THERE IS NO PLUS-CAPPING DEGREE

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Abstract. Answering a question of Per Lindström, we show that there is no "plus-capping" degree, i.e. that for any incomplete r.e. degree $\mathbf{w}$, there is an incomplete r.e. degree $\mathbf{a} > \mathbf{w}$ such that there is no r.e. degree $\mathbf{v} > \mathbf{w}$ with $\mathbf{a} \cap \mathbf{v} = \mathbf{w}$.

1. The Theorem. In [2], Feferman defines relative interpretability for extensions of Peano arithmetic. The induced structure D_T of degrees of interpretability turns out to be a lattice (cf. [4]). Lindström [5] proved a number of algebraic facts about this lattice. He recently raised the question whether one of these, namely

$$\forall w < 1 \, \exists a \, (w < a < 1 \wedge \forall v > w \, \neg(a \wedge v = w)),$$

also holds in the upper semilattice of r.e. degrees. This question turns out to be of independent interest to classical recursion theorists as they try to better understand the algebraic structure of the r.e. degrees. The purpose of this paper is to answer Lindström's question positively by proving the following

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Theorem. *For any incomplete r.e. degree* $\mathbf{w}$*, there is an incomplete r.e. degree* $\mathbf{a} > \mathbf{w}$ *such that there is no r.e. degree* $\mathbf{v} > \mathbf{w}$ *with* $\mathbf{a} \cap \mathbf{v} = \mathbf{w}$.

This theorem can be abbreviated as "There is no plus-capping degree." since it stands in contrast with the following theorem by Harrington on plus-cupping degrees (we quote here a slightly weaker version to point out the duality):

Plus Cupping Theorem (Harrington, see [3, 6])**.** *There is a nonrecursive r.e. degree* $\mathbf{w}$ *such that for any nonrecursive r.e. degree* $\mathbf{a} < \mathbf{w}$ *there is an r.e. degree* $\mathbf{v} < \mathbf{w}$ *with* $\mathbf{a} \cup \mathbf{v} = \mathbf{w}$.

Obviously, our theorem asserts that the dual statement of the Plus Cupping Theorem (as quoted above) fails, and thus, a fortiori, the dual statement of the full version of the Plus Cupping Theorem fails.

A related theorem is the following result by Downey and Stob:

Theorem (Downey, Stob [1])**.** *For any nonrecursive r.e. degree* $\mathbf{w}$*, there is a nonrecursive r.e. degree* $\mathbf{a} < \mathbf{w}$ *such that there is no nonrecursive r.e. degree* $\mathbf{v} \leq \mathbf{w}$ *with* $\mathbf{a} \cap \mathbf{v} = \mathbf{0}$.

Our notation generally follows Soare [7] with the following exceptions: The *use* of a computation $\Phi^X(x)$ is the largest number *actually* used in that computation and is denoted by $\varphi(x)$ (and similarly for other Greek letters). We assume the "hat-trick" for all functionals Φ not constructed by us, i.e.

$$\begin{aligned} &\forall x \forall s (\Phi^X(x)[s] \downarrow \wedge X \restriction (\varphi(x)+1)[s] \neq X_{s+1} \restriction (\varphi_s(x)+1) \rightarrow \\ &\qquad \Phi^X(x)[s+1] \uparrow). \end{aligned} \tag{1}$$

Furthermore, we assume the uses of all functionals Φ not constructed by us to be nondecreasing in the stage and increasing in the argument, i.e.

$$\forall x \forall s (\Phi^X(x)[s] \downarrow \rightarrow \varphi_s(x) \leq \varphi_{s+1}(x) \wedge \varphi_s(x) < \varphi_s(x+1)). \tag{2}$$

We set $\varphi_s(x) = \infty$ if $\Phi^X(x)[s]$ is undefined. Finally, when the oracle of a functional is given as the join of two sets, we assume the use to be computed separately on each set, i.e.

$$\Phi^{(X \oplus Y) \restriction (\varphi(x)+1)}(x) = \Phi^{X \restriction (\varphi(x)+1) \oplus Y \restriction (\varphi(x)+1)}(x). \tag{3}$$

2. The Requirements and the Strategies. We fix an r.e. set W and build an r.e. set A satisfying the following requirements:

$$\mathcal{N}_\Phi : \Phi^{A\oplus W} = K \to \exists\Lambda(\Lambda^W = K)$$

for all partial recursive (p.r.) functionals Φ. (This ensures that $\mathbf{a} = \deg(A \oplus W) < \mathbf{0}'$ unless $\mathbf{w} = \deg W = \mathbf{0}'$.)

Furthermore, for each r.e. set V, we build r.e. sets B $(= B_V)$ and p.r. functionals Γ $(= \Gamma_V)$ and Δ $(= \Delta_V)$ satisfying

$$\mathcal{P}_V : B = \Gamma^{A\oplus W} = \Delta^{V\oplus W}, \text{ and}$$
$$\mathcal{P}_{V,\Psi} : B = \Psi^W \to \exists\Theta(V = \Theta^W)$$

for all p.r. functionals Ψ. (Thus $\mathbf{b} = \deg B \leq \mathbf{a}$; $\mathbf{b} \leq \mathbf{v}(= \deg(V \oplus W))$; and $\mathbf{b} \not\leq \mathbf{w}$ unless $\mathbf{v} \leq \mathbf{w}$, witnessing $\neg(\mathbf{a} \cap \mathbf{v} = \mathbf{w})$ unless $\mathbf{v} \leq \mathbf{w}$.)

