ON THE FILTER OF COMPUTABLY ENUMERABLE SUPERSETS OF AN R-MAXIMAL SET

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Abstract. We study the filter $\mathcal{L}^*(A)$ of computably enumerable supersets (modulo finite sets) of an r-maximal set A and show that, for some such set A, the property of being cofinite in $\mathcal{L}^*(A)$ is still Σ^0_3-complete. This implies that for this A, there is no uniformly computably enumerable "tower" of sets exhausting exactly the coinfinite sets in $\mathcal{L}^*(A)$.

1. The theorem

The computably enumerable (or recursively enumerable) sets form a countable sublattice (denoted by $\mathcal{E}$ in the following) of the power set $\mathcal{P}(\omega)$ of the set of natural numbers. The operations of union and intersection are effective on $\mathcal{E}$ (i.e., effective in the indices of the computably enumerable sets). The complemented elements of $\mathcal{E}$ are exactly the computable sets. The finite sets in $\mathcal{E}$ are definable as those elements bounding only complemented elements; thus studying $\mathcal{E}$ is closely related to studying $\mathcal{E}^*$, the quotient of $\mathcal{E}$ modulo the ideal of finite sets. (From now on, the superscript $*$ will always denote that we are working modulo finite difference.)

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The structure of $\mathcal{E}$ has been the subject of much investigation over the past fifty-five years, starting with Post's seminal 1944 paper [Po44]. The $\forall\exists$-theory of $\mathcal{E}$ (in the language of lattices) was shown to be decidable by Lachlan [La68b], while the full first-order theory was shown to be undecidable independently by Harrington (unpublished) and Herrmann [He84], and in fact is known to be as complicated as full first-order arithmetic by Harrington (see [HN98]).

Much of the work in the past has concentrated on studying various principal filters of $\mathcal{E}$, i.e., final segments of the form

$$\mathcal{L}(A) = \{W \text{ computably enumerable} \mid A \subseteq W\}$$

for various computably enumerable sets A. (Note here that all principal *ideals* of $\mathcal{E}$ are either finite or effectively isomorphic to $\mathcal{E}$ and are thus less interesting.)

A computably enumerable set is called *hyperhypersimple* if A is coinfinite and $\mathcal{L}(A)$ (or, equivalently, $\mathcal{L}^*(A)$) is a Boolean algebra, i.e., if all sets in $\mathcal{L}(A)$ are complemented in $\mathcal{L}(A)$. (This was shown by Lachlan [La68a] to be equivalent to the original definition of hyperhypersimplicity by Post [Po44, p. 313].)

The principal filters $\mathcal{L}^*(A)$ above hyperhypersimple sets A have been studied extensively and shown to be closely connected to, but not always to uniquely determine, the orbit of A in $\mathcal{E}$. On the other hand, if A is not hyperhypersimple then $\mathcal{L}^*(A)$ can be shown to contain a copy of $\mathcal{E}$ as a sublattice — this follows easily from Post's definition of hyperhypersimplicity — and so $\mathcal{L}^*(A)$ is quite complicated.

Among the non-hyperhypersimple sets, the so-called r-maximal sets A have one of the simplest $\mathcal{L}^*(A)$. For this, we first need to give a few definitions.

A coinfinite computably enumerable set A is called *maximal* if $\mathcal{L}^*(A)$ has exactly two elements, i.e., if any computably enumerable superset of A equals (modulo finite difference) A or ω. So A is maximal iff its complement $\overline{A}$ cannot be split by any computably enumerable set into two infinite pieces.

A coinfinite computably enumerable set A is called *r-maximal* if its complement $\overline{A}$ cannot be split by any computable (or recursive) set into two infinite pieces, or, equivalently, if $\mathcal{L}^*(A)$ has no *cuppable* elements, i.e., there are no coinfinite computably enumerable sets $V, W \supseteq A$ with $V \cup W = \omega$.

A related notion is that of a major subset: A computably enumerable set A is called a *major subset* of a computably enumerable set $B \supset_\infty A$ if

$$\forall W\, (B \cup W = \omega \to A \cup W =^* \omega)$$

(where W ranges over computably enumerable sets), or, equivalently, if B is not *cuppable* in $\mathcal{L}^*(A)$, i.e., if there is no computably enumerable set W with $A \subset_\infty W \subset_\infty \omega$ and $B \cup W = \omega$. (We denote this as $A \subset_m B$.)

