Reductions between Types of Numberings*

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Abstract. This paper considers reductions between types of numberings; these reductions preserve the Rogers Semilattice of the numberings reduced and also preserve the number of minimal and positive degrees in their semilattice. It is shown how to use these reductions to simplify some constructions of specific semilattices. Furthermore, it is shown that for the basic types of numberings, one can reduce the left-r.e. numberings to the r.e. numberings and the k-r.e. numberings to the $(k+1)$-r.e. numberings; all further reductions are obtained by concatenating these reductions.

1 Introduction

Uniform computations of families of recursively enumerable sets, also called computable numberings, are a classical object of research in the theory of algorithms. The theory of numberings is one of the fundamental topics in recursion theory and mathematical logic. It is basically due to Gödel's idea to code countable families of objects by numbers, so that objects of the family can be effectively identified with numbers, or indices, and studied from their indices. Given its relevance, the theory of numberings has seen the contributions of many distinguished scholars, including Kleene, Kolmogorov, Uspenskii, Friedberg and Rogers and became a systematic object

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of study due to the work of the Novosibirsk school of algebra and logic, led by Maltsev and Ershov, who dedicated to the theory of numberings a famous monograph [18]. While numberings are a powerful tool to use the set of natural numbers in order to study families of constructive objects (in recursive algebra, recursive model theory, etc.), they are an interesting object of study in themselves: Here, an important device is that of reducibility between numberings, where a numbering is reducible to another numbering if there is an effective way to go from indices of an object in the first numbering to indices of the same object in the second numbering. Thus the relative complexity of numberings of objects of a fixed family can be measured by this notion of reducibility, and gives rise to the so-called Rogers upper semilattice of the family, whose elements are the degrees of numberings, where two numberings have the same degree if they are reducible to each other.

In the last decade, also due to the influence of the proposal of Goncharov and Sorbi for a generalised theory of numberings [24], the objects of study in the theory of numberings have gradually changed, and more and more emphasis has been given, and is being given, to uniform computations of families of sets in the arithmetical, analytical and Ershov hierarchies, which are increasingly attracting the attention of specialists in recursion theory.

A numbering is a family (A_n) of sets which can be enumerated or approximated uniformly in n. Numberings are ordered by many-one reduction; here a numbering (A_n) is many-one reducible to (B_n) iff there is a recursive function f such that $A_n = B_{f(n)}$ for all n. Rogers initiated the study of the semilattice of numberings under many-one reduction and Ershov [15–17] transferred it in particular to the study of the k-r.e. and, more generally, α-r.e. sets where the sets A_n are approximated uniformly by enumerating and taking out elements a limited number of times; this number can either be a constant k or an ordinal which is counted down each time the number is enumerated or taken out. An important question investigated is how the numberings on various levels of the hierarchy differ and which types of semilattices can be realised as the semilattice of all α-r.e. numberings with the same range as a given α-r.e. numbering (A_n). Friedberg showed that the family of all r.e. sets has a numbering in which every set occurs only once [20]; on the other hand, Vyugin [40, 41] showed that there are numberings of families of r.e. sets whose Rogers semilattice does not contain a minimal numbering. Goncharov contributed substantially to the field of r.e. numberings by constructing a numbering which has (up to many-one equivalence) exactly two Friedberg numberings [23].

Recently, the study of numberings has shifted much to the Ershov hierarchy and looked at how results about r.e. numberings can be replicated on the various levels of this hierarchy [8, 6, 25, 29, 33]. The overall goal of this paper is to establish some reductions between various types of numberings that allow to show some basic facts about numberings which contribute to this study.

- If a k-r.e. numbering can realise a certain type of Rogers semilattice, so can a $(k+1)$-r.e. numbering or, more generally, an $(\alpha+k)$-r.e. numbering for a given recursive ordinal α;
- Every type of Rogers semilattice realised by an r.e. numbering is also realised by an α-r.e. for every recursive ordinal α which is not a power of ω and which is not 0 while if α is a power

of ω then there is no α-r.e. numbering without minimal numberings in the Rogers semilattice (which stands in contrast to the r.e. case);
- If a left-r.e. numbering can realise a certain type of Rogers semilattice, so can an r.e. numbering;
- There are also some limitations to the method, for example a direct construction exhibits a $(k+1)$-r.e. numbering which cannot be reduced to a k-r.e. numbering.

More involved constructions show an even harder limitation. For example, Badaev and Lempp [8] constructed a numbering of d.r.e. sets with exactly two Friedberg numberings and no further minimal numberings and showed that their Rogers semilattice does not exist as a Rogers semilattice of an r.e. numbering. This shows in a very strong way that d.r.e. numberings cannot be reduced to r.e. numberings.

The reductions constructed permit to give various alternative proofs to known results about d.r.e. and, in general, α-r.e. numberings. For example, one can show that for every recursive ordinal α which is not an ω-power and every Rogers semilattice of r.e. sets, the corresponding semilattice is also realised within the α-r.e. numberings and that in both semilattices the same numberings are positive and Friedberg and so on. Examples of such results are semilattices with exactly two Friedberg numberings (as first constructed by Goncharov in the r.e. case), lattices with exactly one numbering though not a Friedberg one, lattices with a Friedberg numbering but not a least one, lattices with exactly one positive numbering which is undecidable but not the least member of the lattice and so on. Furthermore, it is shown that this type of transfers from r.e. numberings to d.r.e. numberings can be realised by Rogers semilattices where every numbering of d.r.e. sets actually is an r.e. numbering.

Note the following related findings about numberings in the arithmetical hierarchy — they are about Σ_k-numberings which are for $k=1$ the uniformly r.e. and for $k>1$ the uniformly $K^{(k-1)}$-r.e. numberings. One Σ_k-numbering (A_n) is reducible to an other numbering Σ_k-numbering (B_n) iff there is a recursive function f with $B_{f(n)} = A_n$ for all n. Badaev, Goncharov and Sorbi [7] showed that for all $k \geq 0$ and $h \geq k+3$, every non-trivial Rogers semilattice of a Σ_k-numbering is non-isomorphic to every Rogers semilattice of a Σ_h-numbering. Podzorov [36] looked at the case of $k+2$ versus k and showed that every Rogers semilattice of a non-trivial Σ_k-numbering is nonisomorphic to the Rogers semilattice of every Σ_{k+2}-numbering. Whether one can show the same for Σ_{k+1}-numberings versus Σ_k-numberings is an open problem. In contrast to the results of Badaev, Goncharov and Sorbi [7], one of the above mentioned main findings of the present work is that every numbering which is on the k-th level of the Ershov hierarchy has a Rogers semilattice which is isomorphic to a numbering on the $(k+1)$-st level of the Ershov hierarchy.

The reader is referred to the standard textbooks of Downey and Hirschfeldt [14], Nies [30], Odifreddi [31, 32] and Soare [37] for further information on recursion theory and related notions.

2 Reductions on the numberings

The starting point is the following result, which shows that families of k-r.e. numberings can be reduced to $(k+1)$-r.e. numberings by an operator which preserves the Rogers semilattice of the families and also properties of numberings like being Friedberg and positive. For this, recall the following fundamental definitions.

Definition 1. Let (A_n) denote a family of subsets of natural numbers or a numbering for short and for (A_n), one makes the following definitions:

1. A numbering (A_n) is *k-r.e.* iff there is a uniformly recursive approximation $(A_{n,t})$ to (A_n) such that $A_{n,0} = \emptyset$ and for each n and x there are at most k times t such that $A_{n,t+1}(x) \neq A_{n,t}(x)$ – for these t one says that A_n makes a mind change on x at time t;
2. A numbering (A_n) of subsets of the natural numbers is *ω-r.e.* iff there is a uniformly recursive approximation $(A_{n,t})$ to (A_n) and a recursive function f such that $A_{n,0} = \emptyset$ and for each n and x there are at most $f(n, x)$ times t such that $A_{n,t+1}(x) \neq A_{n,t}(x)$;
3. A numbering (A_n) is *many-one reducible* to a numbering (B_n) iff there is a recursive function f such that
$$\forall n \, [A_n = B_{f(n)}];$$
4. A numbering (A_n) is *positive* iff the set $\{(m, n) : A_m = A_n\}$ is recursively enumerable;
5. A numbering (A_n) is *decidable* iff the set $\{(m, n) : A_m = A_n\}$ is recursive;
6. A numbering (A_n) is *Friedberg* iff there are no distinct n and m with $A_n = A_m$;
7. Two numberings are *equivalent* iff there are recursive functions f and g such that, for all n, $B_{f(n)} = A_n$ and $A_{g(n)} = B_n$;
8. The *Rogers semilattice* of a family (A_n) enumerated by a k-r.e. numbering is the collection of all k-r.e. numberings (B_n) of that family partially ordered by the many-one reducibility $\leq$ where equivalent numberings with respect to $\leq$ are considered equal; similarly for the Rogers semilattice of an ω-r.e. or α-r.e. numbering (A_n) (for the definition of an α-r.e. numbering, see Definition 18 below);
9. A numbering (A_n) is *minimal* / *least* / *greatest* iff it has the corresponding property with respect to the Rogers semilattice of all numberings of the same type and the same family as (A_n); a greatest numbering is also called a *principal numbering* [35];
10. When counting positive, decidable, Friedberg, ... numberings, one actually counts the equivalence classes (i.e., degrees) containing such numberings. For example, one says that a family has five Friedberg numberings iff within the Rogers semilattice of all numberings of that family of the given type, there are five equivalence classes which contain a Friedberg numbering.

Ershov [15–17] and later Goncharov and Sorbi [24] investigated the more general definitions of α-r.e. sets and numberings going beyond the two special cases above; the general definition can be found in Definition 18 below.

The following results show now that one can carry over every Rogers semilattice realised in the k-r.e. numberings to one realised in the $(k+1)$-r.e. numberings. The operator takes one out of two cases for odd and even k and uses a parameter E; it differs between the two cases only when the second coordinate is in E and in that case it takes the characteristic function obtained by k mind changes.

Theorem 2. *Suppose $k \geq 1$ is a natural number. Given any non-recursive r.e. set E, define the operator $\mathit{\Gamma}_k$ by*

$$\begin{aligned}\mathit{\Gamma}_k(A) &= \{\langle x,y\rangle : (x \in A \wedge y \notin E) \vee (k \textit{ is odd } \wedge y \in E)\} \\ &= \begin{cases} (A \times \mathbb{N}) \cup (\mathbb{N} \times E), & \textit{if } k \textit{ is odd;} \\ (A \times \mathbb{N}) - (\mathbb{N} \times E), & \textit{if } k \textit{ is even.} \end{cases}\end{aligned}$$

Then $\mathit{\Gamma}_k$ maps the k-r.e. sets to $(k+1)$-r.e. sets and also translates every numbering (A_n) of k-r.e. sets into a numbering $(\mathit{\Gamma}_k(A_n))$ of $(k+1)$-r.e. sets such that the following conditions hold:

- *Every numbering of $(k+1)$-r.e. sets enumerating the same family as $(\mathit{\Gamma}_k(A_n))$ is the image $(\mathit{\Gamma}_k(B_n))$ of a k-r.e. numbering (B_n) enumerating the same family as the numbering (A_n);*
- *If (A_n) is a k-r.e. numbering, then so is $(\mathit{\Gamma}_k(A_n))$;*
- *For all m,n, $A_n \subseteq A_m \Leftrightarrow \mathit{\Gamma}_k(A_n) \subseteq \mathit{\Gamma}_k(A_m)$ and $A_n = A_m \Leftrightarrow \mathit{\Gamma}_k(A_n) = \mathit{\Gamma}_k(A_m)$; in particular, (A_n) is a Friedberg numbering iff $(\mathit{\Gamma}_k(A_n))$ is a Friedberg numbering, and (A_n) is a positive numbering iff $(\mathit{\Gamma}_k(A_n))$ is a positive numbering;*
- *$(A_n) \leq (B_n)$ in the Rogers semilattice of the k-r.e. numberings iff $(\mathit{\Gamma}_k(A_n)) \leq (\mathit{\Gamma}_k(B_n))$ in the Rogers semilattice of the $(k+1)$-r.e. numberings.*

Proof. First, given a k-r.e. numbering (A_n), note that the numbering $\tilde{A}_n = \mathit{\Gamma}_k(A_n)$ is even a k-r.e. numbering: The approximation of $\tilde{A}_n(\langle x,y\rangle)$ follows the mind changes of $A_n(x)$ until y is enumerated into E; if this happens before $A_n(x)$ has made k mind changes, then up to one further mind change can be used to obtain the value $\tilde{A}_n(\langle x,y\rangle)$ and no further mind change is needed so that the bound of k is kept; in the case that $A_n(x)$ has made k mind changes then no further mind change is needed, as $\tilde{A}_n(\langle x,y\rangle)$ and $A_n(x)$ have then both the same value by the parity of k. In the case that y is never enumerated into E, it is also sufficient to follow the up to k mind changes of $A_n(x)$.

