COUNTABLE AND FINITARY REDUCTIONS ON EQUIVALENCE RELATIONS

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Abstract. Inspired by the very successful study of Borel equivalence relations under Borel reducibility in descriptive set theory and equivalence relations on ω under computable reducibility in computability theory, R. Miller defined a family of reducibility notions. Defined on equivalence relations on Baire space or Cantor space, these reducibilities are required to succeed (uniformly) on all finite or countable subsets of the whole space. In this paper, we combine methods from computability theory and descriptive set theory to study equivalence relations under these reductions. In particular, we show existence and non-existence results of complete equivalence relations in various settings.

1. Introduction

1.1. **Questions.** To illustrate the questions to be addressed in this article, consider the well-known results of Hjorth and Thomas stating that the isomorphism problem TFAb_{r+1} for torsion-free abelian groups of finite rank $r+1$ has no Borel reduction to the corresponding problem TFAb_r for rank r. This theorem formalized a general sense that the problem for rank 1 had specific invariants for which no analogues existed in rank 2, and extended that intuition to the ranks r and $r+1$ in general. We state it here in full.

Theorem 1 ([2, 9]). *Let F be a Borel function such that, given G in TFAb_{r+1} (by naming the Gödel codes of atomic formulas true in that group), $F(G) \in 2^\omega$ similarly codes the atomic diagram of a group in TFAb_r. Then it is impossible to have*

$$(\forall G, G' \in \mathrm{TFAb}_{r+1})\ [G \cong G' \iff F(G) \cong F(G')].$$

From the point of view of computable structure theory, this theorem is particularly surprising. The isomorphism problem for computable torsion-free abelian

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groups of rank r is always Σ_3-complete, for every single $r > 0$. The following result, although technical, is not difficult to prove on its own. (Here we use the usual indexing $\varphi_0, \varphi_1, \ldots$ of all partial computable functions.)

Theorem 2. *There exists a computable partial function ψ such that*

- *whenever φ_e is the characteristic function of the atomic diagram of a torsion-free abelian group of any rank $r+1 \geq 2$ on the domain ω, then ψ computes an index $j = \psi(r+1, e)$ such that φ_j is the characteristic function of the atomic diagram of a torsion-free abelian group of rank r; and*
- *whenever e and e' are two such indices (for two computable torsion-free abelian groups G, G' of the same rank $r+1$), the groups H and H' with indices $\psi(r+1, e)$ and $\psi(r+1, e')$ satisfy*

$$G \cong G' \iff H \cong H'.$$

Moreover, the computability of the groups in question (i.e., the fact that their atomic diagrams are decidable) is not the decisive difference between Theorems 1 and 2. In [3], the latter theorem is adapted to give a Turing functional Ψ which accepts any join $\oplus_{n \in \omega} G_n$ of atomic diagrams of groups in TFAb_{r+1} as an oracle and, given the rank $r+1$ of all of these groups, outputs

$$\Psi^{(\oplus_n G_n)}(r+1) = H_0 \oplus H_1 \oplus H_2 \oplus \cdots,$$

an infinite sequence of groups in TFAb_r satisfying

$$(\forall m \forall n)\ [G_m \cong G_n \iff H_m \cong H_n].$$

Thus, it is possible to reduce the isomorphism problem for arbitrary countable collections of groups in TFAb_{r+1} to the same problem in TFAb_r — and moreover, the reduction is not merely Borel, but actually computable. This indicates that somehow the impossibility result in Theorem 1 relies on the uncountability of the space TFAb_{r+1} of all torsion-free abelian groups of rank $r+1$, thus stemming from semantic or even set-theoretic issues rather than syntactic ones: computable reductions are possible for arbitrary countable collections of elements from the field of the first equivalence relation, in a uniform way, but no computable reduction — indeed, no Borel reduction! — can handle all (uncountably many) of those elements at once.

(From the set-theoretic side, imagine that one extends the model $\mathfrak{N}$ of **ZFC** to a model $\mathfrak{M}$ in which the cardinal 2^ω of $\mathfrak{N}$ collapses to ω. Then, in that newly extended universe $\mathfrak{M}$, there would be a computable reduction that succeeded in reducing the original class $(\mathrm{TFAb}_{r+1})^{\mathfrak{N}}$ to TFAb_r, although of course in $\mathfrak{M}$ each class $(\mathrm{TFAb}_{r+1})^{\mathfrak{M}}$ and $(\mathrm{TFAb}_r)^{\mathfrak{M}}$ would contain many more groups than the original classes in $\mathfrak{N}$ did. This suggests that the failure of $\mathfrak{N}$ to contain a computable reduction was due to the inability of $\mathfrak{N}$ to line up all its ducks in a row, i.e., to put its TFAb_{r+1} in bijection with ω.)

1.2. **Definitions.** Some first steps in examining the distinction between countable and full reductions on Cantor space 2^ω, more generally than just for torsion-free abelian groups, were taken in [6]. In this article, we will continue that work and extend some of the results to Baire space ω^ω as well. To begin with, we recall the following definitions, most of which appear in [6, §1].

Suppose that E and F are equivalence relations on the domains S and T, respectively. A *reduction* of E to F is a function $g : S \to T$ satisfying the property:

$$(\forall x_0, x_1 \in S)\ [x_0\ E\ x_1 \iff g(x_0)\ F\ g(x_1)].$$

If $S = T = \omega$, then it is natural to speak of a *computable reduction*: this simply means that the function g is (Turing-)computable. For equivalence relations on Polish spaces, computability has a related meaning.

Definition 3. *Let E and F be equivalence relations on 2^ω. A* computable reduction *of E to F is a reduction $g : 2^\omega \to 2^\omega$ given by a computable function Φ (that is, an oracle Turing functional) on the reals involved:*

$$(\forall A \in 2^\omega)(\forall x \in \omega)\ g(A) = \Phi^A.$$

If such a reduction exists, then E is computably reducible to F, *denoted $E \leq_0 F$. Equivalence relations on ω^ω may be treated in exactly the same way.*

Borel reductions on these equivalence relations simply require that the relevant function Φ be a Borel function, rather than being given by an oracle Turing machine. Thus Theorem 1 really said that there is no Borel reduction from the isomorphism relation on TFAb_{r+1} to the same relation on TFAb_r. Borel reductions may be stratified as follows.

Definition 4. *Let E and F be equivalence relations on 2^ω. A* jump-reduction *of E to F is a reduction $g : 2^\omega \to 2^\omega$ given by a computable function Φ (that is, an oracle Turing functional) on the jumps of the reals involved:*

$$(\forall A \in 2^\omega)\ g(A) = \Phi^{(A')}.$$

An α-jump-reduction *of E to F is a reduction $g : 2^\omega \to 2^\omega$ given by a computable function Φ (that is, an oracle Turing functional) on the α-jumps of the reals involved:*

$$(\forall A \in 2^\omega)\ g(A) = \Phi^{(A^{(\alpha)})}.$$

Definition 4 becomes awkward when the countable ordinal α in question is not computable. We resolve this by allowing a fixed real $O \subseteq \omega$ in the oracle, which is allowed to be used to compute a specific presentation of α and then to aid the computation as well. Thus we pass from the so-called *lightface hierarchy* to the *boldface hierarchy*, as is common in descriptive set theory. Doing so brings the topological notion of continuity into play.

Definition 5. *Let E and F be equivalence relations on 2^ω. A* continuous reduction *of E to F is a reduction $g : 2^\omega \to 2^\omega$ given by a computable function Φ (that is, an oracle Turing functional) with an additional (arbitrary, but fixed) oracle set $O \subseteq \omega$:*

$$(\forall A \in 2^\omega)\ g(A) = \Phi^{(O \oplus A)}.$$

Similarly, a (boldface) α-jump-reduction *of E to F is a reduction $g : 2^\omega \to 2^\omega$ given by a computable function Φ (that is, an oracle Turing functional), with an additional fixed oracle O, on the α-jumps of the reals involved:*

$$(\forall A \in 2^\omega)\ g(A) = \Phi^{\left((O \oplus A)^{(\alpha)}\right)}.$$

All of the reductions described in Definitions 3, 4, and 5 are *full reductions*: they accept as input any element of S (the domain of E). In Subsection 1.1, we considered functions that could accept arbitrary countable collections of groups from TFAb_{r+1} as input, outputting one group in TFAb_r corresponding to each group from the countable collection. These are not "full" in the sense above. They are formalized as "countable reductions" in the following definition.