The strategy for $\mathcal{N}_\Phi$ is the usual Sacks preservation strategy: As the length of agreement between $\Phi^{A\oplus W}$ and K increases, we extend the domain of Λ^W, and we ensure that if $\Phi^{A\oplus W}(y)$ becomes undefined (and thus $K(y)$ can change and $\Phi^{A\oplus W}(y)$ can be redefined correctly) then $\Lambda^W(y)$ will become undefined also. This will usually be ensured by A-restraint but sometimes we add a new twist to this. If a link (as defined below) is traveled on the tree of strategies skipping over the $\mathcal{N}_\Phi$-strategy then the $\mathcal{N}_\Phi$-strategy's A-restraint (to protect some computation $\Phi^{A\oplus W}(y)$) may be injured by a lower-priority strategy. In this case, we ensure that if the approximation to the true path ever returns to the $\mathcal{N}_\Phi$-strategy then a W-change must have occurred making $\Lambda^W(y)$ undefined. For this sake, we will arrange the outcomes of an $\mathcal{N}_\Phi$-strategy not in the usual way but of order type $2 \cdot \omega$ $(= \omega)$ where the outcome $2j$ roughly denotes that $\Phi^{A\oplus W} \restriction j \downarrow= K \restriction j$ but $\Phi^{A\oplus W}(j)$ is undefined, and the outcome $2j + 1$ roughly denotes that $\Phi^{A\oplus W} \restriction j \downarrow= K \restriction j$ but $\Phi^{A\oplus W}(j) \downarrow\neq K(j)$. (The precise version of the outcomes is more complicated as it depends on the order of events leading up to the least disagreement, which may actually occur at a number $j' < j$.)

The strategy for $\mathcal{P}_V$ simply consists in defining $\Gamma^{A\oplus W}$ and $\Delta^{V\oplus W}$ on larger and larger arguments and in occasionally increasing the use $\delta(x)$ of a computation $\Delta^{V\oplus W}(x)$ when requested by another strategy. The $\mathcal{P}_V$-strategy has only one single outcome.

The $\mathcal{P}_{V,\Psi}$-requirement is too complicated for one strategy and is thus split up into subrequirements

$$\mathcal{P}_{V,\Psi,i} : B \restriction (x + 1) = \Psi^W \restriction (x + 1) \to V(i) = \Theta^W(i)$$

(where $x = x_i$ depends on the $\mathcal{P}_{V,\Psi,i}$-strategy).

The requirement $\mathcal{P}_{V,\Psi}$ is now worked on by a $\mathcal{P}_{V,\Psi}$-strategy and infinitely many $\mathcal{P}_{V,\Psi,i}$-strategies below it, each trying to define $\Theta^W(i)$. The $\mathcal{P}_{V,\Psi}$-strategy has two outcomes, infinite and finite, where the latter denotes the fact that some $\mathcal{P}_{V,\Psi,i}$-strategy below the infinite outcome of the $\mathcal{P}_{V,\Psi}$-strategy has created a disagreement between Ψ^W and B and that therefore there is no reason to work on $\mathcal{P}_{V,\Psi}$ until W changes, allowing Ψ^W to be corrected.

A $\mathcal{P}_{V,\Psi,i}$-strategy will pick a fresh x, wait for $B(x) = \Psi^W(x)$, and then define $\Theta^W(i) = V(i)$ with use $\vartheta(i) = \psi(x)$. If now $V(i)$ changes then we have V-permission to put x into B, and we also put $\gamma(x)$ into A to correct $\Gamma^{A\oplus W}(x)$. There are two complications arising here:

Firstly, V-permissions for x (to correct $\Delta^{V\oplus W}(x)$) are measured at the stages at which the $\mathcal{P}_V$-strategy appears to be on the true path (since the $\mathcal{P}_V$-strategy must keep $\Delta^{V\oplus W}(x)$ defined). The $\mathcal{P}_V$-strategy cannot let the $\mathcal{P}_{V,\Psi,i}$-strategy act before redefining $\Delta^{V\oplus W}(x)$ if the latter appears to be to the left of the true path (i.e. we cannot form a *link* from the $\mathcal{P}_V$-strategy to the $\mathcal{P}_{V,\Psi,i}$-strategy) since the $\mathcal{P}_{V,\Psi,i}$-strategy's enumerating $\gamma(x)$ into A may injure intermediate $\mathcal{N}_\Phi$-strategies. Instead, the $\mathcal{P}_V$-strategy will increase the use $\delta(x)$ when $V(i)$ changes so that when the $\mathcal{P}_{V,\Psi}$-strategy again appears to be on the true path, the corresponding W-change will have made $\Delta^{V\oplus W}(x)$ undefined.

On the other hand, once $\Theta^W(i)$ is defined and $V(i)$ may change, we must form a link from the $\mathcal{P}_{V,\Psi}$-strategy down to the $\mathcal{P}_{V,\Psi,i}$-strategy. This is because there are many $\mathcal{P}_{V,\Psi,i}$-strategies below the $\mathcal{P}_{V,\Psi}$-strategy, all competing to define $\Theta^W(i)$ for the same i. So when the $\mathcal{P}_{V,\Psi}$-strategy appears to be on the true path and has a link down to a $\mathcal{P}_{V,\Psi,i}$-strategy that has the $V \oplus W$-permission to put x into B, the former will let the latter act immediately even if the latter appears to be to the left of the true path. Again there will be $\mathcal{N}_\Phi$-strategies between the $\mathcal{P}_{V,\Psi}$- and the $\mathcal{P}_{V,\Psi,i}$-strategy that are injured by the enumeration of $\gamma(x)$ into A. But the $\mathcal{P}_{V,\Psi}$-strategy now switches outcome since the $\mathcal{P}_{V,\Psi,i}$-strategy has established $\Psi^W(x)\downarrow\neq B(x)$. If this computation $\Psi^W(x)$ ever disappears then the corresponding W-change will allow the injured $\mathcal{N}_\Phi$-strategies to correct their Λ^W as long as these Λ-uses exceed the use $\psi(x)$. We will ensure this to be true.