Second, assume that there is a $(k+1)$-r.e. numbering $(\tilde{B}_n)$ which is for the same family as the numbering $(\tilde{A}_n)$. Let

$$\tilde{E} = \{y : \exists x\, \exists n\, [\tilde{B}_n \text{ makes on } \langle x,y\rangle \text{ exactly } k+1 \text{ mind changes}]\}.$$

The set $\tilde{E}$ is disjoint from E, as $\tilde{B}_n(\langle x,y\rangle)$ takes for all $\langle x,y\rangle$ with $y \in E$ the value which is obtained after k mind changes: 1 in the case of odd and 0 in the case of even k. Hence, after $k+1$ mind changes, the value $\tilde{B}_n(\langle x,y\rangle)$ differs from this value and is final and y cannot be in E. As both sets E and $\tilde{E}$ are r.e. and disjoint and E is not recursive, there is $y \notin E \cup \tilde{E}$. Now fix this y and define

$$B_n(x) = \tilde{B}_n(\langle x,y\rangle)$$

for this fixed y. The family of the (B_n) is a k-r.e. numbering due to the choice of y. Furthermore, as $y \notin E$ and as each $\tilde{B}_n = \mathit{\Gamma}_k(A_m)$ for some m, one has for this m that $A_m(x) = \tilde{A}_m(\langle x,y\rangle) = \tilde{B}_n(\langle x,y\rangle) = B_n(x)$ for all x and therefore $\tilde{B}_n = \mathit{\Gamma}_k(B_n)$ for all n. It follows also that (B_n) and (A_n) enumerate the same family. Note that this reverse translation also ensures that the numbering of the $(\tilde{B}_n)$ is actually a k-r.e. numbering.

Third, note that the defining formula of $\mathit{\Gamma}_k$ directly gives that $A_n \subseteq A_m$ iff $\mathit{\Gamma}_k(A_n) \subseteq \mathit{\Gamma}_k(A_m)$. Thus, (A_n) is a Friedberg numbering iff $(\mathit{\Gamma}_k(A_n))$ is; furthermore, (A_n) is a positive numbering iff $(\mathit{\Gamma}_k(A_n))$ is.

Finally, if f is a recursive function witnessing $(A_n) \leq (B_n)$, that is, satisfying that $A_n = B_{f(n)}$ for all n, then $\mathit{\Gamma}_k(A_n) = \mathit{\Gamma}_k(B_{f(n)})$ for all n and $(\mathit{\Gamma}_k(A_n)) \leq (\mathit{\Gamma}_k(B_n))$; similarly, if $(\mathit{\Gamma}_k(A_n)) \leq (\mathit{\Gamma}_k(B_n))$ via f then $(A_n) \leq (B_n)$ via the same f.

Thus, the Rogers semilattices of the numberings (A_n) and $(\mathit{\Gamma}_k(A_n))$ are the same and one is for k-r.e. numberings while the other is for $(k+1)$-r.e. numberings. Furthermore, $\mathit{\Gamma}_k$ is a degree isomorphism between these numberings and it preserves, in addition to degree-theoretic properties like being least, greatest, or minimal, also properties with respect to the equality relation on the numbered sets like being Friedberg or being a positive numbering. □

Note that one can concatenate the operators. For example the operator

$$A \mapsto \mathit{\Gamma}_5(\mathit{\Gamma}_4(\mathit{\Gamma}_3(A)))$$

maps every 3-r.e. numbering into a 6-r.e. numbering such that its Rogers semilattice in 6-r.e. numberings is the same as the given Rogers semilattice for 3-r.e. numberings and furthermore every 6-r.e. numbering of the family in the range is actually a 5-r.e. numbering. A corollary is that whenever $k < h$ then every semilattice realised by an k-r.e. numbering is also realised by a h-r.e. numbering.

An example of an application of such operators is the following alternative proof (later the first results were generalised by S. Ospichev to all ordinal levels [34]) to results of Goncharov, Lempp and Solomon [25]; the use of the reductions implies that all of the families below can be obtained as families of r.e. sets – while Goncharov, Lempp and Solomon [25] had this only for the first family.

Theorem 3 (Goncharov, Lempp and Solomon [25]). *Considering the Rogers semilattices of families of d.r.e. sets, one can find families and numberings of these families with each of the following properties:*

1. *The numbering is unique and is Friedberg;*
2. *The numbering is a least numbering and not unique and is not equivalent to a Friedberg numbering;*
3. *The numbering is a least numbering and not unique and Friedberg.*

Furthermore, all results are witnessed by families where every d.r.e. numbering of the family is actually a numbering of r.e. sets.

Proof. These tasks are well-known for the case of r.e. numberings; here the families are repeated for the reader's convenience and will be used later in the proof.

1. All r.e. numberings of $A_n = \{n\}$ are equivalent and the given numbering is Friedberg.

2. Let $A_0 = \emptyset$ and $A_{n+1} = \{n\}$ in the case that $n \notin K$ and $A_{n+1} = K$ in the case that $n \in K$. This numbering cannot be made into a Friedberg numbering (B_n), as if such a numbering would exist and K would have index e, then the set $\{n : \exists d \neq e\,[n \in B_d]\}$ would permit to enumerate the complement of K. Furthermore, given any further numbering (C_n) enumerating the same family with $C_m = \emptyset$ for some given m, one can reduce the numbering (A_n) to this one by mapping 0 to m and each number $n+1$ to the first index h found such that $n \in C_h$. Thus the given numbering (A_n) is a minimal numbering.
3. Let $A_0 = \emptyset$ and $A_{n+1} = \{n\}$. This Friedberg numbering is a least numbering and it is not the only numbering, as when taking a one-one enumeration $k_0, k_1, \ldots$ of the halting problem K, one can take $B_{k_n} = \{n\}$ and $B_m = \emptyset$ for $m \notin K$.

Then one invokes Theorem 2 iteratively to show that for $k = 1, 2, 3, \ldots$ there are k-r.e. families with the corresponding properties. □

It is known that for finite r.e. families, the Rogers semilattice of the family only depends on its inclusion structure. The following example shows that this is not true for d.r.e. families.

Example 4. *Let Γ_1 be as in Theorem 2. Let A be the set of even numbers and B the set of odd numbers. The families $\{A, B\}$ and $\{\Gamma_1(A), \Gamma_1(B)\}$ have the same inclusion structure, as the two sets are mutually incomparable under inclusion. But while the class of all d.r.e. numberings of $\{A, B\}$ is isomorphic to the class of all many-one degrees between those of A and of $K \oplus \overline{K}$ for the halting problem K, the family $\{\Gamma_1(A), \Gamma_1(B)\}$ has only one unique d.r.e. numbering.*

The next result provides a list of limitations on the existence of operators as in Theorem 2. Corollary 21 below will also provide the result that the r.e. numberings cannot be reduced to ω-r.e. numberings.

Theorem 5. *There is no operator with the properties from Theorems 2 which reduces $(k+1)$-r.e. numberings to k-r.e. numberings for any $k \geq 1$. Similarly one cannot reduce ω-r.e. numberings to k-r.e. numberings for any k.*

Proof. Consider a family containing $A_0 = \emptyset$ and $A_1 = \{0\}$. Let E be a $(k+1)$-r.e. set which is neither k-r.e. nor the complement of a k-r.e. set. Now let

$$B_{2n} = \begin{cases} A_0, & \text{if } n \notin E; \\ A_1, & \text{if } n \in E; \end{cases}$$
$$B_{2n+1} = A_n.$$

This numbering is also a $(k+1)$-r.e. numbering. Assume now that a reduction Ψ reduces the $(k+1)$-r.e. numberings to the k-r.e. numberings. As Ψ is one-one, there is an x in the symmetric difference of $\Psi(A_0)$ and $\Psi(A_1)$. In the case that $x \in \Psi(A_1) - \Psi(A_0)$ then the set

$$\{n : x \in \Psi(B_{2n})\}$$

is a k-r.e. set which is equal to E; in the case that $x \in \Psi(A_0) - \Psi(A_1)$ then the set

$$\{n : x \in \Psi(B_{2n})\}$$

is a k-r.e. set which is equal to the complement of E. Both cases cannot apply by the choice of E; hence the operator Ψ cannot exist. It is easy to see that a similar construction disproves the possibility to reduce ω-r.e. numberings to k-r.e. numberings for any k.

Goncharov [22] constructed a family (A_n) of r.e. sets which has exactly two non-equivalent Friedberg numberings. Such a family exists by Theorem 2 on all finite levels of the Ershov hierarchy. However, it does not exist on the ω-level, as Theorem 14 below shows. Thus no operator Ψ as in Theorem 2 can map a family with exactly two Friedberg numberings into a family of ω-r.e. numberings with the same property. □

Badaev and Lempp [8] showed in their work an even much stronger result: There is a family of d.r.e. sets which has a Rogers semilattice not occurring within the families of r.e. sets; this strong separation directly implies that one cannot translate d.r.e. numberings into r.e. numberings as it is done for the other direction in Theorem 2.

3 Left-R.E. Numberings

Brodhead and Kjos-Hanssen [11] initiated the study of the numberings of left-r.e. sets. Here the left-r.e. sets [14, 30] are a generalisation of the widely studied k-r.e. sets [4, 12, 13, 15] but in the sense that one does not bound the number of mind changes, but rather postulates that the changes go, lexicographically, always upward.

Definition 6. A set A is left-r.e. iff there is a sequence of uniformly recursive sets $A_0, A_1, \ldots$ converging pointwise to A such that, for each n, $A_n \leq_{lex} A_{n+1}$, that is, either $A_n = A_{n+1}$ or the least element in the symmetric difference of A_n and A_{n+1} is a member of A_{n+1} but not of A_n. In other words, the sums $\sum_m 2^{-m-1} A_n(m)$ are increasing for $n = 0, 1, \ldots$ and converge from below to $\sum_m 2^{-m-1} A(m)$.

Subsequent studies mainly focused on the questions of which families of left-r.e. sets have numberings or Friedberg numberings and what type of index sets such families can have within numberings of all left-r.e. sets [27, 39]. However, prior work on the types of Rogers semilattice realised by left-r.e. numberings is mainly the collection of negative results obtained for r.e. numberings via the reduction Ξ induced by

$$\Xi(A) = \{x : bin(x) \leq_{lex} A\}$$

where $bin(x)$ is the x-th binary string, so $bin(0)$ is the empty string, $bin(1) = 0$, $bin(2) = 1$, $bin(3) = 00$, $bin(4) = 01$ and so on. This reduction maps every left-r.e. set in a one-one way into an r.e. set and similar to the reductions in the previous section, this reduction preserves all properties like being Friedberg or positive, is invertible, and also preserves the structure of the Rogers semilattice of every left-r.e. numbering. Furthermore, $A \leq_{lex} B \Leftrightarrow \Xi(A) \subseteq \Xi(B)$. These properties of this reduction are well-known. The following results establish some further properties of left-r.e. numberings.

Theorem 7. *A left-r.e. family without a greatest set (with respect to lexicographic order) has infinitely many Friedberg numberings. Furthermore, the Rogers semilattice of such a numbering has no greatest element.*

Proof. Let (B_n) be a given numbering of the family. The theorem is proven by producing a Friedberg numbering (A_n) which is not reducible to (B_n) in the Rogers semilattice. As one could choose (B_n) to be the join of k Friedberg numberings, it follows that the lattice must have at least $k+1$ Friedberg numberings and thus infinitely many of them. Furthermore, the construction disproves that the starting numbering (B_n) is the greatest one.

One first constructs a numbering of sets $C_0, C_1, \ldots$ such that all sets occur in the numbering (B_n) and all n satisfy $C_n >_{lex} C_m$ for all $m < n$ and $C_n > B_m$ for all $m \leq c_K(n) + n + 1$, where c_K is the convergence modulus of the halting problem K; note that c_K dominates all recursive functions. The sets C_n are defined using bounds b_n which can go up at certain times t; more precisely, in the definition

$$C_{n,t} = \max{}_{lex}\{B_{m,t} : m \leq b_{n,t}\}$$

the bound is increased whenever it is found out that there is a set

$$E \in \{C_0, C_1, \ldots, C_{n-1}, B_0, B_1, \ldots, B_{c_{K_t}(n)+n+1}\}$$

with

$$E_t(0)E_t(1)\ldots E_t(b_{n,t}) >_{lex} C_{n,t}(0)C_{n,t}(1)\ldots C_{n,t}(b_{n,t})$$

and then the update $b_{n,t+1} = b_{n,t} + t + 1$ is performed; if this condition does not apply, then the bound remains unchanged, that is, $b_{n,t+1} = b_{n,t}$.

The main construction uses movable markers and a set $X = \{n : n$ is not occupied by a marker at some stage $t > n\}$. Let X_t denote X enumerated up to stage t. The marker a_n tracks the set B_n also using its current position $a_{n,t}$ as a bound up to which it compares and starts with a value $a_{n,0} = 2^{n+2}$. Whenever the marker a_n discovers that $B_{n,t}$ coincides either with some $B_{m,t}$ up to the bound $a_{n,t}$ for some $m < n$ or coincides with $C_{m,t}$ up to the bound $a_{n,t}$ for some $m \in X_t$ with $m < n$, then the marker moves to the position $a_{n,t+1} = 2^{n+2} + (t+1) \cdot 2^{n+3}$; note that this position will never be taken by any other marker and is larger than the current time t.

The left enumeration of A_n will stay at $\emptyset$ until time n is reached and then, in the case that some marker a_m sits on n, follows the left enumeration of B_m as long as a_m remains on this location. In the case that a_m moves away, the enumeration of A_n will start to follow the one of C_n. Note that when a_m sits on n at time t then $m \leq n$ and therefore $B_{m,t} \leq_{lex} C_n$ so that whenever n is enumerated into X, the enumeration of A_n can swap from following B_m to following C_n without violating the constraint that the numbering of the (A_n) is a numbering of left-r.e. sets. Furthermore, each C_n is a member of the family, as it is the lexicographical maximum of finitely many members of the family.

It is easy to see that the sets C_n are all different and satisfy the order requirements that they are an ascending chain and that C_n is lexicographic above all B_m with $m \leq c_K(n) + n + 1$.

Furthermore, in the case that n is the least index of the set B_n and that furthermore B_n differs from all C_m with $m \in X$ (by construction this is true for all C_m with $m \geq n$) then the marker a_n moves only finitely often until a time t is reached when the characteristic sequence of B_n bounded by $a_{n,t}$ is different from all the other finitely many sets and t is also large enough that the left-r.e. enumeration of the sets involved has reached its final values on the points which witness the difference. Hence the marker a_n will not move again and $A_{a_n} = B_n$. In the case that B_n equals some C_m with $m \in X$, then $m < n$ and the marker a_n will move infinitely often so that B_n is not copied into the new enumeration; similarly in the case that $B_n = B_m$ for some $m < n$: then it also happens that a_n moves infinitely often. Thus the numbering (A_n) is a Friedberg numbering enumerating the same family as the numbering (B_n).

Furthermore, assume now by way of contradiction that there is a recursive function f with $B_{f(n)} = A_n$ for all n. Let n be so large that $c_K(n) > f(n)$ and n is never occupied by a marker, hence $A_n = C_n$. By construction $C_n >_{lex} B_{f(n)}$ and therefore $A_n \neq B_{f(n)}$. Thus the function f cannot witness that the numbering (A_n) is below (B_n) in the Rogers semilattice; this completes the proof. □

Recall that a numbering is called *decidable* if and only if the set $\{(m, n) : A_m = A_n\}$ is recursive. The numbering is called *positive* iff the set $\{(m, n) : A_m = A_n\}$ is recursively enumerable. Note that positive numberings are minimal.

Theorem 8. *Every infinite left-r.e. family has infinitely many positive but undecidable numberings and also minimal numberings which are not positive.*

Proof. First the result is shown for the existence of infinitely many positive and undecidable numberings.

In the case that some set in the left-r.e. numbering is the lexicographically greatest, then one can use the reduction Ξ of the left-r.e. numberings to the r.e. sets to obtain a numbering of r.e. sets with a greatest set with respect to set-inclusion. Badaev [5] showed that such a family has infinitely many positive undecidable numberings; this result then directly translates back to the left-r.e. numberings.

In the case that the numbering does not have a lexicographically greatest element, Theorem 7 provides a Friedberg numbering (A_n). Now one can define for each r.e. set E with a recursive one-one enumeration $e_0, e_1, \ldots$ a new numbering $(B_{E,n})$ such that the following conditions hold:

- $B_{E,3n} = \max_{lex}(A_{2n}, A_{2n+1})$;
- if $n \notin E$ then $B_{E,3n+1} = \min_{lex}(A_{2n}, A_{2n+1})$ else $B_{E,3n+1} = \max_{lex}(A_{2n}, A_{2n+1})$;
- $B_{E,3n+2} = \min_{lex}(A_{2e_n}, A_{2e_n+1})$.

This numbering is decidable if and only if E is recursive. It is also obvious that the numberings are positive, as the only non-trivial equalities are $B_{3n} = B_{3n+1}$ for $n \in E$. Furthermore, if E and F are r.e. sets of incomparable Turing degree then $(B_{E,n})$ and $(B_{F,n})$ are incomparable numberings which enumerate the same family as (A_n): If $(B_{E,n})$ were many-one reducible to $(B_{F,n})$ via g then $E \leq_T F$ as for each n, $n \in E$ iff $g(3n+1) = 3n \vee (g(3n+1) = 3n+1 \wedge n \in F)$. As there is an infinite antichain of r.e. Turing degrees, the existence of infinitely many incomparable positive numberings follows.

Second, for the case of numberings which have a lexicographically greatest element A_0, say, one can effectively find in the limit lists $I_0, I_1, \ldots, I_n, \ldots$ such that I_n has $n+1$ indices $e_{n,0}, \ldots, e_{n,n}$ with $A_{e_{n,0}} <_{lex} A_{e_{n,1}} <_{lex} \ldots <_{lex} A_{e_{n,n}}$ and the sets being distinct for different sets of indices. Furthermore, each set will eventually have indices showing up in exactly one set I_n except for the greatest element. Now one reserves for each I_n $n+2$ indices in the new numbering of the B_m and whenever the list I_n changes, all the sets of these reserved indices will be made equal to the largest set A_0 and a new set of $n+2$ indices is selected. For these indices $k_{n,0}, \ldots, k_{n,n+1}$, they will be initialised as $B_{k_{n,0}} = A_{e_{n,0}}$ and $B_{k_{n,m+1}} = A_{e_{n,m}}$ for $m = 0, 1, \ldots, n$. Now one chases possible reductions φ_d with $d < n$: These must have all but at most one index $k_{n,m}$ in their range; if it happens that for such φ_d the range contains all numbers but one $e_{k_{n,m}}$ and $B_{k_{n,m}}$ is together with some other set $B_{k_{n,m'}}$ currently equal to a set $A_{e_{n,h}}$, then one adjusts $B_{k_{n,m'}}$ to track the set $A_{e_{n,h+1}}$ and from now on $k_{n,m}$ is the only index of $A_{e_{n,h}}$ in the numbering. This enforces that for every reduction of some numbering to the numbering of the (B_k) almost all indices which are not for A_0 occur in the range of the reduction; together with the fact that the k with $B_k = A_0$ are recursively enumerable and that finitely many missing inverses can be patched by table-lookup, it follows that such a reduction can be inverted and therefore the numbering (B_k) is minimal. Furthermore, it is easy to see from the set-up that the numbering is not positive, otherwise, one would have for all such $n+1$ indices in I_n, after I_n reaches the final stage, that one can identify which two indices are equal and a reduction would leave out one of them what is, as just seen, impossible.

The adjustment to the case that there is no maximal set A_0 again starts with a Friedberg numbering. Then one partitions the natural numbers into intervals I_n of length $n+1$ and also partitions them into J_n of length $n+2$. Now let $\tilde{A}_{e_{n,m}}$ denote the set number m with respect to lexicographic order among the sets (A_e) with $e \in I_n$. Now one initialises the sets $B_{k_{n,0}}, B_{k_{n,1}}, \ldots, B_{k_{n,n+1}}$ in J_n such that initially $B_{k_{n,0}} = \tilde{A}_{e_{n,0}}$ and $B_{k_{n,m+1}} = \tilde{A}_{e_{n,m}}$ as above. Once this is done, the same diagonalisation process is used, and so one enforces that every reduction of a numbering of the (A_e) is mapped by a reduction to the $(B_{k_{n,m}})$ with a cofinite range, thus it can be inverted and the numbering is minimal. As before, one can also show that the numbering of the $B_{k_{n,m}}$ is not positive, as otherwise one could reduce it to itself in a way that for each n one of the $B_{k_{n,m}}$ does not occur in the range; however, this was shown to be impossible. □

Theorem 9. *The left-r.e. family (A_n) with $A_n = \{n\}$ has exactly one Friedberg numbering, infinitely many minimal numberings and one greatest numbering.*

Proof. Clearly the given numbering is Friedberg and can easily seen to be realised by a left-r.e. enumeration. Note that $\{0\} >_{lex} \{1\} >_{lex} \ldots$ and that whenever $\{m\} \leq_{lex} E$ for a left-r.e. set then there is a time t with $\{m\} \leq_{lex} E_t$. Now assume that there is another Friedberg numbering (B_n) of the same class. One defines inductively $f(n)$ to be the first m found such that the following conditions hold:

– $m \notin \{f(o) : o < n\}$ and

– $\{n\} \leq_{lex} B_m$.

Note that for every n the corresponding m exists: There are $n+1$ indices m with $\{n\} \leq_{lex} B_m$ and only n of them can be equal to $f(o)$ for some $o < n$. So each $f(n)$ will be defined and f is a total recursive function. Furthermore, by the Friedberg property of (B_n), one can show inductively that $B_{f(0)} = \{0\}$, $B_{f(1)} = \{1\}$, ..., $B_{f(n)} = \{n\}$, etc., and so the recursive function f witnesses that the numbering (A_n) reduces to (B_n). As (B_n) is Friedberg, the two numberings are then equivalent.

Note that by Theorem 8, there are infinitely many minimal numberings in the Rogers semilattice of this family.