Definition 6. *For equivalence relations E and F on domains S and T, and for any cardinal $\mu \leq |S|$, we say that a function $g : S^\mu \to T^\mu$ is a* μ-ary reduction *of E to F if, for every $\vec{x} = (x_\alpha)_{\alpha \in \mu} \in S^\mu$, we have*

$$(\forall \alpha < \beta < \mu)\ [x_\alpha\ E\ x_\beta \iff g_\alpha(\vec{x})\ F\ g_\beta(\vec{x})],$$

where $g_\alpha : S^\mu \to T$ are the component functions of $g = (g_\alpha)_{\alpha<\mu}$. For limit cardinals μ, a related notion applies with $<\mu$ in place of μ: a function $g : S^{<\mu} \to T^{<\mu}$ which restricts to a ν-ary reduction of E to F for every cardinal $\nu < \mu$ is called a $(<\mu)$-ary reduction. *For $\mu = \omega$, an ω-ary reduction is called a* countable reduction, *and a $(<\omega)$-ary reduction is called a* finitary reduction.

When $S = T = 2^\omega$ and the μ-ary reduction g is computable, we write $E \leq_0^\mu F$, with the natural adaptation $E \leq_\alpha^\mu F$ for α-jump μ-ary reductions. Likewise, when a $(<\mu)$-ary reduction g is α-jump computable, we write $E \leq_\alpha^{<\mu} F$.

When $\alpha > 0$, it is important to note that $\Phi^{((\vec{x})^{(\alpha)})}$ is required to equal $g(\vec{x})$. This oracle provides more information than we would have had if we had required $\Phi^{((x_0^{(\alpha)} \oplus x_1^{(\alpha)} \oplus \cdots)} = g(\vec{x})$, with the jumps of the individual inputs taken separately.

In this paper, when we speak of computable μ-ary reductions, we will also assume that $\mu \leq \omega$. For μ-ary reductions with $\mu \leq \omega$, the oracle is a single real whose columns are the μ-many inputs to the reduction. In a $(<\omega)$-ary reduction, with an input $\vec{A} \in (2^\omega)^n$, the oracle is officially equal to $\{n\} \oplus A_0 \oplus \cdots \oplus A_{n-1} \oplus \emptyset \oplus \emptyset \oplus \cdots$, meaning that the oracle does specify the size of its tuple. (However, we usually gloss over this issue and just write $A = A_0 \oplus \cdots \oplus A_{n-1}$ as the oracle.)

For equivalence relations on ω, countable reducibility is equivalent to full reducibility, since a countable reduction can simply be given the entire set ω as its (countable) input. Finitary reducibility for equivalence relations on ω was first studied by Miller and Ng in [7]. Notice that full reducibility always implies countable and finitary reducibility, just by applying the full reduction to each element of the countable or finite set of inputs.

In practice, the reason why countable reductions can be computable when full reductions (on $T = 2^\omega$ or $T = \omega^\omega$) are not usually involves the ability to "play off" individual elements of T against each other. Given a countable set $\oplus_n X_n$ from T as input, some portion of the output $\oplus_n Y_n$ is devoted to ensuring that if $Y_0\ F\ Y_1$ only if $X_0\ E\ X_1$, and similarly for each other pair (i,j) from ω^2. Thus each X_j actually has only countably many requirements to fulfill. In contrast, in a full reduction (on an uncountable space T), no such strategy is possible. Several of our results on countable reducibility will mirror known results about equivalence relations on ω, where even a full reduction can play each pair of individual elements off against each other.

Very often, the nature of this "playing off" is that, for each X_j, we construct Y_j worrying only about the finitely many pairs (i,j) with $i < j$, building Y_j to satisfy each of those requirements. Reflecting on this phenomenon, we developed another definition to describe it: *sequential computable reducibility*, in which a Turing functional Φ constructs each $Y_j = \Phi^{X_0 \oplus \cdots \oplus X_j}$, without access to those X_k with $k > j$. This appears as Definition 7 below.

1.3. **Summary of the paper.** The rest of the paper is organized as follows: In Section 2, we discuss the sequential reduction and show that the Vitali equivalence relation E_0 is Σ^0_2-complete under sequential computable reductions. In Section 3, we consider the relation between the descriptive complexity of equivalence relations and the μ-reduction. In Section 4 we investigate the Louveau jump of an equivalence relation with respect to a filter. Finally, in Section 5, we focus on the existence and non-existence of complete equivalence relations in different complexity classes and under different reducibilities.

2. Sequential reduction

Recall the Vitali equivalence relation E_0 (on 2^ω or ω^ω):

$$X \; E_0 \; Y \iff (\exists n)(\forall m \geq n) \; X(m) = Y(m).$$

In [6, Proposition 2.2], it is shown that E_0 is complete under $\leq^\omega_0$ among Σ^0_2-definable equivalence relations on 2^ω. We can actually require the reduction to be *sequential*.

Definition 7. *We write $\leq^\mu_c$ for continuous μ-reductions (or $\leq^\mu_k$ for k-computable μ-reductions), meaning Φ is continuous (or computable from $(\bigoplus_{i<\mu} X_i)^{(k)}$, respectively).*

If Φ, when computing the i-th output Y_i, only uses the first $i+1$ inputs $X_0, \ldots, X_i$, namely, $Y_i = \Phi_i(\bigoplus_{j \leq i} X_j)$, then we say the reduction is sequential *and write $\leq^{\mu,s}_c$.*

Theorem 8. *E_0 is complete under $\leq^{\omega,s}_0$ among Σ^0_2-definable equivalence relations on 2^ω. Relativizing, we also have that E_0 is complete under $\leq^{\omega,s}_c$ among $\boldsymbol{\Sigma}^0_2$-definable equivalence relations on 2^ω.*

Proof. Let E be a Σ^0_2-equivalence relation such that AEB iff $(\exists k)(\forall n)\varphi(k, n, A, B)$ where φ is computable. Given $X_0, X_1, \ldots, X_i$, we need to define Y_i, which is a countable reduction. We will actually have $Y_i \in \omega^\omega$, but since E_0 on ω^ω computably reduces to E_0 on 2^ω, such a reduction suffices.

At each stage, we will define some j_s and k_s. Intuitively, j_s is the current $j < i$ such that we are checking if $X_i E X_j$; and k_s is the k that we are checking if it is a (partial) witness of the Σ^0_2-outcome. We start with $j_0 = k_0 = 0$. At each stage s, we check if

$$(\forall n \leq s)\varphi(k_s, n, X_i, X_{j_s}). \tag{1}$$

- If yes, we set $Y_i(s) = Y_{j_s}(s)$, and $j_{s+1} = j_s$, $k_{s+1} = k_s$.
- If no, we set $Y_i(s) = i$. We also set $(j_{s+1}, k_{s+1}) = (j_s + 1, k_s)$ if $j_s \neq i - 1$; and $(j_{s+1}, k_{s+1}) = (0, k_s + 1)$ if $j_s = i - 1$.

The intuition is that we try to guess if X_i is E-equivalent to some X_j by checking the Σ^0_2-outcome up to $n \leq s$ and some guesses at $j = j_s$ and $k = k_s$. If X_i is not equivalent to any earlier X_j, then for every guess (j_s, k_s), it will eventually fail at some stage and we will have infinitely many $Y_i(s) = i$, making Y_i not E_0-equivalent to any earlier Y_j. If it is, then after a certain point we would have guessed the right (j_s, k_s), and so Y_i will copy Y_{j_s} from that point on, so they must be E_0-equivalent.

