 2, let s' be the least stage such that γ is not initialized after stage s'.

First assume that γ is an $\mathcal{R}_e$-strategy. Since $\gamma \subset f$, γ must eventually be in state *waite* or in state *win*. In either case, $\mathcal{R}_e$ is satisfied.

On the other hand, assume that γ is an $\mathcal{S}_{i,j,k}$-strategy and that $A = W_i \sqcup W_j$. Suppose first that $\gamma^\wedge\langle 1\rangle \subset f$. Then, since $\gamma \subset f$ and $A = W_i \sqcup W_j$, α must eventually always be in state *wait*Φ. But then $\lim \Gamma^A_{i,j}(x,s) = 1$ and not $\lim_s \Phi^{W_i}_k(x,v) = 1$.

Finally, assume $\gamma^\wedge\langle 0\rangle \subset f$. Then Δ_γ must be a total recursive function. Suppose $W_j \neq^* \Delta_\gamma$. Then for infinitely many n and s, $a_s(n) = \gamma$, so by Lemma 1 (i) there are infinitely many n such that $a(n) = \alpha$. But then, for all these n, by the construction, no $\beta \in \bigcup_{m<n} \mathcal{C}(m)$ will put a number into A after $t(n)$; every $\beta \in \mathcal{C}_{t(n)}(n)$ is initialized at $t(n)$, so its number $z > t(n)$ if it enters after $t(n)$; and any other $\mathcal{R}_e$-strategy β has $|\beta| > t(n)$. Therefore we have an increasing sequence $\{v_n\}_{n\in w}$ such that $\Phi^A_k(x,v_n)\downarrow= \Phi^{A_{t(n)}}_{k,t(n)}(x,v_n)\downarrow= 1$ while $\Gamma^A(x,s) = 0$ for all s as in the proof of Lemma 2 (ii). This establishes W_j recursive, or $\lim_s \Gamma^A(x,s) = 0$ and not $\lim_v \Phi^{W_i}_k(x,v) = 0$, in the case $\gamma^\wedge\langle 0\rangle \subset f$.

The last two lemmas complete the proof of the theorem.

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Department of Mathematics and Computer Science, State University of New York, College at New Paltz, New Paltz, NY 12561, USA
Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA