PRIORITY ARGUMENTS USING ITERATED TREES OF STRATEGIES

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Abstract. A general framework for priority arguments in classical recursion theory, using iterated trees of strategies, is introduced and used to present new proofs of four fundamental theorems of recursion theory.

1. Introduction. Only a few years after Cohen invented forcing in set theory in 1961, a quite general framework had been developed by Shoenfield, Solovay, Feferman, and others. This framework has since allowed set theorists to prove a wide variety of theorems by defining the appropriate partial order and appealing to the general lemmas on forcing. The lemmas could also be applied to extensions of forcing such as iterated forcing or class forcing.

Recursion theorists, unfortunately, have had a much harder time with the priority argument invented in 1956/57 by Muchnik and Friedberg. Even nowadays and even for finite-injury priority arguments, recursion theorists either reprove the combinatorics of finite injury or simply assume that the reader is familiar enough with the combinatorics to fill in the details. For the most complicated well-understood kind of priority argument, the $\mathbf{0}'''$-priority argument, the whole framework has to be reproved every time.

Of course, there have been numerous attempts at finding a framework: Lachlan [**7**, **8**] tried a game-theoretical approach, and also a topological approach, using an effective version of the Baire Category Theorem. (The true stages method for some infinite-injury priority arguments has its origin there.) Lerman [**11**] devised the pinball machine model for infinite-injury priority arguments. Harrington conceived the tree of strategies method as a way to understand Lachlan's Nonsplitting Theorem, an approach that was then worked out in detail and popularized by Soare [**18**, **19**]; this method is the most widely used today.

In the 1980's, Harrington introduced the "worker at level n" approach, later widely used in recursive model theory by Knight and others. Ash [**1**, **2**] gave a more detailed version of this, working out a general framework in terms of iterated trees, which he and Knight then extended and used to prove results in recursive model theory. Groszek and Slaman [**4**] attempted a tree of trees approach in their work on reverse mathematics in recursion theory. Finally, Kučera [**5**, **6**] introduced the construction of recursively

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enumerable degrees below degrees of diagonally nonrecursive functions as a way of eliminating one level of injury and of separating negative and positive strategies in the construction.

The framework that we would like to introduce here grew out of our work on the decidability of the existential theory of the recursively enumerable (r.e.) degrees with nth jump reducibility predicates [**9**], which, by the Shore Noninversion Theorem [**17**], requires a general $\mathbf{0}^{(n)}$-priority argument for arbitrarily large n. Of course, this framework was inspired by many of the approaches above, especially Ash's, and Groszek's and Slaman's. However, we see our approach as the most promising at this time to reach the goal set above, namely to prove combinatorial lemmas about the framework and to eliminate these from the individual priority arguments, which we hope to achieve in [**10**]. In this paper, however, we will prove the combinatorial lemmas in each instance separately to better give the intuition for the framework.

Our notation generally follows Chapter XIV of Soare [**19**]. We denote the *use* (the largest number *actually* used) of a partial recursive functional by the corresponding lower-case Greek letter, so e.g. φ is the use function of Φ. With the "opponent's" (i.e. the given) functionals, we will always assume that the use of a computation is bounded by the stage at which the computation first appears.

2. The Framework. We will define, for fixed $n > 0$, a sequence of trees of strategies $T_0, T_1, \ldots, T_n$ such that, roughly speaking, the construction taking place on T_i is recursive in $\mathbf{0}^{(i)}$, and for $i > 0$ is actually a finite-injury construction relative to $\mathbf{0}^{(i-1)}$, and such that each strategy $\alpha \in T_{i+1}$ is split up into ω many substrategies $\alpha_k \in T_i$, working on the kth instance of α's requirement. By the way the strategies will be arranged on the tree, the construction on T_i, which is recursive in $\mathbf{0}^{(i)}$, will work to satisfy not only (sub)requirements on T_i, but also higher-level (sub)requirements from $T_{i+1}, \ldots, T_n$. The assignment of (sub)requirements to nodes on the trees is thus at the heart of the framework. The splitting-up of requirements will automatically ensure the negative requirements and will follow the same pattern for all four theorems presented here. (For some other theorems, such as the Minimal Pair Theorem, the splitting-up would have to be modified slightly, however, to ensure that "at most one side of the minimal pair" is injured.)