We now verify that the reduction works. Suppose first that $X_i \not\!E \; X_j$ for every $j < i$. Then for every (j, k), there is some s such that Equation (1) fails on $(j_s, k_s) = (j, k)$. This will give $Y_i(s) = i$ for such s. As (j_s, k_s) cycles through all possible guesses, we will have $Y_i(s) = i$ for infinitely many s. But by induction, $Y_j(s) \leq j < i$ for every $j < i$, so we must have $Y_i \not\!E_0 \; Y_j$ for every $j < i$.

On the other hand, suppose $X_i E X_j$ for some $j < i$. We will prove that there is some s such that for every $s' > s$, $Y_i(s') = Y_j(s')$. Pick the j with $X_i E X_j$ and with the corresponding k least (if for several such j, the corresponding k is the same, then pick the least j for that k). As any (k', j') lex-smaller than (k, j) is not a correct guess, there will be a stage s such that we land on the correct guess $(j, k) = (j_s, k_s)$. Then this guess will stabilize, and for any $s' > s$, we will have $Y_i(s') = Y_{j_{s'}}(s') = Y_j(s')$. Now by induction, if j' is the least member of $[i]_E$, then there is some t such that for every $t' > t$, we have $Y_i(t') = Y_{j'}(t')$. In particular, we will have $Y_i E_0 Y_j$ for every $j < i$ with $X_i E X_j$.

So we have $X_i E X_j$ if and only if $Y_i E_0 Y_j$, as needed. □

Recall that E_∞ is the universal countable Borel equivalence relation. When $\mathbb{F}_2$ is the free group on two generators, E_∞ can be represented as the shift action of $\mathbb{F}_2$ on the space $2^{\mathbb{F}_2}$. It is a Σ^0_2-equivalence relation, and is universal in the sense that every countable Borel equivalence relation Borel reduces to it (in fact, embeds in it, i.e., by a one-to-one reduction). In fact, if E is an equivalence relation on a 0-dimensional Polish space given as the orbit equivalence relation from the continuous action of a countable group, then E continuously embeds into E_∞. (By the Feldman–Moore theorem, every countable Borel equivalence relation on a Polish space is the orbit equivalence relation of a Borel action of a countable group.) The reader can consult [8] for further details.

Corollary 9. $E_\infty \leq^\omega_0 E_0$. □

From the Harrington–Kechris–Louveau theorem we have that if F is a non-smooth equivalence relation then E_0 continuously embeds into F. So, from the previous corollary and preceding remarks we have the following

Corollary 10. *If E is a countable Borel equivalence relation induced by the action of a countable group on a 0-dimensional space, and F is a non-smooth equivalence relation, then $E \leq^{\omega,s}_c F$.* □

Note that the condition of 0-dimensionality seems to be needed, since otherwise E may be induced by the action of $\mathbb{Z}$ on $\mathbb{R}$, making every continuous map from $\mathbb{R}$ to 2^ω constant, so not a reduction. We implicitly used 0-dimensionality in the proof as we are thinking of continuous as meaning computable relative to an oracle.

The following corollary is immediate from Theorem 8.

Corollary 11. *If E is the orbit equivalence relation induced by the shift action of a computable group G on 2^G, then $E \leq^{\omega,s}_0 E_0$.* □

We do not know if the computability of G is necessary here. The following question would yield an obstacle to this:

Question 12. *Is there a recursively presented group G such that the orbit equivalence relation of the shift action of G on 2^G is Σ^0_3-complete?*

Note that this question is sensitive to how 2^G is presented. Indeed, if G is not computable, then we cannot find a computable set of words such that every element of G is uniquely presented.

On the other hand, if F is not smooth, then by Harrington–Kechris–Louveau (or Glimm–Effros if F is countable), E_0 must continuously reduce to F, so in particular, we have $E_0 \leq^{\omega,s}_c F$. Thus, we ask the following.

Question 13. *Suppose F is smooth. When do we have $E_0 \leq_c^{\omega,s} F$?*

3. Descriptive complexity and μ-reduction

Every (computable) μ-reduction (for $\mu \geq 2$) naturally induces an m-reduction of the equivalence relations. Thus, it is natural to look at the structure of the μ-degrees within the m-degrees:

Question 14. *Are there equivalence relations E and F in the same proper boldface (or lightface, respectively) complexity class that are incomparable under $\leq_c^\omega$ (or $\leq_0^\omega$, respectively)?*

We first observe that 2-reductions are the same as Wadge reductions.

Proposition 15. *$E \leq_c^2 F$ if and only if E Wadge-reduces to F as subsets of $(2^\omega)^2$.* □

We also note that k-computable reductions induce a bound on the descriptive complexity.

Remark 16. *If $E \leq_k^2 F$, and F is Σ_n^0 (or Π_n^0), then E is Σ_{n+k}^0 (or Π_{n+k}^0, respectively).*

Proposition 17. *Let $\mu \leq \omega$, let $k, \ell \in \omega$, and let E, F, G be equivalence relations. If $E \leq_k^\mu F$ and $F \leq_\ell^\mu G$, then $E \leq_{k+\ell}^\mu G$.* □

For an equivalence relation E on 2^ω or ω^ω, let E^ω be the ω-product of E. That is, E^ω is the equivalence relation on 2^ω or ω^ω given by: $AE^\omega B$ iff $\forall c \in \omega\ A^{[c]} = B^{[c]}$, where $A^{[c]}$ denotes the c-th column of A when A is viewed as an array. (So $k \in A^{[c]}$ iff $\langle k, c\rangle \in A$.)

Proposition 18. *Let E be a Δ_{k+1}^0-equivalence relation. Then $E^\omega \leq_k^\omega =$.*

Proof. We follow the same proof as in [6, Proposition 3.5].

Let $X = \bigoplus_i X_i$. Define Y_i by putting $\langle j, c\rangle \in Y_i$ if and only if $X_i^{[c]} E X_j^{[c]}$, where we think of X_i as an array and $X_i^{[c]}$ is the c-th column of X_i. Note that this is computable from $X^{(k)}$ since E is Δ_{k+1}^0.

To see this is an $\leq_k^\omega$-reduction, first suppose that $X_i E^\omega X_j$. Then we have that for every c, $X_i^{[c]} E X_j^{[c]}$, so $\langle j, c\rangle \in Y_i$. By symmetry, we have $\langle i, c\rangle \in Y_j$, and by definition we have $\langle i, c\rangle \in Y_i$ and $\langle j, c\rangle \in Y_j$. Thus Y_i and Y_j agree on the i-th and j-th columns. For $m \neq i, j$, we have $\langle m, c\rangle \in Y_j \Leftrightarrow X_m^{[c]} E X_j^{[c]} E X_i^{[c]} \Leftrightarrow \langle m, c\rangle \in Y_i$. So Y_i and Y_j agree on each m-th column for $m \neq i, j$ as well, making $Y_i = Y_j$.

If $\neg(X_i E^\omega X_j)$ then there is some c such that $X_i^{[c]} \not\!E\ X_j^{[c]}$. So $\langle j, c\rangle \notin Y_i$ but $\langle j, c\rangle \in Y_j$, and we have $Y_i \neq Y_j$. □

Example 19. *Recall that two reals X and Y are E_3-equivalent if and only if for every k, the k-th column of X is E_0-equivalent to the k-th column of Y. That is, $E_3 = E_0^\omega$.*

Since E_0 is Σ_2^0 (hence Δ_3^0), we have $E_3 = E_0^\omega \leq_2^\omega =$ from the previous proposition.

4. Louveau jumps and μ-reduction

In this section, we consider a definition of Louveau [5, Definition 4] and investigate its interaction with μ-reductions.

Definition 20. *Let $\mathcal{F}$ be a filter and E an equivalence relation. Define $E^{\mathcal{F}}$ to be the* jump of E with respect to $\mathcal{F}$, *defined by $AE^{\mathcal{F}}B$ iff $\{j \mid A_j E B_j\} \in \mathcal{F}$, where $A = \bigoplus_i A_i$ and $B = \bigoplus_i B_i$.*

Proposition 21. *Let $\mu \leq \omega$ and $\mathcal{F}$ a filter. If $E \leq_k^\mu F$, then $E^{\mathcal{F}} \leq_k^\mu F^{\mathcal{F}}$.*

Proof. Suppose Φ is the $\leq_k^\mu$-reduction from E to F, namely, given $X = \bigoplus_{i\in\mu} X_i$, we have $Y = \bigoplus_{i\in\mu} Y_i = \Phi^{X^{(k)}}$ such that $X_i E X_j$ if and only if $Y_i F Y_j$.

We now describe a $\leq_k^\mu$-reduction Ψ from $E^{\mathcal{F}}$ to $F^{\mathcal{F}}$. Given $A = \bigoplus_{i\in\mu} A_i$, denote by $\hat{A}_{\langle i,c\rangle} = A_i^{[c]}$ the columns of A_i. Fixing c, let $\hat{\hat{A}}_c = \bigoplus_{i\in\mu} \hat{A}_{\langle i,c\rangle}$. Let $\hat{\hat{B}}_c = \bigoplus_{i\in\mu} \hat{B}_{\langle i,c\rangle} = \Phi^{\hat{\hat{A}}_c^{(k)}}$, so $\hat{A}_{\langle i,c\rangle} E \hat{A}_{\langle j,c\rangle}$ if and only if $\hat{B}_{\langle i,c\rangle} F \hat{B}_{\langle j,c\rangle}$. For every i and c, define B_i by columns via $B_i^{[c]} = \hat{B}_{\langle i,c\rangle}$. Note that $B = \bigoplus_{i\in\mu} B_i$ is uniformly computable from $A^{(k)}$, and we define $\Psi^{A^{(k)}} = B = \bigoplus_i B_i$.

To show this is indeed a $\leq_k^\mu$-reduction, we consider the following:

$$\begin{aligned} A_i E^{\mathcal{F}} A_j &\Leftrightarrow \left\{c \mid A_i^{[c]} E A_j^{[c]}\right\} \in \mathcal{F} \\ &\Leftrightarrow \left\{c \mid \hat{A}_{\langle i,c\rangle} E \hat{A}_{\langle j,c\rangle}\right\} \in \mathcal{F} \\ &\Leftrightarrow \left\{c \mid \hat{B}_{\langle i,c\rangle} F \hat{B}_{\langle j,c\rangle}\right\} \in \mathcal{F} \\ &\Leftrightarrow \left\{c \mid B_i^{[c]} F B_j^{[c]}\right\} \in \mathcal{F} \\ &\Leftrightarrow B_i E^{\mathcal{F}} B_j \end{aligned}$$

□

Definition 22. *Let* cof *be the cofinite filter, i.e., $S \in \mathrm{cof}$ if and only if S is cofinite. For an equivalence relation E, the* Louveau jump E^{cof} *of E is the jump of E with respect to* cof. *Namely, $AE^{\mathrm{cof}}B$ iff $\{j \mid A_j E B_j\}$ is cofinite, where $A = \bigoplus_i A_i$ and $B = \bigoplus_i B_i$.*

In [5, Theorem 5], Louveau proved that this is strictly increasing for Borel reductions, namely, for any Borel E with at least two classes, $E <_B E^{\mathrm{cof}}$.

Recall that two reals are E_1-equivalent if and only if all but finitely many of their columns are equal, namely, E_1 is identical to $=^{\mathrm{cof}}$.

Corollary 23. *$E_3^{\mathrm{cof}} \leq_3^\omega E_1$. Thus, we have $E_0^{\mathrm{cof}} \leq_0^\omega E_3^{\mathrm{cof}} \leq_3^\omega E_1 \equiv_0^\omega E_0$.*

Proof. Note that $E_3 = E_0^\omega$ and E_0 is Σ_2^0. So by Proposition 18, we have $E_3 \leq_3^\omega =$, and thus $E_3^{\mathrm{cof}} \leq_3^\omega E_1$. □

In [6], the equivalence relations E_{set} and Z_0 are introduced. The relation E_{set} is also known as $E_{=}^{+}$, the Friedman–Stanley jump of $=$. We recall that $AE_{\mathrm{set}}B$ iff $\{A^{[c]} \colon c \in \omega\} = \{A^{[c]} \colon c \in \omega\}$, that is, A and B when viewed as arrays have the same set of columns. Also, for $A, B \in 2^\omega$, AZ_0B iff $A \triangle B$ has density 0, viewing A and B as sets of integers.

Question 24. *In* [6, Theorem 3.9], *it is shown that $E_{\mathrm{set}}, Z_0 \not\leq_0^\omega E_3$. Do we have $E_{\mathrm{set}} \leq_1^\omega E_3$?*

We observe that if E is Σ^0_n, then E^{cof} is Σ^0_{n+2}; and if E is Π^0_n, then E^{cof} is Σ^0_{n+1}.

Conjecture 25. *For every k, there is some equivalence relation E such that $E^{\mathrm{cof}} \not\leq^\omega_k E$.*

Question 26. *Let E be a Σ^0_3-equivalence relation. Does $E^{\mathrm{cof}} \leq^\omega_2 E$?*

5. Complete equivalence relations

What does the degree structure look like for μ-reducibility? In particular, we ask whether there are (natural) equivalence relations that are complete within their complexity class. There are some known results for complete equivalence relations on ω, and E_0 on Cantor space is Σ^0_2-complete under countable reduction, while the isomorphism relation on subgroups of $\mathbb{Q}$ is Σ^0_3-complete under countable reduction.

Question 27. *Let μ be ω, "$< \omega$", or a natural number. Is there an equivalence relation on 2^ω (or ω^ω) that is Σ^0_n- (or Π^0_n-)complete under $\leq^\mu_0$?*

In this section, we will show that there are Σ^0_n-complete equivalence relations under countable reduction (Corollary 45) and Π^0_n-complete equivalence relation under finitary reduction (Theorem 34). However, we will also show that there is no Π^0_n-complete equivalence relation (for $n > 1$) under countable reduction (Theorem 39). For $n = 1$, the situation is somewhat reversed.

5.1. Π^0_1- and Σ^0_1-equivalence relations.

Studying complete equivalence relations at the lowest complexity level, we obtain:

Proposition 28.
1. *The equivalence relation $E_=$ is a Π^0_1-complete equivalence relation under $\leq^\omega_0$ (in fact, even under $\leq^{\omega,s}_0$).*
2. *Each Σ^0_1-equivalence relation on 2^ω (lightface or boldface) has at most finitely many equivalence classes. Thus, there is no Σ^0_1-complete equivalence relation on 2^ω under $\leq^\omega_0$.*
3. *There is a Σ^0_1-complete equivalence relation on ω^ω under $\leq^\omega_0$.*

Proof. (1): Clearly, the equivalence relation $E_=$ is Π^0_1. Let E be any Π^0_1-equivalence relation, say, AEB iff $\forall x R(A,B,x)$ for a computable relation R. Then, given $\bigoplus_i X_i$, we need to effectively build $\bigoplus_i Y_i$ so that for all i and j, $X_i E X_j$ iff $Y_i = Y_j$, and such that Y_i only depends on $\bigoplus_{j\leq i} X_j$. Since E is an equivalence relation, we may assume a slight "speed-up" in R by requiring that for all $i, j < k$,

$$\exists y \leq x\, \neg R(X_i, X_j, y) \text{ and } \forall y \leq x\, R(X_i, X_k, y) \implies \exists y \leq x\, \neg R(X_j, X_k, y).$$

We start by setting $Y_0(m) = 0$ for all m. For Y_{i+1}, define $Y_{i+1}(m) = Y_j(m)$ for the least $j \leq i$ such that $\forall x \leq m\ R(X_j, X_{i+1}, x)$ if such j exists; otherwise set $Y_{i+1}(m) = i + 1$.

To show this is a reduction, suppose $i < j$ and $\neg X_i E X_j$ and $Y_i = Y_j$. Since $\neg X_i E X_j$, there is some x so that $\neg R(X_i, X_j, x)$. On the other hand, $Y_i = Y_j$, so in particular, $Y_i(x) = Y_j(x)$. But $\neg\forall y \leq x\ R(X_i, X_j, y)$, so we must have some $k < i, j$ such that $Y_i(x) = Y_j(x) = Y_k(x)$ and $\forall y \leq x\ R(X_k, X_i, y)$ and $\forall y \leq x\ R(X_k, X_j, y)$. This contradicts the speed-up assumption.

