SPECTRA OF COMPUTABLE MODELS OF STRONGLY MINIMAL DISINTEGRATED THEORIES IN RANK 1 LANGUAGES

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Abstract. We study, for a fixed first-order theory T, which countable models of T can be presented effectively. We consider this question for the class of strongly minimal disintegrated theories, where the countable models can be characterized by their dimension. The spectrum of computable models of T is the subset S of $\omega+1$ such that $\alpha \in S$ if and only if the αth model of T can be effectively presented.

We examine the class of strongly minimal disintegrated theories in computable relational languages where each relation symbol defines a set of Morley rank at most 1. We characterize the spectra of computable models of such theories (exactly, with the exception of three sets) under the assumption of bounded arity on the language, and (with the exception of one specific set and one specific class of sets) without that assumption. We also determine the exactly seven possible spectra for strongly minimal theories in binary relational languages and show that there are at least nine but no more than eighteen spectra of disintegrated theories in ternary relational languages.

1. Introduction

Classifying the computable models of a given first-order theory (in a computable language) has been a long-standing problem in computable model theory, going back at least four decades. Here, a model is called computable if there is an isomorphic copy of it with universe ω such that the quantifier-free diagram of the model forms a computable set, and a language is called computable if the signature is computable and for each symbol in the signature, we can effectively determine its arity and whether it is a relation, function or constant symbol.

Our problem can be stated particularly succinctly for uncountably but not totally categorical theories T, since in that case, the countable models of T form an elementary chain $\mathcal{M}_0 \prec M_1 \prec \cdots \prec \mathcal{M}_\omega$, where $\mathcal{M}_\alpha$ is the model of dimension $p+\alpha$ and where p is the dimension of the prime model. Classifying the computable models of such T can now be phrased in terms of subsets of $[0,\omega] = \omega+1 = \omega \cup \{\omega\}$; namely, we call the set $\mathrm{SRM}(T) = \{\alpha \leqslant \omega \mid \mathcal{M}_\alpha \text{ is computable}\}$ the *spectrum of computable models* of T. (Note that for historical reasons, there is a slight mismatch in notation: The dimension of the prime model can be positive, but we still refer to it as $\mathcal{M}_0$, etc. In the setting of our paper, we will always have $p \leqslant 1$.)

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This spectrum problem has been a motivating question in the field since Goncharov [Go78] proved that there are nontrivial spectra. Since then, only incremental progress has been made in classifying the spectra of uncountably but not totally categorical theories. In fact, very few negative results are known; and no results are known separating uncountably categorical theories from the more restrictive notion of strongly minimal theories; so in this paper, we will restrict our attention to the latter. (We will also tacitly assume from now on that all languages are computable.)

What was known before the current paper is contained in the following theorems:

Theorem 1. *The following are spectra of computable models of strongly minimal theories:*

- $\varnothing$ *and* $[0, \omega]$,
- $\{0\}$ *(Goncharov* [Go78]*)*,
- $[0, n]$ *for any* $n \in \omega$ *(Kudaibergenov* [Ku80]*)*,
- $[0, \omega)$ *and* $[1, \omega]$ *(Khoussainov/Nies/Shore* [KNS97]*)*,
- $\{1\}$ *(Nies* [Ni99]*)*,
- $[1, \alpha)$ *for any* $\alpha \in [2, \omega]$ *(Hirschfeldt/Nies, see* [Ni99, p. 314]*)*,
- $\{\omega\}$ *(Hirschfeldt/Khoussainov/Semukhin* [HKS06]*)*,
- $\{0, \omega\}$ *(Andrews* [An11]*), and*
- $[0, n] \cup \{\omega\}$ *for any* $n \in \omega$ *(Andrews/Mermelstein* [AM21]*)*.

Despite the relatively few sets known to be spectra, there were also very few general upper bounds on the complexity of spectra.

Theorem 2. *Let* T *be a strongly minimal theory.*

- *(Nies* [Ni99]*) The spectrum of computable models of an uncountably categorical theory* T *is* $\Sigma^0_3(\varnothing^{(\omega)})$.
- *(Nies* [Ni99]*) The spectrum of computable models of a model complete uncountably categorical theory is* Σ^0_4.
- *(Goncharov/Harizanov/Laskowski/Lempp/McCoy* [GHLLM03]*) The spectrum of computable models of a strongly minimal disintegrated theory* T *is* Σ^0_5.

Even in the geometrically simplest case of a strongly minimal theory, namely, that of a binary relational language, the gap remained very large:

Theorem 3. *The following are spectra of computable models of strongly minimal theories in binary relational languages:*

- $\varnothing$ *and* $[0, \omega]$,
- $\{0\}$ *(Goncharov* [Go78]*)*,
- $[0, 1]$ *Kudaibergenov* [Ku80]*)*,
- $[1, \omega]$ *(Khoussainov/Nies/Shore* [KNS97]*)*,
- $\{1\}$ *(Nies* [Ni99]*), and*
- $\{\omega\}$ *(Hirschfeldt/Khoussainov/Semukhin* [HKS06]*)*.

The only known upper bound is the above result that the spectrum must be Σ^0_5 *(since for binary relational languages, a strongly minimal theory must be disintegrated).*

Further specializing from strongly minimal to strongly minimal disintegrated theories, a much stronger result is possible in the case of finite languages:

Theorem 4 (Andrews/Medvedev [AM14]). *The only possible spectra of computable models of a strongly minimal disintegrated theory in a finite language are $\varnothing$, $[0, \omega]$ and $\{0\}$. (They also show the same result for modular groups in a finite language and note that it follows from Poizat* [Po88] *that the only spectrum for field-like strongly minimal theories in finite languages is $[0, \omega]$.)*

The positive direction, showing that $\{0\}$ is a possible spectrum of a strongly minimal theory in a finite binary relational language, was obtained by Herwig, Lempp and Ziegler [HLZ99].

All known examples seemed to suggest that one needs to increase the arity more and more to obtain more spectra. We explore this point of view in a number of directions and obtain a number of fairly sharp results as well as a number of previously unknown spectra of disintegrated strongly minimal theories.

In Section 2, we will first show that in the case of (infinite) binary relational languages (in which case the theory must be disintegrated), the seven spectra in Theorem 3 are the only possible spectra for strongly minimal (and thus disintegrated) theories.

In the subsequent sections, we will extend our techniques to a larger class of strongly minimal disintegrated theories in infinite relational languages, but with restrictions on either the Morley rank or the arity of each relation:

In Section 3, we will show that if the arity of the (relational) language is bounded and the theory forces each relation to have Morley rank at most 1, then there are at least seven and at most ten possible spectra.

In Section 4, we will expand the work from the previous section by showing that removing the bound on the arity of the relations (but still assuming that the theory forces each relation to have Morley rank at most 1), the only possible new spectra are all the initial segments of ω, and possibly also the set $\{1, \omega\}$ and the sets of the form $S \cup \{\omega\}$ where S is a finite initial segment of ω.

In Section 5, we will turn to strongly minimal disintegrated theories in ternary relational languages (without assumptions on Morley rank) and show that there are at least nine but no more than eighteen possible spectra.

These results also lead us to the following sweeping

Conjecture 5. *For $m \geqslant 1$, there are only finitely many spectra of computable models of strongly minimal disintegrated theories T in computable relational languages of arity at most m.*

1.1. **Model-theoretic notions.** In this subsection, we briefly review the model-theoretic notions most pertinent to this paper.

Definition 6. *A theory T is* strongly minimal *if for every formula $\phi(x, \bar{y})$, there is an $N \in \omega$ so that in any model $\mathcal{M} \models T$ and for any tuple $\bar{a} \in M^n$, either $\{x \mid \mathcal{M} \models \phi(x, \bar{a})\}$ or $\{x \mid \mathcal{M} \models \neg\phi(x, \bar{a})\}$ has size $< N$.*

Definition 7. *For any structure $\mathcal{M}$ and elements A, b from M, we say b is* algebraic over A *($b \in \mathrm{acl}(A)$) if there is a formula with parameters from A which defines a finite set containing b.*

If A, b are from M, we say b is generic over A *if it is not algebraic. Note that if* $\mathrm{Th}(\mathcal{M})$ *is strongly minimal then there is a single type of a generic element over A.*

If a set has the property that no element is algebraic over the remainder of the set, then the set is said to be independent.

For any tuple $\bar{a}$ in a strongly minimal theory, the dimension *of the tuple is the maximal size of an independent subset.*

For a formula $\phi(\bar{x})$, possibly with parameters, we say the Morley rank *of the formula ϕ is the maximal dimension of a tuple $\bar{a}$ in a model of T so that $\phi(\bar{a})$ holds.*

Definition 8. *A strongly minimal theory T is* disintegrated *if, for any $A \subseteq M$ for $\mathcal{M} \models T$, $\operatorname{acl}(A) = \bigcup_{a \in A} \operatorname{acl}(\{a\})$.*

The following theorem is folklore, though it does fall out of the analysis in [AM14, Claim in the proof of Proposition 2.6].

Theorem 9. *If T is a strongly minimal theory in a relational language and each relation symbol defines a formula of Morley rank 1, then T is disintegrated.*

These should be seen as the canonical disintegrated strongly minimal theories. In fact, Andrews and Medvedev [AM14] show that every disintegrated strongly minimal theory is Δ_1^0-interdefinable with a relational strongly minimal theory where each relation is Morley rank 1.

2. Binary Languages

The purpose of this section is to prove the following

Theorem 10. *The following are exactly the seven possible spectra of computable models of strongly minimal disintegrated theories in binary relational languages: $\varnothing$, $[0, \omega]$, $\{0\}$, $\{1\}$, $\{0, 1\}$, $[1, \omega]$, and $\{\omega\}$.*

Proof. First of all, Theorem 3 shows that the seven spectra above are indeed realized in binary relational languages; recall that a binary relational language forces a strongly minimal theory to be disintegrated.

We now first reduce the number of possible spectra to a small finite list and then analyze cases. Throughout this section, we will assume an infinite computable binary relational language $\mathcal{L}$ which (in any $\mathcal{L}$-theory T considered) is effectively closed under permutation of variables, i.e., such that from an index of $R(x, y) \in \mathcal{L}$, we can effectively find an index for $R(y, x)$. This causes no loss of generality, as we can always computably add to any relational language new relations which are permutations of the variables of old relations, and computability of models is preserved.

Lemma 11. *For a strongly minimal $\mathcal{L}$-theory T in a binary relational language, if $k \in \mathrm{SRM}(T)$ for some $k \in [2, \omega]$, then the set of relations in $\mathcal{L}$ which are of rank 2 is a computable set. Thus T is computably interdefinable with a theory all relation symbols of which have rank at most 1.*

Proof. Let $\mathcal{N}$ be a computable presentation of $\mathcal{M}_k$ for some $k \geqslant 2$, and let $a, b \in N$ be a generic pair in $\mathcal{N}$. Then a relation R has rank 2 if and only if $\mathcal{N} \models R(a, b)$. So the rank-2 relations R form a computable set, and each such R can be replaced by $\neg R$, which by strong minimality has rank at most 1. □

Thus, in the case where $\mathrm{SRM}(T)$ contains an element $\geqslant 2$, we will assume from now on that every relation has rank $\leqslant 1$.

For the remainder of the proof, we need the following

Definition 12. *Fix a model $\mathcal{M}$ in a binary relational language $\mathcal{L} = \{R_0, R_1, \ldots\}$ such that the Morley rank of each relation is at most 1.*

(1) *The* 0-neighborhood *of an element $a \in M$ is* $\mathrm{Nbh}_0(a) = \{a\}$.
(2) *The* $(n+1)$-neighborhood $\mathrm{Nbh}_{n+1}(a)$ *of $a \in M$ is the union of the n-neighborhood* $\mathrm{Nbh}_n(a)$ *and the set of all elements $d \in M$ such that there are $c \in \mathrm{Nbh}_n(a)$ and a relation R_i (with $i \leqslant n+1$) with*

$$\big[R_i(c,d) \wedge \neg\exists^\infty y\, R_i(c,y) \wedge \neg\exists^\infty x\, R_i(x,d)\big] \vee$$
$$\big[\neg R_i(c,d) \wedge (\exists^\infty y\, R_i(c,y) \vee \exists^\infty x\, R_i(x,d))\big].$$

(3) *The* neighborhood $\mathrm{Nbh}(a)$ *of an element $a \in M$ is the union of all n-neighborhoods of a.*

(Recall that we assume the language $\mathcal{L}$ to be closed under permutation of variables.)

Before we embark on the rest of our proof, recall the structure of a strongly minimal model $\mathcal{M}$ in our language $\mathcal{L}$: Fixing a basis B of $\mathcal{M}$, the model will be the disjoint union of the algebraic closure $\mathrm{acl}(\varnothing)$ and the interalgebraic closures $\mathrm{iacl}(a) = \mathrm{Nbh}(a)$ for each $a \in B$ (by Herwig/Lempp/Ziegler [HLZ99] and Andrews/Medvedev [AM14]). Note that all sets $\mathrm{iacl}(a)$ for $a \notin \mathrm{acl}(\varnothing)$ will be isomorphic, and 1-transitive in $\mathcal{M}$; they may be finite or countably infinite. The algebraic closure $\mathrm{acl}(\varnothing)$ may be finite or countably infinite; it may even be empty. If $\mathrm{acl}(\varnothing)$ is finite, then the interalgebraic closures $\mathrm{iacl}(a)$ of all generic elements a must be infinite since T is assumed to be not totally categorical, and $\mathcal{M}_k$ has a basis of size $1+k$; otherwise, $\mathcal{M}_k$ has a basis of size k.

Throughout this section, we will repeatedly use the following

Observation 13. *If $\mathcal{M}_k$ is a computable model of T for $k \geqslant 2$ and c is any element of $\mathcal{M}$, then $\mathrm{Nbh}(c)$ is a Σ^0_1-set.*

Proof. The definition of $\mathrm{Nbh}(c)$ is Σ^0_1 over the set of (R, x) so that $\exists^\infty y\, R(x,y)$ (using the effective closure of $\mathcal{L}$ under permutation of variables). To see that this set is computable, we fix a pair of mutually generic elements a, b and claim that $\mathcal{M} \models \exists^\infty y\, R(x,y)$ if and only if $\mathcal{M} \models R(x,a) \wedge R(x,b)$.

Suppose that $\mathcal{M} \models \exists^\infty y\, R(x,y)$. Then, $x \in \mathrm{acl}(\varnothing)$ since R has rank 1. Thus both a and b are generic over x, so $\mathcal{M} \models R(x,a) \wedge R(x,b)$. Next, suppose that $\mathcal{M} \models R(x,a) \wedge R(x,b)$. If it were true that $\mathcal{M} \models \neg\exists^\infty y\, R(x,y)$, then both a and b would be algebraic over x, so the dimension of the pair $\{a, b\}$ would be $\leqslant 1$, but this contradicts a and b being mutually generic. $\square$

We now isolate a key step of Lemma 15 below, as it will reappear later on.