We define the trees of strategies by induction as follows:

$$T_0 = \{\,\infty, 0\,\}^{<\omega},$$
$$T_{i+1} = \Big(\{\,\infty\,\} \cup \bigcup_{j\le i} T_j\Big)^{<\omega}. \tag{1}$$

Here ∞ and 0 are distinct symbols, and we tacitly assume that the nodes of each tree (including the empty node) are appropriately tagged so that the trees form pairwise disjoint sets, and we can thus tell which tree a particular node comes from. (Intuitively, for $\alpha \in T_i$, $\alpha\frown\langle\infty\rangle$ denotes the Π_i-outcome of α, and $\alpha\frown\langle\beta\rangle$ denotes a Σ_i-outcome witnessed by the Π_{i-1}-outcome of β on T_{i-1}.)

Suppose we are given two partial functions $\mathrm{up}\colon T_i \to T_{i+1}$ and $\mathrm{lev}\colon T_i \to \{\, j \mid j \le n\,\}$. (In the following, these will denote that $\alpha \in T_i$ is a substrategy of $\mathrm{up}(\alpha) \in T_{i+1}$ and works for a requirement of complexity level $\mathrm{lev}(\alpha)$. We do not split up a requirement in going from T_{i+1} to T_i if $i \geq \mathrm{lev}(\alpha)$, but only if $i < \mathrm{lev}(\alpha)$.) We define the *approximation function* $\lambda\colon T_i \to T_{i+1}$ by induction on m as follows (until $\lambda(\alpha)(m)\uparrow$):

$$\lambda(\alpha)(m) = \begin{cases} \alpha(k), & \text{if } \mathrm{lev}(\lambda(\alpha) \upharpoonright m) \leq i \text{ and} \\ & \quad k = \mu k' < |\alpha| \, (\mathrm{up}(\alpha \upharpoonright k') = \lambda(\alpha) \upharpoonright m), \\ \beta, & \text{if } \mathrm{lev}(\lambda(\alpha) \upharpoonright m) > i \text{ and} \\ & \quad \beta = \mu\beta' \subset \alpha \, (\mathrm{up}(\beta') = \lambda(\alpha) \upharpoonright m \;\&\; \beta'^\frown\langle\infty\rangle \subseteq \alpha), \\ \infty, & \text{if } \mathrm{lev}(\lambda(\alpha) \upharpoonright m) > i \text{ and} \\ & \quad \forall \beta \subset \alpha \, (\mathrm{up}(\beta) = \lambda(\alpha) \upharpoonright m \to \beta^\frown\langle\infty\rangle \not\subseteq \alpha) \text{ and} \\ & \quad \exists \beta \subset \alpha \, (\mathrm{up}(\beta) = \lambda(\alpha) \upharpoonright m), \\ \uparrow, & \text{if } \forall \beta \subset \alpha \, (\mathrm{up}(\beta) \neq \lambda(\alpha) \upharpoonright m). \end{cases} \tag{2}$$

(Intuitively, α believes that if α is an initial segment of the true path on T_i then $\lambda(\alpha)$ is an initial segment of the true path on T_{i+1}. We notice that, by the remark at the end of the preceding paragraph, we need not specify the i for T_i on up and λ; and we denote by up^j and λ^j the j-fold iteration of up and λ, respectively.)

The definition of the approximation function naturally extends to infinite paths $\Lambda \in [T_i]$; here $\lambda(\Lambda)$ can be either a path through T_{i+1} or a node on T_{i+1}.

Next, we define the concept of consistency. We say that *an instance of* $\beta \in T_{i+1}$ *is consistent at* $\alpha \in T_i$ iff

$$\forall j < n - i\big(\mathrm{up}^j(\beta) \subseteq \lambda^{j+1}(\alpha)\big), \tag{3.1}$$

$$\mathrm{lev}(\beta) \leq i \to \forall \alpha' \subset \alpha\big(\mathrm{up}(\alpha') \neq \beta\big), \text{ and} \tag{3.2}$$

$$\mathrm{lev}(\beta) > i \to \forall \alpha' \subset \alpha\big(\alpha'^\frown\langle\infty\rangle \subseteq \alpha \to \mathrm{up}(\alpha') \neq \beta\big). \tag{3.3}$$

(Intuitively, we state in (3.1) that $\mathrm{up}^j(\beta)$ is consistent with α's guess at the true path through T_{i+j+1}; in (3.2) that β's requirement need not be split up from T_{i+1} to T_i and has not already been taken care of before α; and in (3.3) that along α, no substrategy working for β has had a Π_i-outcome and that therefore at α we must still search for a Σ_{i+1}-witness for β's requirement. There are theorems (not presented here) in the proofs of which this notion of consistency has to be defined in a less restrictive way.)