Now let $C_{e,m}$ be the m-th member of the e-th enumeration of some left-r.e. sets. There is a recursive enumeration $(e_0, m_0), (e_1, m_1), \ldots$ of all pairs (e, m) such that $C_{e,m}$ is not empty. Now let $B_n = \{h\}$ for the minimal h such that $\{h\} \leq_{lex} C_{e_n,m_n}$. Given now any left-r.e. numbering (E_n) enumerating the same family as (A_n), there is an index e with $C_{e,n} = E_n$ for all n and therefore there is a recursive function f such that $e_{f(n)} = e \wedge m_{f(n)} = n$ for all n and f witnesses that (E_n) is below (B_n) in the Rogers semilattice. Hence the (B_n) form a greatest numbering of this family. $\square$

Example 10. *Let A be a dense simple set with $0 \notin A$. The numbering assigning to A_n the set $\{m\}$ for the maximal $m \leq n$ with $m \notin A$ is a left-r.e. numbering of $\{\{m\} : m \notin A\}$; this family has, however, no Friedberg numbering.*

Remark 11. The Rogers semilattice of a left-r.e. family consisting of n sets is equivalent to the many-one degrees of those functions f which have the range $\{0, 1, \ldots, n-1\}$ and for which $\{(x, y) : y < f(x)\}$ is recursively enumerable. Note that for $n = 2$, these degrees coincides with the many-one degrees of the r.e. sets different from $\emptyset$ and $\mathbb{N}$.

4 Numberings which are ω-r.e.

There are several definitions of the notion of ω-r.e. sets in the literature and not all of them are equivalent. The one used for the present work is the following, which is also used by Nies [30, Definition 1.4.3]. Note that the ω-r.e. sets, as defined here, coincide with those which are truth-table reducible to the halting problem K and also with those which are weakly truth-table reducible to K [30, Proposition 1.4.4]. See Remark 17 below for further explanations.

Definition 12. A family (A_n) of sets is *ω-r.e.* if and only if there is a recursive function f and a uniformly recursive approximation $A_{n,t}$ to each A_n such that, for all n, $A_{n,0} = \emptyset$ and, for all x, $|\{t : A_{n,t+1}(x) \neq A_{n,t}(x)\}| < f(n, x)$.

Furthermore, a numbering (A_n) is called *uniformly ω-r.e.* if and only if the underlying approximation of the numbering (A_n) and the recursive function f from the definition of an ω-r.e. family can be chosen such that $f(m, x) = f(n, x)$ for all m, n and x.

Proposition 13. *Assume that a family (A_n) consists only of c many sets with $1 \leq c < \infty$. Then the Rogers semilattice of the ω-r.e. numberings of A is equivalent to the semilattice of the* *many-one degrees of all ω-r.e. functions f with range $\{0, 1, \ldots, c-1\}$ and the Rogers semilattice of the uniformly ω-r.e. numberings of A is equivalent to the semilattice of the many-one degrees of all functions f which have the range $\{0, 1, \ldots, c-1\}$ and which are k-r.e. for some k.*

Theorem 14. *The following applies to the Rogers semilattice of ω-r.e. numberings of a family given by an ω-r.e. numbering (A_n).*

1. *If (A_n) contains infinitely many sets, then there are infinitely many positive undecidable numberings and infinitely many minimal numberings which are not positive.*
2. *If (A_n) is a Friedberg numbering, then there are infinitely many Friedberg numberings.*
3. *If $A_n = \{n\}$ for all n, then there is no greatest numbering.*

Proof. For item 1, note that there is a K-recursive enumeration of the minimal indices in the numbering (A_n); that is, there are sequences $n_{e,t}$ converging to a sequence $n_0, n_1, \ldots$ such that an n appears in this sequence iff $A_m \neq A_n$ for all $m < n$. Furthermore, there is a maximal set S with complement $b_0, b_1, \ldots$ such that each b_e is so large that $n_{e,t}$ converges to n_e in time smaller than b_e and that $b_e > n_e$. Let $f(n, x)$ be the recursive bound on the number of mind changes of the approximation to $A_n(x)$. Now one defines a new numbering $(\tilde{A}_n)$ with bound $\tilde{f}(n, x) = \max\{f(m, x) + 1 : m \leq n\} \cdot (n + 2)$, and for each m and x, one defines $\tilde{A}_{m,t}(x)$ as follows:

1. If $t = 0$ then $\tilde{A}_{m,t}(x) = 0$;
2. If $t > 0$ and $m \notin S_t$ and $\{0, 1, \ldots, m\} - S_t$ has e elements and $n_{e,m} \leq m$, then $\tilde{A}_{m,t}(x) = A_{n_{e,m},t}(x)$;
3. If $t > 0$ and $m \notin S_t$ and the second case does not apply then $\tilde{A}_{m,t}(x) = \tilde{A}_{m,t-1}(x)$.
4. If $t > 0$ and $m \in S_t$ then $\tilde{A}_{m,t}(x) = A_{0,t}(x)$;

The value of e starts with m and goes down to 1 as long as the second or third case applies and for each fixed value of e with $n_{e,m} \leq m$, at most $f(n_{e,m}, x)$ mind changes are made plus one more which might occur when e goes down further or when m is enumerated into S; furthermore $f(0, x)$ many mind changes might be made after m is enumerated into S. This shows that the overall number of mind changes given by $\tilde{f}(m, x)$ is kept and the numbering is an ω-r.e. numbering. Furthermore, due to the maximality of S, given a numbering (B_n) and a many-one reduction g from (B_n) to $(\tilde{A}_n)$, almost all non-members of S are in the range of g. If now (B_n) enumerates the same family as $(\tilde{A}_n)$, then one can invert g by $\tilde{g}(n) = m$ for the first m such that either $g(m) = n$ or $g(m) \in S \wedge n \in S$; this is defined and correct on all but finitely many n; these finitely many cases can be adjusted by a finite table look-up. It is well-known that for this method, one obtains infinitely many incomparable numberings by exchanging in the construction A_0 with A_{n_d} for $d = 1, 2, \ldots$ and making the corresponding adjustments on the bounds; the rest of the proof remains the same.

If one wants a minimal numbering which is not positive, the same construction applies as above with the only difference being that in the second case, one does not choose e to be $|\{0, 1, \ldots, m\} - S_t|$ but takes instead half this value (rounded down). This makes every index appear at least twice in the numbering; however, one cannot connect the equivalent indices in the complement of S by a positive equivalence relation due to S being maximal.

For item 2, the proof can be simplified by assuming that the indices of the Friedberg numbering are not numbers but pairs (i,j), with $i \in \mathbb{N}$ and $j \in \mathbb{Z}$. Now given $(A_{(i,j)})$ and a parameter $h > 0$, one considers a new numbering of

$$\tilde{A}_{i,j} = \begin{cases} A_{i,j}, & \text{if } i \notin K, \\ A_{i,j+h}, & \text{if } i \in K, \end{cases}$$

where K is the halting problem. These Friedberg numberings can be realised as one changes at most one time the set which is followed and therefore the mind change bound $f((i,j),x)$ from the original numbering has to be replaced by $\tilde{f}((i,j),x) = f((i,j),x) + f((i,j+h),x) + 1$ for the new numbering. However, any reduction between the two Friedberg numberings of this type would solve the halting problem, as $A_{i,j}$ has to be mapped to $\tilde{A}_{i,j}$ and vice versa iff $i \notin K$. Note that, similarly, the versions obtained for two different values of h are also incomparable for the same reason; thus there are infinitely many incomparable Friedberg numberings.

For item 3, given a numbering (B_n) of the same family, there is a K-recursive function g such that $g(n)$ gives the maximum x such that $B_m = \{x\}$ for some $m \leq c_K(n)$ where c_K is the convergence modulus of the halting problem K; c_K dominates every recursive function. Now let $\tilde{A}_{2n} = \max\{g_t(n) + 1 : t \in \mathbb{N}\}$ for a recursive approximation g_t to g and let $\tilde{A}_{2n+1} = \{n\}$. Then this numbering is a d.r.e. and thus ω-r.e. numbering which is not many-one reducible to (B_n). Thus the family of all singleton sets does not have a greatest numbering in its Rogers semilattice. □

Theorem 15. *The following applies to the Rogers semilattice of uniformly ω-r.e. numberings of a family given by a uniformly ω-r.e. numbering (A_n).*

1. *If (A_n) contains infinitely many sets, then there are infinitely many positive undecidable numberings.*
2. *If (A_n) is a Friedberg numbering, then there are infinitely many Friedberg numberings.*
3. *If (A_n) has at least two different members, then there is no greatest numbering.*

Proof. When manipulating uniformly ω-r.e. numberings to get new uniformly ω-r.e. numberings, one can simulate entries in the given old numbering and follow them; however, the technique permits in general only constantly many mind changes on which member of the given numbering is followed. Thus some adjustments to the corresponding proofs of item 1 of Theorem 14 have to be made.

For item 1, one has to work with a co-retraceable set S in place of a maximal set S (as used there). Again one has that b_e is an upper bound to the convergence of $n_{e,t}$ to n_e. Now one defines the numbering $(\tilde{A}_m)$ as follows:

1. If $t = 0$ then $\tilde{A}_{m,t}(x) = 0$;
2. If $t > 0$ and $m \notin S_t$ and the retracing function of the complement of S tells that "m is the e-th non-element of S unless $m \in S$" then $\tilde{A}_{m,t}(x) = A_{n_{e,m},t}(x)$ for this e;

3. If $t > 0$ and $m \in S_t$ then $\tilde{A}_{m,t}(x) = A_{0,t}(x)$.

The bound on the number of mind changes at x is $2 \cdot f(0,x)+1$, as there is only one mind change with respect which member of (A_n) is followed. As all the A_{n_e} are different, it follows that the range of a many-one reduction to $(\tilde{A}_m)$ has to cover the whole complement of S and this permits to invert the function as above. All other parts of the proof are as in item 1 of Theorem 14.

For item 2, one can carry over the proof of item 2 of Theorem 14 without any change.

For item 3, assume that one has at least 2 different sets, say A_0 and A_1 with $0 \in A_1 - A_0$. Then the set $\{n : 0 \in A_n\}$ is at most $f(0,0)$-r.e., and the same applies to all numberings which are many-one reducible to (A_n). However, one can, as in Theorem 5, make another uniformly ω-r.e. numbering (B_n) enumerating the same family as (A_n), where the set $\{n : 0 \in B_n\}$ is not $f(0,0)$-r.e. and therefore the Rogers semilattice of the uniformly ω-r.e. numberings does not have a greatest element unless the whole numbering is the numbering of a single set. □

Corollary 16. *There is no operator with the properties from Theorems 2 between any two different types of numberings among the following: ω-r.e. numberings, uniformly ω-r.e. numberings and left-r.e. numberings.*

Proof. The Rogers semilattice of the ω-r.e. numberings of two different sets has a least and a greatest numbering and is infinite; such a numbering does not exist in the Rogers semilattices of any uniformly ω-r.e. numberings; thus there is no reduction in this direction. By Theorem 5, there is no reduction from the ω-r.e. numberings to the r.e. numberings and as one can reduce the left-r.e. numberings to the r.e. ones, there is, by transitivity, also no reduction from the ω-r.e. numberings to the left-r.e. ones.

Furthermore, in the uniformly ω-r.e. numberings, a Rogers semilattice can have a least numbering while there is no greatest numbering; this happens exactly when the numbering has at least two and at most finitely many sets. However, there is no such numbering in the case of the ω-r.e. numberings and also not in the case of left-r.e. numberings; thus there is no reduction from the uniformly ω-r.e. numberings into the left-r.e. or the ω-r.e. numberings.