We are now ready to formally describe the construction.

3. The Construction. We use a tree of strategies $T \subseteq \omega^{<\omega}$, which we will define inductively. We first define satisfaction of a requirement along a node of T:

Definition 1. Let ξ be a node on T.

(i) Requirement $\mathcal{N}_\Phi$, $\mathcal{P}_V$, or $\mathcal{P}_{V,\Psi}$ is *satisfied along* ξ if there is a

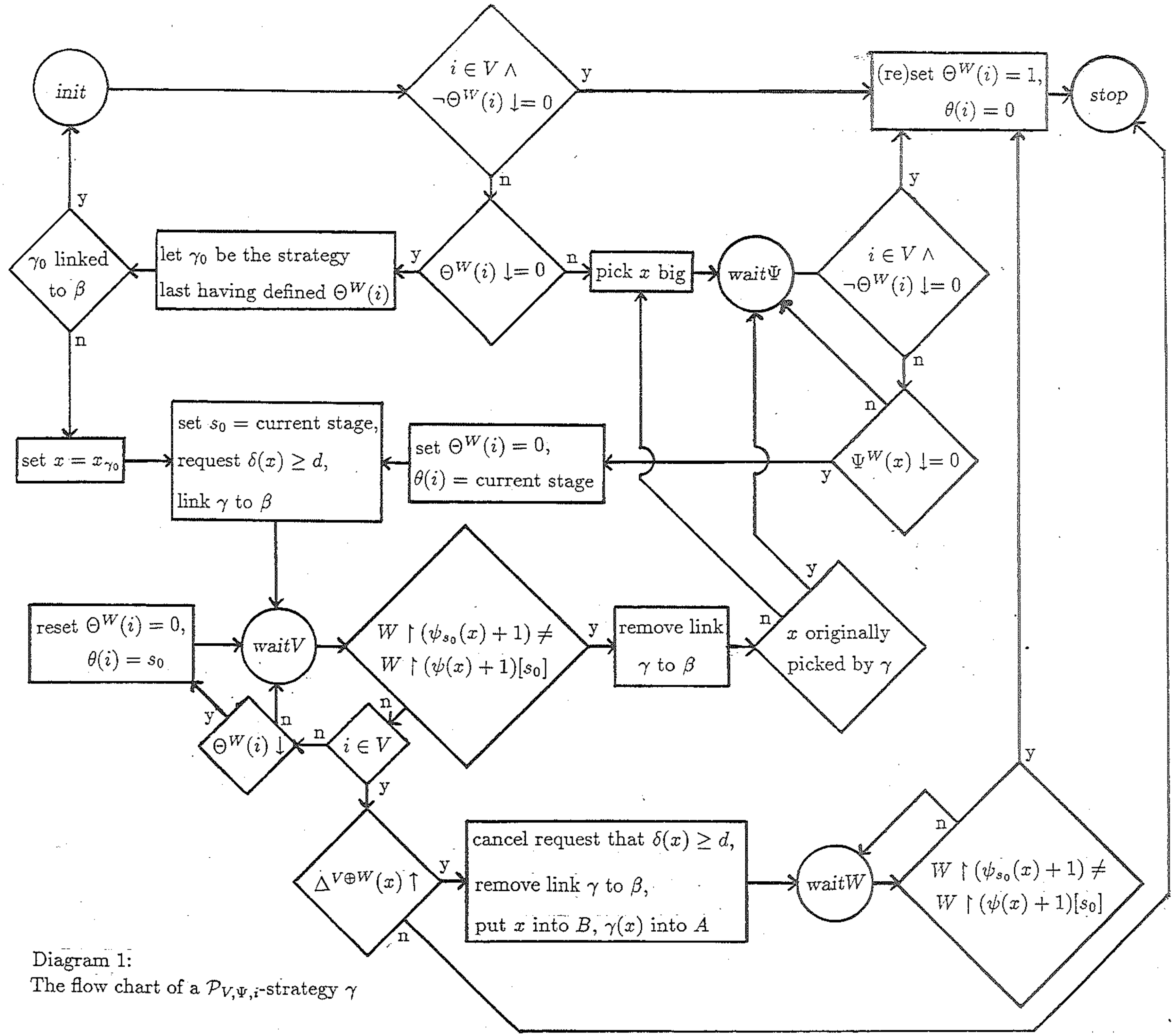

Diagram 1:
The flow chart of a $\mathcal{P}_{V,\Psi,i}$-strategy γ

strategy $\eta \subset \xi$ working on $\mathcal{N}_\Phi$, $\mathcal{P}_V$, or $\mathcal{P}_{V,\Psi}$, respectively.

(ii) Subrequirement $\mathcal{P}_{V,\Psi,i}$ is *satisfied along* ξ if there is a $\mathcal{P}_{V,\Psi}$-strategy β with $\beta\hat{\ }\langle 1\rangle \subseteq \xi$ or if there is a $\mathcal{P}_{V,\Psi,i}$-strategy $\gamma \subset \xi$. (The first case here corresponds to β having found a disagreement between Ψ^W and B.)

We now fix an effective ω-ordering of all requirements $\mathcal{N}_\Phi$, $\mathcal{P}_V$, $\mathcal{P}_{V,\Psi}$, and all subrequirements $\mathcal{P}_{V,\Psi,i}$ such that

(i) $\mathcal{P}_V$ precedes all $\mathcal{P}_{V,\Psi}$;
(ii) $\mathcal{P}_{V,\Psi}$ precedes all $\mathcal{P}_{V,\Psi,i}$; and
(iii) $\mathcal{P}_{V,\Psi,i}$ precedes $\mathcal{P}_{V,\Psi,i'}$ if $i < i'$.

We define the tree of strategies $T \subseteq \omega^{<\omega}$ inductively as follows:

Definition 2. Let ξ be a node on T. Then ξ *works on* the requirement $\mathcal{R}$ of highest priority that is not satisfied along ξ. (We call ξ an $\mathcal{R}$-*strategy.*) The *immediate successors* of ξ on T are the following:

(i) $\xi\hat{\ }\langle j\rangle$ for all $j \in \omega$ if ξ is an $\mathcal{N}_\Phi$-strategy;
(ii) only $\xi\hat{\ }\langle 0\rangle$ if ξ is a $\mathcal{P}_V$- or $\mathcal{P}_{V,\Psi,i}$-strategy; and
(iii) $\xi\hat{\ }\langle 0\rangle$ and $\xi\hat{\ }\langle 1\rangle$ if ξ is a $\mathcal{P}_{V,\Psi}$-strategy (denoting ξ's infinite and finite outcome, respectively).

We next define initialization. An $\mathcal{N}_\Phi$-strategy is *initialized* by making its functional Λ^W totally undefined. A $\mathcal{P}_V$-strategy is *initialized* by making its functionals $\Gamma^{A\oplus W}$ and $\Delta^{V\oplus W}$ totally undefined and by discarding its set B. A $\mathcal{P}_{V,\Psi}$-strategy β is *initialized* by making its functional Θ^W totally undefined and by initializing all the $\mathcal{P}_{V,\Psi,i}$-strategies $\gamma \supseteq \beta\hat{\ }\langle 0\rangle$. A $\mathcal{P}_{V,\Psi,i}$-strategy is *initialized* by making all its parameters undefined, by removing any link to it, and by canceling all its requests.

A parameter is *defined big* by setting it to a number greater than any number mentioned so far.

The construction now proceeds in stages.

At stage 0, we initialize all strategies and let A be the empty set.

A stage $s > 0$ consists of finitely many substages t with some additional action at the end of the stage. At each substage t, a strategy of length t is eligible to act and will, after it acts as described below, determine a strategy ξ of length $t+1$ eligible to act at the next substage (unless we end the stage) and initialize all strategies $>_L \xi$. We *end the stage* after substage t if $s \leq t$ and for any $\mathcal{P}_{V,\Psi}$-strategy β eligible to act at a substage $\leq t$ of stage s which let $\beta\hat{\ }\langle 0\rangle$ be eligible to act next, and for any i such that $\Theta^W_\beta(i)$ is currently defined, some $\mathcal{P}_{V,\Psi,i}$-strategy was eligible to act at a substage $\leq t$ of stage s.

We now describe the action of the strategy eligible to act at substage t of stage s. We distinguish cases depending on the type of strategy eligible to act.

Case 1: An $\mathcal{N}_\Phi$-strategy δ is eligible to act. Define the *length of agreement*

$$\ell[s] = \max\{x \mid \Phi^{A\oplus W} \upharpoonright x\,[s]\downarrow = K_s \upharpoonright x\} \tag{3}$$

and the *maximum length of agreement*

$$\begin{aligned} m[s] = \max\{\ell[s'] \mid s' \leq s \wedge (A\oplus W) \upharpoonright (\varphi(x-1)+1)[s'] = \\ (A\oplus W)_s \upharpoonright (\varphi_{s'}(x-1)+1)\}\,. \end{aligned} \tag{4}$$

Let s^* be the last stage at which δ was eligible to act (or let $s^* = 0$ if this is the first stage since δ's most recent initialization at which δ is eligible to act). Let $m^* = \min\{m[s'] \mid s^* \leq s' \leq s\}$. For all $y < m^*$ such that $\Lambda^W(y)$ is now undefined, set $\Lambda^W(y) = K_s(y)$ with use

$$\begin{aligned} \lambda(y) = \min\big\{s' \mid \delta \text{ eligible to act at } s' \wedge \Phi^{A\oplus W}(y)[s']\downarrow \wedge y < m^*[s'] \wedge \\ (A\oplus W) \upharpoonright (\varphi(y)+1)[s'] = (A\oplus W)_s \upharpoonright (\varphi_{s'}(y)+1)\}\,. \end{aligned} \tag{5}$$

If $\Phi^{A\oplus W}(m^*)[s']$ was defined for all $s' \in [s^*, s]$ then $\delta\hat{\ }\langle 2m^*+1\rangle$ is *eligible to act next*, otherwise $\delta\hat{\ }\langle 2m^*\rangle$ is *eligible to act next.*

Case 2: A $\mathcal{P}_V$-strategy α is eligible to act. Since the definition of $\Gamma^{A\oplus W}$ and $\Delta^{V\oplus W}$ can only occur at the end of the stage (to allow the $\mathcal{P}_{V,\Psi}$-strategies to act and to allow the lifting of uses), α merely lets $\alpha\hat{\ }\langle 0\rangle$ be *eligible to act next.*

Case 3: A $\mathcal{P}_{V,\Psi}$-strategy β is eligible to act. First, check if there is some $\mathcal{P}_{V,\Psi,i}$-strategy $\gamma \supseteq \beta\hat{\ }\langle 0\rangle$ such that β has let γ diagonalize before (as defined below) at a stage s', say, γ has not been initialized since stage s', and γ's computation $\Psi^W(x_\gamma)$ has not been destroyed since stage s' (i.e. γ is not ready to move from state *waitW* to state *stop*). If there is such a γ then let $\beta\hat{\ }\langle 1\rangle$ be *eligible to act next.*