Now suppose $i < j$ and $X_i E X_j$. For any m, we have $\forall x \leq m\ R(X_i, X_j, x)$. For any k, if $\forall x \leq m\ R(X_k, X_i, x)$, then by the speed-up assumption we also have $\forall x \leq m\ R(X_k, X_j, x)$. Thus, we must have either $Y_i(x) = Y_k(x) = Y_j(x)$ for the least $k < i$ with $\forall x \leq m\ R(X_k, X_i, x) = k$ (and hence $\forall x \leq m\ R(X_k, X_j, x)$), or $Y_i(x) = Y_j(x) = i$. So $Y_i = Y_j$.

(2): The equivalence classes of a Σ^0_1-equivalence relation on 2^ω induce a partitioning of the compact space 2^ω into open sets. This is an open cover and therefore must be a finite cover.

(3): Fix a Σ^0_1-complete D on ω (such equivalence relations are called *universal ceers*, see [1]) and use it to define a Σ^0_1-equivalence relation E on ω^ω by setting AEB iff $A(0)DB(0)$. Now let F be any Σ^0_1-equivalence relation on ω^ω. We may assume there is a computable relation R such that AFB iff $\exists m\, \exists n\, \exists x\ R(A \restriction m, B \restriction n, x)$. We can form a c.e. set $S \subseteq \omega^{<\omega} \times \omega^{<\omega} \times \omega$ such that AFB iff there is $(\alpha, \beta, x) \in S$ with $\alpha \prec A$ and $\beta \prec B$. In fact, by possibly extending the lengths of α and β, we may assume that S is computable, and that $\min\{|\alpha|, |\beta|\} \geq x$, so the coordinate x is not needed and S is just a computable set of pairs of strings (α, β).

Suppose we are given $X = \bigoplus_i X_i$. For each i, let $s_i = X_i \restriction n_i$ where n_i is least so that $(s_i, s_i) \in S$. Clearly the map $i \mapsto s_i$ is computable from X_i, and these maps are uniformly computable from X. For any i, j we have $X_i F X_j$ iff there are t_i, t_j extending s_i, s_j, respectively, with $S(t_i, t_j)$. We may identify the set $A = \{s \colon S(s, s)\}$ with ω, and so the equivalence relation on A given by

$$s_1 \sim s_2 \iff (\exists t_1 \supseteq s_1)(\exists t_2 \supseteq s_2)\ S(t_1, t_2)$$

is identified with a ceer. The map $X = \bigoplus X_i \mapsto Y = \bigoplus s_i$ is a $\leq^\omega_0$ reduction of F to this ceer. So, we can reduce F to D and thus to E. □

With the completeness results at the Σ^0_1- and Π^0_1-levels, one wonders whether there is a jump-like operator that preserves completeness. The operator $E \mapsto E^\omega$ and the Louveau jump $E \mapsto E^{\text{cof}}$ are possible candidates but we do not know if they preserve completeness. Note that if E is Σ^0_α (or Π^0_α), then E^ω is $\Pi^0_{\alpha+1}$ (or Π^0_α, respectively), and E^{cof} is $\Sigma^0_{\alpha+2}$ (or $\Sigma^0_{\alpha+1}$, respectively).

5.2. **Projections of equivalence relations.** One possibility is to look at projections. Recall from [6, Definition 4.1] that for an equivalence relation E, the equivalence relation E^{ec} is defined by $AE^{\text{ec}}B$ iff A^pEB^p, where A^p is defined by $A^p(m) = 1$ iff $A^{[m]} \neq \emptyset$, and $A^p(m) = 0$ otherwise.

Proposition 29. *If E is a Π^0_n-equivalence relation (Σ^0_n-equivalence relation, respectively), then E^{ec} is a Π^0_{n+1}-equivalence relation (Σ^0_{n+1}-equivalence relation, respectively).*

Proof. Suppose E is Π^0_n (the other case is similar), say, AEB iff

$$\forall x_1 \exists x_2 \dots Q_n x_n R(A, B, \overline{x})$$

for a computable relation R. Then $AE^{\text{ec}}B$ iff A^pEB^p iff

$$\forall x_1 \exists x_2 \dots Q_n x_n R(A^p, B^p, \overline{x}).$$

Now $R(A^p, B^p, \overline{x})$ is a Δ^0_2-relation in the parameters A, B and $\overline{x}$, so $AE^{\text{ec}}B$ is Π^0_{n+1}. □

Remark 30. *In the definition of E^{ec}, the extra quantifier is on the inside, as opposed to E^ω or E^{cof}, where the extra quantifier is on the outside.*

This naturally begs

Question 31. *Let μ be ω, "$<\omega$", or a natural number. If E is a Π^0_n-complete equivalence relation (Σ^0_n-complete equivalence relation, respectively) on Cantor space under $\leq^\mu_0$, then must E^{ec} be a Π^0_{n+1}-complete equivalence relation (Σ^0_{n+1}-complete equivalence relation, respectively) under $\leq^\mu_0$?*

As we will see later, Theorem 39 gives a negative answer to this question on the Π^0_n-side under computable countable reduction.

We let $E^{\mathrm{ec}}_=$ abbreviate $(E_=)^{\mathrm{ec}}$, that is, it is the equivalence relation given by Definition 31 starting from the equality equivalence relation $E = E_=$ on Cantor space. We give a direct proof that $E^{ec}_=$ is Π^0_2-complete for finitary reductions.

Theorem 32. *If E is a Π^0_2-equivalence relation on ω^ω, then $E \leq^{<\omega}_0 E^{\mathrm{ec}}_=$.*

Proof. Fixing $n \in \omega$ and $X_0, \dots, X_{n-1} \in \omega^\omega$, we effectively compute from them $Y_0, \dots, Y_{n-1} \in \omega^\omega$ such that for all $i < j < n$, $X_i E X_j$ iff $Y_i E^{\mathrm{ec}}_= Y_j$. As E is Π^0_2, we have AEB iff $\forall m\ \exists n\ R(A, B, m, n)$ where R is computable. Say that, for distinct $i, i' < n$, the pair (i, i') *receives a token at step k* if

$$\max\{m \leq k \mid \forall m' \leq m\ \exists n \leq k\ R(X_i, X_{i'}, m', n)\} > \max\{m \leq k-1 \mid \forall m' \leq m\ \exists n \leq k-1\ R(X_i, X_{i'}, m', n)\}.$$

(Note that (i, i') will receive tokens at infinitely many steps iff $X_i E X_{i'}$.) Let Y^j_i for $i < n$ and $j \in \omega$ refer to the jth column of the real Y_i we are building. We start with initial approximations $Y^j_{i,0}$ to the reals Y^j_i. To define these, let $A_0, \dots, A_{n-1}$ partition ω with each A_i infinite and computable. Let $Y^j_{i,0}(0) = 1$ iff $j \in A_i$. At step $k > 0$, assume $Y^j_{i,k-1}(\ell)$ has been defined for all $i < n$, all $j \in \omega$, and all $\ell < k$. Fix $i < n$ and define $Y^j_{i,k}(k)$ for all $j \leq k$ as follows: For each $i' < n$ with $i' \neq i$, if (i, i') receives a token at step k, then $Y^j_{i,k}(k) = 1$ iff $\exists \ell \leq k\ Y^j_{i',k-1}(\ell) = 1$; otherwise set $Y^j_{i,k}(k) = 0$. For all other j and k, we let $Y^j_{i,k}(k) = Y^j_{i,k-1}(k-1)$. Let $Y^j_i = \bigcup_k Y^j_{i,k}$. This completes the definition of the Y^j_i. Clearly, the map $(X_0, \dots, X_{n-1}) \mapsto (Y_0, \dots, Y_{n-1})$ is computable.