There is a left-r.e. family with exactly one Friedberg numbering in its Rogers semilattice; as this cannot happen for ω-r.e. or uniformly ω-r.e. numberings, there is no reduction from the left-r.e. numberings into either of them. □

5 Beyond ω-r.e. Numberings

A system of notations of ordinals is a way to represent the ordinals. There are various measures of the quality of such a system. For example,

1. One can recognise whether an ordinal is a limit ordinal or a successor ordinal;
2. One can recognise its Cantor normal form (where the system of notations for the exponents might be different from the given one);
3. All recursive ordinals have a unique representation in the system;
4. All other representations can be many-one reduced to the given one.

The fourth postulate stands in a contradiction to the third one. Hence each approach to representations has its advantages and disadvantages. One can realise items 1–3 as follows. Harrison [26] showed that one can have all recursive ordinals without losing uniqueness, i.e., there is a recursive linear ordering with an initial segment of type ω_1^{CK}, where ω_1^{CK} is the first nonrecursive ordinal, named after Church and Kleene; this linear ordering has no hyperarithmetic infinite descending chains. Now one can order lexicographically the strings $\alpha_n^{a_n}\alpha_{n-1}^{a_{n-1}}\ldots\alpha_1^{a_1}\alpha_0^{a_0}$ (with exponentiation meaning repetition) where $\alpha_n >_{Ha} \alpha_{n-1} >_{Ha} \ldots >_{Ha} \alpha_1 >_{Ha} \alpha_0$ and interpret the α_m as ordinals (or pseudoordinals in the case that they are in the non-well-ordered part of the ordering), and one obtains an ordering where strings as the one mentioned before represent

$$\omega^{\alpha_n} \cdot a_n + \omega^{\alpha_{n-1}} \cdot a_{n-1} + \ldots + \omega^{\alpha_1} \cdot a_1 + \omega^{\alpha_0} \cdot a_0$$

and the α_n represent the ordinal of the same order-type as their representative in the ordering. Now one can carry out ordinal addition on these representations, decide what is a limit ordinal and what is a successor ordinal, compare the ordinals and so on. A notation system with the first three properties will from now on be called a *Harrison notation* of ordinals and it is assumed that such a system is fixed throughout this section. Furthermore, for ease of notation, when dealing with ordinal counters, the notation "$f(x,t) = \alpha$" is used in place of "$f(x,t)$ has the value corresponding to the ordinal α in the fixed notation of ordinals".

Remark 17. Note that several authors [1, 15–18, 33, 38] use a slight variation of this definition and the Σ_α^{-1}-sets of Ershov [15–17] are, for $\alpha \geq 1$, the $(\alpha+1)$-r.e. sets as used here. They have, however, other ways to refer to the limit levels. For this paper, the authors chose the also widespread version [9, 10, 14, 19, 21, 28, 30] of α-r.e. sets where in particular the notion of ω-r.e. sets is used for those sets where the number of mind changes are bounded by a recursive function. Some results in the literature follow Ershov's notation and therefore one always has to be careful which type of terminology is used when certain results are stated.

Definition 18. Let α be a recursive ordinal and $\alpha \geq \omega$. A numbering (A_n) is called *α-r.e.* if and only if there is a recursive three-place function f such that the following conditions hold:

- $\alpha >_{Ha} f(n,x,0) \geq_{Ha} f(n,x,1) \geq_{Ha} \ldots$ for all n, x;
- $x \in A_n$ if and only if there is an odd number of t with $f(n,x,t+1) <_{Ha} f(n,x,t)$.

A family is called *α-r.e.* iff it has an α-r.e. numbering. Depending on f, one defines an approximation $A_{n,t}$ to A_n such that $A_{n,t}(x) = 1$ iff there is an odd number of $s < t$ with $f(n,x,s) >_{Ha} f(n,x,s+1)$.

There are various facts about α-r.e. families. For example, in the case that α is a successor ordinal, there is an α-r.e. many-one complete set K_α such that for every α-r.e. numbering (A_n), there is a one-one recursive function g with $A_n(x) = K_\alpha(g(n,x))$ for all n and x; such a complete set does not exist for limit ordinals.

Theorem 19. *Let α be a recursive ordinal with $\alpha \geq \omega$. There is a reduction $\Theta_{\alpha,k}$ from the k-r.e. numberings into the $(\alpha+k)$-r.e. numberings which is derived from a relation on sets as follows. If k is odd then*

$$\Theta_{\alpha,k}(A) = (A \times \mathbb{N}) \cup (\mathbb{N} \times K_{\alpha+1})$$

else

$$\Theta_{\alpha,k}(A) = (A \times \mathbb{N}) \cap (\mathbb{N} \times (\mathbb{N} - K_{\alpha+1}))$$

and this reduction is one-one and preserves the inclusion relation. Furthermore, as a reduction of numberings, $\Theta_{\alpha,k}$ preserves the Rogers semilattice of numberings as well as the property of whether an individual numbering in the semilattice is Friedberg or positive.

Proof. First the proof is carried out for the case that k is odd. Now it is verified that whenever (A_n) is a k-r.e. numbering, then $(B_n) = \Theta_{\alpha,k}(A_n)$ is an $(\alpha+k)$-r.e. numbering. Let $A_{n,t}$ be the approximation of A_n, with $f^A(n,x,t)$ being the corresponding countdown function. Let $K_{\alpha+1,t}$ be the approximation of $K_{\alpha+1}$, with $f^K(x,t)$ being the corresponding countdown function. Without loss of generality it is assumed that $f^K(x,0) = \alpha$ and $f^K(x,t)$ is never 0. The $(\alpha+k)$-enumeration of B_n on a pair (x,y) is defined by using the functions $f^A(n,x,t)$ and $f^K(y,t)$. These two functions are combined to define inductively $f^B(n,x,y,t)$ via the condition that $B_{n,t}(x,y) = 1$ iff there are an odd number of $s < t$ with $f^B(n,x,y,s+1) <_{Ha} f^B(n,x,y,s)$. The definition follows now the first case which applies:

- If $t = 0$, then let $f^B(n,x,y,0) = \alpha + k - 1$;
- If $\max\{A_{n,t}(x), K_{\alpha+1,t}(y)\} = \max\{A_{n,t-1}(x), K_{\alpha+1,t-1}(y)\}$, then let $f^B(n,x,y,t) = f^B(n,x,y,t-1)$;
- If $\max\{A_{n,t-1}(x), K_{\alpha+1,t-1}(y)\} = 0$ and $f^A(n,x,t) = 0$ (causing $A_{n,t}(x) = 1$), then let $f^B(n,x,y,t) = 0$;
- If $\max\{A_{n,t}(x), K_{\alpha+1,t}(y)\} \neq \max\{A_{n,t-1}(x), K_{\alpha+1,t-1}(y)\}$, then let $f^B(n,x,y,t) = f^K(y,t) + f^A(n,x,t) - 1$.

Note that in the last case, the expression $f^K(y,t)+f^A(n,x,t)-1$ is well-defined as $f^A(n,x,t) > 0$ and furthermore a mind change occurs as either $f^A(n,x,t) < f^A(n,x,t-1)$ or $f^K(y,t) < f^K(y,t-1)$. In addition, the mind change in the third case is always possible when this case applies, as $f^K_{\alpha+1}(y,t)$ never takes the value 0; once this case has applied, no further mind change is needed, as $A_n(x)$ can no longer become 0.

Next, one has to show how to reconstruct the k-r.e. preimage of an $(\alpha+k)$-r.e. numbering (B_n) enumerating the same family as $(\Theta_{\alpha,k}(A_n))$. As $K_{\alpha+1}$ is many-one complete for $(\alpha+1)$-r.e. numberings, there is a recursive function g such that the approximation to $K_{\alpha+1}(g(n,x))$ behaves as follows:

$$f^K(g(n,x),t) = \begin{cases} \alpha & \text{if there are at most } k-1 \text{ stages } s < t \text{ with } \\ & f^B(n,x,g(n,x),s+1) <_{Ha} f^B(n,x,g(n,x),s); \\ f^B(n,x,g(n,x),t) & \text{if there are at least } k \text{ stages } s < t \text{ with } \\ & f^B(n,x,g(n,x),s+1) <_{Ha} f^B(n,x,g(n,x),s). \end{cases}$$

Note that in the second case, the k mind changes cause that $f^B(n,x,g(n,x),t) <_{Ha} \alpha$, as the starting value is $f^B(n,x,g(n,x),0) \leq_{Ha} \alpha + k - 1$. Assume now that $B_n = \Theta_{\alpha,k}(E)$. Then the following holds:

- If $f^K(g(n,x),t) = \alpha$ for all t, then $\Theta_{\alpha,k}(E)(x,g(n,x)) = E(x)$;
- If $f^K(g(n,x),t) < \alpha$ for some t, then $\Theta_{\alpha,k}(E)(x,g(n,x)) = \max\{E(x), 1 - B_n(x,g(n,x))\}$ and this can be equal to $B_n(x,g(n,x))$ only if $E(x) = 1$.

In the first case, $E(x) = B_n(x,g(n,x))$ and the approximation to $B_n(x,g(n,x))$ makes at most $k-1$ mind changes; in the second case, $E(x) = 1$ and the approximation to $B_n(x,g(n,x))$ makes at least k mind changes. Hence, one can define a numbering $C_n(x)$ as follows:

$$C_n(x) = \begin{cases} B_n(x,g(n,x)) & \text{if the approximation of } B_n(x,g(n,x)) \\ & \text{makes at most } k-1 \text{ mind changes;} \\ 1 & \text{if the approximation of } B_n(x,g(n,x)) \\ & \text{makes at least } k \text{ mind changes.} \end{cases}$$

Now $(B_n) = (\Theta_{\alpha,k}(C_n))$ and the numbering (C_n) is a k-r.e. numbering enumerating the same family as (A_n).

Second, for the case of even k, the above proofs are similar. However, after k mind changes, the elements are outside the k-enumerated set and not inside; therefore one has to make some adjustments which are reflected by the choice of the reduction as

$$\Theta_{\alpha,k}(A) = (A \times \mathbb{N}) \cap (\mathbb{N} \times (\mathbb{N} - K_{\alpha+1})),$$

and the fact that the default state of $(x,g(n,x))$ in the subsequent proof needs to be inside B_n and not outside, therefore, $\mathbb{N} - K_{\alpha+1}$ is used in place of $K_{\alpha+1}$. The proof of this case is very similar and therefore not repeated.

Third, having these forth and back translations and using from the definition that $A \subseteq B$ iff $\Theta_{\alpha,k}(A) \subseteq \Theta_{\alpha,k}(B)$, one can conclude the following facts about this reduction in a way similar to Theorem 2:

- $\Theta_{\alpha,k}$ is a one-one translation from k-r.e. numberings to $(\alpha+k)$-r.e. numberings;
- Basic properties about the family like whether there are sets in it ordered by inclusion and how many different sets it contains are preserved;
- The structure of the Rogers semilattice is preserved, as $(A_n) \leq (B_n)$ if and only if $(\Theta_{\alpha,k}(A_n)) \leq (\Theta_{\alpha,k}(B_n))$;
- The properties of whether a numbering is positive or Friedberg are preserved.

This completes the theorem. □

One might ask for which further ordinals α the α-r.e. numberings can realise every Rogers semilattice of the r.e. numberings. The next theorem shows that this is also true for limit ordinals which are the non-absorbing sum of at least two other nonzero ordinals.