Lemma 14. *Let $\mathcal{N}$ be a computable strongly minimal structure. Let $\mathcal{M} \preceq \mathcal{N}$ be such that M is a Δ^0_2-subset of N, and let $A \subseteq M$ be any infinite Σ^0_1-set. Then there is a computable copy of $\mathcal{M}$.*

Proof. We may assume that $N \smallsetminus M$ is infinite, else the result is trivial. Let M_s be the set of elements believed to be in M at stage s via Δ^0_2-approximation. We may assume that if $z \in A_s$, the Σ^0_1-approximation to A, then $z \in M_s$. We construct a copy of $\mathcal{M}$ by copying elements and relations from $\mathcal{N}$. At stage s, we will have copied a finite set $X \subseteq N$ and the quantifier-free diagram of X in a finite sublanguage $\mathcal{L}_0$. At this stage s, we then want to reassign which elements we are copying so as to only be copying elements in M_s. Let $Y = X \smallsetminus M_s$. For each $y_0 \in Y$, either the approximation to M_s changes, or else

$$\phi(z) := \exists Z\ [(Z, X \smallsetminus Y, z) \cong_{\mathcal{L}_0} (Y, X \smallsetminus Y, y_0)]$$

is an existential first-order formula in the type of y_0 over $X \smallsetminus Y$. But if y_0 is not in M, then, by strong minimality, all but finitely many elements in N satisfy $\phi(z)$. Thus we eventually see an element $a \in A$ which satisfies ϕ. We reassign y_0 to be copying a. Repeating this process, eventually Y is empty.

At every stage s, we have built a finite structure X in a finite language $\mathcal{L}_0$ and have an embedding $f_s : X \to M_s$. If $x \in M$, then from some point onward $x \in \operatorname{ran}(f_s)$. If $x \notin M$, then from some point onwards $x \notin \operatorname{ran}(f_s)$. We argue by induction that for every x in our copy, $\lim_s f_s(x)$ is defined (and thus in M). We may suppose that this is the case for every $y < x$, and we choose a stage large enough so that $f_s(y)$ has stabilized for all $y < x$. Finally, if we ever see $f_t(x) \notin M$ for any $t > s$, we either see the approximation change back to showing $f_t(x) \in M$, or we set $f_{t+1}(x) \in A$. Once $f_{t+1}(x) \in A$, we know that the approximation will never say that it is out of M, and thus $\lim_s f_s(x) = f_{t+1}(x)$. Thus, in the limit, we construct a copy of $\mathcal{M}$. $\square$

Our next lemma reduces the number of possible spectra to a small finite number of possibilities:

Lemma 15. *For a strongly minimal $\mathcal{L}$-theory T in a binary relational language, if $k \in \mathrm{SRM}(T)$ for $2 \leqslant k < \omega$, then $[1, \omega] \subseteq \mathrm{SRM}(T)$.*

Proof. Fix a computable presentation $\mathcal{N}$ of the model $\mathcal{M}_k$. We first show that $k+1 \in \mathrm{SRM}(T)$. Fix a generic pair of elements a, b from $\mathcal{N}$. Then $\operatorname{iacl}(a) = \operatorname{Nbh}(a)$ is a Σ^0_1-set by Observation 13; so there is a computable structure $\mathcal{C}$ isomorphic to $\operatorname{Nbh}(a)$. We create a computable presentation $\mathcal{M}$ of $\mathcal{M}_{k+1}$, which is the disjoint union of a copy of $\mathcal{C}$ along with a copy of $\mathcal{N}$. We declare a relation R to hold on the pair (c, d) for $c \in C$ and $d \in N$ iff $\mathcal{N} \models R(a, d) \wedge R(b, d)$ (which extends to a full definition of all relations by our convention that $\mathcal{L}$ is effectively closed under permutation of variables). It is easy to see that $\mathcal{M}$ is isomorphic to $\mathcal{M}_{k+1}$, thus $k + 1 \in \mathrm{SRM}(T)$. As the procedure to take a presentation of $\mathcal{M}_k$ and produce a presentation of $\mathcal{M}_{k+1}$ is uniformly computable, we see that $\omega \in \mathrm{SRM}(T)$ as well.

It thus remains to verify that $\mathrm{SRM}(T)$ contains $k - 1$. The proof distinguishes two cases:

Case 1: $\operatorname{Nbh}(a)$ is finite: In this case, there is a computable (even cofinite) subset of $\mathcal{N}$ isomorphic to $\mathcal{M}_{k-1}$.

Case 2: $\operatorname{Nbh}(a)$ is infinite: In this case, $\mathcal{M}_{k-1}$ is a Δ^0_2- (in fact, a Π^0_1-)subset of $\mathcal{M}_k$. Furthermore, it contains an infinite Σ^0_1-set $\operatorname{Nbh}(b)$ for some generic element $b \in M_{k-1}$ (using $k-1 > 0$). Thus, by Lemma 14, $\mathcal{M}_{k-1}$ has a computable copy. $\square$

We immediately conclude the following

Corollary 16. *Every possible spectrum of a binary strongly minimal theory is among the following ten: $\varnothing$, $[0, \omega]$, $[1, \omega]$, $\{\omega\}$, $\{0\}$, $\{1\}$, $\{0, 1\}$, $\{0, \omega\}$, $\{1, \omega\}$, and $\{0, 1, \omega\}$.* $\square$

The first seven are known to be spectra of binary strongly minimal theories by Theorem 3. We will now show that the last three, $\{0, \omega\}$, $\{1, \omega\}$ and $\{0, 1, \omega\}$, are not spectra of binary strongly minimal theories:

Lemma 17. *If $\omega \in \mathrm{SRM}(T)$, and one of $0 \in \mathrm{SRM}(T)$ or $1 \in \mathrm{SRM}(T)$, then $2 \in \mathrm{SRM}(T)$, and so $[1, \omega] \subseteq \mathrm{SRM}(T)$ by Lemma 15.*

Proof. We fix a computable presentation $\mathcal{N}$ of the model $\mathcal{M}_\omega$ and a computable presentation $\mathcal{M}$ of the model $\mathcal{M}_0$ or $\mathcal{M}_1$. We distinguish two cases.

Case 1: There is no element $c \in \mathrm{acl}_{\mathcal{M}}(\varnothing)$ so that for each relation R, both $\neg\exists^\infty x\, R(x,c)$ and $\forall x\, (\exists^\infty y\, R(x,y) \to R(x,c))$ hold. (Recall again that we assume $\mathcal{L}$ to be closed under permutation of variables.)

In this case, we fix generic elements a and b in $\mathcal{N}$. We then have

$$\mathrm{acl}_{\mathcal{N}}(\varnothing) = \{d \mid \exists R \in \mathcal{L}\, [(R(a,d) \wedge R(b,d)) \vee \exists x\, (R(x,a) \wedge R(x,b) \wedge \neg R(x,d))]\}.$$

Thus $\mathrm{acl}_{\mathcal{N}}(\varnothing)$ is a Σ^0_1-subset of N. We have already seen, in Observation 13, that the inter-algebraic closure of a generic singleton in $\mathcal{N}$ is Σ^0_1, thus a computable presentation of $\mathcal{M}_2$ can be given as $\mathrm{acl}_{\mathcal{N}}(\varnothing) \cup \mathrm{iacl}_{\mathcal{N}}(a) \cup \mathrm{iacl}_{\mathcal{N}}(b)$, which is a Σ^0_1-subset of $\mathcal{N}$ and thus a computably presentable model.

Case 2: Otherwise, there is an element $c \in \mathrm{acl}_{\mathcal{M}}(\varnothing)$ so that for each relation R, both $\neg\exists^\infty x\, R(x,c)$ and $\forall x\, (\exists^\infty y\, R(x,y) \to R(x,c))$ hold. We fix such an element c and further distinguish two subcases.

Subcase 2.1: There is an element $d \in M$ so that for each relation R,

$$\mathcal{M} \models \forall x\ (\exists^\infty y\, R(x,y) \leftrightarrow (R(x,c) \wedge R(x,d)))\,.$$

This will in particular be the case if $\mathcal{M}$ is a presentation of $\mathcal{M}_1$ since we can choose a generic element d. If $\exists^\infty y\, R(x,y)$, then $x \in \mathrm{acl}(\varnothing)$ since R has rank 1, and so d is generic over x and $R(x,d))$ must hold. If $(R(x,c) \wedge R(x,d))$, then since $c \in \mathrm{acl}(\varnothing)$ and $\neg\exists^\infty y\, R(y,c)$, we again have that $x \in \mathrm{acl}(\varnothing)$, so d is generic over x and $R(x,d)$ implies $\exists^\infty y\, R(x,y)$.

We now expand $\mathcal{M}$ by defining new computable unary predicates A_i by letting $A_i(x)$ hold for $x \in M$ iff $\mathcal{M} \models R_i(x,c) \wedge R_i(x,d)$. Then by the assumptions of Subcase 2.1, $A_i(x)$ holds iff $\mathcal{M} \models \exists^\infty y\, R_i(x,y)$. As in Case 1, we can fix computable presentations of $\mathrm{iacl}(a)$ and $\mathrm{iacl}(b)$. We now define a computable presentation of $\mathcal{M}_2$ as the disjoint union of $\mathcal{M}$ and $\mathrm{iacl}(a)$ (as well as $\mathrm{iacl}(b)$ if $\mathcal{M}$ is a computable presentation of $\mathcal{M}_0$), letting $R_i(x,y)$ hold for $x \in M$ and $y \in \mathrm{iacl}(a)$ or $y \in \mathrm{iacl}(a) \cup \mathrm{iacl}(b)$, respectively, iff $A_i(x)$ holds, and letting $R_i(x,y)$ never hold for $x \in \mathrm{iacl}(a)$ and $y \in \mathrm{iacl}(b)$.

Subcase 2.2: Otherwise,

$$M = \{e \in N \mid \exists R \in \mathcal{L}\, \exists x\, (\mathcal{M} \models \exists^\infty y\, R(x,y) \not\leftrightarrow (R(x,c) \wedge R(x,e)))\}$$

(since, as noted in Subcase 2.1, we must have $\mathcal{M} = \mathcal{M}_0$ in Subcase 2.2). Let $c' \in N$ be the image of c under an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$, and again fix a pair of generic elements a and b in $\mathcal{N}$. Then

$$\mathrm{acl}_{\mathcal{N}}(\varnothing) = \{e \mid \exists R \in \mathcal{L}\, \mathcal{N} \models \exists x\, ((R(x,a) \wedge R(x,b)) \not\leftrightarrow (R(x,c') \wedge R(x,e)))\},$$

which is a Σ^0_1-set. But then, as in Case 1, the disjoint union of $\mathrm{acl}_{\mathcal{N}}(\varnothing)$, $\mathrm{Nbh}(a)$ and $\mathrm{Nbh}(b)$ is a Σ^0_1-subset of N isomorphic to $\mathcal{M}_2$. ☐

This concludes the proof of Theorem 10. ☐

3. Rank 1 Languages of bounded arity

For this section, we always assume T to be a strongly minimal disintegrated theory of relations of Morley rank at most 1. (We will specify in each statement whether it has bounded arity when we use that assumption.)

3.1. **The negative results.** We will first show that there are at most ten possible spectra in this setting. We begin with a definition generalizing Definition 12 to this setting.

Definition 18. *Fix a model $\mathcal{M}$ in a relational language $\mathcal{L} = \{R_0, R_1, \dots\}$ such that the Morley rank of each relation in $\mathcal{M}$ is at most 1. For each relation symbol $R \in \mathcal{L}$ of arity m, say, define $R^{j,l}$ (for $1 \leqslant j < l \leqslant m$) as the projection of R onto its jth and lth coordinates, i.e., $R^{j,l}(x_j, x_l)$ holds iff*

$$\exists x_1 \dots \exists x_{j-1} \exists x_{j+1} \dots \exists x_{l-1} \exists x_{l+1} \dots \exists x_m \, R(\bar{x}).$$

We now define the neighborhood of $a \in M$ by recursion as follows:

(1) *The* 0-neighborhood *of an element $a \in M$ is $\mathrm{Nbh}_0(a) = \{a\}$.*
(2) *The* $(n+1)$-neighborhood $\mathrm{Nbh}_{n+1}(a)$ *of $a \in M$ is the union of the n-neighborhood $\mathrm{Nbh}_n(a)$ and the set of all elements $d \in M$ such that there are $c \in \mathrm{Nbh}_n(a)$, a relation R_i (with $i \leqslant n+1$) of arity m, say, and $1 \leqslant j < l \leqslant m$ with*

$$\begin{aligned}&\big[R_i^{j,l}(c,d) \wedge \neg\exists^\infty y\, R_i^{j,l}(c,y) \wedge \neg\exists^\infty x\, R_i^{j,l}(x,d)\big] \vee \\ &\big[\neg R_i^{j,l}(c,d) \wedge (\exists^\infty y\, R_i^{j,l}(c,y) \vee \exists^\infty x\, R_i^{j,l}(x,d))\big].\end{aligned} \tag{$*$}$$

(3) *The* neighborhood $\mathrm{Nbh}(a)$ *of an element $a \in M$ is the union of all n-neighborhoods of a.*

(Recall that we assume the language $\mathcal{L}$ to be closed under permutation of variables.)

The main difficulty in working with rank 1 relations, over the binary case in the previous section, is that these projections inherently define uniformly Σ^0_1 subsets of models rather than uniformly computable subsets of models.

We first state a useful technical

Lemma 19. *Fix a model $\mathcal{M}$ of T of dimension at least 2 with a computable presentation in a relational language $\mathcal{L} = \{R_0, R_1, \dots\}$ such that the Morley rank of each relation is at most 1. Then the set of elements d so that $\exists^\infty y\, R_i^{j,l}(d,y)$ is a finite Σ^0_1-set. In particular, $\mathbf{0}'$ can compute the canonical index of the finite set of all elements $d \in M_k$ so that $\exists^\infty y\, R_i^{j,l}(d,y)$, uniformly in i,j,l.*

Proof. Note that, as in the proof of Observation 13, an element d satisfies the formula $\exists^\infty y\, R_i^{j,l}(d,y)$ if and only if it satisfies $R_i^{j,l}(d,a)$ and $R_i^{j,l}(d,b)$ for two mutually generic elements a and b. Thus the set of elements d so that $\exists^\infty y\, R_i^{j,l}(d,y)$ is a finite Σ^0_1-set. Thus $\mathbf{0}'$ can compute the canonical index of the set (uniformly in i,j,l). □

We note for future reference that Lemma 19 is uniform in the indices of relations, so in any model of dimension at least 2, the set $\mathcal{B}$ is a Σ^0_1-subset of $\mathrm{acl}(\varnothing)$.

We can now prove some analogs of lemmas from the binary case.

Lemma 20. *Let $\mathcal{M}$ be a model of a disintegrated strongly minimal theory with a computable presentation in a relational language $\mathcal{L} = \{R_0, R_1, \dots\}$ such that the Morley rank of each relation is at most 1. Suppose that $\mathbf{0}'$ can compute canonical indices for the sets $\{d \in M \mid \exists^\infty y\, R_i^{j,l}(d,y)\}$ uniformly in i,j,l. Then for each generic element a, $\mathrm{Nbh}(a) = \mathrm{iacl}(a)$ is a Σ^0_2-subset of M; in fact, $\mathbf{0}'$ can compute the canonical index of the n-neighborhood of a, uniformly in any generic element a and $n \in \omega$.*

Proof. Each one of the binary projections $R_i^{j,l}$ is $\mathbf{0}'$-computable. Andrews and Medvedev [AM14, Claim in the proof of Proposition 2.6] showed that if the language only consists of relations of rank at most 1, then $\mathrm{iacl}(a)$ is exactly $\mathrm{Nbh}(a)$.