Suppose $i < i' < n$ and $X_i E X_{i'}$. So there are infinitely many steps k at which (i, i') receives a token. At such a step k, we have that for all $j \leq k$, $Y^j_{i,k}(k) = 1$ iff $Y^j_{i',k-1}(\ell) = 1$ for some $\ell \leq k$. Since there are infinitely many such k, we have for all j that if $Y^j_{i'}(\ell') = 1$ for some ℓ', then $Y^j_i(\ell) = 1$ for some ℓ. A symmetrical argument shows that if $Y^j_i(\ell) = 1$ for some ℓ then $Y^k_{i'}(\ell') = 1$ for some ℓ'. So $\exists \ell\ Y^j_i(\ell) = 1$ iff $\exists \ell'\ Y^j_{i'}(\ell') = 1$. Since this holds for all columns j, we have $Y_i E^n_= Y_{i'}$.

Suppose that $i < i' < n$ and $\neg X_i E X_{i'}$. Then for some k_0 and all steps $k \geq k_0$, tokens are only added to (i, p) for p with $X_p E X_i$ and likewise only added to (i', q) for $X_q E A_{i'}$. This implies that for $j \geq k_0$ with $j \in A_i$, $Y^j_{i',k}(\ell) = 0$ for all ℓ. So, $Y^j_{i'}(\ell) = 0$ for all ℓ, but $A^j_i(0) = 1$ since $j \in A_i$. So, $\neg Y_i E^{\mathrm{ec}}_= Y_j$. □

We next generalize Theorem 32 to arbitrary levels Π^0_m, for $m \geq 2$. For $A \in 2^\omega$, and for $j_1, \dots, j_m \in \omega$, let $A^{j_1,\dots,j_m} \in 2^\omega$ be given by

$$A^{j_1,\dots,j_m}(j) = A(\langle j, j_1, \dots, j_m \rangle),$$

where $(j, j_1, \dots, j_m) \mapsto \langle j, j_1, \dots, j_m \rangle$ is a recursive bijection between ω^{m+1} and ω.

Definition 33. *Let E be an equivalence relation on 2^ω. We define the equivalence relation E^{ec_m} on 2^ω by $Y_1 E^{\mathrm{ec}_m} Y_2$ iff $y_1 E y_2$ where $y_1(j) = 1$ iff*

$$\exists j_1\, \forall j_2\, \cdots\, Q j_m\, Y_1^{j_1,\dots,j_m}(j) = 1,$$

and similarly for y_2.

As before, we let $E_=^{\mathrm{ec}_m}$ abbreviate $(E_=)^{\mathrm{ec}_m}$, that is, it is the equivalence relation given by Definition 33 starting from the equality equivalence relation $E = E_=$. Note that $E^{\mathrm{ec}_1} = E^{\mathrm{ec}}$.

Theorem 34. *For all $m \geq 2$, for any Π^0_{m+1}-equivalence relation E on $X \leq \omega^\omega$, we have $E \leq_0^{<\omega} E_=^{\mathrm{ec}_m}$.*

Proof. Fix a computable R so that for all $A, B \in \omega^\omega$, we have AEB if and only if $\forall a\, \exists a_1\, \cdots\, Q a_m\, R(A, B, a, a_1, \dots, a_m)$. By modifying R, we may assume that the first two quantifiers are $\exists^\infty \exists$ instead of $\forall\exists$, namely, we have AEB iff

$$\exists^\infty a\, \exists a_1\, \cdots\, Q a_m\, R(A, B, a, a_1, \dots, a_m).$$

Fix $n \in \omega$ and $X_0, \dots, X_{n-1} \in \omega^\omega$ and we effectively compute $Y_0, \dots, Y_{n-1} \in 2^\omega$ from these such that for all $i < i' < n$, we have $X_i E X_{i'}$ iff $Y_i E_=^{\mathrm{ec}_m} Y_{i'}$.

First, set $Y_i(j, 0, j_2, \dots, j_m) = 1$ iff $j \in A_i$, where $A_0, \dots, A_{n-1}$ partition ω and each A_i is infinite and computable (as in the proof of Theorem 32). It might be helpful to note that in the following definition, the j and k correspond to the first two of the $m+1$ variables of R, which defines the Π^0_{m+1}-relation E. Set

$$\begin{aligned} &Y_i(j, \langle j', k, i', j_1\rangle, j_2, \dots, j_m) = 1 \\ \leftrightarrow\ &(j \leq j') \wedge (i' \neq i) \wedge (0 \leq i, i' \leq n-1) \wedge S_i(\langle j, j', k, i', j_1\rangle, j_2, \dots, j_m) \end{aligned}$$

where S_i is computable and such that $\forall j_2 \cdots Q j_m\ S_i(\langle j, j', k, i', j_1\rangle, j_2, \dots, j_m)$ iff

$$\forall j_2 \cdots Q j_m\ Y_{i'}(j, j_1, j_2, \dots, j_m) = 1 \text{ and } \forall j_2 \cdots Q j_m\ R(X_i, X_{i'}, j', k, j_2, \dots, j_m).$$

This can be done since the two conditions are both Π^0_{m-1}, which is closed under conjunction. (We assume here that our coding of tuples is such that $j_1 < \langle j, j', k, i', j_1\rangle$.) Now, by induction on the second argument, it easily follows that this definition of Y_i is well-defined, and also $Y_0, \dots, Y_{n-1}$ are computable from $X_0, \dots, X_{n-1}$.

Fixing $i \neq i'$ with $0 \leq i, i' \leq n-1$, we show that $X_i E X_{i'}$ iff $Y_i E_=^{\mathrm{ec}_m} Y_{i'}$.

Suppose first that $X_i E X_{i'}$, so $\exists^\infty j\, \exists j_1\, \cdots Q j_m\, R(X_i, X_{i'}, j, j_1, \dots, j_m)$. Fix any j such that $\exists j_1 \forall j_2 \cdots Q j_m\ Y_{i'}(j, j_1, \dots, j_m) = 1$; we will show that

$$\exists j_1 \forall j_2 \cdots Q j_m\ Y_i(j, j_1, \dots, j_m) = 1.$$

(The proof for the reverse direction is symmetric.) Fix $j' \geq j$ and k such that $\forall j_2 \cdots Q j_m\ R(X_i, X_{i'}, j', k, j_2, \dots, j_m)$. Fix j_1 such that

$$\forall j_2 \cdots Q j_m\ Y_{i'}(j, j_1, j_2, \dots, j_m) = 1.$$

For these fixed values of j, j', k, i', j_1, we then have by the definition of Y_i that

$$\forall j_2 \cdots Q j_m\ Y_i(j, \langle j', k, i', j_1\rangle, j_2, \dots, j_m) = 1.$$

Conversely, suppose that $i < i' < n$ and $\neg X_i E X_{i'}$. Fix some j so that $j \in A_i$ and for any $0 \leq p, q \leq n-1$ with $\neg X_p E X_q$, no $j' \geq j$ is a "false witness", i.e., $(\forall j' \geq j)\neg(\exists k \forall j_2 \cdots Q j_m\ R(X_p, X_q, j', k, j_2, \dots, j_m))$. Therefore, for any $j' \geq j$ and any k, we have $\neg\forall j_2 \cdots Q j_m\ R(X_p, X_q, j', k, j_2, \dots, j_m))$.

As $j \in A_i$, by definition, we have that $Y_i(j, 0, j_2, \ldots, j_m) = 1$ for all $j_2, \ldots, j_m$, so $\exists j_1 \forall j_2 \cdots Qj_m\, Y_i(j, j_1, j_2, \ldots, j_m) = 1$. We will show $\neg Y_i E_{=}^{\mathrm{ec}_m} Y_{i'}$ by showing that $\neg \exists j_1 \forall j_2 \cdots Qj_m\, Y_{i'}(j, j_1, j_2, \ldots, j_m) = 1$.