Next, we fix an effective 1–1 poset homomorphism par from $\cup_{i\in\omega} T_i$ *onto* ω with the following property:

$$\forall \alpha, \beta \in T_{i+1} \forall \gamma \in T_i\big(\beta^\frown\langle\gamma\rangle \subseteq \alpha \to \mathrm{par}(\gamma) < \mathrm{par}(\alpha)\big). \tag{4}$$

We call $\mathrm{par}(\alpha)$ the *parameter* of α. (The intuition is that $\mathrm{par}(\alpha)$ is a number reserved for α and "big" relative to α's predecessors.)

We now assign requirements to nodes. Fix n to be the highest (quantifier) complexity level of any requirement for the theorem at hand. Given an effective ω-ordering of all requirements $\{\,\mathcal{R}_k\,\}_{k\in\omega}$ for the theorem, we assign $\mathcal{R}_k$ to all nodes $\alpha \in T_n$ with $|\alpha| = k$ (or a proper subset thereof if certain outcomes contradict the hypotheses of the theorem, as in the Sacks Splitting and the Sacks Density Theorems, but always such that the nodes that are assigned requirements form a recursive subtree of T_n). We also define the complexity level $\mathrm{lev}(\alpha)$ to be the (quantifier) complexity level of requirement $\mathcal{R}_k$. (This assignment of requirements on T_n will have to be modified slightly for the Sacks Splitting Theorem.)

We then proceed by reverse induction on $i < n$ and by induction on $|\alpha|$ for $\alpha \in T_i$. Given an effective ω-ordering $\preceq$ of $\{\, \beta \in T_{i+1} \mid \text{some requirement is assigned to } \beta \,\} \times \omega$ with the property

$$\big(\beta = \beta' \ \&\ k \le k'\big) \text{ or } \big(\beta \subset \beta' \ \&\ k \le k' + \mathrm{par}(\beta')\big) \to \langle \beta, k\rangle \preceq \langle \beta', k'\rangle, \tag{5}$$

and given an assignment to all $\alpha' \subset \alpha$, we let $\langle \beta, k\rangle$ be $\preceq$-minimal such that

$$\exists^{\le k} \alpha' \subset \alpha\big(\mathrm{up}(\alpha') = \beta\big), \text{ and} \tag{6.1}$$

$$\text{an instance of } \beta \text{ is consistent at } \alpha. \tag{6.2}$$

We then assign β's requirement to α (if $\mathrm{lev}(\beta) \le i$) or the kth instance of β's requirement to α (if $\mathrm{lev}(\beta) > i$), respectively, and we set $\mathrm{up}(\alpha) = \beta$ and $\mathrm{lev}(\alpha) = \mathrm{lev}(\beta)$. If $\langle \beta, k\rangle$ does not exist then we assign no requirements to any $\alpha' \supseteq \alpha$.

We will now prove a general lemma about the splitting-up of requirements.

Lemma 1 (Splitting-Up Lemma). *Assume $\Lambda_0 \in [T_0]$ and $\Lambda_j = \lambda^j(\Lambda_0)$ for $0 < j \le n$. Then:*

(i) If $0 < j \le n$ and $|\Lambda_j| < \infty$ then no requirement is assigned to the node Λ_j.

(ii) If $0 < j \le n$ then for all t (with $t \le |\Lambda_j|$ if Λ_j is a node) there is some t' such that $\lambda^j(\Lambda_0 \restriction t') = \Lambda_j \restriction t$ and for all $j' < j$, $\lambda^{j'}(\Lambda_0 \restriction t') \subseteq \Lambda_{j'}$.

(iii) If $\mathcal{R}$ is a requirement of complexity level i assigned to a (necessarily unique) node on Λ_n, say $\alpha_n^{\emptyset}$, then for all $j < n$:

(a) If $j \ge i$ then $\mathcal{R}$ is assigned to a (necessarily unique) node on Λ_j, say $\alpha_j^{\emptyset}$.