Maass and Stob [MS83] have shown that for $A \subset_m B$, the interval

$$[A, B] = \{W \text{ computably enumerable} \mid A \subseteq W \subseteq B\}$$

is uniformly effectively unique, i.e., for any two pairs of sets $A \subset_m B$ and $C \subset_m D$, there is an isomorphism from $[A, B]$ onto $[C, D]$ which is effective in the index of $W \in [A, B]$, and uniformly so in the indices of A, B, C, and D. We denote the unique isomorphism type of the lattice $[A, B]$ for $A \subset_m B$ by $\mathcal{M}$.

We now observe the following close connection between r-maximality and major subsets, which follows directly from the definitions: If A is r-maximal, and B is computably enumerable with $A \subset_\infty B \subset_\infty \omega$, then $A \subset_m B$.

We can thus distinguish three types of r-maximal sets A:

(1) A is a maximal set: In that case, $\mathcal{L}^*(A)$ is the two-element Boolean algebra.

(2) A is not maximal but has a maximal superset B: Since $\mathcal{L}^*(A)$ has no cuppable element, any computably enumerable set C with $A \subseteq C \subset_\infty \omega$ satisfies $C \subseteq^* B$. Together with the above observation that $A \subset_m B$, this uniquely determines the isomorphism type of $\mathcal{L}^*(A)$ as the disjoint union $\mathcal{M}^* \cup \{1\}$.

(3) A is *atomless*, i.e., has no maximal superset: Since $\mathcal{L}^*(A)$ has no cuppable element and also no coatom, we can noneffectively construct a *weak tower* $\{H_i\}_{i\in\omega}$ in $\mathcal{L}(A)$, i.e., a nondecreasing sequence of coinfinite computably enumerable sets $H_i \in \mathcal{L}(A)$ such that for any coinfinite set $W \in \mathcal{L}(A)$,

$$\exists i\,(W \subseteq^* H_i).$$

(To see this, simply set $H_0 = A$ and $H_{e+1} = H_e \cup W_e$ iff this set is coinfinite, and $H_{e+1} = H_e$ otherwise. A weak tower is called a *tower* if furthermore $H_i \subset_\infty H_{i+1}$ for all i, a property less relevant to our considerations here.)

For an atomless r-maximal set A, $\mathcal{L}^*(A)$ can vary. One can construct (see Soare [So87, X.5.8]) two atomless r-maximal sets A and B such that $\mathcal{L}^*(B)$ contains a noncappable element (i.e., there is a computably enumerable set W with $B \subset_\infty W \subset_\infty \omega$ such that there is no computably enumerable set V with $B \subset_\infty V \subset_\infty \omega$ and $W \cap V = B$) but $\mathcal{L}^*(A)$ contains only cappable elements. Recently, Cholak and Nies [CNta] have extended this by showing that there are infinitely many atomless r-maximal sets A with pairwise non-elementarily equivalent $\mathcal{L}^*(A)$. However, their proof shows that we are far from a general classification of all possible isomorphism types of $\mathcal{L}^*(A)$ for atomless r-maximal sets A.

One way to approach this question is to study possible towers $\{H_i\}_{i\in\omega}$ inside $\mathcal{L}^*(A)$. By the above observation, the interval $[A, H_i]$ is always isomorphic to $\mathcal{M}$; so since each interval $[A, H_i]$ is a subinterval of $[A, H_{i+1}]$, we have an induced embedding ι_i of $\mathcal{M}$ onto an initial segment of $\mathcal{M}$. Obviously, the isomorphism type of $\mathcal{L}^*(A)$ is determined by this sequence $\{\iota_i\}_{i\in\omega}$ of embeddings.

For each of the atomless r-maximal sets constructed above (i.e., Soare [So87, X.5.8] and Cholak and Nies [CNta]), there exists a uniformly computably enumerable weak tower $\{H_i\}_{i\in\omega}$ in $\mathcal{L}^*(A)$. It turns out that this implies that the index set

$$\mathrm{Cof}_A = \{e \in \omega \mid A \cup W_e \text{ cofinite}\}$$

is Δ^0_3. (To see this, note that Cof_A is obviously always Σ^0_3. Now, if there is a uniformly computably enumerable weak tower, then $e \notin \mathrm{Cof}_A$ iff $W_e \subseteq^*$ one of the sets of this weak tower.) Observe that in the above cases (1) and (2) (i.e., when A is maximal, or is not maximal but r-maximal with a maximal superset), Cof_A is trivially Δ^0_3. Finally note that Cof_A need not be Δ^0_3 for hyperhypersimple sets A by Lachlan's characterization of $\mathcal{L}^*(A)$ for A hyperhypersimple (cf. Soare [So87, Chapter X.7]).)