Otherwise, check if there is a link to a $\mathcal{P}_{V,\Psi,i}$-strategy $\gamma \supseteq \beta\hat{\ }\langle 0\rangle$ which is ready to proceed to state *waitW*. If so then pick γ leftmost such, *let γ diagonalize* according to Case 4 below, initialize all $\mathcal{N}_\Phi$-, $\mathcal{P}_{\hat{V}}$-, and $\mathcal{P}_{\hat{V},\hat{\Psi}}$-strategies $> \gamma$, and let $\beta\hat{\ }\langle 1\rangle$ be *eligible to act next.*

Otherwise, i.e. if there is no such γ, let $\beta\hat{\ }\langle 0\rangle$ be *eligible to act next.*

Case 4: A $\mathcal{P}_{V,\Psi,i}$-strategy γ is eligible to act. We describe γ's action using the flow chart in Diagram 1. After each initialization, γ starts in stage *init*, and at each substage at which it is eligible to act, it proceeds from one state (denoted by a circle) to the next, following the arrows and along the way executing the instructions (in rectangular boxes) and deciding the truth of statements (in diamonds, following the y-arrow iff the statement is true). Some parameters formally defined in the flow chart have the following intuitive meaning: i is the argument at which γ is trying to define Θ^W; x is the witness at which γ is trying to diagonalize B against Ψ^W; and s_0 is the (most recent) stage at which γ found a computation $\Psi^W(x)\downarrow= 0$. α and β are the $\mathcal{P}_V$- and $\mathcal{P}_{V,\Psi}$-strategy $\subset \gamma$, respectively. The *delayed permission parameter* d is defined by

(6)
$$\begin{aligned} d = \min(\{\varphi_\delta(y) \mid \alpha \subset \delta\hat{}\langle 2j\rangle \subseteq \beta \vee \alpha \subset \delta\hat{}\langle 2j+1\rangle \subseteq \beta) \wedge \\ \delta \text{ is an } \mathcal{N}_\Phi\text{-strategy}\} \cup \\ \{\psi_{\beta_0}(i_0) \mid \alpha \subset \beta_0\hat{}\langle 0\rangle \subseteq \beta \wedge \beta_0 \text{ is a } \mathcal{P}_{V_0,\Psi_0}\text{-strategy} \wedge i_0 \in V_0 \wedge \\ \beta_0 \text{ is linked to a } \mathcal{P}_{V_0,\Psi_0,i_0}\text{-strategy } \gamma_0 \wedge \\ \Psi_0^W(x_{\gamma_0}) \text{ has not been destroyed since stage } s_{0,\gamma_0}\}). \end{aligned}$$

(We allow $d=\infty$ if none of the computations is defined. Intuitively, the delayed permission parameter d is such that a W-change on a number $\leq d$ will allow the correction of $\Delta^{V\oplus W}(x)$ even if the initial V-permission (via i entering V) has occurred long before.) The phrase "request $\delta(x) \geq d$" will be explained below when we specify the definition of $\Delta^{V\oplus W}$ by α.

The strategy eligible to act next is $\gamma\hat{}\langle 0\rangle$.

At the end of stage s, i.e. after we end the stage, we let all $\mathcal{P}_V$-strategies $\alpha \subseteq \delta_t$ define their functionals $\Gamma^{A\oplus W}$ and $\Delta^{V\oplus W}$ as follows. For all $x \leq s$ such that $\Gamma^{A\oplus W}(x)$ (or $\Delta^{V\oplus W}(x)$) is currently undefined, α sets $\Gamma^{A\oplus W}(x) = B(x)$ (or $\Delta^{V\oplus W}(x) = B(x)$, respectively). α sets the Γ-use $\gamma(x) = x$ and defines the Δ-use by

$$\begin{aligned} \delta(x) = \max\big(\{x\} \cup \{\delta_{s'}(x') \mid s' < s \wedge x' \leq x \wedge \Delta^{V\oplus W}(x')[s']\downarrow\} \cup \\ \{d_\gamma \mid \gamma \text{ is a } \mathcal{P}_{V,\Psi,i}\text{-strategy requesting} \\ \text{“}\delta(x') \geq d_\gamma\text{”} \wedge i_\gamma \in V_\gamma \wedge x' \leq x\}\big). \end{aligned} \tag{7}$$

This completes the description of the construction.

4. The Verification. We define the *true path* $f \in [T]$ of the construction by induction on n as follows:

$$f(n) = \mu z \in \omega((f \restriction n)\hat{}\langle z\rangle \text{ is eligible to act infinitely often}).$$

(Notice that conceivably $f(n)$ may not be defined for some (least) n in which case we assume $f = f \restriction n \in T$. We will show in Lemma 2 that if $W <_{\mathrm{T}} K$ then this will not happen.)

We begin with a lemma on the $\mathcal{N}_\Phi$-strategies.