Theorem 20. *Let α be a recursive ordinal such that $\alpha \geq \omega$ and $\alpha = \omega^\beta \cdot k + \gamma$ for a natural number $k \geq 1$ and ordinals β, γ with $\beta \geq 1$ and $1 \leq \gamma \leq \omega^\beta$. Then the r.e. numberings can be reduced to the α-r.e. numberings.*

Proof. Let $Y = K_{\omega^\beta \cdot k+1}$ and $Z = K_{\omega^\beta+1}$ be many-one complete sets for the $(\omega^\beta \cdot k+1)$-r.e. and $(\omega^\beta+1)$-r.e. sets, respectively. Let f^Y and f^Z be the corresponding counters starting with $\omega^\beta \cdot k$ and ω^β, respectively, and assume that f^Y counts ordinals down only at steps of the form $3t+1$ and f^Z only at steps of the form $3t+2$ — so that the two counters never act at the same time and also the times of the form $3t$ are reserved for other things to act. Let (A_n) be a given r.e. numbering containing at least two different sets. Without loss of generality, $0 \notin A_0$ and $0 \in A_1$. Now define the sets $S_{n,x}$ as follows:

- $S_{0,0} = Y$;
- $S_{1,0} = \{y : \exists t\, [f^Y(y,t) < \omega^\beta \cdot k \wedge f^Z(y,t) = \omega^\beta \wedge y \in Y] \vee \exists t\, [f^Y(y,t) = \omega^\beta \cdot k \wedge f^Z(y,t) < \omega^\beta \wedge y \in Z]\}$;
- If $x \in A_n$ then $S_{n,x} = S_{1,0}$ else $S_{n,x} = S_{0,0}$.

These sets permit to introduce a reduction Υ from the numbering (A_n) to $(\Upsilon(A_n))$ by letting

$$\Upsilon(A_n) = \{(x,y) : y \in S_{n,x}\} = \{(x,y) : y \in S_{A_n(x),0}\}$$

First one has to show that there is an α-r.e. numbering of (B_n) with $B_n = \Upsilon(A_n)$. The numbering (B_n) is defined using a counter $f^B(n,x,y,t)$ which works as follows and is always in the first of these cases which applies. By delaying, one can assume that elements of A_n are enumerated only at steps which are multiples of 3.

1. Initially, as long as $x \notin A_{n,t}$ and $f^Y(y,t) = \omega^\beta \cdot k$, the counter is at $\omega^\beta \cdot k$, which is strictly below α.
2. If t is the first stage with $f^Y(y,t) < \omega^\beta \cdot k$ and $f^Z(y,t) = \omega^\beta$, then $f^B(n,x,y,t') = f^Y(y,t')$ for all $t' \geq t$, as $B_n(x,y) = Y(y)$ in this case.
3. If t is the first stage with $f^Z(y,t) < \omega^\beta$, $f^Y(y,t) = \omega^\beta \cdot k$, $x \in A_{n,t}$ and $Z_t(y) = 1$, then $f^B(n,x,y,t') = f^Z(y,t')$ for all $t' \geq t$, as $B_n(x,y) = Z(y)$ in this case.
4. If t is the first stage with $f^Z(y,t) < \omega^\beta$, $f^Y(y,t) = \omega^\beta \cdot k$, $x \in A_{n,t}$ and $Z_t(y) = 0$, then $f^B(n,x,y,t') = \omega^\beta \cdot k$ for all $t' \geq t$ with $f^Z(y,t') = f^Z(y,t)$ and $f^B(n,x,y,t') = f^Z(y,t')$ for all $t' \geq t$ with $f^Z(y,t') < f^Z(y,t)$, as again $B_n(x,y) = Z(y)$ in this case.
5. The remaining case is that t is the first stage with $f^Y(y,t) < \omega^\beta \cdot k$, $f^Z(y,t) < \omega^k$ and $A_{n,t}(x) = 0$. Note that $f^Y(y,t-1) = \omega^\beta \cdot k$, as otherwise some prior cases would have applied. Now, as long as $t' \geq t$ and $A_{n,t'}(x) = 0 \vee Y_{t'}(y) = Z_{t'}(y)$, one has $f^B(n,x,y,t') = f^Z(y,t) + 1 + f^Y(y,t')$. Note that this value is below $\omega^\beta \cdot k$, as $f^Z(y,t) < \omega^\beta$ and $f^Y(y,t') < \omega^\beta \cdot k$. From the first $t' \geq t$ with $A_{n,t'}(x) = 1 \wedge Y_{t'}(y) \neq Z_{t'}(y)$ onwards, $f^B(n,x,y,t'') = f^Z(y,t'')$ for $t'' \geq t'$; note that this t' satisfies $t' > t$.

The basic idea behind this algorithm is to wait until either $f^Y(y,t)$ goes below $\omega^\beta \cdot k$ or x enters A_n and $f^Z(y,t)$ goes below ω^β. Although the general idea would then be just to follow each case, one has to take special care in the case that it happens that first $f^Z(y,t)$ goes below ω^β and then second $f^Y(y,t')$ goes below $\omega^\beta \cdot k$ at $t' > t$ though x is not yet in $A_{n,t'}$. In this case, the algorithm has to reserve some additional amount $f^Z(y,t')+2$ in the mind change counter for being able to handle the possibility of x at some later time entering A_n; this can, however, be done without violating the $\omega^\beta \cdot k+1$ bound on the number of mind changes. So one can see that f^B defines a $(\omega^\beta \cdot k+1)$-enumeration of the numbering (B_n).

Consider $(\tilde{B}_n)$ enumerating the same family as (B_n); now it is shown that there is an r.e. numbering $(\tilde{A}_n)$ with $\Upsilon(\tilde{A}_n) = \tilde{B}_n$ for all n. As Y and Z are many-one complete sets in their numberings, there is a recursive function $g(n,x)$ such that the algorithm does the following for $f^{\tilde{B}}$:

- As long as $f^{\tilde{B}}(n,x,g(n,x),t)$ is of the form $\omega^\beta \cdot k+\delta$, $f^Z(g(n,x),t') = \delta$, where an additional mind change from α to the first value of δ is included in order to have a different parity and to get that $Z_t(g(n,x)) \neq \tilde{B}_{n,t}(x,g(n,x))$.
- Once $f^{\tilde{B}}(n,x,g(n,x),t) < \omega^\beta \cdot k$, the counter $f^Y(g(n,x),t)$ goes down from the value $\omega^\beta \cdot k$ to the value $f^{\tilde{B}}(n,x,g(n,x),t)$ with perhaps one additional mind change in order to achieve opposite parity so that $Y_t(g(n,x)) \neq \tilde{B}_{n,t}(x,g(n,x))$.

Minor adjustments (delays of updates up to three steps) might be needed in order to meet the above requirements on the updating behaviour of f^Y and f^Z; these are here skipped.

Now one has achieved the following property: If $f^{\tilde{B}}(n,x,g(n,x),t) \geq \omega^\beta \cdot k$ for all t, then $Z(g(n,x)) \neq \tilde{B}_n(x,g(n,x))$, else $Y(g(n,x)) \neq \tilde{B}_n(x,g(n,x))$. Furthermore, there is t with $f^Z(g(n,x),t) < \omega^\beta \wedge f^Y(g(n,x),t) = \omega^\beta \cdot k$. As $\tilde{B}_n = \Upsilon(E)$ for a set E, one has that if $x \in E$, then the counter $f^{\tilde{B}}(x,g(n,x),t)$ must go below $\omega^\beta \cdot k$ for some t, else the counter must remain above $f^{\tilde{B}}(x,g(n,x),t)$ for all t. Thus one can define the sets

$$\tilde{A}_n = \{x : \exists t\,[f^{\tilde{B}}(n,x,g(n,x),t) < \omega^\beta \cdot k]\},$$

and these sets form an r.e. numbering and satisfy $\Upsilon(\tilde{A}_n) = \tilde{B}_n$ for all n. Furthermore, note that due to this equivalence, the numbering $(\tilde{B}_n)$ is actually an $(\omega^\beta+1)$-r.e. numbering. □

The two preceding results showed that whenever an ordinal is not an ω-power, then it can realise all Rogers semilattices in α-r.e. numberings which are realised by r.e. numberings. Furthermore, the next corollary shows that there are no further such ordinals.

Corollary 21. *The following conditions are equivalent for every ordinal $\alpha \geq \omega$:*

(a) *α is not an ω-power, that is, α is not of the form ω^β;*
(b) *The r.e. numberings can be reduced to the α-r.e. numberings;*
(c) *There is an α-r.e. family (A_n) without minimal numberings;*
(d) *There is an α-r.e. family (A_n) with exactly one Friedberg numbering;*
(e) *There is an α-r.e. family (A_n) with exactly two Friedberg numberings.*

Proof. If α is not an ω-power, then by either Theorem 19 or Theorem 20, one can reduce the r.e. numberings to the α-r.e. numberings and therefore carry over the constructions of a family of (A_n) without a minimal numbering [2, 3, 40, 41] or a family of (A_n) with exactly ℓ Friedberg numberings [22], where $\ell \in \{1, 2, 3, \ldots\}$. On the other hand, when α is an ω-power, then one can carry over the proofs from Theorem 14 to the α-r.e. numberings in order to show that the family has a minimal numbering and that, whenever the family has Friedberg numberings, then it has infinitely many of them. The latter is carried out explicitly for the ω-r.e. numberings, but uses only that the sum of two ordinals properly below ω^β is again properly below ω^β; thus these results carry over to all ω-powers. □

Note that the k-r.e. levels are off by one if one considers the extension of the definition of α-r.e. levels downwards. Then $1 = \omega^0$ would only be the level of the 0-r.e. sets which just contains $\emptyset$ and level 2 would contain the r.e. sets. Thus the above characterisation can be extended to $\alpha < \omega$, however, such a way to handle the finite levels looks a bit "odd" and is therefore excluded from the above corollary. For the next result, recall that $\omega^0 = 1$ and that therefore 1 counts as an ω-power.

Theorem 22. *If $\alpha = \omega^{\beta_1} + \omega^{\beta_2} + \omega^{\beta_3}$ with $\beta_1 \geq \beta_2 \geq \beta_3$ and $\beta_2 \geq 1$ then there exists an operator Γ_α such that for every d.r.e. numbering (A_n), the numbering $(\Gamma_\alpha(A_n))$ is an α-r.e. numbering which has in the α-r.e. degrees the same Rogers semilattice as (A_n) in the d.r.e. degrees.*

Proof. First one defines sets X and Y which are $(\omega^{\beta_1} + \omega^{\beta_2} + 1)$-r.e., satisfy $X \subseteq Y$ and are explained below; second, one defines Γ_α by the formula

$$\Gamma_\alpha(A_n)(\langle x, y\rangle) = \begin{cases} X(\langle x, y\rangle) & \text{if } x \notin A_n; \\ Y(\langle x, y\rangle) & \text{if } x \in A_n. \end{cases}$$

For the first, one defines $X(\langle x, y\rangle)$ and $Y(\langle x, y\rangle)$ as follows: Let E be the y-th α-r.e. set and define X and Y and their ordinals depending on the current value and ordinal of E according to the following table. The underlying assumption on γ for E is $\gamma < \omega^{\beta_1}$; that the form $\omega^{\beta_1} + \gamma$ implies $\gamma < \omega^{\beta_2}$; and that the form $\omega^{\beta_1} + \omega^{\beta_2} + \gamma$ implies $\gamma < \omega^{\beta_3}$. If defined, the current value of $E(\langle x, y\rangle)$ is denoted by e.