In particular, the so-called triangle sets (see Cholak and Nies [CNta, Def. 4.1]) form an interesting class of atomless r-maximal sets where Cof_A is always Δ^0_3. For suppose A is a triangle set, and fix any B such that $A \subset_\infty B \subset_\infty \omega$. Then

$$e \notin \mathrm{Cof}_A \Leftrightarrow \exists \widetilde{B} \exists C [B \subseteq \widetilde{B} \wedge \widetilde{B} \cap C = A \wedge W_e \subseteq \widetilde{B} \cup C],$$

which is a Σ_3^0 condition on e.

The purpose of this paper is to show that Cof_A is not always Δ_3^0 for r-maximal sets A:

Theorem. *There is an r-maximal set A for which*

$$\mathrm{Cof}_A = \{e \in \omega \mid A \cup W_e \textit{ cofinite}\}$$

is Σ_3^0-complete. (So there is no uniformly computably enumerable weak tower in $\mathcal{L}^*(A)$.)

This theorem provides additional evidence that the classification of the isomorphism types of $\mathcal{L}^*(A)$ for r-maximal sets A is likely to be quite difficult.

The rest of this paper is devoted to the proof of this theorem. Our notation generally follows Soare [So87].

2. The proof

Fix a Σ_3^0-complete predicate P. We construct an r-maximal set A and a uniformly computably enumerable sequence of sets $\{C_e\}_{e \in \omega}$ such that $P \leq_m \mathrm{Cof}_A$ via $e \mapsto C_e$. To this end, we need to meet, for all pairs $(V_{e,0}, V_{e,1})$ of computably enumerable sets, the following

Requirements:

$$\mathcal{P}_e : V_{e,0} \sqcup V_{e,1} = \omega \to \overline{A} \subseteq^* V_{e,0} \text{ or } \overline{A} \subseteq^* V_{e,1}, \text{ and}$$
$$\mathcal{N}_e : A \cup C_e =^* \omega \leftrightarrow P(e).$$

(Here $V_{e,0} \sqcup V_{e,1}$ denotes the disjoint union of $V_{e,0}$ and $V_{e,1}$.)

Note that the $\mathcal{N}$-requirements ensure that A is coinfinite, so together with the $\mathcal{P}$-requirements, they ensure that A is r-maximal.

$\mathcal{P}$-strategy: This strategy necessarily makes essential use of the fact that $V_{e,0} \sqcup V_{e,1} = \omega$. It proceeds as follows:

(1) Reserve a fresh number x.
(2) Wait for $(V_{e,0} \sqcup V_{e,1}) \upharpoonright (x+1) = [0, x]$.
(3) Dump all unreserved numbers $< x$ into A. If $x \in V_{e,0}$ then also dump all reserved numbers in $V_{e,1} \upharpoonright x$ into A.
(4) Return to Step (1).

$\mathcal{P}$-outcomes: We distinguish three possible outcomes for the $\mathcal{P}_e$-strategy:

fin: The strategy is eventually stuck waiting at Step (2) forever: Then $V_{e,0} \sqcup V_{e,1} = \omega$ fails, and the strategy only dumps finitely many numbers into A.

∞_0: The strategy finds infinitely many of its reserved numbers to be in $V_{e,0}$: Then $\overline{A}$ contains only reserved numbers, all of which are in $V_{e,0}$. Note that the strategy leaves infinitely many reserved numbers in $\overline{A}$ for lower-priority strategies to work with.

∞_1: Otherwise: Then $\overline{A}$ contains only reserved numbers, almost all of which are in $V_{e,1}$. Note again that the strategy leaves infinitely many reserved numbers in $\overline{A}$ for lower-priority strategies to work with.

$\mathcal{N}$**-strategy:** We represent the Σ^0_3-complete predicate by a uniformly computably enumerable array of set $Q_{e,i}$ such that for all e,

$$P(e) \text{ iff } \exists i\,(Q_{e,i} \text{ is infinite}).$$

For each i, we now have an $\mathcal{N}_{e,i}$-substrategy which will be finitary if $Q_{e,i}$ is finite but will enumerate all but finitely many elements of $\overline{A}$ into C_e otherwise. More precisely, the $\mathcal{N}_{e,i}$-substrategy proceeds as follows:

(1) Reserve a fresh number y from $\overline{A}$ and keep it out of A and C_e for now.
(2) Wait for the size of $Q_{e,i}$ to increase.
(3) Add y into C_e and add all unreserved numbers $< y$ into A.
(4) Return to Step (1).