Lemma 1 ($\mathcal{N}_\Phi$-Strategy Lemma). *Suppose δ is an $\mathcal{N}_\Phi$-strategy that is initialized at most finitely often.*

(i) *There is a stage s_δ after which δ is no longer initialized and such that for all stages $s_2 > s_1 \geq s_\delta$ the following holds: If δ defines $\Lambda^W(y)$ (for some y) at stage s_1 with use $\lambda(y) = s_1$ and at stage s_2 at which δ is eligible to act again we still have $\Lambda^W(y)\downarrow$ with use $\lambda(y) = s_1$ then the computation $\Phi^{A\oplus W}(y)$ from stage s_1 has also not been destroyed by an $(A \oplus W)$-change by stage s_2.*

Assume furthermore that δ is eligible to act infinitely often. Then

(ii) *δ satisfies its requirement; and*
(iii) *if $W <_{\mathrm{T}} K$ then $\liminf_s m^*_\delta$ is finite.*

Proof. (i) We proceed by induction on $|\delta|$ and assume (i) fails for δ. Since $\Lambda^W(y)$ is defined with use $\lambda(y) \geq \varphi_{s_1}(y)$ at stage s_1, the destruction of the computation $\Phi^{A\oplus W}(y)$ at some (least) stage $s_3 \in [s_1, s_2]$ must be caused by the enumeration of some number $\gamma(x) = x \leq \varphi_{s_1}(y)$ into A by some $\mathcal{P}_{V,\Psi,i}$-strategy γ_0. This x was originally picked by some $\mathcal{P}_{V,\Psi,i}$-strategy γ at some stage $s_4 < s_3$. Then γ must also have defined $\Theta^W(i) = 0$ at some stage $s_5 \in (s_4, s_3)$. Since $x \leq \varphi_{s_1}(y) \leq s_1$, we have $s_4 < s_1$. If $\gamma \neq \gamma_0$ then $\gamma_0 <_L \gamma$ since whenever a $\mathcal{P}_{V,\Psi,i}$-strategy γ' is initialized (without the corresponding $\mathcal{P}_{V,\Psi}$-strategy β being initialized) while $\Theta^W(i)\downarrow= 0$ then some $\mathcal{P}_{V,\Psi,i}$-strategy $\gamma'' <_L \gamma'$ is linked to β at the same stage by the way we end the stage. Furthermore, by the same reasoning,

$$\forall s \in [s_5, s_2](\gamma_0 \leq \delta_s). \tag{8}$$

We now compare the relative locations of γ (or γ_0) and δ.

Case 1: $\gamma_0 < \delta$: This is impossible since then δ is initialized by γ_0 at s_3.

Case 2: $\delta < \gamma_0$ and $\delta <_L \gamma$: Let ξ be the longest common substring of γ and δ. Then ξ is an $\mathcal{N}_{\hat\Phi}$- or a $\mathcal{P}_{\hat V,\hat\Psi}$-strategy, and $s_5 < s_1$ by initialization of γ at s_1.

Subcase 2a: ξ is an $\mathcal{N}_{\hat\Phi}$-strategy: Pick $\hat y$ such that $\xi\hat{\ }\langle 2\hat y\rangle \subseteq \delta$ or $\xi\hat{\ }\langle 2\hat y+1\rangle \subseteq \delta$. Then $\hat\Phi^{A\oplus W}(\hat y)$ is defined at stage s_5 but is destroyed by stage s_1. By induction, we may assume $s_5 \geq s_\xi$ (excluding only finitely many s_1). Then by (i) applied to ξ, we have $W \upharpoonright (\lambda_\xi(\hat y)+1)[s_5] \neq W_{s_1} \upharpoonright (\lambda_{\xi,s_5}(\hat y)+1)$ and $\lambda_{\xi,s_5}(\hat y) \leq s^*_\xi[s_5] < s_5 < s_1 = \lambda_{s_1}(y)$ as desired.

Subcase 2b: ξ is a $\mathcal{P}_{\hat V,\hat\Psi}$-strategy. Pick $\hat x$ such that ξ lets $\xi\hat{\ }\langle 1\rangle$ be eligible to act next at stage s_5 because of a diagonalization $\hat\Psi^W(\hat x) \neq$

$B_\xi(\hat{x})$. Since $\xi\,\hat{}\,\langle 0\rangle \subseteq \delta \subseteq \delta_{s_1}$, we have $W \restriction (\hat{\psi}(\hat{x})+1)[s_5] \neq W_{s_1} \restriction (\hat{\psi}_{s_5}(\hat{x})+1)$ and $\hat{\psi}(\hat{x})[s_5] < s_5 < s_1 = \lambda_{s_1}(y)$ as desired.

Case 3: $\delta < \gamma_0$ and $\delta \subset \gamma$: Then $\delta \subset \gamma_0$, say $\delta\,\hat{}\,\langle j\rangle \subseteq \gamma_0$. Since $\delta_{s_1^*} <_L \delta\,\hat{}\,\langle 2y+1\rangle$ (for $s_1^* = s^*[s_1]$) and by (8), we have $j \leq 2y$; fix y' such that $j = 2y'$ or $= 2y'+1$. Since the computation $\Phi^{A\oplus W}(y)$ from stage s_1 is first destroyed at s_3, we have that $\gamma_0 \not\subseteq \delta_s$ for any stage $s \in [s_1, s_3]$, and thus a $\mathcal{P}_{V,\Psi}$-strategy β (necessarily $\subset \delta$) lets γ_0 diagonalize at s_3. But, by $\beta\,\hat{}\,\langle 0\rangle \subseteq \delta \subseteq \delta_{s_2}$, the computation $\Psi^W(x)$ from stage s_5 is destroyed between stages s_3 and s_2, so $W_{s_3} \restriction (\psi_{s_5}(x)+1) \neq W_{s_2} \restriction (\psi_{s_5}(x)+1)$. Thus $\psi_{s_5}(x) \leq s_5 < s_1 = \lambda_{s_1}(y)$ establishes the claim in this case.