E: ordinal and value		X: ordinal and	value	Y: ordinal and value	
undefined	undefined	$\omega^{\beta_1} + \omega^{\beta_2}$	0	$\omega^{\beta_1} + \omega^{\beta_2}$	0
$\omega^{\beta_1} + \omega^{\beta_2} + \gamma$	e	$\omega^{\beta_1} + \omega^{\beta_2}$ or ω^{β_1}	0	$\omega^{\beta_1} + \gamma + 1$	$1 - e$
$\omega^{\beta_1} + \gamma$	e	$\omega^{\beta_1} + \gamma + 1$ or $\gamma + 1$	$1 - e$	ω^{β_1}	1
γ	e	ω^{β_1} or 0	0	$\gamma + 1$	$1 - e$

In the entry of X, the "or" shows two possibilities and it will be seen later that both are needed to realise the operator Γ_α. Note that the "undefined" entries are needed for the case of $\beta_3 > 0$, as when α is a limit ordinal, any enumeration of all α-r.e. sets has some partially or incompletely defined members, as otherwise it would not cover all α-r.e. sets, the situation is similar to a recursive enumeration covering all recursive functions which then must also contain some partial ones.

Now the translation $\Gamma_\alpha(A_n)$ is realised by the following modified table, where the entry in the first column shows the number of mind changes, and thus whether $x \in A_n$. When either $A_n(x)$

or the ordinal of $E(x,y)$ change, then one goes down to the first row which applies; in the case that none applies, one waits until one applies and leaves for the time being $\Gamma_\alpha(A_n)$ unchanged. One makes the same assumptions on the entries as in the previous table.

Mind changes of $A_n(x)$	$E(\langle x,y\rangle)$: ord	and val	$\Gamma_\alpha(A_n)(\langle x,y\rangle)$: ord	and val	Tracking
anything	undefined	undefined	$\omega^{\beta_1}+\omega^{\beta_2}$	0	none
none	$\omega^{\beta_1}+\omega^{\beta_2}+\gamma$	e	$\omega^{\beta_1}+\omega^{\beta_2}$	0	X
none	$\omega^{\beta_1}+\gamma$	e	$\omega^{\beta_1}+\gamma+1$	$1-e$	X
none	γ	e	ω^{β_1}	0	X
one	$\omega^{\beta_1}+\omega^{\beta_2}+\gamma$	e	$\omega^{\beta_1}+\gamma+1$	$1-e$	Y
one	$\omega^{\beta_1}+\gamma$	e	ω^{β_1}	1	Y
one	γ	e	$\gamma+1$	$1-e$	Y
two	$\omega^{\beta_1}+\omega^{\beta_2}+\gamma$	e	ω^{β_1}	0	X
two	$\omega^{\beta_1}+\gamma$	e	$\gamma+1$	$1-e$	X
two	γ	e	0	0	X

Note that when $\Gamma_\alpha(A_n)(\langle x,y\rangle) = E(\langle x,y\rangle)$ for the y-th α-r.e. set E, then

- either $\Gamma_\alpha(A_n)(\langle x,y\rangle) = X(\langle x,y\rangle)$ and its ordinal is strictly below ω^{β_1} or above $\omega^{\beta_1}+\omega^{\beta_2}$
- or $\Gamma_\alpha(A_n)(\langle x,y\rangle) = Y(\langle x,y\rangle)$ and its ordinal is above ω^{β_1} and strictly below $\omega^{\beta_1}+\omega^{\beta_2}$.

Given now a numbering $(\tilde{B}_n)$ of α-r.e. sets which enumerates the same family as the numbering $(\Gamma_\alpha(A_n))$, one can compute for each n the index y of $\tilde{B}_n$ in the list of all α-r.e. sets and then track the ordinal of $E(\langle x,y\rangle)$ in order to determine a value $B_n(x)$ by having $B_n(x)=0$ while the ordinal is above $\omega^{\beta_1}+\omega^{\beta_2}$ and then $B_n(x)=1$ while the ordinal is strictly below $\omega^{\beta_1}+\omega^{\beta_2}$ and above ω^{β_1} and then again having $B_n(x)=0$ after the ordinal goes strictly below ω^{β_1}. This gives a d.r.e. numbering (B_n) translated into $(\tilde{B}_n)$ with $\Gamma_\alpha(B_n)=\tilde{B}_n$; the numbering (B_n) enumerates the same family as (A_n). Thus every α-r.e. numbering of the family $(\Gamma_\alpha(A_n))$ is the image of a d.r.e. numbering of the family (A_n) and so the Rogers semilattices of both numberings are the same. □

One can see that in the proof of Theorem 22, the value ω^{β_1} could be replaced by any fixed sum of finitely many ω-powers greater or equal to ω^{β_2} without any further change; so the only restriction would be that $\beta_2 \geq 1$. Theorem 19 covers the missing case where one has a sum with the two last ω-powers being 1. Thus one needs only that α is in the Cantor normal form the sum of at least three ω-powers. Furthermore, one can see that Γ_α is in Theorem 22 actually an isomorphism of the Rogers semilattices which preserves Friedberg numberings, minimal numberings, positive numberings, and least and greatest numberings.

Corollary 23. *If α is the sum of at least three ω-powers in the Cantor normal form, then there is a reduction Γ_α from the d.r.e. numberings to the α-r.e. numberings such that (A_n) and $\Gamma_\alpha(A_n)$ have the same Rogers semilattice and the operator Γ_α is an isomorphism between the semilattices, which also preserves Friedberg numberings, positive numberings, minimal numberings, and least numberings and greatest numberings.*

6 Khutoretskii's Lemma

Badaev and Lempp [8] studied whether Khutoretskii's Lemma for the r.e. sets can be extended to d.r.e. sets. They showed that there is a family of d.r.e. sets which does not satisfy this lemma. Thus, for the k-r.e. degrees with $2 \leq k < \omega$, Khutoretskii's Lemma does not hold; by Theorem 19, it also does not hold for the α-r.e. numberings for ordinals of the form $\alpha = \beta + 2$. The next result shows that this is different for ω-powers. Khurotevskii's Lemma states, for the α-r.e. case, the following.

Theorem 24. *If α is an ω-power then for every α-r.e. family and all α-r.e. numberings (A_n) and (B_n) of this family, either $(A_n) \leq (B_n)$ or there is a further numbering (C_n) of the same family satisfying the following two conditions:*

- $(C_n) \not\leq (B_n)$;
- $(A_n) \not\leq (B_n) \oplus (C_n)$.

Proof. The proof follows a priority construction similar to the proof of Khutoretskii's Theorem. There is a recursive function which picks for each n and x, an ordinal $f(n+x) < \alpha$ so that the approximations $A_{n,t}(x)$ to $A_n(x)$ and $B_{n,t}(x)$ to $B_n(x)$ do not make $f(n+x)$ mind changes or more. The corresponding ordinal for $C_n(x)$ is $(f(n+x)+1) \cdot (n+x)$, which is also below α due to α being an ω-power. Here $(f(n+x)+1) \cdot (n+x)$ means the sum of $(n+x)$ many copies of the ordinal $f(n+x)+1$. One performs a priority construction with requirements R_{2e} and R_{2e+1} where

- R_{2e} aims at achieving that φ_e does not map (A_n) to $(B_n) \oplus (C_n)$ unless $(A_n) \leq (B_n)$ and
- R_{2e+1} aims at achieving that φ_e does not map (C_n) to (B_n) unless $(A_n) \leq (B_n)$.

For the following, one denotes the requirements by R_{2e+d}, where it is always understood that $e \in \mathbb{N}$ and $d \in \{0,1\}$. Each requirement R_{2e+d} only manipulates the indices above the restraint r_{2e+d-1}, and whenever such a restraint changes at stage s, it directly goes to $2e+d+s$. The overall strategy of R_{2e+d} is to monitor φ_e and whenever this becomes newly defined up to an element $n > r_{2e+d-1}$ and there is some evidence that φ_e is correct on a sufficiently large initial segment, then the requirement seeks attention to carry out some activities. The least requirement which infinitely often requires attention will show that $(A_n) \leq (B_n)$, and in the case that this does not happen, (C_n) will satisfy all the other desired properties. Now, for the action of all the requirements, without loss of generality it is assumed that $\varphi_e(m)$ only becomes defined when $\varphi_e(n)$ has already been defined for all $n < m$; this can be achieved by a suitable delaying in the simulation of φ_e. In the following, s denotes the number of the current stage. Initially every restraint r_c is on c.

- R_{2e}: One tries to enforce that whenever φ_e reduces (A_n) to $(B_n) \oplus (C_n)$, then actually it also reduces (A_n) to (B_n). More precisely, the requirement requires attention whenever m is the current r_{2e} and one has that $\varphi_e(k)$ is defined for all $k < m$ by stage s and

$$\forall k < m \, \forall x < m \, [A_{k,s}(x) = (B \oplus C)_{\varphi_e(k),s}(x)].$$

In the case that the above holds and attention is granted, R_{2e} defines for all k with $r_{2e-1} + k + 1 \leq s$ that

$$C_{r_{2e-1}+k+1} = B_e,$$

except for those k where this was already done in a previous stage; in other words, this has to be done for all k with $r_{2e} + 1 \leq r_{2e-1} + 1 + k \leq s$. The restraints r_{2e+c} are then updated to the value $s + c$ for all $c \in \mathbb{N}$.

- R_{2e+1}: One tries to enforce that whenever φ_e reduces (C_n) to (B_n) then actually it also reduces (A_n) to (B_n). More precisely, the requirement requires attention whenever m is the current r_{2e+1} and one has that $\varphi_e(k)$ is defined for all $k < m$ by stage s and

$$\forall k < m \, \forall x < m \, [B_{\varphi_e(k),s}(x) = C_{k,s}(x)].$$

  In the case that the above holds and attention is granted, R_{2e+1} defines for all k with $r_{2e} + 1 + k \leq s$ that

$$C_{r_{2e}+1+k} = A_k,$$

  except for those k where this was already done in a previous stage; in other words, this has to be done for all k with $r_{2e+1} + 1 \leq r_{2e} + 1 + k \leq s$. The restraints r_{2e+1+c} are then updated to the value $s + c$ for all $c \in \mathbb{N}$.
- If no requirement acts in stage s and C_s is owned by the requirement R_{2e+d}, that is, r_{2e+d} is the least restraint to be at least s, then C_s is updated as follows:
  - if $d = 0$ then $C_s = B_e$;
  - if $d = 1$ then $C_s = A_{s-r_{2e}-1}$.

  This is to avoid that sets different from those in (A_e) and (B_e) creep into the numbering (C_n).
- Whenever a new C_k is defined by requirement R_{2e+d}, for each x, the counter of C_k at x is updated to $(f(k + x) + 1) \cdot (2e + d) + f(k + x)$ and from then on, the last term initialised as $f(k + x)$ in the sum is modified according to the simulation of the set $A_{k'}$ or $B_{k'}$ which is assigned to C_k. As $k' \leq k$ for this k', the allowance of $f(x + k)$ is sufficient to make the modifications inside C_k until a higher order requirement takes over and for that case, the remaining $(f(k + x) + 1) \cdot (2e + d)$ are sufficient to carry out the further mind changes. Furthermore, whenever a requirement R_{2e+d} acts in stage s, the restraint is set to s in order to protect the sets thus constructed against lower-priority requirements. One initialises r_{2e+d} beyond $2e + d$ in order to make sure that the allowance of $(n + x) \cdot f(n + x)$ is enough in order to realise all activities on the set C_n. The sets with indices beyond s can only become non-empty when the stage goes beyond their index.

For the verification, one considers the first of the following cases which applies:

- There are requirements which act infinitely often and R_{2e} is the least of them.
  In this case, φ_e is a many-one reduction from (A_n) to $(B_n) \oplus (C_n)$, as initial parts of this reduction are verified to a larger and larger extent during the construction. However, as r_{2e-1} stabilises to some value, now denoted by the same expression, all sets C_k with $k > r_{2e-1}$ are

equal to B_e. For that reason, one can obtain from φ_e a new reduction h from (A_n) to (B_n) as follows:

$$h(n) = \begin{cases} \varphi_e(n)/2 & \text{if } \varphi_e(n) \text{ is even;} \\ e & \text{if } \varphi_e(n) \text{ is odd and } \varphi_e(n) \geq 2r_{2e-1}+1; \\ h'((\varphi_e(n)-1)/2) & \text{otherwise.} \end{cases}$$

where h' is a finite table which maps $C_0, C_1, \ldots, C_{r_{2e-1}}$ to the corresponding $B_{h'(0)}, B_{h'(1)}, \ldots, B_{h'(r_{2e-1})}$.