$\mathcal{N}$**-outcomes:** We distinguish two possible outcomes for an $\mathcal{N}_{e,i}$-strategy:

fin: $Q_{e,i}$ is finite: Then the $\mathcal{N}_{e,i}$-strategy enumerates finitely many numbers and eventually keeps one number permanently out of A and C_e.

∞: $Q_{e,i}$ is infinite: Then the $\mathcal{N}_{e,i}$-strategy enumerates all numbers into A or C_e (but leaves infinitely many numbers in $\overline{A}$ for lower-priority strategies to work with).

So if $P(e)$ fails then each $\mathcal{N}_{e,i}$-strategy will keep one (distinct) number out of $A \cup C_e$, whereas if $P(e)$ holds then some $\mathcal{N}_{e,i}$-strategy will enumerate all numbers into $A \cup C_e$. This clearly meets the $\mathcal{N}_e$-requirement.

Interaction between strategies: The interaction between the various strategies is essentially that of an infinite injury priority argument. A typical case here is that of a $\mathcal{P}_e$-strategy α, say, above other strategies which (individually or collectively) enumerate an infinite set of numbers into A. Then it is conceivable that the strategies below α might enumerate into A all the numbers which α later sees enter $V_{e,i}$ (for some fixed $i < 2$) and would therefore want to keep out of A.

Note that we can rule out this potential conflict as follows: α only works with numbers which have not been "handled" by strategies below α before, and which are not handled by strategies below α until α "decides" what to do with these numbers. This is exactly where the hypothesis of the $\mathcal{P}_e$-requirement comes in: α can afford to handle "fresh" numbers (coming from above α) and not let strategies below α handle these numbers: If α has the finite outcome then this will only constitute a finite restraint on strategies below α. On the other hand, if α has the infinite outcome then the strategies below the true infinite outcome of α will only handle numbers that have already appeared in $V_{e,0}$ or $V_{e,1}$, and all numbers handled by strategies below the finite outcome of α will eventually be dumped by α into A.

Similar remarks apply to an $\mathcal{N}_{e,i}$-strategy above other strategies.

Streaming: We formalize the above by introducing streaming into our construction: Each strategy α works with a set S_α of numbers such that

(1) $S_\emptyset = \omega$;
(2) if α is not the empty node then S_α is a subset of S_{α^-} (where α^- is the immediate predecessor of α);
(3) no number is ever in $S_\alpha \cap S_\beta$ for incomparable α and β;
(4) at the time a number x first enters S_α, x is not in A;

(5) each time α is initialized, S_α is made empty;
(6) S_α is enlarged only at stages at which α appears to be on the true path; and
(7) if α is along the true path of the construction then S_α is an infinite computable set.

Thus S_α can be thought of as a set d. c. e. uniformly in α; and S_α is finite if α is to the left of the true path of the construction; an infinite computable set if S_α is along the true path; and empty if α is to the right of the true path.

Tree of strategies: We define the *set of outcomes* as

$$\Lambda = \{\infty <_\Lambda \infty_0 <_\Lambda \infty_1 <_\Lambda \mathit{fin}\}.$$

We now inductively define the *tree of strategies* $T \subseteq \Lambda^{<\omega}$ and the assignment of requirements to nodes on T as follows: Fix an effective ordering $\{\mathcal{R}_j\}_{j\in\omega}$ (of order type ω) of all $\mathcal{P}_e$-requirements and all $\mathcal{N}_{e,i}$-subrequirements such that $\mathcal{R}_0 = \mathcal{P}_0$ and such that $\mathcal{N}_{e,i}$ precedes $\mathcal{N}_{e,j}$ whenever $i < j$.

To the empty node $\emptyset \in T$, we assign requirement $\mathcal{R}_0 (= \mathcal{P}_0)$.

Now assume that α has been declared to be on T. We distinguish two cases:

Case 1: Some $\mathcal{P}_e$-requirement has been assigned to α: We declare its immediate successors on T to be $\alpha\hat{}\langle o\rangle$ for all $o \in \{\infty_0, \infty_1, \mathit{fin}\}$ and declare $\mathcal{P}_e$ to be *satisfied along* $\alpha\hat{}\langle o\rangle$ for any $o \in \{\infty_0, \infty_1, \mathit{fin}\}$. Furthermore, for any $o \in \{\infty_0, \infty_1, \mathit{fin}\}$, we declare all requirements satisfied along α to be also *satisfied along* $\alpha\hat{}\langle o\rangle$, and assign to $\alpha\hat{}\langle o\rangle$ the highest-priority requirement not satisfied along $\alpha\hat{}\langle o\rangle$.