For each $p \leq n-1$, by definition of Y_p, if $\forall j_2 \cdots Qj_m\, Y_p(j, t, j_2, \ldots, j_m) = 1$, then t must be of the form $t = \langle j', k, q, j_1 \rangle$ for some $j' \geq j$ and $q \neq p$ with $0 \leq q \leq n-1$ and $\forall j_2 \cdots Qj_m\, S_p(\langle j, j', k, q, j_1 \rangle, j_2, \ldots, j_m)$. By definition of S_p, this implies that $\forall j_2 \cdots Qj_m\; R(X_p, X_q, j', k, j_2, \ldots, j_m)$. Finally, by the choice of j, we have that $X_p E X_q$. However, the recursive definition of Y shows $Y_i(j, 0, j_2, \ldots, j_m) = 1$ iff $j \in A_i$, so by induction, if $\forall j_2 \cdots Qj_m\, Y_p(j, t, j_2, \ldots, j_m) = 1$ for some t, then we must have $X_p E X_i$. As we have $\neg X_i E X_{i'}$, this shows that

$$\neg \exists j_1 \forall j_2 \cdots Qj_m\, Y_{i'}(j, j_1, j_2, \ldots, j_m) = 1.$$

□

Here is an analogous definition on Baire space, together with related propositions and questions.

Definition 35.
- *For $A \in \omega^\omega$, define $A^p(m) = \min_n A^{[m]}(n)$.*
- *Let E be an equivalence relation on ω^ω. Define $A E^{\mathrm{ec}} B$ iff $A^p E B^p$.*

We have the analogous proposition and questions.

Proposition 36. *If E is a Π^0_n-equivalence relation (Σ^0_n-equivalence relation, respectively) on ω^ω, then E^{ec} is a Π^0_{n+1}-equivalence relation (Σ^0_{n+1}-equivalence relation, respectively).* □

The projections of the equality relations $=_C$ on Cantor space and $=_B$ on Baire space turn out to be bi-reducible.

Theorem 37. $(=_B)^{\mathrm{ec}} \equiv (=_C)^{\mathrm{ec}}$.

Proof. The reduction $\geq$ is clear. The idea for $\leq$ is that for every column i in X, we use countably many columns $\langle i, j \rangle$ in Y to code if $\min(X^{[i]}) \leq j$. Given $X = \bigoplus X^{[i]}$, define $Y(\langle i, j, m \rangle) = 1$ if and only if $X(\langle i, m \rangle) \leq j$.

Suppose $X =^{\mathrm{ec}}_B X'$, we need to show $Y =^{\mathrm{ec}}_C Y'$. Fixing i, let $j_0 = \min(X^{[i]})$. Then for every $j < j_0$, we have $X(\langle i, m \rangle) \geq j_0 > j$ for every m, so $Y(\langle i, j, m \rangle) = 0$ and $Y^{[\langle i, j \rangle]} = \emptyset$, and thus $Y^p(\langle i, j \rangle) = 0$. For $j \geq j_0$, we have $X(\langle i, m \rangle) = j_0 \leq j$ for some m, so $Y(\langle i, j, m \rangle) = 1$ and $Y^p(\langle i, j \rangle) = 1$. This is the same for Y', and since $\min(X^{[i]}) = \min(X'^{[i]})$, we have that $Y^p(\langle i, j \rangle) = Y'^p(\langle i, j \rangle)$. Thus $Y =^{\mathrm{ec}}_C Y'$.

Conversely, suppose $X \neq^{\mathrm{ec}}_B X'$. Without loss of generality, suppose there is some i with $\min(X^{[i]}) = j < \min(X'^{[i]})$. Then $X'(\langle i, m \rangle) > j$ for every m, so $Y'(\langle i, j, m \rangle) = 0$ and $Y'^p(\langle i, j \rangle) = 0$. However, we have $X(\langle i, m' \rangle) \leq j$ for some m', so $Y(\langle i, j, m' \rangle) = 1$ and $Y^p(\langle i, j \rangle) = 1 \neq Y^p(\langle i, j \rangle)$. Thus $Y \neq^{\mathrm{ec}}_C Y'$. □

Question 38. *Let μ be ω, "$< \omega$", or a natural number. If E is a Π^0_n-complete equivalence relation (Σ^0_n-complete equivalence relation, respectively) on ω^ω under $\leq^\mu_0$, must E^{ec} then be a Π^0_{n+1}-complete equivalence relation (Σ^0_{n+1}-complete equivalence relation, respectively) under $\leq^\mu_0$?*

Again, Theorem 39 gives a negative answer to this question on the Π^0_n-side under countable reduction.

5.3. **Transferring completeness from ω to Baire space.** For equivalence relations on ω, there are no Π^0_n-complete equivalence relations under countable reduction [4]. The proof there exploits the fact that when one closes a binary Π^0_2-relation under transitivity, the resulting relation may no longer be Π^0_2. (Reflexivity and symmetry do not have the same problem.) Using the idea in the proof, we show this is also the case for ω^ω under computable countable reduction.

Theorem 39. *For $n \geq 2$, there is no Π^0_n-complete equivalence relation on ω^ω under computable countable reduction.*

Proof. We first prove the statement for $n = 2$. Given a Π^0_2-equivalence relation E, we will build a Π^0_2-equivalence relation F such that $F \not\leq^\omega_0 E$.

Fix a computable predicate R such that AEB if and only if $\forall i \exists j R(A, B, i, j)$. We build F by diagonalizing against every possible computable reduction Φ_e. Fix a computable family of distinct computable reals $\{X_e, Y_e, Z_{k,e}\}_{e,k\in\omega}$. The diagonalization for each Φ_e will be independent, and we use $X_e, Y_e, Z_{0,e}, Z_{1,e}, \ldots$ for the diagonalization against Φ_e. For simplicity, we will write $X = X_e$, $Y = Y_e$, and $Z_k = Z_{k,e}$. We also define $\hat{X}, \hat{Y}, \hat{Z}_k$ so that $\Phi_e(X \oplus Y \oplus (\bigoplus Z_k)) = \hat{X} \oplus \hat{Y} \oplus (\bigoplus \hat{Z}_k)$. Note that $\hat{X}, \hat{Y}, \hat{Z}_k$ are all computably enumerable.

Intuitively, we think of a Π^0_2-event happening if and only if it acts infinitely often. We proceed to make XFZ_0 until E acts on $\hat{X}E\hat{Z}_0$, then we switch making YFZ_0 until E acts on $\hat{Y}E\hat{Z}_0$, then go back to XFZ_0 and loop. We break the loop when E acts on $\hat{X}E\hat{Y}$, and get into another loop by replacing Z_0 with Z_1. In the end, every Z_i will force $\hat{X}E\hat{Y}$ to act once, so it will act infinitely often, while we never acted on XFY. This yields $\neg XFY$ but $\hat{X}E\hat{Y}$, diagonalizing against Φ_e being a reduction.

Formally, we define F by defining a computable predicate S and declaring AFB if and only if $\forall i \exists s[S(A, B, i, s) \vee S(B, A, i, s)]$ or $A = B$. We will define S so that for every stage s, there is at most one tuple (A, B, i, s) with $A, B \in \{X, Y, Z_0, Z_1, \ldots\}$ that S holds on, and it is computable to find this tuple. At stage s, define k to be the largest number such that $k \leq s$ and $\forall i < k \exists j \leq s\ R(\hat{X}, \hat{Y}, i, j)$ (this includes requiring that $\hat{X}$ and $\hat{Y}$ have converged enough for R to converge and hold, similarly for $\hat{Z}_k$ below). We then define ℓ_X to be the largest number such that $\ell_X \leq s$ and $\forall i < \ell_X \exists j \leq s\ R(\hat{X}, \hat{Z}_k, i, j)$, and similarly define ℓ_Y. Define i_X be the largest number so that $\exists j' < s\ S(X, Z_k, i', j')$ for every $i' < i_X$, and similarly define i_Y. If $\ell_X \leq \ell_Y$, define $S(X, Z_k, i_x, s)$; and if $\ell_X > \ell_Y$, define $S(Y, Z_k, i_x, s)$. This completes the definition of S and thus F.