Clearly, the second claim of our lemma implies the first, so fix n and a generic element a. The claim is trivial for $n = 0$, so assume we are given a canonical index for $\mathrm{Nbh}_n(a)$. Fix any element $c \in \mathrm{Nbh}_n(a)$. Since a and thus also c is generic, the second line of $(*)$ cannot apply to c and any d. The first conjunct of the first line of $(*)$ is Σ_1^0 and can apply to at most finitely many d, so $\mathbf{0}'$ can effectively find all possible d. But then for each of these finitely many d, we can $\mathbf{0}'$-effectively check the second and third conjunct by the assumption that $\mathbf{0}'$ computes $\{d \in M \mid \exists^\infty y\, R_i^{j,l}(d,y)\}$. Thus $\mathbf{0}'$ can compute a canonical index for $\mathrm{Nbh}_{n+1}(a)$. $\square$

Definition 21. *In any model $\mathcal{M}$ of T, we let $\mathcal{B}_n \subseteq M$ be the set $\bigcup_{i<n,j,l}\{d \mid \exists^\infty y\, R_i^{j,l}(d,y)\}$ and $\mathcal{B} = \bigcup_n \mathcal{B}_n \subseteq M$.*

Lemma 22. *If $\mathcal{B}$ is finite in $\mathcal{M}$ and $a \in M$ is generic, then $\mathrm{iacl}(a)$ is Σ_1^0.*

Proof. We enumerate $\mathrm{Nbh}(a)$ as follows: If $c \in \mathrm{Nbh}(a)$ and $R_i^{j,l}(c,e)$, then we enumerate e into $\mathrm{Nbh}(a)$ unless $e \in \mathcal{B}$. $\square$

The counterpart of Lemma 20 for $\mathrm{Nbh}(a)$ for $a \in \mathrm{acl}(\varnothing)$ requires a more careful proof since now the second line of $(*)$ can hold. However, we still have:

Lemma 23. *Fix a model $\mathcal{M}$ of a disintegrated strongly minimal theory with a computable presentation in a relational language $\mathcal{L} = \{R_0, R_1, \dots\}$ such that the Morley rank of each relation is at most 1. Suppose that $\mathbf{0}'$ can compute canonical indices for the sets $\{d \mid \exists^\infty y\, R_i^{j,l}(d,y)\}$ uniformly in i,j,l. Then $\mathrm{acl}(\varnothing)$ is a Σ_2^0-subset of $\mathcal{M}$.*

Proof. We may assume M contains a generic element, since otherwise $M = \mathrm{acl}(\varnothing)$, and we are done. By Lemma 20, we can fix a generic element b and uniformly in $\mathbf{0}'$ compute canonical indices of its n-neighborhoods. By Andrews and Medvedev [AM14, Proof of Proposition 2.6], $a \in \mathrm{acl}(\varnothing)$ iff for some n, the n-neighborhoods of a and b are not isomorphic over $\mathcal{B}_n$. So we need to verify that $\mathbf{0}'$ can enumerate all such a by enumerating enough of the neighborhood of a until we know that the n-neighborhoods of a and b are not isomorphic over $\mathcal{B}_n$ for some n. By assumption, $\mathbf{0}'$ can compute the sets $\mathcal{B}_n$. Clearly, the 0-neighborhoods of a and b are isomorphic, so fix n and assume we have $c \in \mathrm{Nbh}_n(a)$. For any fixed R_i with $i \leqslant n+1$ of arity m, say, and any j,l with $1 \leqslant j < l \leqslant m$, first check, whether $\exists^\infty y\, R_i^{j,l}(c,y)$. If so, then $a \in \mathrm{acl}(\varnothing)$. If not, then find the finite set of d so that $R_i^{j,l}(c,d)$ and $\neg\exists^\infty x\, R_i^{j,l}(x,d)$, or so that $\neg R_i^{j,l}(c,d)$ and $\exists^\infty x\, R_i^{j,l}(x,d)$. Doing this for all $c \in \mathrm{Nbh}_n(a)$ and all i,j,l with $i \leqslant n+1$ and $1 \leqslant j < l \leqslant m$, we either find out that $a \in \mathrm{acl}(\varnothing)$, or we compute a canonical index for $\mathrm{Nbh}_{n+1}(a)$. $\square$

Corollary 24. *If T is a relational strongly minimal disintegrated theory such that each relation has rank at most 1, and if $k \in [2,\omega)$, then for each $a \in M_k$, $\mathrm{iacl}(a)$ is Δ_2^0. In particular, $\mathrm{acl}(\varnothing)$ is a Δ_2^0-set.*

Proof. Let $\mathcal{M}_k$ have basis B. Then M_k is the finite disjoint union $\mathrm{acl}(\varnothing) \cup \bigcup_{b\in B} \mathrm{iacl}(b)$. Each piece is Σ_2^0 by Lemmas 19, 20, and 23; thus, each piece is Δ_2^0. $\square$

The following general theorem yields a strong upper bound for spectra of relational strongly minimal theories of rank 1, down from an upper bound of Σ^0_5 to a small collection of possible spectra.

Theorem 25. *Let T be a strongly minimal disintegrated theory in a relational language $\mathcal{L}$ consisting of relations $\{R_0, R_1, \dots\}$ of Morley rank at most 1 (closed under permutations of variables), and assume that $k \in \mathrm{SRM}(T)$ for some $k \in [2, \omega)$. Then $[1, k] \subseteq \mathrm{SRM}(T)$. If, in addition, there is a bound on the arities of the relations in $\mathcal{L}$, then $[1, \omega] \subseteq \mathrm{SRM}(T)$.*

Proof. We again proceed in two parts, the Up part and the Down part. We fix mutually generic elements $a, b \in M_k$.

Down: Fix $j \in [1, k)$.

Case 0: $\mathrm{Nbh}(a)$ is finite. Then M_j is a cofinite, thus computable, subset of M_k.

Case 1: $\mathcal{B}$ is finite. In this case, $\mathrm{Nbh}(g)$ is Σ^0_1 for any generic element g by Lemma 22. Then M_j is a Δ^0_2-subset of M_k by Corollary 24, and it contains the infinite Σ^0_1-set $\mathrm{Nbh}(a)$. Thus, by Lemma 14, $\mathcal{M}_j$ has a computable copy.

Case 2: $\mathcal{B}$ is infinite. In this case, $\mathrm{Nbh}(g)$ will generally only be Δ^0_2 for a generic element g by Corollary 24, but we can use the infinite Σ^0_1-subset $\mathcal{B}$ of $\mathrm{acl}(\varnothing)$ (from the remark following Lemma 22). Again, M_j is a Δ^0_2-subset of M_k containing an infinite Σ^0_1-set $\mathcal{B}$, so by Lemma 14, $\mathcal{M}_j$ has a computable copy.

Note that in Cases 0 and 2, in fact $[0, k] \subseteq \mathrm{SRM}(T)$, as the argument works for $j = 0$ as well.

Up: We again work in two cases. In the first case, we will be able to find an infinite Σ^0_1-set C of tuples $\bar{x}$ so that in each tuple at least one element is generic and such that C contains an infinite set of disjoint tuples. This will be useful in ensuring that what we build, in attempting to build a new generic neighborhood, is generic enough to handle further changes in our approximations to $\mathrm{Nbh}_n(g)$. We will show that if the first case does not hold, then for every generic g, $\mathrm{iacl}(g)$ is finite, and we prove the result directly in this case.

Case 1: There is a Σ^0_1-set C of tuples $\bar{e}$ (of varying sizes) so that at least one of the elements in each tuple is a generic element and so that C contains an infinite set of disjoint tuples.

This condition allows us to conclude that if we see every e in some $\bar{e} \in C$ satisfy some fixed $\exists$-formula $\phi(x)$, then we know that ϕ is generic. We want to "copy" $\mathrm{iacl}(g)$ for some generic g in $\mathcal{M}_k$ to build the model $\mathcal{M}_{k+1}$. The main difficulty is that $\mathrm{iacl}(g)$ is a Δ^0_2-subset of M_k. The source of movement out of $\mathrm{iacl}(g)$ is observing an element being enumerated into $\mathcal{B}$; i.e., if the Δ^0_2-approximation to the isomorphism type of $\mathrm{Nbh}_n(g)$ changes, it is because some element of the current approximation to $\mathrm{Nbh}_n(g)$ is enumerated into $\mathcal{B}$.

We fix a basis $\bar{b}$ of $\mathcal{M}_k$. At each stage, we will have copied a finite subset X of M_k and built an additional finite subset Y containing a specified element d, where d is a new element intended to be generic over M_k, and where Y is, in the limit, a Π^0_1-set, intended to be the interalgebraic closure of d. At any stage, elements in X come in three flavors:

- $\mathcal{B}_c$: These elements have already been enumerated into $\mathcal{B}$.
- X_{ialg}: These elements currently appear to be in $\bigcup_{b \in \bar{b}} \mathrm{Nbh}_n(b)$ and we see some relations apparently witnessing this. Note that they may stop being in X_{ialg} if they appear currently in X_{ialg} due to some apparent connection to some $b \in$

$\bar{b}$ and some element on the path from b to the element later moves into $\mathcal{B}$, thus breaking the path.

X_{free}: These are elements which currently appear to satisfy no relation with any elements of X_{ialg}, nor are they currently in $\mathcal{B}$.

At any stage, we let n be least so that the above partition exists. The only reason a choice of n might fail to give a partition as above at stage s is if there are elements $x, y \in X$ so that

$$x \in \bigcup_{b \in \bar{b}} \mathrm{Nbh}_n(b)$$

and

$$y \in \bigcup_{b \in \bar{b}} \mathrm{Nbh}_{n+1}(b) \smallsetminus \bigcup_{b \in \bar{b}} \mathrm{Nbh}_n(b).$$

Thus n can be found and will be $\leqslant |X| + 1$. We also use a speed-up to ensure that at each stage s, the apparent neighborhoods $\mathrm{Nbh}^s_n(b_i)$ and $\mathrm{Nbh}^s_n(b_j)$ for $b_i, b_j \in \bar{b}$ are isomorphic over $\mathcal{B}_c$.

At various stages, we work with finite fragments $\mathcal{L}_0$ of $\mathcal{L}$. If at stage s, we are working with $\mathcal{L}_0$, then when we discuss neighborhoods of elements, we mean to consider the $\mathcal{L}_0$-neighborhood.

We say that a quantifier-free $\mathcal{L}_0$-diagram $\chi(X, Y)$ (for a finite fragment $\mathcal{L}_0$ of $\mathcal{L}$) is *allowed* at stage s if the following hold:

(1) For each $b_i \in \bar{b}$, there is an $\mathcal{L}_0$-isomorphism ι_i over $\mathcal{B}_c$ between Y and a subset A of $\mathrm{Nbh}^s_n(b_i)$ sending d to b_i.
(2) For each tuple $\bar{x} \subseteq X$ which is not contained in $\mathcal{B}_c$, each tuple $\bar{y}$ in Y, and each $R \in \mathcal{L}_0$, $\neg R(\bar{x}, \bar{y}) \in \chi$.
(3) There is a tuple $\bar{e} \in C$ (where C is the set given by our assumption of being in Case 1) such that for each $e \in \bar{e} \smallsetminus \mathcal{B}_c$, there is an $\mathcal{L}_0$-isomorphism h_e of Y with some $A' \subseteq \mathrm{Nbh}^s_n(e)$ over all of X so that $h_e(d) = e$. Note in particular that for each $e \in \bar{e} \smallsetminus \mathcal{B}_c$, $\mathrm{Nbh}^s_n(e)$ is disjoint from X, no elements in $\mathrm{Nbh}^s_n(e)$ realize any relations with $X \smallsetminus \mathcal{B}_c$, and $\chi(X, A')$ holds.

In building the new generic element d and its interalgebraic closure, we will ensure that at all stages, we have committed to an allowed diagram. The idea is that if our approximations to $\mathrm{Nbh}^s_n(b)$ for $b \in \bar{b}$ change by elements entering $\mathcal{B}$, then there are elements in $\mathrm{Nbh}^s_n(e)$ for a generic element $e \in \bar{e}$ which can be used as images of the corresponding elements in Y.

We now check that once our approximations to $\mathrm{Nbh}_m(b)$ for each $b \in \bar{b}$ have settled, every allowed configuration is correct on the m-neighborhood of d over all of X. In particular, if we always commit to allowed diagrams, then for every $m \in \omega$, there is a stage after which we will not make a mistake on the m-neighborhood of d.

Claim 26. *Let χ be an allowed $\mathcal{L}_0$-diagram at stage s and $m \leqslant n$. If $\mathrm{Nbh}^s_m(b_i) = \mathrm{Nbh}_m(b_i)$ for each $b_i \in \bar{b}$, then $\mathrm{Nbh}^s_m(d) \subseteq Y$ is $\mathcal{L}_0$-isomorphic over X to a subset of $\mathrm{Nbh}_m(g)$ for an element g generic over M_k.*

Proof. Since b_i is generic and $\mathrm{Nbh}^s_m(b_i) = \mathrm{Nbh}_m(b_i)$, $\mathrm{Nbh}^s_m(b_i) \cong_{\mathcal{L}_0} \mathrm{Nbh}_m(g)$ over $\mathcal{B}_c$. By (1), $\mathrm{Nbh}^s_m(d)$ is $\mathcal{L}_0$-isomorphic to a subset of $\mathrm{Nbh}^s_m(b_i)$ over $\mathcal{B}_c$, thus $\mathrm{Nbh}^s_m(d)$ is $\mathcal{L}_0$-isomorphic to a subset of $\mathrm{Nbh}_m(g)$ over $\mathcal{B}_c$. Let f be such an isomorphism.

It remains to see that this is an $\mathcal{L}_0$-isomorphism over X. Given a tuple $\bar{c}\bar{d}$, where $\bar{c} \in X$, $\bar{d} \in \mathrm{Nbh}^s_m(d)$, and $R \in \mathcal{L}_0$, we need to show that $R(\bar{c}, \bar{d})$ if and only if $R(\bar{c}, f(\bar{d}))$. We first consider the case where $\bar{c} \nsubseteq \mathcal{B}_c$. This implies that $\neg R(\bar{c}, \bar{a})$ for every tuple $\bar{a} \in \mathrm{Nbh}^s_m(b_i) = \mathrm{Nbh}_m(b_i)$. Thus $\bar{c} \notin \mathcal{B}_m$. It is assumed in (2) that $\neg R(\bar{c}, \bar{d})$, and since $\bar{c} \notin \mathcal{B}_m$, we also have $\neg R(\bar{c}, f(\bar{d}))$.

Now consider the case where $\bar{c} \subseteq \mathcal{B}_c$, then since f is a $\mathcal{L}_0$-isomorphism over $\mathcal{B}_c$, we have $R(\bar{c}, \bar{d})$ if and only if $R(\bar{c}, f(\bar{d}))$. □

Next we see that every correct n-neighborhood will be allowed from some stage onwards.