Case 2: Some $\mathcal{N}_{e,i}$-requirement has been assigned to α: We declare its immediate successors on T to be $\alpha\hat{}\langle o\rangle$ for $o \in \{\infty, \mathit{fin}\}$. We declare $\mathcal{N}_{e,i}$ to be *satisfied along* $\alpha\hat{}\langle \mathit{fin}\rangle$, and we declare $\mathcal{N}_{e,j}$ (for *all* j) to be *satisfied along* $\alpha\hat{}\langle\infty\rangle$. Furthermore, for $o \in \{\infty, \mathit{fin}\}$, we declare declare all requirements satisfied along α to be also *satisfied along* $\alpha\hat{}\langle o\rangle$, and then assign to $\alpha\hat{}\langle o\rangle$ the highest-priority requirement not satisfied along $\alpha\hat{}\langle o\rangle$.

Lemma 1. *Given any path $p \in [T]$ and any requirement $\mathcal{R}$ (of the form $\mathcal{P}_e$ or $\mathcal{N}_{e,i}$), $\mathcal{R}$ is satisfied along all sufficiently long $\alpha \subset p$.*

Proof. Immediate.

Construction: The construction proceeds as usual in stages $s \in \omega$ which are subdivided into substages $t \leq s$. At each substage t, we let the strategy α of length t act whose guess about the outcomes of the higher-priority strategies $\beta \subset \alpha$ "currently appears correct", and then choose the strategy $\alpha' \supset \alpha$ for the next substage $t+1$ as well as its stream $S_{\alpha'}$ (or decide to end the stage). At substage 0 of each stage s, we set the *stream* $S_\emptyset$ of the empty node $S_\emptyset = [0, s)$.

Any parameter will remain unchanged unless explicitly changed. A strategy is *initialized* by making all its parameters undefined or empty. If the description below directs α to pick a certain number from its stream but no such number exists, then we agree to end the stage. (We will see later that since any strategy along the true path has an infinite stream, any such delay for a strategy along the true path will be finite.)

Stage 0: Initialize all strategies.

Stage $s > 0$*:* In order to describe the action of the strategy α eligible to act at substage t of stage s, we now distinguish two cases:

Case 1: α is a $\mathcal{P}_e$-strategy: Pick the first applicable subcase:

Subcase 1.1: α currently has no reserved number x: Then α chooses a number x from its stream S_α to be its *reserved number* which is greater than any stage $< s$ at which α had a reserved number or at which α was initialized, and ends the stage.

Subcase 1.2: $(V_{e,0} \sqcup V_{e,1}) \upharpoonright (x+1) = [0,x]$ and $x \in V_{e,0}$: Let s_0 be greatest stage $< s$ at which α was initialized. Then α adds x to the stream of $\alpha\hat{\ }\langle\infty_0\rangle$ and *dumps* into A all numbers in the interval (s_0, x) which are not in $S_{\alpha\hat{\ }\langle\infty_0\rangle}$, and lets $\alpha\hat{\ }\langle\infty_0\rangle$ be eligible to act next.

Subcase 1.3: $(V_{e,0} \sqcup V_{e,1}) \upharpoonright (x+1) = [0,x]$ and $x \in V_{e,1}$: Let s_0 be greatest stage $< s$ at which α was initialized or at which $\alpha\hat{\ }\langle\infty_0\rangle$ was eligible to act. Then α adds x to the stream of $\alpha\hat{\ }\langle\infty_1\rangle$ and *dumps* into A all numbers in the interval (s_0, x) which are not in $S_{\alpha\hat{\ }\langle\infty_1\rangle}$, and lets $\alpha\hat{\ }\langle\infty_1\rangle$ be eligible to act next.

Substage 1.4: Otherwise: Let s_0 be the greatest stage $< s$ at which $\alpha\hat{\ }\langle \mathit{fin}\rangle$ was initialized. Then α sets the stream of $\alpha\hat{\ }\langle \mathit{fin}\rangle$ to be

$$S_{\alpha\hat{\ }\langle \mathit{fin}\rangle} = S_\alpha \cap (s_0, s),$$

and lets $\alpha\hat{\ }\langle \mathit{fin}\rangle$ be eligible to act next.