(ii) Suppose $\Phi^{A\oplus W} = K$. Fix the stage s_δ from part (i) and an arbitrary y. By (5) and since $\Phi^{A\oplus W}(y)$ is defined and $\liminf_s m_\delta^* > y$, $\Lambda^W(y)$ is defined with (eventually) constant use at almost all the stages at which δ is eligible to act unless $\Lambda^W(y)$ is still defined from before. Thus $\Lambda^W(y)$ is defined. By (i) and our assumption on stage s_δ, any permanent definition $\Lambda^W(y)$ made after stage s_δ satisfies $\Lambda^W(y) = \Phi^{A\oplus W}(y)$.

We have thus established that $\Phi^{A\oplus W} = K$ implies $\Lambda^W(y) = K(y)$ for cofinitely many y as desired.

(iii) If $W <_{\mathrm{T}} K$ then $\Phi^{A\oplus W}(y) \neq K(y)$ for some (least) y by (ii). If $\Phi^{A\oplus W}(y)$ is undefined, clearly $\liminf_s m_\delta^* = 2y$ by (4). If $\Phi^{A\oplus W}(y)\downarrow \neq K(y)$ then $\lim_s \ell_\delta = y$, and so $\lim_s m_\delta^*$ must exist and be finite. ■

We can now show that the true path is well-defined and has nice properties, assuming that W is incomplete.

Lemma 2 (True Path Lemma). *Assume that $W <_{\mathrm{T}} K$, and set $\xi = f \restriction n$ for any integer n. Then:*

(i) *ξ is initialized at most finitely often;*
(ii) *ξ is eligible to act infinitely often; and*
(iii) *$f(n)$ is well-defined, and there is a stage after which no strategy $<_L \xi\,\hat{}\,\langle f(n)\rangle$ is eligible to act.*

Proof. We proceed by simultaneous induction on n:

(i) This vacuous for $n = 0$, so assume the lemma for $n-1$ where $n > 0$. Fix a stage $s_0 > |\xi|$ such that no strategy $\eta <_L \xi$ is eligible to act after stage s_0. Then ξ can be initialized after stage s_0 only if some $\mathcal{P}_{V,\Psi}$-strategy $\beta \subset \xi$ lets a $\mathcal{P}_{V,\Psi,i}$-strategy $\gamma < \xi$ diagonalize. Each time this happens, the corresponding i_γ must have entered V_γ, so this can happen at most once for each $\mathcal{P}_{V,\Psi,i}$-strategy γ. Since there are only finitely many $\mathcal{P}_{V,\Psi,i}$-strategies γ such that $\gamma \subset \xi$ or γ is eligible to act by stage s_0, ξ cannot be initialized after some stage $s_1 \geq s_0$, say.

(ii) This holds by the definition of $f(n-1)$.

(iii) Fix a stage s_1 after which no $\xi' \subseteq \xi$ is no longer initialized. If ξ is not an $\mathcal{N}_\Phi$-strategy then the first half of the claim is clear since each time ξ is eligible to act (except possibly the first finitely many times), one of the finitely many immediate successors of ξ is eligible to act next. So suppose ξ is an $\mathcal{N}_\Phi$-strategy. By Lemma 1, $\liminf_s m_\xi$ is finite. Thus, in this case, $f(n) = 2m_0$ or $= 2m_0 + 1$ where $m_0 = \liminf_s m_\xi$. The second half of the claim is now obvious since whenever $\eta \supset \xi$ is eligible to act, $\eta \restriction (|\xi| + 1)$ is eligible to act also. ■

It is now easy to show the following

Lemma 3 ($\mathcal{N}_\Phi$-Satisfaction Lemma). *If $W <_\mathrm{T} K$ then all $\mathcal{N}_\Phi$-requirements are satisfied, i.e. $A \oplus W <_T K$.*

Proof. Fix Φ and apply Lemmas 1 and 2 to the $\mathcal{N}_\Phi$-strategy $\delta \subset f$. ■

We next turn to the $\mathcal{P}_V$-requirements.

Lemma 4 ($\mathcal{P}_V$-Satisfaction Lemma). *If $W <_\mathrm{T} K$ then each $\mathcal{P}_V$-requirement is satisfied.*

Proof. Fix the $\mathcal{P}_V$-strategy $\alpha \subset f$ and an arbitrary x. At each stage $\geq x$ at which α is eligible to act, $\Gamma^{A\oplus W}(x)$ is (re)defined with the use of $\gamma(x) = x$ if necessary, and x is put into B only if $\gamma(x)$ enters A at the same time, destroying the old computation $\Gamma^{A\oplus W}(x)$ if necessary. Thus $\Gamma^{A\oplus W} = B$ as desired.

For the sake of a contradiction, assume $\Delta^{V\oplus W}(x) \neq B(x)$ for some (least) x. First suppose that $\Delta^{V\oplus W}(x)$ is undefined. Since $\Delta^{V\oplus W}(x)$ is (re)defined at each stage $\geq x$ at which α is eligible to act we have $\lim_s \delta(x) = \infty$. By (7) and the minimality of x, there must be a $\mathcal{P}_{V,\Psi,i}$-strategy $\gamma \supset \alpha$ with $x = \lim_s x_\gamma$ and

$$\limsup_s d_\gamma = \infty \tag{9}$$

such that γ requests $\delta(x) \geq d_\gamma$. Since $x = \lim_s x_\gamma$, necessarily $\gamma \leq f$. Since γ's request is responsible for $\lim_s \delta(x) = \infty$, we have $i_\gamma \in V_\gamma$ (using (7)). Once $i_\gamma \in V_\gamma$, γ cannot cancel its request since it would then never again request $\delta(x) \geq d_\gamma$. Therefore, $\beta <_L f$ for the $\mathcal{P}_{V,\Psi}$-strategy $\beta \subset \gamma$. Let $\xi \subset f$ be maximal with $\xi \subset \beta$. Since β and f split at ξ, ξ must be an $\mathcal{N}_\Phi$- or a $\mathcal{P}_{V_0,\Phi_0}$-strategy.