– There are requirements which act infinitely often and R_{2e+1} is the least of them.
  In this case, $C_{r_{2e}+k} = A_k$ for the final value of the restraint r_{2e} of the lower marker, and by assumption, φ_e reduces the numbering of (C_n) to (B_n). This reduction directly implies also a reduction from the (A_n) to (B_n), that is, $(A_n) \leq (B_n)$.
– All requirements act only finitely often. Then one has the following: $(A_n) \not\leq (B_n) \oplus (C_n)$, as the requirements of all witnesses φ_e get stuck at some inconsistency and require attention only finitely often. Similarly, $(C_n) \not\leq (B_n)$. However, due to the copying of (B_e) in the requirement R_{2e} and each of these owning at least one set, each B_e occurs in the numbering. Furthermore, only sets of the type A_n or B_n occur in the numbering. Thus (C_n) is a numbering of the same family as (A_n) and (B_n). Furthermore, (C_n) satisfies the conditions of the theorem.

This case distinction completes the proof. □

Note that by Corollary 23 one has that no recursive ordinal α which is the sum of at least three ω-powers in its Cantor normal form can satisfy Khutoretskii's Lemma, as the counterexample of Badaev and Lempp [8] transfers to these α. The following theorem provides an alternative proof for one main claim of their result.

Theorem 25 (Badaev and Lempp [8]). *There is a d.r.e. family (A_n) with a second numbering (B_n) such that both numberings (A_n) and (B_n) are minimal and incomparable and every further numbering (C_n) satisfies $(A_n) \leq (C_n)$.*

Proof. Let S be a given maximal set and f a K-recursive function which is approximable from below, which dominates all recursive functions and which satisfies $f(x) \geq 2$ for all x. Now one defines a family consisting of the following sets:

$$A_{2n} = D \cup \{\langle x, y, z\rangle : x = n \wedge ((z < f(x) \wedge x \notin S) \vee z = 0)\};$$
$$A_{2n+1} = D \cup \{\langle x, y, z\rangle : x = n \wedge ((z = f(x) \wedge x \notin S) \vee z = 0)\}.$$

Here D is a so-called "dirt set" which is recursively enumerable and contains only tuples of the form $\langle x, y, z\rangle$ where y is not the index of a numbering of the family. The set D makes numberings illegal which have enumerated and then deleted entries $\langle x, y, z\rangle$ with y being the index of the numbering in a way which is either not permitted at all or is not permitted at the time when the deletion was carried out. For example, all entries of the form $\langle x, y, 0\rangle$ are not supposed to be deleted after being enumerated once and so whenever such an entry is deleted by the y-th numbering, then the corresponding coordinate $\langle x, y, 0\rangle$ is enumerated into D and is from then on in all sets of the family; so the y-th numbering itself cannot be a d.r.e. numbering for the family, as it had already enumerated and then deleted this entry and now would have to enumerate it again.

So the forbidden operations are to extract a triple $\langle x, y, z\rangle$ already enumerated under the circumstances below, and this is checked for each y with respect to the y-th numbering; whenever it is observed that the protocol is broken for some $\langle x, y, z\rangle$ by the y-th numbering, then $\langle x, y, z\rangle$ is enumerated into D. Here is a list of the forbidden "deletions" (actions of extracting an element already enumerated):

1. It is forbidden to delete $\langle x, y, 0\rangle$ for any x;
2. It is forbidden to delete $\langle x, y, 1\rangle$ unless x has already been enumerated into S;
3. It is forbidden to delete $\langle x, y, z\rangle$ with $z \geq 2$ unless either x has already been enumerated into S or the approximation of $f(x)$ is already strictly larger than z at the current stage.

The numbering of all (A_n) is d.r.e.: For A_{2x}, one continues to enumerate all $\langle x, y, z\rangle$ with z seen to be strictly below $f(x)$ and all triples in D into A_{2x} until x happens to be enumerated into S; if at some time it occurs that x is enumerated into S then those $\langle x, y, z\rangle$ which satisfy $z > 0$ and are not already in D at the time when x is enumerated into S will be deleted from A_{2x}. For A_{2x+1}, one first enumerates all $\langle x, y, 0\rangle$ into A_{2x+1} and for each current value of the approximation of $f(x)$ from below, one enumerates all $\langle x, y, z\rangle$ into A_{2x+1} and keeps them until either the approximation of $f(x)$ changes to a value larger than z or x is enumerated into S; in that case, those $\langle x, y, z\rangle$ which satisfy $z > 0$ and are not yet enumerated into D will be deleted from A_{2x+1}; in addition, all elements from D are copied into A_{2x} or A_{2x+1} whenever they are enumerated into D. Note that, for $z \geq 2$, once $f(x)$ grows beyond z or x is enumerated into S, no tuples of the form $\langle x, y, z\rangle$ will be enumerated into D and thus one does not need to take care of elements of D enumerated in the future in the case of permitted deletions.

The numbering (B_n) is defined as follows: It contains for every x and all $z = 1, 2, \ldots, f(x)$ an index $n_{x,z}$ such that $B_{n_{x,z}}$ is equal to A_{2x} in the case that $z < f(x)$ and equal to A_{2x+1} in the case that $z = f(x)$. The sets $B_{n_{x,z}}$ are all r.e. for those x which are not in S: one enumerates into $B_{n_{x,z}}$ first all elements of the form $\langle x, y, 0\rangle$ and $\langle x, y, z\rangle$; in the case that it turns out that $z < f(x)$, one adds also all those elements $\langle x, y, 1\rangle, \ldots, \langle x, y, f(x) - 1\rangle$ into $B_{n_{x,z}}$ which have not yet been enumerated; furthermore, one adds all elements of D. For those x which are in S, one enumerates $B_{n_{x,z}}$ as above until x is enumerated into S; once that has happened, one knows that no further $\langle x, y, z'\rangle$ with $z' > 0$ will be enumerated into D and so one can extract all those $\langle x, y, z'\rangle$ from $B_{n_{x,z}}$ which have not yet been enumerated into D. Thus the numbering of the (B_n) is a d.r.e. numbering and enumerates the same family as (A_n).

Now consider any further numbering (C_n) of the same class and fix y as its index. As this numbering is not destroyed by an illegal deleting operation, no triple of the form $\langle x, y, z\rangle$ is enumerated into D for this y. There are now two cases:

First, the case where for almost all $x \notin S$ it holds that $\langle x, y, z\rangle$ is not deleted from any C_n for any $z < f(x)$. Then the numbering is equivalent to (B_n) and one can reduce the numberings to each other by the following argument: Let E be the finite exception set where $x \notin S$ and some $\langle x,y,z\rangle$ is deleted for some $z < f(x)$; note that the algorithm knows for these finitely many x what $f(x)$ is. Now, for each C_n, one searches until x and z are found satisfying one of the following conditions (with the waiting time large enough to verify the r.e. conditions):

1. $\langle x,y,z\rangle \in C_n$ and $x \in S$: Then one maps the set C_n to $B_{n_{x,1}}$ as all $B_{n_{x,z'}}$ are the same.
2. $\langle x,y,z\rangle \in C_n$ and $x \in E$ and $z \in \{1, f(x)\}$: Then one maps C_n to $B_{n_{x,z}}$ for exactly this z; the conditions on the family prohibit that $\langle x,y,z\rangle$ is deleted and therefore C_n and $B_{n_{x,z}}$ must be the same.
3. $\langle x,y,0\rangle, \langle x,y,z\rangle \in C_n$ with $z \geq 1$ and $x \notin E$ and some $B_{n_{x,z}}$ with $\langle x,y,0\rangle, \langle x,y,z\rangle \in B_{n_{x,z}}$ (at that stage) have been found: Then $C_n = B_{n_{x,z}}$ in the case that $x \notin S$; furthermore, both will be mapped to the union of D and the set $\{\langle x,y',0\rangle : y' \in \mathbb{N}\}$ in the case that $x \in S$.

Similar arguments also show that (B_n) can be reduced to (C_n). Again the mapping follows the above cases, only that this time one searches a partner for $B_{n_{x,z}}$ by searching for some C_n which satisfies the same conditions as above and then mapping $n_{x,z}$ to that n.

Second, the case where for almost all $x \notin S$ it holds that there are n_x and z such that $\langle x,y,z\rangle$ gets deleted from any C_{n_x} for some $z < f(x)$. Then the numbering (A_n) is reducible to (C_n) as follows. Let E be the finite subset of the complement of S containing those x for which the corresponding C_{n_x} do not exist. Let y be the index of the numbering (C_n). Now one maps A_{2x} and A_{2x+1} as follows.

1. Case $x \in E$: The images for A_{2x} and A_{2x+1} are chosen according to a finite table.
2. Case $x \notin E$ and mapping of A_{2x}: The set A_{2x} is mapped to the first C_n found such that $\langle x,y,0\rangle \in C_n$ and either $x \in S$ or $\langle x,y,1\rangle \in C_n$.
3. Case $x \notin E$ and mapping of A_{2x+1}: The set A_{2x+1} is mapped to the first C_n found such that $\langle x,y,0\rangle \in C_n$ and either $x \in S$ or there is z with $2 \leq z < f(x)$ such that $\langle x,y,z\rangle$ was once enumerated and later deleted from C_n; in the case that $x \notin S$ this is the set C_{n_x} from above.

One can see that the algorithm is in all three cases effective. Furthermore, if $\langle x,y,0\rangle$ and $\langle x,y,1\rangle$ are both in C_n at some stage, then C_n must be equal to A_{2x}, as $\langle x,y,0\rangle$ cannot be deleted and $\langle x,y,1\rangle$ can only be deleted after x is enumerated into S and A_{2x}, A_{2x+1} are both mapped to the same set. If $\langle x,y,0\rangle$ and $\langle x,y,z\rangle$ for some $z < f(x)$ are at some stage both enumerated and later $\langle x,y,z\rangle$ is deleted then either $x \notin S$ and $C_n = A_{2x+1}$, as A_{2x} contains $\langle x,y,z\rangle$, or $x \in S$ and $C_n = A_{2x} = A_{2x+1}$, as this is then the only set in the family containing $\langle x,y,0\rangle$. So $(A_n) \leq (C_n)$.

It remains to show that (A_n) and (B_n) are incomparable. Let y be the default index of the numbering (B_n). In the case that $(A_n) \leq (B_n)$, one could compute $f(x)$ for each $x \notin S$ by finding for these x the image B_n of A_{2x+1} and then determine the unique $z \geq 2$ such that $\langle x,y,z\rangle$ is enumerated into B_n; this z is then $f(x)$. For those x which are enumerated into S before such z is found, one could assign an arbitrary value in order to make the algorithm total. Then the algorithm would compute a recursive function which is infinitely often equal to f, but such a function does not exist. Hence $(A_n) \not\leq (B_n)$.

If $(B_n) \leq (A_n)$ via h then one could search for each x until either x is enumerated into S or a value $n_{x,z}$ is found such that $h(n_{x,z})$ is odd and therefore either $x \in S$ or $f(x) = z$. Hence, when assigning some arbitrary value in the first case and z in the second case, one would again get a recursive function which coincides with f on the infinite complement of S and such a recursive function does not exist by the choice of f; thus $(B_n) \not\leq (A_n)$.

So (A_n) and (B_n) are incomparable and every further numbering (C_n) of the family is either above (A_n) or equivalent to (B_n). Thus this family witnesses that Khutoretskii's Lemma is false for this numbering. □

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