Claim 27. *Fix $X \subseteq M_k$, and let $\chi(X, Y)$ describe the quantifier-free $\mathcal{L}_0$-type of $\mathrm{Nbh}_n(g)$ over X for a g generic over M_k. Then χ is allowed at any sufficiently large stage.*

Proof. Since g is generic, any sufficiently generic element will satisfy the existence of such an n-neighborhood satisfying χ. At every late enough stage, the n-neighborhoods of each $b \in \bar{b}$ will be isomorphic to this Y over B_c, so condition (1) holds. Further, at every late enough stage, we have $\mathcal{B}_c \cap X = \mathcal{B} \cap X$, so condition (2) holds. Finally, using the assumption that we are in Case 1, and specifically that C contains an infinite set of disjoint tuples, there are tuples $\bar{e} \in C$, where each $e \in \bar{e}$ satisfies the generic formula declaring that it has a neighborhood satisfying $\chi(X, -)$. □

Next we see that we can recover from errors to move from one allowed configuration to another even after elements leave X_{ialg}.

Claim 28. *Suppose $\chi(X, Y)$ is an allowed configuration at stage s and some elements are removed from $\bigcup_{b \in \bar{b}} \mathrm{Nbh}^s_n(b)$ (due to enumeration into $\mathcal{B}$) and move into $\mathcal{B}_c$ or into X_{free}. Then there is a partition of Y into $Z \cup W$ with $d \in W$, and there is a set Z' in $M_k \smallsetminus X$ so that $\chi(X \cup Z', W)$ is an allowed configuration at some stage $r > s$.*

That is, if we identify the elements of Z as copying the elements of Z', we once again have an allowed configuration.

Proof. We will consider stages larger than t, which is chosen to be a stage large enough that $\mathrm{Nbh}^t_n(b)$ is correct for each $b \in \bar{b}$, as is $\mathrm{Nbh}^t_n(e)$ for a generic $e \in \bar{e}$, where $\bar{e}$ witnessed condition (3) that $\chi(X, Y)$ was allowed at stage s. Let Z' be the set of elements in $h_e(Y)$ which are removed from $\mathrm{Nbh}^s_n(e)$ by stage t, and let W' be the set of elements in $h_e(Y)$ which have remained in $\mathrm{Nbh}^t_n(e)$. We let Z be $h_e^{-1}(Z') \subseteq Y$ and W be $h_e^{-1}(W') \subseteq Y$.

We will verify that then $\chi(X \cup Z', W)$ is a fragment of the type of $\mathrm{Nbh}_n(g)$ over $X \cup Z'$. Then Claim 27 guarantees that this is an allowed configuration at some stage $r > t$. Note that after identifying Z with Z', we have that h_e is an isomorphism of W with W' over $X \cup Z'$. In particular, the only relations holding between W and $X \cup Z'$ are between W and $\mathcal{B}^t_c$.

Since $e \notin \mathrm{acl}(\varnothing)$, we have that W', and thus W, is isomorphic to a fragment of $\mathrm{Nbh}_n(g)$ for a generic g over $\mathcal{B}^t_c$, say, via f. As in Claim 26, we argue that this isomorphism extends to being over $X \cup Z'$. In particular, if a tuple $\bar{c} \notin \mathcal{B}^t_c$, and $\bar{d} \in W$, we have $\neg R(\bar{c}, \bar{d})$. Similarly, since t is large enough that $\mathcal{B}^t_c \cap (X \cup Z') = \mathcal{B}_c \cap (X \cup Z')$, we also have $\neg R(\bar{c}, f(\bar{d}))$.

Thus the quantifier-free type of W over $X \cup Z'$ is a fragment of the type of the neighborhood of a generic element, and thus we will eventually see an allowed configuration extending it by Claim 27. □

Using the claims above, we give the construction of $\mathcal{M}_{k+1}$. At each step s of our construction, we will have a finite $\mathcal{L}_0 \subseteq \mathcal{L}$, a set $X \subseteq M_k$ which we are copying along with a set Y and a stage $t > s$ so that the diagram $\chi(X, Y)$ which we have built is allowed at stage t.

To ensure that we build a copy of $\mathcal{M}_{k+1}$, we must infinitely often increase X to copy an additional element from M_k, increase $\mathcal{L}_0$ to encompass a new relation symbol, and we must ensure that for every n, from some stage onwards the n-neighborhood of d has correctly settled down. That is, for sufficiently large t, we must have $\mathrm{Nbh}_n^t(d) \cong \mathrm{Nbh}_n^t(g)$ over all of X.

To add an element to X or to add a relation to $\mathcal{L}_0$, we use Claim 27. That is, we let N be large enough that Y appears to be in the N-neighborhood of d. Claim 27 guarantees that we will either see our approximation to $\mathrm{Nbh}_N(g)$ change or we will see the full N-neighborhood over X allowed. We apply this either to the larger set X or the larger language $\mathcal{L}_0$. In either case, if we see an allowed configuration, we can expand X or $\mathcal{L}_0$ as needed. If our approximation to $\mathrm{Nbh}_N(g)$ changes, then we simply use Claim 28 and try again. We note that either the parameter N here has not increased or Y has strictly decreased in size. Thus eventually we will have a correct approximation to $\mathrm{Nbh}_N(g)$ and will be allowed to increase X or $\mathcal{L}_0$.

Finally, to ensure that the n-neighborhood of d is correct at stage s, we infinitely often use Claim 27 as follows: Let $N \geqslant n$ be least so that Y currently appears to be contained in the N-neighborhood of d. Then (as above, allowing for necessary corrections) copy the full N-neighborhood of d into Y. By Claim 26, this ensures that at cofinitely many stages, Y contains a correct copy of the n-neighborhood of d. Finally, we must see that the n-neighborhood of d stabilizes as a set. Once the finite set $\mathcal{B}_n$ has been enumerated into $\mathcal{B}_c$, there is no way for an element to be removed from $\mathrm{Nbh}_n(d)$. Thus, at any stage after the full correct n-neighborhoods of the basis elements are discovered and all of $\mathcal{B}_n$ is enumerated into $\mathcal{B}_c$ when we place the full n-neighborhood of d into Y, the n-neighborhood of d must stabilize as a set. Further, by Claim 26, $\mathrm{Nbh}_n(d)$ is then $\mathcal{L}_0$-isomorphic with $\mathrm{Nbh}_n(g)$ over X where g is generic over $\mathcal{M}_k$. Since this is true at all larger stages, in particular as X and $\mathcal{L}_0$ grow, we have that $\mathrm{Nbh}_n(d)$ is $\mathcal{L}$-isomorphic with $\mathrm{Nbh}_n(g)$ over M_k.

Thus we have built a copy of $\mathcal{M}_{k+1}$.

This suffices to show in Case 1 that $[1, \omega) \subseteq \mathrm{SRM}(T)$. To see that $\omega \in \mathrm{SRM}(T)$, we run the previous construction, building more and more generic neighborhoods. For a configuration to be allowed when building l many new neighborhoods, we need l many tuples $\bar{e} \in C$ whose n-neighborhoods are disjoint as protection. Everything else remains the same.

Case 2: We suppose we are not in Case 1, and we assume a bound on the arities of the relations in $\mathcal{L}$. We note first that for a generic element g, there are only finitely many pairwise disjoint tuples $\bar{x}$ so that

$$\exists i\,[R_i(g, \bar{x}) \wedge \neg\exists^{\infty} y\, R_i(y, \bar{x})].$$

Otherwise, the collection of tuples $\bar{x}$ so that $\exists i\,[R_i(g, \bar{x}) \wedge \neg\exists^{\infty} y\, R_i(y, \bar{x})]$ would witness that we are in Case 1.

Fix $h_1, \ldots, h_n$ so that $\forall \bar{x}\, \forall i\, (R_i(g, \bar{x}) \rightarrow \bar{x} \cap \{h_1, \ldots, h_n\} \neq \varnothing)$. Fix s so that for each $l \leqslant n$, there is an $i \leqslant s$ so that $R_i^{12}(g, h_l)$ (again using the fact that our relations are closed under permutations of variables). For $i \leqslant s$, let N_i be the finite number of tuples $\bar{y}$ so that $R_i(g, \bar{y})$ (using that g is generic and R_i has rank at most 1). We consider the new language

$$\mathcal{L}' := \{R_i^{12} \mid i \leqslant s\} \cup \{R_m^i \mid i \leqslant s \text{ and } m \in \omega\},$$

where

$$R_m^i(z, \bar{x}) \text{ if and only if } \exists y\,(R_i^{12}(z, y) \wedge R_m(z, y, \bar{x})).$$

Note that, using the fixed numbers N_i, if we know that z is generic, we can computably find all y so that $R_i^{12}(z, y)$ hold, and thus we can determine whether $R_m^i(z, \bar{x})$ holds. Further, we can determine for a tuple $\bar{x}$ whether or not $R_m^i(a, \bar{x}) \wedge R_m^i(b, \bar{x})$ for fixed generics a and b; and thus we can computably determine whether $\exists^\infty z R_m^i(z, \bar{x})$. This is not enough to get a computable interdefinability of the two languages, but we have that the neighborhoods $\mathrm{Nbh}(g)$ in $\mathcal{L}$ and in $\mathcal{L}'$ are the same, and this will suffice for our needs.

Note that for a generic g, it cannot be that there are infinitely many pairwise disjoint tuples $\bar{x}$ so that

$$\exists R \in \mathcal{L}'\left[R(g, \bar{x}) \wedge \neg\exists^\infty y R(y, \bar{x})\right],$$

as then we would again be in Case 1.

So, we can suppose that there are only finitely many disjoint tuples satisfying

$$\exists R \in \mathcal{L}'\left[R(g, \bar{x}) \wedge \neg\exists^\infty y R_i(y, \bar{x})\right].$$

We proceed to define another language $\mathcal{L}''$, and we repeat this process, each time reducing the arity of the language until we reach a binary language $\mathcal{L}_b$. Note that this is the use of the assumption that we have an upper bound on the arity of the relations.

Note that we have maintained that the set of x so that $R(z, x)$ holds is uniformly computable for every generic z and $R \in \mathcal{L}_b$ and that the neighborhood $\mathrm{Nbh}(g)$ of a generic g is the same in $\mathcal{L}_b$ as in $\mathcal{L}$. Thus we see that the neighborhood $\mathrm{Nbh}(g)$ for a generic g is Σ_1^0 as in Observation 13. If this Σ_1^0-set is infinite, then we are in Case 1 again using $C = \mathrm{iacl}(a)$, so we may suppose that $\mathrm{iacl}(g)$ is finite for any generic g.

Now we build a copy of $\mathcal{M}_{k+1}$ (in the original language $\mathcal{L}$) by copying M_k along with a copy A of $\mathrm{iacl}(g)$ for a generic element g. We need to determine, for $\bar{x} \in \mathcal{M}_k$ and $\bar{y} \in A$ and relation $R \in \mathcal{L}$ whether to define $R(\bar{x}\bar{y})$ or $\neg R(\bar{x}\bar{y})$. We fix an isomorphism f between A and $\mathrm{iacl}(a)$. We define $R(\bar{x}, \bar{y})$ if and only if $\bar{x}$ is disjoint from $\mathrm{iacl}(a)$ and $\mathcal{M}_k \models R(\bar{x}, f(\bar{y}))$. This yields a copy of $\mathcal{M}_{k+1}$.

We finally note that in this last case, we obtained $\mathcal{M}_{k+1}$ from $\mathcal{M}_k$ uniformly with M_k as a computable subset of M_{k+1}, and so we can also build a computable copy of $\mathcal{M}_\omega$.

This concludes the proof of Theorem 25. $\square$

Corollary 29. *Let T be a strongly minimal disintegrated theory in a relational language $\mathcal{L}$ consisting of relations $\{R_0, R_1, \dots\}$ of rank at most 1 (closed under permutations of variables), and assume that $k \in \mathrm{SRM}(T)$ for some $k \in [1, \omega)$. Assume further that the set $\mathcal{B}$ (defined in Lemma 22 and at the beginning of the proof of Theorem 25) is finite. Then $[1, \omega] \subseteq \mathrm{SRM}(T)$.*

Proof. The down direction follows from the above. For the up direction, we use the fact that the neighborhood of a generic element is a Σ_1^0-set. Let A be a copy of $\mathrm{Nbh}(g)$ for a generic element g. We consider the structure $M_k \cup A$ where we make relations $R_i(\bar{c}_0\bar{c}_1)$ hold for $\bar{c}_0 \subseteq M_k$ and $\bar{c}_1 \subseteq A$ if and only if $R_i(\bar{c}_0\bar{g}_1)$ (where $\bar{g}_1$

is the isomorphic image of $\bar{c}_1$ in $\mathrm{Nbh}(g)$) and $\bar{c}_0 \subseteq \mathcal{B}$. This gives a presentation of $\mathcal{M}_{k+1}$, and using the uniformity in this construction, we get a presentation of $\mathcal{M}_\omega$ as well. □

We can immediately conclude:

Corollary 30. *Every spectrum of a strongly minimal disintegrated theory with relations of rank at most 1 and of bounded arity is among the following:*

(1) $\varnothing$, $[0,\omega]$, $[1,\omega]$, $\{0\}$, $\{\omega\}$,
(2) $\{0,1\}$, $\{1\}$,
(3) $\{0,\omega\}$, $\{0,1,\omega\}$, *and*
(4) $\{1,\omega\}$. □

We will show below that every set in (1) and (2) is indeed such a spectrum, but whether the sets in (3)-(4) are such spectra is still open.

The spectra in clause (1) of Corollary 30 were already known to be spectra of such theories. The spectra in clause (2) of Corollary 30 were known to be spectra, but so far only with relations of Morley rank 2. The spectra in clause (3) of Corollary 30 were again known to be spectra, constructed using a Hrushovski construction, thus they are not known to be spectra of a theory satisfying the Zilber trichotomy.

3.2. **The positive results.** In the remainder of this section, we exhibit two new spectra in our setting, namely, the two sets in clause (2) of Corollary 30.

The following Lemma is useful for verifying that a constructed theory is strongly minimal.

Lemma 31 (Similar to Lemma of Herwig/Lempp/Ziegler [HLZ99] and the Claim in Proposition 2.6 of Andrews/Medvedev [AM14])**.** *Let $\mathcal{M}$ be an $\mathcal{L}$-structure where $\mathcal{L}$ is relational and every symbol has Morley rank $\leqslant 1$ in $\mathcal{M}$. For every finite $\mathcal{L}' \subseteq \mathcal{L}$, let $\mathcal{B}_{\mathcal{L}'}$ be the set of elements in M contained in infinitely many tuples satisfying a relation in $\mathcal{L}'$.*

Suppose that for every finite $\mathcal{L}'$ and $n \in \omega$ there is an isomorphism type $\mathcal{C}_n^{\mathcal{L}'}$ and $c \in \mathcal{C}_n^{\mathcal{L}'}$ so that $(\mathrm{Nbh}_n^{\mathcal{L}'}(x), x)$ is isomorphic to $(\mathcal{C}_n^{\mathcal{L}'}, c)$ over $\mathcal{B}_{\mathcal{L}'}$ for almost every $x \in M$. Then $\mathcal{M}$ is strongly minimal.

Further, $a \in \mathrm{acl}(\varnothing)$ if and only if there are some n and $\mathcal{L}'$ so that $(\mathrm{Nbh}_n(a), a) \ncong (\mathcal{C}_n^{\mathcal{L}'}, c)$ over $\mathcal{B}_{\mathcal{L}'}$.