Case 2: α is an $\mathcal{N}_{e,i}$-strategy: Pick the first applicable subcase:

Subcase 2.1: α currently has no reserved number y: Then α chooses a number y from its stream S_α to be its *reserved number* which is greater than any stage $< s$ at which α had a reserved number or at which α was initialized, and ends the stage.

Subcase 2.2: The size of $Q_{e,i}$ has increased since the last stage at which α was eligible to act: Let s_0 be greatest stage $< s$ at which α was initialized. Then α adds y to the stream of $\alpha\hat{\ }\langle\infty\rangle$ and *dumps* into A all numbers in the interval (s_0, y) which are not in $S_{\alpha\hat{\ }\langle\infty\rangle}$, and lets $\alpha\hat{\ }\langle\infty\rangle$ be eligible to act next.

Subcase 2.3: Otherwise: Let s_0 be the last stage at which $\alpha\hat{\ }\langle \mathit{fin}\rangle$ was initialized. Then α sets the stream of $\alpha\hat{\ }\langle \mathit{fin}\rangle$ to be

$$S_{\alpha\hat{\ }\langle \mathit{fin}\rangle} = S_\alpha \cap (s_0, s),$$

and lets $\alpha\hat{\ }\langle \mathit{fin}\rangle$ be eligible to act next.

At the end of stage s, we set f_s to be longest node eligible to act at stage s, and initialize all strategies $>_L f_s$.

This ends the description of the construction.

Verification: We now verify that the above construction meets our requirements.

We define *the true path of the construction* $f \in [T]$ inductively by

$$f(n) = \liminf_s \{o \in \Lambda \mid (f \upharpoonright n)\hat{\ }\langle o\rangle \subseteq f_s\}.$$

Lemma 2. *Fix* n *and set* $\alpha = f \upharpoonright n$*.*

(1) α *exists.*
(2) α *is initialized at most finitely often, say, never after a (least) stage* s_α*.*
(3) *The stream* S_α *is an infinite computable set, and after stage* s_α*, elements enter* S_α *in increasing order and while they are not yet in* A*.*

Proof. We proceed by induction on n. The claim is clear for $n = 0$, so assume $n > 0$.

(1) Immediate by (2) and (3) for $n - 1$ since Λ is finite.

(2), (3) Immediate by (1) for n and by the construction. $\square$

Lemma 3. *All $\mathcal{P}_e$-requirements are satisfied.*

Proof. Suppose $V_{e,0} \sqcup V_{e,1} = \omega$ and, by Lemma 1, fix the $\mathcal{P}_e$-strategy $\alpha \subset f$. Then α never waits forever for $(V_{e,0} \sqcup V_{e,1}) \restriction (x+1) = [0, x]$ for a fixed reserved number x. There are now two cases:

Case 1: α finds infinitely many of its reserved numbers to be in $V_{e,0}$: Then the stream $S_{\alpha\hat{}\langle\infty_0\rangle}$ consists only of numbers in $V_{e,0}$, and all but finitely many numbers outside of $V_{e,0}$ are dumped into A.

Case 2: Otherwise: Then the stream $S_{\alpha\hat{}\langle\infty_1\rangle}$ consists only of numbers in $V_{e,1}$, and all but finitely many numbers outside of $V_{e,1}$ are dumped into A. $\square$

Lemma 4. *All $\mathcal{N}_e$-requirements are satisfied.*

Proof. We distinguish two cases:

Case 1: $P(e)$ fails: Then for all i, the set $Q_{e,i}$ is finite. We note that for all $\mathcal{N}_{e,i}$-strategies $\alpha_i \subset f$, α_i has outcome *fin* along the true path; and so, by Lemma 1, α_i exists for all i. Each α_i eventually has a (distinct) fixed reserved number, which it permanently keeps out of $A \cup C_e$.

Case 2: $P(e)$ holds: Then for some i, the set $Q_{e,i}$ is infinite, and all sets $Q_{e,j}$ (for $j < i$) are finite. As in Case 1, we see that each $\mathcal{N}_{e,j}$-strategy (for $j < i$) along the true path has outcome *fin*; so, by Lemma 1, there is an $\mathcal{N}_{e,i}$-strategy $\alpha_i \subset f$. Then α_i must have outcome ∞ along the true path, and will ensure that almost all numbers are in $A \cup C_e$.

This establishes our Theorem.

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