We now show that Φ_e is not a reduction from F to E. Suppose, toward a contradiction, that Φ_e is a computable countable reduction. Note that we never have $S(X, Y, i, s)$, so we must have $\neg XFY$, and hence $\neg\hat{X}E\hat{Y}$. Let k be the largest number so that $\forall i < k\ \exists j R(\hat{X}, \hat{Y}, i, j)$, and fix a stage s_0 so that $\forall i < k\ \exists j < s_0\ R(\hat{X}, \hat{Y}, i, j)$. Considering $\hat{Z}_k$, as E is an equivalence relation, we must have $\neg\hat{X}E\hat{Z}_k$ or $\neg\hat{Y}E\hat{Z}_k$. Thus, at least one of ℓ_X and ℓ_Y will converge, so we can fix a stage $s_1 > s_0$ such that (without loss of generality, assume) $\ell_X \leq \ell_Y$ for any stage $s > s_1$. From then on, we will have $S(X, Z_k, i, s)$ for each stage. Indeed, considering a particular stage s, the choice of i_X at that stage shows we have $\exists j' < s\ S(X, Z_k, i', j')$ for every $i' < i_X$, and at the subsequent stages we will increase i_X by 1 at every stage. Thus, we have $\forall i \exists j S(X, Z_k, i, j)$, showing that

XFZ_k. But the fact that ℓ_X converges means that $\neg \hat{X} E \hat{Z}_k$, a contradiction, so Φ_e is not a reduction.

We note that F is an equivalence relation: If $\hat{X} E \hat{Y}$, we will never have AFB for any $A \neq B$ and F is just the equality relation; and if $\neg \hat{X} E \hat{Y}$, we have seen in the proof above that we make exactly one of XFZ_k and YFZ_k but nothing else equivalent.

The above proof constructs, for every Π^0_2-equivalence relation E, a Π^0_2-equivalence relation F such that $F \not\leq^\omega_0 E$. Relativizing, we get that for $n \geq 2$, for every Π^0_n-equivalence relation E, there is a Π^0_n-equivalence relation F such that $F \not\leq^\omega_{n-2} E$. In particular, this means that we also have $F \not\leq^\omega_0 E$, completing the proof. □

On the other hand, we are able to transfer completeness results from ω assuming enough uniformity.

Definition 40. *Let μ be ω, "$< \omega$", or a natural number, and let E be a Σ^0_α-equivalence relation on ω such that the Σ^0_α-definition of E can be uniformly relativized to any oracle $X \in \omega^\omega$. We say that E is* uniformly Σ^0_α-complete with oracles under $\leq^\mu_0$ *if for every equivalence relation F that is Σ^0_α in X, there is an X-computable μ-reduction from F to E^X such that this reduction is uniform in X. (We define this similarly in the Π^0_α-case.)*

Under this definition, the Σ^0_α-complete equivalence relation that is the join of all Σ^0_α-equivalence relations is uniformly Σ^0_α-complete with oracles (under full reduction).

Question 41. *Is the equivalence relation $E^n_=$ (defined in R. Miller and Ng [7] as $i\ E^n_=\ j$ iff $W_i^{\emptyset^{(n)}} = W_j^{\emptyset^{(n)}}$) uniformly Π^0_{n+2}-complete with oracles under finitary reduction?*

Definition 42. *Let E be an equivalence relation on ω that is uniformly Σ^0_α- (or Π^0_α-)complete with oracles under $\leq^\mu_0$. We define E_B to be the equivalence relation on Baire space, which we think of as $\omega^\omega \times \omega^\omega$, by setting $(Y, X) E_B (Y', X')$ iff $X = X'$ and $Y(0) E^X Y'(0)$.*

Remark 43. *In Definition 42, note that if $\alpha \geq 2$ (or $\alpha \geq 1$ in the case of Π^0_α), then E_B is Σ^0_α (or Π^0_α, respectively).*

Theorem 44. *Suppose E is an equivalence relation on ω which is uniformly Σ^0_α-complete (or Π^0_α-complete, respectively) with oracles under $\leq^\mu_0$ among equivalence relations on ω, where $\alpha \geq 2$ (or $\alpha \geq 1$ in the case of Π^0_α). Then E_B is Σ^0_α-complete (or Π^0_α-complete, respectively) under $\leq^\mu_0$ among equivalence relations on Baire space.*

Proof. We will prove the theorem for Σ^0_α. The proof for Π^0_α is identical.

Given a Σ^0_α-equivalence relation G on ω^ω, we need to produce a μ-ary reduction from G to E_B. That is, given $X = \bigoplus_{i<\mu} X_i$, we need to define a Turing functional Φ that uses X as an oracle and outputs $Y = \bigoplus_{i<\mu} Y_i$ so that $X_i G X_j$ iff $Y_i E_B Y_j$.

Given $X = \bigoplus_{i<\mu} X_i$, we first define a Σ^0_α-equivalence relation G' on ω via $iG'j$ iff $X_i G X_j$ for $i, j < \mu$, letting any $i \geq \mu$ form a singleton class if $\mu \in \omega$. Then G' is Σ^0_α relative to X, and this definition is uniform in X. Thus, since E is uniformly

Σ^0_α-complete with oracles, there is a (uniformly) X-computable μ-ary reduction Φ from G' to E^X.

For any oracle X, there is a (full) reduction Ψ^X from E^X to E_B, namely, $\Psi^X(i) = ((i)^\omega, X)$. It is straightforward to see that $\Psi^X(i) E_B \Psi^X(j)$ iff $iE^X j$, and Ψ^X is uniform in X.

We now have X-computable μ-reductions $\Phi : \omega^\mu \to \omega^\mu$ from G' to E^X, and $\Psi : \omega^\mu \to (\omega^\omega \times \omega^\omega)^\mu$ from E^X to E_B. Write $\tau = (0, 1, \dots, \mu - 1)$ if μ is a natural number, $\tau = (0, 1, 2, \dots)$ if $\mu = \omega$, and let $\tau \in \omega$ vary if μ is "$< \omega$". Then $\Psi \circ \Phi(\tau)$ is an element of $(\omega^\omega \times \omega^\omega)^\mu$, so we will write $Y = \Psi \circ \Phi(\tau) = \bigoplus_{i<\mu} Y_i$. Define $\Phi(X) = Y$, noting that Y is uniformly computable from X.

Finally, since Φ and Ψ are both μ-ary reductions, we have for all appropriate i and j that $X_i G X_j$ iff $iG'j$ iff $\pi_i(\Phi(\tau)) E \pi_j(\Phi(\tau))$ iff $Y_i = \pi_i(\Psi(\Phi(\tau)))$ is E_B-equivalent to $Y_j = \pi_j(\Psi(\Phi(\mu)))$. This shows that Φ is a μ-ary reduction from G to E_B. □

The definition of E_B and the proof can be modified to work for Cantor space. Using the theorem, we can transfer completeness results from equivalence relations on ω to equivalence relations on Baire space (or Cantor space).

Recall that for every $\alpha \geq 1$, there is a Σ^0_α-complete equivalence relation on ω. Indeed, for any Σ^0_α-relation E, its closure $\bar{E}$ under reflexivity, symmetry, and transitivity is also Σ^0_α. Thus, listing all Σ^0_α-relations as $E_0, E_1, \dots$, the uniform join of their closure $\bigoplus \bar{E}_i$ is a Σ^0_α-complete equivalence relation on ω. This construction can be relativized to get that E is uniformly Σ^0_α-complete with oracles under $\leq^\omega_0$. So from the previous theorem, we obtain the following.

Corollary 45. *For every $\alpha \geq 2$, there are Σ^0_α-complete equivalence relations on ω^ω under countable reducibility.*

6. Further questions

We end the paper with the following question. Intuitively, it asks how much uniformity is needed to "upgrade" n-reductions for every $n \in \omega$ to a finitary reduction.

Question 46. *Suppose E and F are two equivalence relations such that $E \leq^n_0 F$ holds for all n. Must it be the case that $E \leq^{<\omega}_0 F$?*

Notice that there is an obvious uniform procedure which, for all n and all $m < n$, creates an m-ary reduction from an n-ary one. Therefore, if we had (for example) n-ary reductions uniformly for every $n \in \emptyset'$, we would also have a finitary reduction. The same comment applies with any infinite non-immune set (i.e., any set with an infinite c.e. subset) in place of $\emptyset'$. Thus a negative answer to Question 46 would require substantial non-uniformity among the n-ary reductions.

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