Proof. Showing strong minimality for every finite sublanguage is equivalent to showing strong minimality of $\mathcal{M}$, so fix a finite $\mathcal{L}' \subseteq \mathcal{L}$ and work in $\mathcal{L}'$. We want to verify that over any set X in a model $\mathcal{M}' \succeq \mathcal{M}$, there is a unique non-algebraic type over X.

If $a \in \bigcup_{x\in X} \mathrm{Nbh}(x) \cup \mathrm{acl}(\varnothing)$, then $a \in \mathrm{acl}(X)$. Suppose $a, b \notin \bigcup_{x\in X} \mathrm{Nbh}(x) \cup \mathrm{acl}(\varnothing)$. Then since they are non-algebraic elements, we must have $(\mathrm{Nbh}_n(a), a) \cong (\mathcal{C}, c) \cong (\mathrm{Nbh}_n(b), b)$ for every n. By König's Lemma, $(\mathrm{Nbh}(a), a) \cong (\mathrm{Nbh}(b), b)$ over $\mathcal{B}_{\mathcal{L}'}$. We verify that this is an isomorphism over $\bigcup_{x\in X} \mathrm{Nbh}(x)$ and thus over X. For any $R \in \mathcal{L}'$, tuple $\bar{c} \in \bigcup_{x\in X} \mathrm{Nbh}(x)$, and $\bar{d} \in \mathrm{Nbh}(a)$, we have $M' \models \neg R(\bar{c}, \bar{d})$ unless $\bar{c} \in \mathcal{B}_{\mathcal{L}'}$. We have the same condition for $\bar{d} \in \mathrm{Nbh}(b)$. Thus the same isomorphism extends to an isomorphism over X showing that the type of a and b are the same over X.

For the final statement: If there are no n and $\mathcal{L}'$ so that $(\mathrm{Nbh}_n(a), a) \ncong \mathcal{C}_n^{\mathcal{L}'}$ over $\mathcal{B}_{\mathcal{L}'}$, then the same argument as above with $X = \varnothing$ shows that $a \notin \mathrm{acl}(\varnothing)$.

If there is, then there are assumed to be only finitely many such elements showing $a \in \mathrm{acl}(\varnothing)$. $\square$

Throughout the constructions in Theorems 33 and 34, we will use the following notion:

Definition 32 (see Khoussainov/Nies/Shore [KNS97])**.** *For every $n \in \omega$, we define an* n-cube *to be a finite structure in the language comprising binary relations $\{R_i \mid i \in \omega\}$.*

A 0-cube *is a single point without any relation. An* $(n+1)$-cube *is a pair of disjoint n-cubes with each pair of corresponding points in the two n-cubes connected by the relation R_{n+1}.*

An ω-cube *is an increasing union of n-cubes for all $n \in \omega$.*

We begin with the spectrum $\{0, 1\}$. Note that Kudaibergenov's construction [Ku80] of this spectrum proceeds by coding the halting set into the set of i so that $R_i(x, y)$ has Morley rank 2. No variant of this construction could work in our context.

Theorem 33. *$\{0, 1\}$ is the spectrum of computable models of a strongly minimal disintegrated theory with relations of rank at most 1 and of bounded arity.*

Proof. In a finite-injury construction, we will construct a Π^0_1-set S. Our model $\mathcal{M}_1$ will be an expansion of the structure $\mathcal{A}$ containing one n-cube for every $n \in \omega$, one additional n-cube for every $n \in S$ (called the *n-S-cube*), and one ω-cube.

Our structure $\mathcal{M}_1$ will, model-theoretically speaking, be a definitional expansion of $\mathcal{A}$; but in order to show that it is computable, we present here an effective construction of $\mathcal{M}_1$ and then show how to modify it to obtain an effective construction of $\mathcal{M}_0$. The models $\mathcal{M}_1$ and $\mathcal{M}_0$ have additional, symmetric binary relations V^i_n (for $i, n \in \omega$). In order to prevent the models of dimension $\geqslant 2$ of the theory of $\mathcal{M}_1$ from being computable, we will have requirements $\mathcal{R}_m$, associated with the mth triple $(\mathcal{N}_m, a_m, b_m)$ consisting of a potential computable model $\mathcal{N}_m$ and two distinguished elements $a_m, b_m \in N_m$. The requirement is to ensure that either $\mathcal{N}_m \not\equiv \mathcal{M}_1$ or that a_m, b_m are not mutually generic in $\mathcal{N}_m$.

Each requirement will eventually be associated with a fixed number n which is initially in S until we see a certain configuration of elements and relations in the model $\mathcal{N}_m$; when we do see this configuration, we remove n from S (and thus the n-S-cube from $\mathcal{M}_1$) and have a permanent win against the model $\mathcal{N}_m$.

At stage 0, we let $S = \omega$ and start by setting the *threshold* $t^0_0 = 1$ and adding to $\mathcal{M}_1$ one n-cube for each $n \leqslant t^0_0$, one additional t^0_0*-cube destined to become the ω-cube*, and one 0-S-cube. (The thresholds t^i_s here have a subscript for the stage s and, loosely speaking, a superscript i related to the requirement R_m, and help us organize the construction to keep it finite injury.) We also connect the point in the 0-S-cube to each point in both t^0_0-cubes by the relation V^0_0 and call this relation *active*. Finally, we associate requirement $\mathcal{R}_0$ with the number 0; and we set the *counter* $c_0 = 0$ for the highest requirement currently assigned to a number potentially in S.

At stage $s+1$, we have from stage s one n-cube for each $n \leqslant t^s_s$, one additional t^s_s-cube which is destined to become the ω-cube, and an n-S-cube for each $n \leqslant s$ which is in S_s (with some active and inactive relations V^i_n such that each point in each n-S-cube is connected to points in (larger) cubes by an active relation V^i_n for some i, and some points inside cubes are connected by inactive relations V^i_n for some n and i). We also have an association of all requirements $\mathcal{R}_m$ for $m \leqslant c_s$ with numbers $\leqslant s$.

Stage $s + 1$ now proceeds in two phases, the *correction phase* and then the *expansion phase*. For the *correction phase*, we check if there is a (least) $n \in S_s \cap [0, s]$ such that

- some requirement $\mathcal{R}_m$ for $m \leqslant c_s$ is associated with n, and
- there are two n-cubes in $\mathcal{N}_{m,s+1}$ (which are not currently part of $(n + 1)$-cubes) such that each point of one of these n-cubes is connected by an active relation V_n^i to both a_m and b_m.

If there is no such n, then we immediately move on to the expansion phase below (setting $n = s$ and the *counter* $c_{s+1} = c_s + 1$). Otherwise, we extract n from the set S, set the *counter* $c_{s+1} = m$, and perform the *correction phase* for this requirement $\mathcal{R}_m$ in the following steps for $l \in S_s \cap [n, s]$ in decreasing order: At the beginning of Step l, we have a setup as described in the previous paragraph but with possibly increased values of the thresholds t_s^k. We now perform the following actions:

(1) Declare the active relation V_l^i to be *inactive*;
(2) enlarge each cube currently involved in the relation V_l^i (which are exactly the l-S-cube and all k-cubes for $k \geqslant t_s^l$) to a t_s^s-cube by adding new elements to M;
(3) add new points to merge all the points from the enlarged t_s^s-cubes from clause (2) (except for the enlarged t_s^s-cube from the previous l-S-cube) into a single cube, and redefine t_s^l to be the size of this cube;
(4) increase the value of the thresholds t_s^k for $k \in (l, s]$ to values greater than three times the value of t_s^l (preserving their ordering, and with a slight abuse of notation by not introducing new names for these parameters in order not to clutter the indexing of our parameters even more);
(5) enlarge the previous l-S-cube even further to a different new t_s^l-cube;
(6) merge these two t_s^l-cubes from clauses (3) and (5) into a single $(t_s^l + 1)$-cube, which is now the new *cube destined to become the ω-cube* (note that this new cube contains the previous cube that was destined to become the ω-cube); and
(7) declare the relation V_l^i to hold between any point in one of these two t_s^l-cubes and any point in the other t_s^l-cube; from now on, the relation V_l^i will be declared to hold between any new points in distinct t_s^l-subcubes *inside the same* (t_s^l+1)-(sub)cube without our further mentioning this. For future reference, we set $v_l^i = t_s^l$.

After the correction phase, or if the correction phase did not apply, we let t_{s+1}^l be the current values of the thresholds t_s^l (either defined during the correction phase, or from stage s if $l < n$ or there was no correction phase), and we pick the threshold t_{s+1}^{s+1} large.

We then proceed to the *expansion stage*:

- For each $l \in S \cap (n, s+1]$, create a new l-S-cube and connect each point in it by a new relation V_l^i (calling this relation *active*, for a new number i) to each point in each current k-cube for all $k \geqslant t_{s+1}^l$ (including k-S-cubes);

- create a new l-cube (which is not an l-S-cube) not destined to become an ω-cube for each $l \leqslant t_{s+1}^{s+1}$ for which no such cube currently exists, and enlarge the current cube destined to become the ω-cube to be a t_{s+1}^{s+1}-cube;
- connect each point in each l-S-cube (for $l \in S \cap [0, n)$) by its active relation V_l^i to each new point in each k-cube for all $k \geqslant t_{s+1}^l$ (including k-S-cubes); and
- if no number is currently associated with requirement $\mathcal{R}_{c_{s+1}}$, then associate this requirement with the number $s + 1$.

We can now easily check the following properties of the construction of $\mathcal{M}_1$: It is an expansion of the model $\mathcal{M}$ described at the beginning of the proof. Each relation V_l^i declared inactive at some time during the construction connects all points between distinct v_l^i-subcubes (for v_l^i as defined in clause (7) above) within $(v_l^i + 1)$-(sub)cubes. Each relation V_l^i declared active but never declared inactive during the construction connects each point in the l-S-cube with all points in all sufficiently large n-cubes (both n-cubes and n-S-cubes, including the ω-cube; note that this is definable in $\mathcal{M}_1$ using a parameter from the l-S-cube). Thus $\mathcal{M}_1$ is a definitional expansion of $\mathcal{M}$, and therefore $\mathcal{M}_1$ is strongly minimal.

These properties make it easy to check that we could have built the model $\mathcal{M}_0$ effectively as well, without the ω-cube: In clause (6) above, we would simply make the new large cube a new large finite cube. Since all new thresholds are chosen to be larger than this cube, this cube cannot be infinitely often forced to grow. So the theory of $\mathcal{M}_1$ has two computable models, $\mathcal{M}_0$ and $\mathcal{M}_1$, of dimension 0 and 1, respectively.

Finally, note that no higher-dimensional model can have a computable presentation $\mathcal{N}_m$ with mutual generics a_m and b_m, say: Let n be such that the requirement $\mathcal{R}_m$ is eventually permanently associated with the number n. Then $n \in S$ implies that $\mathcal{N}_m$ is not isomorphic to a model of $\mathrm{Th}(\mathcal{M}_1)$ since $\mathcal{N}_m$ is missing an n-cube connected by the active relation V_n^i to all but finitely many points in $\mathcal{N}_m$, and thus to both a_m and b_m. And $n \notin S$ implies that in $\mathcal{N}_m$, a_m and b_m are mutually algebraic via an inactive relation V_n^i. $\square$

A fairly easy modification of the above construction will yield the next spectrum, $\{1\}$:

Theorem 34. $\{1\}$ *is the spectrum of computable models of a disintegrated theory with relations of rank at most 1 and of bounded arity.*

Proof. The construction from Theorem 33 needs to be modified as follows: We add additional requirements $\mathcal{S}_p$ associated with a potential computable model $\mathcal{P}_p$ and will show that $\mathcal{P}_p$ cannot be isomorphic to $\mathcal{M}_0$.

To satisfy one such requirement, we fix a large number n (specifically larger than the thresholds of higher-priority requirements) and wait until the structure $\mathcal{P}_p$ constructs an n-cube. Then, using the algorithm in the correction phase above, we make $\mathcal{M}_1$ have no n-cube (including no n-S-cube). If, at some later stage, the n-cube in $\mathcal{P}_p$ grows to become an n'-cube, we re-instate the n-cube (and an n-S-cube if $n \in S$). We then again, via the same algorithm as in the correction phase above, declare that there is no n'-cube in $\mathcal{M}_1$.

In the long run, there are now two possibilities: Either $\mathcal{P}_p$ contains an n-cube for some n where $\mathcal{M}_1$ contains no n-cube, and so $\mathcal{P}_p$ is not isomorphic (in fact, not even elementarily equivalent) to $\mathcal{M}_0$; the effect of this outcome on the rest of the construction is finite. Or some cube in $\mathcal{P}_p$ keeps growing without bound and so $\mathcal{P}_p$ contains an ω-cube C, while $\mathcal{M}_0$ does not; the effect of this outcome on the rest of the construction is infinite, but for each fixed n, the cube C will be bigger than an n-cube, so other requirements can eventually work with any number $\leqslant$ any fixed n.

It is now fairly standard to implement the modification of the construction from Theorem 33, adding the requirements $\mathcal{P}_p$ to make $\mathcal{M}_0$ noncomputable. □

We are now able to summarize our results and nail down the at least seven and at most ten possible spectra in the disintegrated rank 1 case of bounded arity in the following

Theorem 35. *The following are the spectra of computable models of a disintegrated strongly minimal theory with relations of rank at most 1 and of bounded arity:* $\varnothing$, $[0,\omega]$, $[1,\omega]$, $\{0\}$, $\{1\}$, $\{\omega\}$, $\{0,1\}$, *and possibly* $\{0,\omega\}$, $\{0,1,\omega\}$ *and* $\{1,\omega\}$. *Furthermore, these seven can be achieved in binary relational languages.*

Proof. By Theorem 25, if any $k \in [2,\omega)$ is in $\mathrm{SRM}(T)$, then $[1,\omega] \subseteq \mathrm{SRM}(T)$. This leaves the possibilities of $[0,\omega]$, $[1,\omega]$, and subsets of $\{0,1,\omega\}$. The original constructions show that $\varnothing$, $[0,\omega]$, $\{0\}$, $[1,\omega]$, and $\{\omega\}$ are spectra of binary rank 1 theories. By Theorems 33 and 34, $\{0,1\}$ and $\{1\}$ are spectra of binary (rank 1) theories. □

4. Rank 1 relational languages in unbounded arities

Recall the definition of $\mathcal{B}$ in Definition 21 and the following from Corollary 29.

Theorem 36. *If* $\mathcal{B}$ *is finite and* $k \in [2,\omega)$ *is in* $\mathrm{SRM}(T)$, *then* $[1,\omega] \subseteq \mathrm{SRM}(T)$.

This allows us to limit the possible spectra to a small (but infinite) collection of sets:

Theorem 37. *If* T *is a strongly minimal theory in a relational language of rank at most 1, then* $\mathrm{SRM}(T)$ *is among the following:* $\varnothing$, $[0,\omega]$, $[1,\omega]$, $[0,n]$ *and* $[0,n]\cup\{\omega\}$ *for* $n \in \omega$, $[0,\omega)$, $\{1\}$, $\{\omega\}$, *and* $\{1,\omega\}$.