Case 1: ξ is an $\mathcal{N}_\Phi$-strategy. Fix j such that $\xi\hat{\ }\langle 2j\rangle \subseteq \beta$ or $\xi\hat{\ }\langle 2j+1\rangle \subseteq \beta$. By (4) and $\beta <_L f$, $\Phi^{A\oplus W}(j)$ is defined, and thus $\limsup_s d_\gamma \leq \varphi(j)$ by (6), contradicting (9) as desired.

Case 2: ξ is a $\mathcal{P}_{V_0,\Phi_0}$-strategy. Then $\xi\hat{\ }\langle 0\rangle \subseteq \beta$ and $\xi\hat{\ }\langle 1\rangle \subset f$. Thus there is a permanent computation $\Psi_0^W(x_{\gamma_0})$ (for some $\mathcal{P}_{V_0,\Psi_0,i_0}$-strategy

$\gamma_0 \supseteq \xi\hat{\ }\langle 0\rangle$). So again $\limsup_s d_\gamma \leq \psi_0(x_{\gamma_0})$ by (6), contradicting (9) as desired.

We have thus established that $\Delta^{V\oplus W}(x)$ is defined. Now by the construction, x enters B only if $\Delta^{V\oplus W}(x)$ is currently undefined, so $\Delta^{V\oplus W}(x) \neq B(x)$ is impossible as desired. ∎

We finally prove the satisfaction of the $\mathcal{P}_{V,\Psi}$-requirements.

Lemma 5 ($\mathcal{P}_{V,\Psi}$-Satisfaction Lemma). *If $W <_{\mathrm{T}} K$ then each $\mathcal{P}_{V,\Psi}$-requirement is satisfied.*

Proof. Fix the $\mathcal{P}_{V,\Psi}$-strategy $\beta \subset f$ and assume $\Psi^W = B$.

First suppose that $\beta\hat{\ }\langle 1\rangle \subset f$. Then there is a fixed $\mathcal{P}_{V,\Psi,i}$-strategy $\gamma \supseteq \beta\hat{\ }\langle 0\rangle$ such that γ has a (permanent) computation $\Psi^W(x_\gamma)\downarrow = 0 \neq 1 = B(x_\gamma)$ (i.e. γ is eventually permanently in state *waitW*). So $\mathcal{P}_{V,\Psi}$ is clearly satisfied.

So we may assume that $\beta\hat{\ }\langle 0\rangle \subset f$. By Definitions 1 and 2, for each $i \in \omega$ fix the $\mathcal{P}_{V,\Psi,i}$-strategy γ_i with $\beta \subset \gamma_i \subset f$. By Lemma 2(i), each γ_i is initialized at most finitely often and will thus eventually have a permanent witness $x_{\gamma_i} = x_i$ (unless some $\mathcal{P}_{V,\Psi,i}$-strategy $\hat{\gamma}_i <_L \gamma_i$ acts for it). We will show that $\Theta^W(i)\downarrow = V(i)$ for almost all i. Since $\Psi^W = B$ and $\beta\hat{\ }\langle 0\rangle \subset f$, each γ_i will eventually be permanently in state *waitV* or permanently in state *stop* or permanently in state *init*. In the last case some $\mathcal{P}_{V,\Psi,i}$-strategy $\hat{\gamma}_i$ is permanently linked to β and in state *waitV* or state *stop*. We distinguish two cases:

Case 1: γ_i (or $\hat{\gamma}_i$) is eventually permanently in state *waitV*: Then $\Theta^W(i)$ must be defined since almost every time γ_i is eligible to act it attempts to redefine $\Theta^W(i)$ with fixed use s_0 (= the last stage γ_i enters state *waitV*), or else some $\mathcal{P}_{V,\Psi,i}$-strategy $< \gamma_i$ permanently defined $\Theta^W(i)$. Suppose $\Theta^W(i)$ is permanently defined at stage s_1 by some $\mathcal{P}_{V,\Psi,i}$-strategy γ and i enters V at some stage $s_2 > s_1$. Then $\Psi^W(x_\gamma)$ cannot be defined at stage s_1 with a permanent computation, and so eventually γ_i (or $\hat{\gamma}_i$) will proceed to state *stop* forever.

Case 2: γ_i (or $\hat{\gamma}_i$) is eventually permanently in state *stop*: Then $i \in V$. If $\Theta^W(i)$ is ever set to 1 with use 0, we are done. Again suppose some $\mathcal{P}_{V,\Psi,i}$-strategy γ defines $\Theta^W(i) = 0$ at some stage s_1, and this computation is valid at stage $s_2 > s_1$ when i enters V. Then γ_i (or $\hat{\gamma}_i$) will eventually proceed to state *stop* directly from state *waitV* (at a stage $s_3 \geq s_2$, say) with witness $x = x_{\gamma_i}$ (or $x_{\hat{\gamma}_i}$) $= x_\gamma[s_1]$ since $\Delta^{V\oplus W}(x)$ is defined at stage s_3. We will show that this can happen for at most finitely many i, namely that this cannot happen if $s_1 \geq s_\delta$ (as defined in Lemma 1(i)) for all $\mathcal{N}_\Phi$-strategies $\delta \subset \beta$.

Let $\alpha \subset \beta$ be the $\mathcal{P}_V$-strategy, let s_4 be the least stage $\geq s_2$ at which α is eligible to act. Then at any stage $s \in [s_4, s_3)$, there will be a request