Proof. By Theorem 25, $\mathrm{SRM}(T) \cap [1,\omega)$ is an initial interval in $[1,\omega)$. Thus, we are limited to $[0,\omega]$ or $[1,\omega]$, subsets of $\{0,1,\omega\}$, or sets of the form $[0,\alpha)$ or $[1,\alpha)$ for $\alpha \leqslant \omega$, or $[0,n] \cup \{\omega\}$ or $[1,n] \cup \{\omega\}$ for $n \in \omega$. The possibilities $[1,\alpha)$ (for $3 \leqslant \alpha \leqslant \omega$) and $[1,n]\cup\{\omega\}$ (for $n \in \omega \smallsetminus \{0,1\}$) are ruled out by cases. If $\mathcal{B}$ is finite, then they are ruled out by Theorem 36. If $\mathcal{B}$ is infinite, then, in the model $\mathcal{M}_2$, $\mathcal{B}$ is an infinite Σ^0_1-subset of the prime model of T (by the remark following Lemma 20), so $0 \in \mathrm{SRM}(T)$ by Corollary 24 and Lemma 14. □

In the remainder of this section, we establish one new class of spectra and one new spectrum. In the following theorem, we employ a finite injury argument, while Theorem 41 will modify this basic construction using an infinite injury construction.

Theorem 38. *For each* $n \in \omega$, $[0,n]$ *is the spectrum of a disintegrated strongly minimal theory in a relational language of rank at most 1.*

Proof. We assume $n \geqslant 2$, since the cases $\{0\}$ and $[0,1]$ were already handled in Theorems 3 and 33, respectively. We fix the language $\{R_i \mid i \in \omega\} \cup \{V_k \mid k \in \omega\}$, where each R_i is a 2^{2i+2}-ary relation symbol and each V_k is unary. At each stage of the construction, we will have some relations R_i active. We begin with all relations R_i being inactive. We will ensure that each relation is symmetric and holds only on tuples of pairwise distinct elements, so we will write $R_i(A)$ for a set A of size 2^{2i+2} to mean that $R_i(\bar{a})$ holds on some (or, equivalently, every) tuple so that $\{a_k \mid k < |\bar{a}|\} = A$.

We fix an enumeration of all structures $(\mathcal{N}_\ell, \bar{a}_\ell)$ with a specified $(n+1)$-tuple $\bar{a}_\ell$. Our goal is to build an n-dimensional computable model $\mathcal{M}$ of a strongly minimal disintegrated theory in a rank 1 language while ensuring that any model of dimension $> n$ of the same theory has no computable presentation. This ensures that $[n+1, \omega] \cap \mathrm{SRM}(T) = \varnothing$, and thus, by Theorem 37 and our assumption that $n > 1$, this implies that $\mathrm{SRM}(T) = [0, n]$.

We begin the construction at stage 0 with the following intended interpretation for each symbol: We have an infinite set of commuting involutions f_k and g_k, generating a group isomorphic to $\bigoplus_{k \in \omega} \mathbb{Z}/2\mathbb{Z}$. Each $R_m(\bar{x})$ will be interpreted as saying that $\bar{x}$ is a 2^{2m+2}-tuple of distinct elements closed under $\{f_j, g_j \mid j < m+1\}$ (and so is a connected component when viewed as a graph under these functions).

We keep a fixed set $\{b_0, \ldots, b_{n-1}\}$ throughout the construction intended to be a basis for the model $\mathcal{M}$ which we construct.

The requirements are as follows: Ensure that either $\mathcal{N}_\ell \not\equiv \mathcal{M}$ or the $(n+1)$-tuple $\bar{a}_\ell$ is not independent in $\mathcal{N}_\ell$. We describe the action for a single strategy for a single model $(\mathcal{N}_\ell, \bar{a}_\ell)$; we will put these strategies together via a finite-injury priority argument.

Step 0: Choose a new integer $\ell' > \ell$ (so, in particular, we have made no commitments about realizations of $R_{\ell'}$) and activate $R_{\ell'}$. From before, we have a set F of commuting involutions so that $R_{\ell'-1}(\bar{w})$ is interpreted as $\bar{w}$ being closed under F. We introduce a pair of further commuting involutions $f_{\ell'}, g_{\ell'}$ on our model. We interpret $R_{\ell'}(\bar{x})$ to mean that $\bar{x}$ is closed under all involutions in $F \cup \{f_{\ell'}, g_{\ell'}\}$.

Step 1: Wait until in $\mathcal{N}_\ell$, we see disjoint tuples $\bar{z}_i$ for $i \leqslant n$ so that, for each $a_i \in \bar{a}_\ell$, we have $\mathcal{N}_\ell \models R_{\ell'}(a_i, \bar{z}_i)$ (i.e., each $a \in \bar{a}_\ell$ has the full $2^{2\ell'+2}$-dimensional cube). When this is seen, go to Step 2.

Step 2: In $\mathcal{M}$, for each $i < n$, let S_i be the unique set containing b_i satisfying $R_{\ell'-1}(S_i)$. Let Q_i be the image of S_i under the function $f_{\ell'}$. For each element d currently in $M \smallsetminus \bigcup_{i<n} (S_i \cup Q_i)$, we take a new unary relation V_k and determine that $V_k^{\mathcal{M}} = \{d\}$.

We now give new interpretations for each of the active symbols R_p for $p \geqslant \ell'$. Let X_i^p be the unique set satisfying $R_p(S_i \cup Q_i \cup X_i^p)$. We define a new function h_i^ℓ to be an involution on the set of tuples satisfying $R_{\ell'-1}$. In particular, for each $i < n$, we define $h_i^\ell(S_i)$ to be Q_i. For each set X_i^p, let $\psi_i^p(\bar{x})$ say that $\bar{x}$ is named by the same unary relations as X_i^p.

For $p \geqslant \ell'$, we give the new interpretation for R_p to be the symmetric closure of the relation $\theta_p(\bar{x}\bar{y}\bar{z}) \vee \bigvee_{i<n}(\psi_i^p(\bar{z}) \wedge h_i^\ell(\bar{x}) = \bar{y})$ where θ_p hold on a tuple $\bar{x}\bar{y}\bar{z}$ if every element of the tuple is already named by a relation V_k and R_p holds on $\bar{x}\bar{y}\bar{z}$ (i.e., θ_p describes via the V_k-relations the finite atomic diagram of R_p on the shaded elements in Figure 1). We de-activate all active relations R_p with $p \geqslant \ell'$.

In particular, we are making the following sentence true in the theory:

$$\rho := \exists x_0, \ldots, x_{n-1} \forall \bar{z} \left[R_{\ell'}(\bar{z}) \to \left(\theta_{\ell'}(\bar{z}) \vee \bigvee_{i<n} x_i \in \bar{z} \right) \right].$$

This holds for a tuple $\bar{x}$ chosen so that $x_i \in X_i$. Note that $\mathcal{N}_\ell$ has constructed $n+1$ disjoint tuples satisfying $R_{\ell'}$ with each containing one of the elements a_j for $j \leqslant n$. If $\mathcal{N}_\ell$ satisfies ρ, then one of the elements a_j for $j \leqslant n$ will have to satisfy

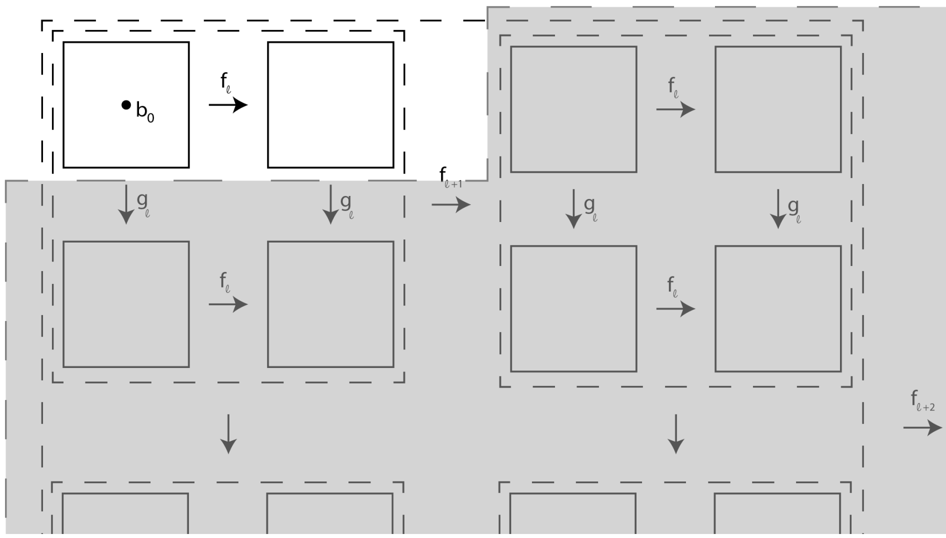

FIGURE 1. S_0 containing b_0, Q_0 being the f_ℓ-image of S_0 and the shaded elements get named by unary relations.

a relation V_k and so will be a definable element. Thus, we have permanently satisfied the requirement. When we go to Step 2, we also re-initialize all lower-priority strategies, and de-activate all active R_i for $i \geqslant \ell'$. We note that higher-priority strategies are concerned with sets closed under involutions which are entirely contained in our sets S_i. Thus they are not affected by the action taken by the strategy to defeat the model $(\mathcal{N}_\ell, \bar{a}_\ell)$.

At a given stage s, each active relation has an interpretation in terms of commuting involutions f_i and g_i. At each stage s, if we have an involution f and an element x (or an appropriate tuple $\bar{x}$ in the case when f is of the form h_i^ℓ) not satisfying any unary predicate V_k and there is no y which we have designated to be the image $f(x)$, then we create such y. The interpretation of the relation symbols is then given by their descriptions.

We verify that the construction works in two lemmas:

Lemma 39. *The model $\mathcal{M}$ which we construct is strongly minimal and of dimension n, and every symbol in the language is interpreted to hold on a subset of M of rank at most 1.*

Proof. By examining the construction, each relation R_ℓ has the property that for each x, either x is named by a unary relation symbol or there are only finitely many tuples $\bar{z}$ so that $R_\ell(x\bar{z})$. Thus we only need to show that $\mathcal{M}$ is strongly minimal, and we will have that each symbol defines a set of rank at most 1. As adding constants to a strongly minimal theory preserves strong minimality, we can consider the structure $\mathcal{M}$ without the unary predicates V_k. For each set $\mathcal{L}_k = \{R_\ell \mid \ell \leqslant k\}$ of relation symbols, fix B_k to be the set of elements y so that for some $R \in \mathcal{L}_k$, there are infinitely many $\bar{z}$ so that $\mathcal{M} \models R(y, \bar{z})$. We need to show that $\mathrm{Nbh}_{\mathcal{L}_k}(x) \cong_{B_k} \mathrm{Nbh}_{\mathcal{L}_k}(b_0)$ for all but finitely many elements $x \in M$. Let $s > k$ be a stage large enough that for each relation R in $\mathcal{L}_k$, if R is ever activated, it is activated before stage s, and if R is ever de-activated, it is de-activated before stage s. Then B_k is contained in the set of elements which are named via a unary predicate before stage s. Also, any element a created after stage s has $\mathrm{Nbh}_{\mathcal{L}_k}(a) \cong_{B_k} \mathrm{Nbh}_{\mathcal{L}_k}(b_0)$. Thus, as in Lemma 31, we see that $\mathrm{Th}(\mathcal{M})$ is strongly minimal. Examining the construction, we can see that each b_i has $\mathrm{Nbh}_{\mathcal{L}_k}(b_i) \cong_{B_k} \mathrm{Nbh}_{\mathcal{L}_k}(b_0)$, thus each b_i is generic. Similarly, the neighborhoods $\mathrm{Nbh}_{\mathcal{L}_k}(b_i)$ are pairwise disjoint, thus the elements b_i are mutually generic. Since every element in $M \smallsetminus \bigcup_{i<n} \mathrm{Nbh}(b_i)$ satisfies some unary relation V_k, we see that no such element can be generic. Thus $\{b_0, \ldots, b_{n-1}\}$ is a basis for $\mathcal{M}$, so $\mathcal{M}$ is n-dimensional. (And clearly $\mathrm{acl}(\varnothing)$ is infinite.) $\square$

Let T be the theory of $\mathcal{M}$.

Lemma 40. *There is no computable presentation of a model of T of dimension $> n$.*

Proof. Suppose $(\mathcal{N}_\ell, \bar{a}_\ell)$ is a computable presentation of a model of T where $\bar{a}_\ell$ is an independent $(n+1)$-tuple in $\mathcal{N}_\ell$. Let s be a stage where the requirement for $(\mathcal{N}_\ell, \bar{a}_\ell)$ is activated for the last time. Then we consider cases based on the outcome for the strategy for the requirement. If the strategy waits in Step 1 with parameter ℓ' indefinitely, then

$$\exists \text{ disjoint } \bar{z}_0 \ldots \bar{z}_n \bigwedge_{i \leqslant n} R_{\ell'}(x_i \bar{z}_i)$$

is in the generic $(n+1)$-type of T, but is not true of $\bar{a}_\ell$, which is a contradiction. If the strategy goes to Step 2, then we have that

$$\rho := \exists x_0, \ldots, x_{n-1} \forall \bar{z} \left[R_{\ell'}(\bar{z}) \to \left(\theta_{\ell'}(\bar{z}) \vee \bigvee_{i<n} x_i \in \bar{z} \right) \right]$$

is in our theory T. Thus either $\mathcal{N}_\ell \not\models \rho$, or one of the $a_i \in \bar{a}_\ell$ satisfies a unary relation V_k and thus $\bar{a}$ is not a basis of $\mathcal{N}_\ell$. $\square$

This concludes the proof of Theorem 38. $\square$

A modification of the above proof gives us the next

Theorem 41. $[0, \omega)$ *is the spectrum of a disintegrated strongly minimal theory in a relational language of rank at most 1.*

Proof. We fix the same language as in the previous construction, and we have the same system of determining active requirements.

We will also proceed similarly to the previous construction. We have the following requirements for all n and all computable models $\mathcal{N}_\ell$:

P_n: There is a computable presentation $\mathcal{M}_n$ of the n-dimensional model of our theory T.

N_ℓ: $\mathcal{N}_\ell$ is not a computable presentation of a model of T of dimension $> \ell$.

The requirements P_n clearly ensure that $[0, \omega) \subseteq \mathrm{SRM}(T)$. The cumulative effect of the N_ℓ-requirements ensures that no computable model can have infinite dimension.

We order the requirements by $P_n < N_n < P_{n+1}$. For this argument, we will need an infinite-injury priority argument, so we will have a tree of strategies for these requirements, defined by the possible outcomes. For a P_n-strategy, we have only one outcome. For an N_ℓ-strategy, the possible outcomes are $d < \infty < \ldots < 3 < 2 < 1 < 0$. The priority ordering and the possible outcomes specify our tree of strategies Λ, assigning each requirement to a full level of the tree. The outcome of an N_ℓ-strategy is determined by the number of times the N_ℓ-strategy is incremented, or it is d or ∞.

The strategy for a P_n-requirement is straightforward: At every stage, we will have (globally) an interpretation for each relation symbol. Each P_n-strategy simply constructs its model accordingly, over a basis of size n, in the style of the global strategy building the computable model in Theorem 38. Note that the interaction between constructing the computable model in Theorem 38 and the requirements is in defining the set of unary relations V_k and the interaction of their named elements in formulas θ_p in Step 2. The strategy in Theorem 38 does this to accommodate the construction of the computable model. In our construction, where we have many constructions of computable models, an N_ℓ-strategy must accommodate the constructions of each P_n-strategy above it on the tree.

For an N_ℓ-strategy σ, we act as follows. In the present theorem, instead of stopping at the end of Step 2, as we did in Theorem 38, we *increment the strategy*, which means that we return to Step 0 when σ is next visited. At any stage when σ is visited, we perform the actions in one of the following "steps". If we are still waiting in Step 1, we take a finite outcome. If we complete the wait in Step 1, we will do one of three things: We might decide to move on to Step 2 at the next stage when σ is visited, in which case we take a finite outcome. We might increment the strategy, in which case we take the outcome ∞. The last possibility is that we see a permanent diagonalization ensuring that N_ℓ does not satisfy the correct theory, in which case we will take the outcome d. Each time we take the action of Step 2, we will take the outcome ∞ once.

Step 0: Pick $\bar{a}$ to be the least $(\ell+1)$-tuple not previously considered by σ. Choose a new integer $\ell' > \ell$, so, in particular, we have made no commitments about realizations of $R_{\ell'}$. We call $R_{\ell'}$ active. From before, we have a set F of commuting involutions so that $R_{\ell'-1}(\bar{w})$ is interpreted as $\bar{w}$ being closed under F. (Again, $R_{\ell'-1}$ may be empty, in which case we make the obvious adjustments to skip over all the empty relation symbols.) We introduce a pair of further commuting involutions $f_{\ell'}$ and $g_{\ell'}$ on our model. We interpret $R_{\ell'}(\bar{x})$ to mean that $\bar{x}$ is closed under all involutions in $F \cup \{f_{\ell'}, g_{\ell'}\}$. Go to Step 1 (at the same stage).

Step 1: Wait until in $\mathcal{N}_\ell$, we see tuples $\bar{z}_i$ for $i \leqslant l$ so that, for each $a_i \in \bar{a}$, we have $\mathcal{N}_\ell \models R_{\ell'}(\bar{z}_i) \wedge a_i \in \bar{z}_i$ (i.e., each $a \in \bar{a}$ has the full $2^{2\ell'+2}$-dimensional cube) or until we see one of the $a \in \bar{a}$ satisfy a unary relation V_k. If we see $V_k(a)$ for any k and any $a \in \bar{a}$ or we see a single tuple $\bar{w}$ containing two members of $\bar{a}$ so that $R_{\ell'}(\bar{w})$, then we increment the strategy and take outcome ∞. If the found tuples $\bar{z}_i$ are non-disjoint (but not the above case), we take outcome d. If the tuples $\bar{z}_i$ are disjoint, then we will go to Step 2 when σ is next visited and we take a finite outcome.

Note that we increment the strategy and take outcome ∞ if we see a permanent way to witness that $\bar{a}$ is not independent in $\mathcal{N}_\ell$. We take outcome d when we see non-disjoint and non-equal tuples realizing $R_{\ell'}$; so we will have a permanent win for σ by simply never changing the interpretation of $R_{\ell'}$, as such tuples do not exist in models of our theory.

Step 2: Let $\mathcal{M}_\ell$ be the model being built by the P_ℓ-strategy $\tau = \sigma^-$, i.e., τ is the immediate predecessor of σ on the tree of strategies. Let S_i be the unique set containing b_i with $\mathcal{M}_\ell \models R_{\ell'-1}(S_i)$. Let Q_i be the image of S_i under the function f_ℓ. For each element d currently in $M_\ell \smallsetminus \bigcup_{i<\ell}(S_i \cup Q_i)$, we take a new unary relation V_k. The P_ℓ-strategy τ will determine that $V_k^{\mathcal{M}_\ell} = \{d\}$ in its *repair* module (described below). For the remainder of this description of Step 2, we will say V_k is *associated with* d.

We now give new interpretations for each of the active symbols R_p for $p \geqslant \ell'$. Let X_i^p be the unique set satisfying $R_p(S_i \cup Q_i \cup X_i^p)$. We define a new function h_i^ℓ for each $i < \ell$ to be an involution on the set of tuples satisfying $R_{\ell'-1}$. In particular, for each $i < \ell$, we define $h_i^\ell(S_i)$ to be Q_i. For each set X_i^p, we let the formula $\psi_i^p(\bar{x})$ say that $\bar{x}$ is named by the same unary relations as are associated with X_i^p. For $p \geqslant \ell'$, we give a new interpretation for active R_p to be the symmetric closure of the relation

$$\theta_p(\bar{x}\bar{y}\bar{z}) \vee \bigvee_{i<n}(\psi_i^p(\bar{z}) \wedge h_i^\ell(\bar{x}) = \bar{y}),$$

where θ_p states that every element in the tuple $\bar{x}\bar{y}\bar{z}$ satisfies one of the finitely many relations V_k associated with a tuple of elements on which we have already made R_p hold. We de-activate all active relations R_p with $p \geqslant \ell'$. We also now call the *repair* module of every P_k-strategy ρ with $\rho \prec \sigma$ (these are described below in the description of the P_n-strategies).

In particular, we are making the following sentence true in the theory:

$$\rho := \exists x_0, \ldots, x_{\ell-1} \forall \bar{z} \left[R_{\ell'}(\bar{z}) \rightarrow \left(\theta_{\ell'}(\bar{z}) \vee \bigvee_{i<\ell} x_i \in \bar{z} \right) \right]. \tag{1}$$

This holds for a tuple $\bar{x}$ chosen so that $x_i \in X_i$. Note that $\mathcal{N}_\ell$ has constructed $\ell + 1$ disjoint tuples satisfying $R_{\ell'}$ with each containing one of the elements a_j for $j \leqslant \ell$. If $\mathcal{N}_\ell$ satisfies ρ, then one of the elements a_j for $j \leqslant \ell$ will have to satisfy a relation V_k and so will be a definable element. Thus, we have permanently ensured that either $\mathcal{N}_\ell \not\models T$ or $\bar{a}$ cannot be an independent tuple. When we go to Step 2, we also initialize all strategies $\tau >_L \sigma^\frown\langle\infty\rangle$ and de-activate all active R_i for $i \geqslant \ell'$. We then increment the strategy σ, i.e., go back to finite outcome and step 0.

We note that N-strategies not re-initialized at this point, i.e., any strategy not to the right of $\sigma^\frown\langle\infty\rangle$, are concerned with sets closed under involutions which are entirely contained in our sets S_i. Thus they are not affected by the action taken by the strategy to defeat the model $(\mathcal{N}_\ell, \bar{a}_\ell)$.

For a P_n-strategy τ, we act as follows. At every stage, for every active relation R, we have an interpretation. We describe three modules that τ can run: The *initialization module*, the *update module*, and the *repair module*. We will run the initialization module when τ is first visited after the strategy is initialized. We will run the update module when τ is later visited on the tree of strategies. We will run the repair module when some N_ℓ-strategy σ with $\tau \prec \sigma$ performs its Step 2 action.

The initialization module: Build n chosen elements $b_1, \ldots, b_n$ intended as the basis of our model. Then run the update module.

The update module: We have n finite disjoint configurations A_i for $i = 1, \ldots, n$ around the basis elements b_i representing their current neighborhoods, i.e., excluding the elements named by V_k-relations. We consider the set of relations which have become active (including those no longer active) since we have last run the update module. For each of these relations R, say, considered in order of arity, we expand the finite configurations A_i according to the current interpretation of R. If R is still active, this means we close A_i under a set of involutions. If R is now inactive, this means we add new elements to satisfy the unary relations V_k used in the interpretation of R as well as closing under the appropriate involutions.

The repair module: We now describe the repair modules which P_n-strategies τ perform when a successor N_ℓ-strategy $\sigma \succ \tau$ acts. The P_ℓ-strategy σ^- plays a special role, but we run this repair module for every P_n-strategy $\tau \prec \sigma$ simultaneously. Each P_n-strategy has a finite structure $\mathcal{M}_n$ constructed so far. Further, there is a natural embedding f_n of $\mathcal{M}_n$ into $\mathcal{M}_\ell$ (the structure built so far by σ^-). This just sends the n basis elements to the first n basis elements in $\mathcal{M}_\ell$, and the corresponding neighborhoods along with them, as well as the elements named by unary predicates to those named by the same unary predicates (allowing $n = 0$). N_ℓ associates some unary relations V_k to elements $d \in \mathcal{M}_\ell$. The P_n-strategy then interprets V_k to name any element x so that V_k is associated with $f_n(x)$. For each V_k not interpreted to be an element in $\mathcal{M}_n$ by this process, the P_n-strategy then creates a new element and interprets V_k to be this new element. We then run the update module for each P_n-strategy $\tau \prec \sigma$. In particular, this adds h_i^ℓ-images as given by the new interpretation of $R_{\ell'}$ in the N_ℓ-strategy.

The construction is now put together as a typical infinite-injury argument on a tree of strategies, assigning the strategies to nodes on the tree in the priority ordering specified above.

We verify that this construction works in the following four lemmas:

Lemma 42. *The model $\mathcal{M}_0$, which is constructed by the unique P_0-strategy, is strongly minimal, and every symbol in the language is interpreted to hold on a subset of M_0 of rank $\leqslant 1$.*

Proof. In $\mathcal{M}_0$, every element is named by a unary predicate V_k which holds on only that element. We consider a relation R_p and want to show that there are only finitely many x so that $\mathcal{M}_0 \models \exists^\infty \bar{z} R_p(x, \bar{z})$. If R_p permanently remains active, then there is no such x. If R_p becomes inactive, then R_p is interpreted by the symmetric closure of the relation $\theta_p(\bar{x}\bar{y}\bar{z}) \vee \bigvee_{i<n}(\psi_i^p(\bar{z}) \wedge h_i^\ell(\bar{x}) = \bar{y})$. It is immediate that the only elements contained in infinitely many tuples satisfying R_p are those contained in a tuple satisfying some $\psi_i^p(\bar{z})$, which is a finite set. Thus, once we show that $\mathcal{M}_0$ is strongly minimal, it follows that each R-relation defines a set of rank 1, and each V-relation defines a set of rank 0.

For each k, fix $\mathcal{L}_k = \{R_i \mid i \leqslant k\}$ and B_k to be the set of elements y satisfying some V_i with $i \leqslant k$. Since every element contained in infinitely many R_i-tuples is contained in $\bigcup_k B_k$, we need to show that cofinitely many elements x have the same isomorphism type of $\mathrm{Nbh}_{\mathcal{L}_k}(x)$ over B_k. Let $s > k$ be a stage large enough so that for each relation R in $\mathcal{L}_k$, if R is ever activated, it is activated before stage s, and if R is ever de-activated, it is de-activated before stage s. Further, let s be large enough so that $\mathcal{M}_0$ has constructed every element in B_k by stage s.

Then any element a created after stage s is created in a repair module because $\mathcal{M}_0$ is constructed with an empty basis, so the update modules do nothing. Suppose this repair module was prompted by an N_ℓ-strategy σ acting. The P_ℓ-strategy σ^- has just performed its own update module, so each element in its structure $\mathcal{M}_\ell$ aside from those already named by V-relations have isomorphic $\mathcal{L}_k$-neighborhoods over B_k before the repair module begins. Then the model $\mathcal{M}_0$ copies the elements associated with new relations V_i, which includes the full $\mathcal{L}_k$-neighborhoods for each of these elements. Thus, each a created after stage s has the same isomorphism type of $\mathrm{Nbh}_{\mathcal{L}_k}(a)$ over B_k. Thus, as in Lemma 31, we see that $\mathrm{Th}(\mathcal{M}_0)$ is strongly minimal. $\square$

Lemma 43. *For any finite collection of unary relations $V_{i_1} \dots V_{i_k}$ ($i_1 < \dots < i_k$), any relation R_n, any stage s, and any pair of P-strategies α, β, if $\mathcal{M}_\alpha$ and $\mathcal{M}_\beta$ both have tuples $\bar{x}$ satisfying $\bigwedge_{j \leqslant k} V_{i_j}(x_j)$ at stage s, then $\mathcal{M}_\alpha \models R(\bar{x})$ at stage s if and only if $\mathcal{M}_\beta \models R(\bar{x})$ at stage s.*

Proof. Consider the strategy γ which defines V_{i_k}. This γ is an N_ℓ-strategy which creates the predicate V_{i_k} via its Step 2 action. At this stage, we consider the structure $\mathcal{M}_\ell$ being built by the P_ℓ-strategy $\tau = \gamma^-$. At this stage, in the repair module, τ will determine whether or not $R_n(\bar{x})$ will hold for its tuple $\bar{x}$ satisfying $\bigwedge_{j \leqslant k} V_{i_j}(x_j)$. Every other strategy which takes a repair module at this stage will follow τ's lead in determining whether or not $R_n(\bar{x})$ holds for its own tuple satisfying $\bigwedge_{j \leqslant k} V_{i_j}(x_j)$. Any other P-strategy which eventually builds a tuple $\bar{x}$ satisfying $\bigwedge_{j \leqslant k} V_{i_j}(x_j)$ will do so in an update module, and it will choose whether to satisfy R_n on this tuple so as to agree with τ's choice. $\square$

Let T be the theory of $\mathcal{M}_0$.

Lemma 44. *Fix $n \in \omega$. Let σ be the P_n-strategy along the true path. Then the P_n-strategy at node σ builds a computable presentation of the model $\mathcal{M}_n$ of T of dimension n.*

Proof. We want to show that $\mathcal{M}_0$ elementarily embeds in $\mathcal{M}_n$. For every element $x \in \mathcal{M}_0$, there is a unary relation V_k so that x is the unique element satisfying V_k. We define a map from $\mathcal{M}_0$ to $\mathcal{M}_k$ by letting f send each element x to the element satisfying the same relation V_k in $\mathcal{M}_n$.

To show that f is an elementary embedding, it suffices to show that f is an embedding and that every element in $\mathcal{M}_n$ which does not satisfy any relation V_i satisfies the generic $\mathcal{L}_k$-neighborhood over B_k for each k. Here, $\mathcal{L}_k = \{R_i \mid i \leqslant k\}$ and B_k is the set of elements in $\mathcal{M}_n$ which satisfy a relation V_i for some $i \leqslant k$.

This suffices since the fact about neighborhoods implies that $\mathcal{M}_n$ is strongly minimal (as in Lemma 31), and the algebraic closure of the empty set is exactly the set of elements which satisfy some relation V_i. Thus, since f is an embedding, $\mathcal{M}_0 \cong \mathrm{acl}_{\mathcal{M}_n}(\varnothing) \preceq \mathcal{M}_n$.

The fact that f is an embedding follows from Lemma 43 and the fact that σ is visited infinitely often, so the update modules will ensure it has copies of all the elements named by V-relations.

It remains to show that every element which does not satisfy any relation V_i has the generic $\mathcal{L}_k$-neighborhood over B_k. To see this, we will consider a relation R_i and see that σ builds $\mathcal{M}_n$ handling R_i appropriately. The proper interpretation of this relation is determined by a single N_ℓ-strategy ρ. We will consider cases based on how σ and ρ are located on the tree of strategies.

Let s be the stage at which σ is last initialized. If $\rho^\frown\langle\infty\rangle <_L \sigma$, then ρ does not act and is not re-activated after stage s, so the interpretation of R_i does not change after σ begins building $\mathcal{M}_n$.

If $\sigma <_L \rho$, then ρ is reinitialized at any stage when σ is visited. Thus whatever interpretation R_i has at any stage when σ is visited is permanently correct (since only ρ could change the interpretation and ρ is initialized).

If $\rho^\frown\langle\infty\rangle \preceq \sigma$, then at any stage at which σ is visited, ρ has just been incremented and so will later pick a large ℓ' beyond the indices of all relations with which σ is now working. In particular, any relation that ρ had been working with has just had its interpretation finalized, so σ sees the correct interpretation of R_i.

Finally, if $\sigma \prec \rho$, then any stage at which ρ changes the interpretation of R_i is a stage at which σ runs the repair module, which ensures that each element has the appropriate neighborhood. □

Lemma 45. *Let τ be the N_ℓ-strategy along the true path. Then the N_ℓ-strategy τ either ensures that $N_\ell \not\models T$ or that N_ℓ has dimension $\leqslant \ell$. So, in particular, N_ℓ is not the ω-dimensional model of T.*

Proof. We consider the possible outcomes of the strategy at node τ after its last initialization.

If the true outcome of τ is d, then we have found two non-disjoint distinct tuples $\bar{z}_i$ and $\bar{z}_j$ in $\mathcal{N}_\ell$ so that $R_{\ell'}(\bar{z}_i)$ and $R_{\ell'}(\bar{z}_j)$. We note that in the theory T, if $R \in \mathcal{L}$ is active and $R(\bar{x}) \wedge R(\bar{y})$, then either $\bar{x} = \bar{y}$ or $\bar{x} \cap \bar{y} = \varnothing$. So we need to check that $R_{\ell'}$ remains active. No node left of τ is ever visited again since τ is not initialized again. Since τ is only now taking outcome d, every previously visited node below τ is being injured, as is every node to the right of τ. Since these are then all initialized, they will at future stages only work with relations that are of higher arity than $R_{\ell'}$ and will not de-activate $R_{\ell'}$. We consider nodes $\beta \prec \tau$. If τ extends a finite outcome of β or the outcome d, then β never again acts (otherwise, τ would be initialized again). If $\beta^\frown\langle\infty\rangle \preceq \tau$, then at the moment when τ takes outcome d, β has just performed its Step 2, so it has no relation R in hand. When β is next visited, it will work with relations of arity larger than that of $R_{\ell'}$ and cannot de-activate $R_{\ell'}$.

If the true outcome of τ is ∞, then every $(\ell + 1)$-tuple $\bar{a}$ is considered by τ at some stage. We either find some unary relation V_k and $a \in \bar{a}$ so that $\mathcal{N}_\ell \models V_k(a)$, or we perform the action in Step 2. In the former case, if $\mathcal{N}_\ell \models T$, then $a \in \mathrm{acl}(\varnothing)$, so $\bar{a}$ is not an independent $(\ell+1)$-tuple. In the latter case, we have made T contain the sentence ρ defined in (1). If $\mathcal{N}_\ell \models T$, then, since each of the $\bar{z}_i$ are disjoint (otherwise, we would have taken outcome d), it follows that $\theta(\bar{z}_i)$ holds for some i. But then a_i satisfies a unary predicate V_k, so once again $\bar{a}$ is not an independent $(\ell + 1)$-tuple in N_ℓ if $N_\ell \models T$.

If the true outcome of τ is finite, then this is because the strategy gets stuck in Step 1 since, for some ℓ' and some element a, $N_\ell \models \neg\exists\bar{x}R_{\ell'}(a\bar{x})$ and a does not satisfy any unary relation V_k. But in $\mathcal{M}_0$, there is a finite set Y of elements, each satisfying some unary relation V_k, so that $\mathcal{M}_0 \models \forall y(y \in Y \vee \exists\bar{x}R_{\ell'}(y\bar{x}))$. Thus $\mathcal{N}_\ell \not\models T$. □

This concludes the proof of Theorem 41. □

Combining Theorems 35, 37, 38 and 41, we obtain

Theorem 46. *The following are the spectra of disintegrated strongly minimal theories in a relational language of rank at most 1: $\varnothing$, $[0,\omega]$, $\{1\}$, $\{\omega\}$, $[1,\omega]$, $[0,n]$ for $n \in \omega$, $[0,\omega)$, and possibly $[0,n] \cup \{\omega\}$ for $n \in \omega$ as well as $\{1,\omega\}$.* □

5. Ternary Languages and Languages of Higher Arity

The purpose of this last section is to show in Theorem 47 that, as for spectra of computable models of strongly minimal disintegrated theories in binary relations, we have only finitely many possible spectra of computable models in the case of ternary relations as well. We emphasize that these results do not make assumptions about the ranks of the relations. On the other hand, Kudaibergenov [Ku80] showed that $\{0,1,2\}$ is the spectrum of a strongly minimal disintegrated theory in a ternary relational language, and Nies/Hirschfeldt [Ni99, p.314] showed that $\{1,2\}$ is the spectrum of a strongly minimal disintegrated theory in a ternary relational language, neither of which, by our Theorem 10, is possible in the binary case. This leads to our Corollary 48, giving the current best upper and lower bounds on the number of spectra of computable models of strongly minimal disintegrated theories in ternary relations.

Theorem 47. *If T is a strongly minimal disintegrated theory in a relational language of ternary relations and $\mathrm{SRM}(T) \cap [3,\omega) \neq \varnothing$, then $[1,\omega] \subseteq \mathrm{SRM}(T)$. Thus there are at most eighteen spectra of computable models of disintegrated strongly minimal theories in ternary relational languages.*

Theorems 10 and 47 as well as the above-mentioned results of Kudaibergenov and of Nies/Hirschfeldt allow us to immediately conclude the following

Corollary 48. *There are at least nine but at most eighteen subsets of $[0,\omega]$ which are spectra of computable models of a strongly minimal disintegrated theory T in a language of ternary relations.* □

Proof of Theorem 47. The proof proceeds in a number of lemmas. The first lemma is the analog of Lemma 11, and the proof is the same.

Lemma 49. *For a strongly minimal $\mathcal{L}$-theory T, if $k \in \mathrm{SRM}(T)$ and $k \in [3,\omega]$, then the set of relations in $\mathcal{L}$ which are of rank 3 is a computable set. Thus T is computably interdefinable with a theory all relation symbols of which have rank at most 2.* □

Using this lemma, whenever we have $\mathrm{SRM}(T) \cap [3,\omega] \neq \varnothing$, we will assume that all relation symbols in $\mathcal{L}$ have Morley rank $\leqslant 2$.

We now use a new trick, which does not seem to generalize to arity > 3, to get down to Morley rank at most 1. Note that a similar reduction to rank 1 for arbitrary theories of bounded arity would, by Theorem 25, suffice to prove Conjecture 5.

Lemma 50. *For each ternary relation R of Morley rank $\leqslant 2$ in a strongly minimal disintegrated theory, the three relations $\exists^\infty w\, R(w,y,z)$, $\exists^\infty w\, R(x,w,z)$, and $\exists^\infty w\, R(x,y,w)$ all have Morley rank $\leqslant 1$, as does the symmetric difference between $R(x,y,z)$ and $\exists^\infty w\, R(w,y,z) \vee \exists^\infty w\, R(x,w,z) \vee \exists^\infty w\, R(x,y,w)$.*

Proof. If y,z were mutually generic, where $\exists^\infty w\, R(w,y,z)$, then for any x generic over y and z, we would have $R(x,y,z)$, but then R would have Morley rank 3. Thus $\exists^\infty w\, R(w,y,z)$ has Morley rank $\leqslant 1$. A symmetric argument works for $\exists^\infty w\, R(x,w,z)$ and $\exists^\infty w\, R(x,y,w)$.

Suppose

$$R(x,y,z) \wedge \neg[\exists^\infty w\, R(w,y,z) \vee \exists^\infty w\, R(x,w,z) \vee \exists^\infty w\, R(x,y,w)]$$

has Morley rank 2. Then, by symmetry, there would be two mutually generic elements x and y as well as an element z so that $R(x,y,z)$ and $\neg\exists^\infty w\, R(x,y,w)$. Thus $z \in \operatorname{acl}(x,y)$. Using the fact that our theory is disintegrated, we may assume without loss of generality that $z \in \operatorname{acl}(y)$. Then x is generic over y, z, and so $\exists^\infty w\, R(w,y,z)$, contradicting our above hypothesis.

Next, suppose

$$[\exists^\infty w\, R(w,y,z) \vee \exists^\infty w\, R(x,w,z) \vee \exists^\infty w\, R(x,y,w)] \wedge \neg R(x,y,z)$$

has Morley rank 2. Then, without loss of generality, for some mutually generic x and y and some element z,

$$[\exists^\infty w\, R(w,y,z) \vee \exists^\infty w\, R(x,w,z) \vee \exists^\infty w\, R(x,y,w)] \wedge \neg R(x,y,z)$$

holds. The third disjunct is impossible, as then R would be rank 3. So, without loss of generality, $\exists^\infty w\, R(w,y,z) \wedge \neg R(x,y,z)$. Thus $x \in \operatorname{acl}(y,z)$. Since x and y are mutually generic, $x \in \operatorname{acl}(z)$. Thus z is generic over y. But then $\exists^\infty w\, R(w,y,z)$ implies that R has rank 3, again contradicting our above hypothesis. □

Given a language $\mathcal{L}$ comprised of ternary relations of rank $\leqslant 2$ (which we may assume if $[3,\omega] \cap \mathrm{SRM}(T) \neq \varnothing$), we define

$$\begin{aligned}\mathcal{L}' = \{&\exists^\infty w\, R(w,y,z) \mid R \in \mathcal{L}\} \cup \\ \{&R(x,y,z) \wedge \neg[\exists^\infty w\, R(w,y,z) \vee \exists^\infty w\, R(x,w,z) \vee \exists^\infty w\, R(x,y,w)] \mid R \in \mathcal{L}\} \cup \\ &\{[\exists^\infty w\, R(w,y,z) \vee \exists^\infty w\, R(x,w,z) \vee \exists^\infty w\, R(x,y,w)] \wedge \neg R(x,y,z) \mid R \in \mathcal{L}\}.\end{aligned}$$

Then $\mathcal{L}'$ is a language inter-definable with $\mathcal{L}$; in fact, every relation in $\mathcal{L}$ is even a Boolean combination of relations in $\mathcal{L}'$, and all relation symbols in $\mathcal{L}'$ have Morley rank $\leqslant 1$. (Recall again here that we assume, as usual, that our language is effectively closed under permutation of variables.) Moreover, we have the following

Lemma 51. *If $\mathcal{M}_k$ is a computable model for $k \geqslant 3$ of a strongly minimal disintegrated theory T, then each relation in $\mathcal{L}'$ is uniformly computable in $\mathcal{M}_k$.*

Proof. Fix mutually generic elements a, b, c in $\mathcal{M}_k$. Without loss of generality, we only need to computably determine whether a pair a', b' satisfies $\exists^\infty w\, R(a',b',w)$. Note that at least one of a, b, c is generic over a', b'.

Case 0: For each $e \in \{a,b,c\}$, $\neg R(a',b',e)$. Then $\neg\exists^\infty w\, R(a',b',w)$, since an element which is generic over a', b' satisfies $\neg R(a',b',e)$.

Case 1: For exactly one $e \in \{a,b,c\}$, $R(a',b',e)$ holds. If a' and b' are mutually generic, then $\neg\exists^\infty w\, R(a',b',w)$ since R has rank $\leqslant 2$. If a' and b' are not mutually generic, then at least two elements in $\{a,b,c\}$ are generic over a', b'. Thus these two elements either both realize $R(a',b',z)$, or neither does. Since exactly one element of $\{a,b,c\}$ realizes $R(a',b',z)$, neither of these two elements which is generic over a', b' can realize $R(a',b',z)$. Thus again $\neg\exists^\infty w\, R(a',b',w)$. So $\neg\exists^\infty w\, R(a',b',w)$ holds independently of whether a' and b' are mutually generic or not.

Case 2: For exactly two $e \in \{a,b,c\}$, $R(a',b',e)$ holds; say,

$$R(a',b',a) \wedge R(a',b',b) \wedge \neg R(a',b',c).$$

In this case, we need to further distinguish subcases in order to determine whether $\exists^\infty w\, R(a',b',w)$ holds or not.

Suppose that $\neg\exists^\infty w\, R(a',b',w)$. Then both a and b are algebraic over a', b'; and so a', b' must be mutually generic as well. So, without loss of generality, by symmetry, a' is interalgebraic with a and b' is interalgebraic with b. Thus $R(a',b',a)$ implies that $R(a',b,a)$, and $R(a',b',b)$ implies that $R(a,b',b)$. So, we have $R(a',b,a) \wedge R(a,b',b)$. (Symmetrically, if a' is interalgebraic with b and b' is interalgebraic with a, then we have $R(a',a,b) \wedge R(b,b',a)$.)

Next, suppose that $\exists^\infty w\, R(a',b',w)$. Then a',b' must have rank at most 1 since the rank of R is at most 2; and c is algebraic over a',b', so by exchange, $a',b' \in \mathrm{acl}(c)$, whereas a and b are both generic over a',b'. Thus if $R(a',b,a)$ holds, then $a' \in \mathrm{acl}(a,b) \cap \mathrm{acl}(c) = \mathrm{acl}(\varnothing)$. Similarly, if $R(a,b',b)$ holds, then $b' \in \mathrm{acl}(\varnothing)$. These cannot both happen since then $c \in \mathrm{acl}(a',b') = \mathrm{acl}(\varnothing)$. Symmetrically, $R(a',a,b)$ and $R(b,b',a)$ cannot both happen.

In conclusion, we have that $\neg\exists^\infty w\, R(a',b',w)$ holds iff

$$(R(a',b,a) \wedge R(a,b',b)) \vee (R(a',a,b) \wedge R(b,b',a)).$$

Case 3: For each $e \in \{a,b,c\}$, $R(a',b',e)$ holds. Then since one of them is generic over a',b', $\exists^\infty w\, R(a',b',w)$ holds.

Thus, in each case, we can computably in the atomic diagram of $\mathcal{M}_k$ determine whether or not $\exists^\infty w\, R(a',b',w)$ holds for any pair of elements a',b'. $\square$

We can now finish the proof of Theorem 47:

Fix a computable copy $\mathcal{M}$ of the model $\mathcal{M}_l$ of T for some $l \in [3,\omega)$. By Lemma 51, we have a computable presentation of $\mathcal{M}'$, the structure $\mathcal{M}$ seen as an $\mathcal{L}'$-structure, where $\mathcal{L}'$ is given by Lemma 51. For each $k \in [1,\omega]$, Theorem 25 gives a computable copy $\mathcal{N}'$ of $\mathcal{M}_k$ in the language $\mathcal{L}'$. Since each relation symbol in $\mathcal{L}$ is uniformly a Boolean combination of relation symbols in $\mathcal{L}'$, we obtain a computable copy $\mathcal{N}$ of $\mathcal{M}_k$. $\square$

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