(b) If $j < i$ and $\alpha_{j+1}^{\sigma}{}^\frown\langle\beta\rangle \subseteq \Lambda_{j+1}$ for some $\beta \in T_j$ and some $\sigma \in \omega^{i-j-1}$ then the kth instance of α_{j+1}^{σ}'s requirement (for some k) is assigned to β with $\beta^\frown\langle\infty\rangle \subseteq \Lambda_j$, and no instance of α_{j+1}^{σ}'s requirement is assigned to any β' with $\beta \subset \beta' \subseteq \Lambda_j$. We denote by $\alpha_j^{\sigma^\frown\langle k'\rangle}$ the (necessarily unique) $\beta' \subset \Lambda_j$ to which the k'th instance of α_{j+1}^{σ}'s requirement is assigned (for $k' \le k$).

(c) If $j < i$ and $\alpha_{j+1}^{\sigma}{}^\frown\langle\infty\rangle \subseteq \Lambda_{j+1}$ for some $\sigma \in \omega^{i-j-1}$ then for all k, the kth instance of α_{j+1}^{σ}'s requirement is assigned to a (necessarily unique) node on Λ_j, say $\alpha_j^{\sigma^\frown\langle k\rangle}$.

Proof. If no requirement is assigned to some $\alpha \subset \Lambda_0$ then we redefine Λ_0 to be the least such α, otherwise we leave α as is. This will not affect the definition of Λ_j for $j > 0$.

(i) We proceed by induction on j. Suppose (i) fails for some (least) $j \le n$. Then Λ_j is finite, and some requirement is assigned to it. Since $\Lambda_j = \lambda(\Lambda_{j-1})$, we have, by the definition of λ, that $\Lambda_j = \lambda(\Lambda_{j-1} \restriction t)$ for all $t \ge t_0$ (for some t_0). Furthermore, by (3.1), $\mathrm{up}^l(\Lambda_j) \subseteq \Lambda_{j+l}$ for all $l \le n-j$, and thus $\langle \Lambda_j, 0\rangle$ must eventually be the least pair $\langle \beta, k\rangle$ used in splitting up T_j's requirements along Λ_{j-1}. Therefore (the 0th instance of) Λ_j's requirement must be assigned to some $\alpha \subset \Lambda_{j-1}$ whence $\Lambda_j \subset \lambda(\Lambda_{j-1})$, a contradiction.

(ii) We proceed by induction on j. Set $\alpha = \Lambda_j \restriction (t-1)$. If $\mathrm{lev}(\alpha) < j$ then there is a unique $\beta \subset \Lambda_{j-1}$ with $\mathrm{up}(\beta) = \alpha$, and we have $\Lambda_{j-1}(|\beta|) = \Lambda_j(|\alpha|)$. If $\mathrm{lev}(\alpha) \ge j$ and $\alpha^\frown\langle\beta\rangle = \Lambda_j \restriction t$ for some β then $\beta^\frown\langle\infty\rangle \subseteq \Lambda_{j-1}$. If $\mathrm{lev}(\alpha) \ge j$ and $\alpha^\frown\langle\infty\rangle = \Lambda_j \restriction t$ then there is a (least) $\beta \subset \Lambda_{j-1}$ with $\mathrm{up}(\beta) = \alpha$, and we have $\beta^\frown\langle\gamma\rangle \subseteq \Lambda_{j-1}$ for some γ. In all these cases, set $t'' = |\beta| + 1$. Since $\mathrm{up}(\beta) = \alpha$, we have

$\alpha \subseteq \lambda(\beta)$ and thus $\Lambda_j \restriction t \subseteq \lambda(\Lambda_{j-1} \restriction t'')$. By (3.2) and (3.3), it is also easy to see that $\Lambda_j \restriction t = \lambda(\Lambda_{j-1} \restriction t'')$. Now apply (ii) with t'' and $j-1$ in place of t and j to get t' which will then make (ii) true for t and j.

(iii) We proceed by reverse induction on j. By the hypothesis, (a) holds for $\alpha_n^{\emptyset}$. Suppose one of (a), (b), or (c) holds for $j+1$, and fix $\alpha_{j+1}^{\sigma} \subset \Lambda_{j+1}$ for some $\sigma \in \omega^{<i}$ in our notation. If $j < i$ and $\alpha_{j+1}^{\sigma}\widehat{\ }\langle\beta\rangle \subseteq \Lambda_{j+1}$ for some $\beta \in T_j$ then (b) holds since $\Lambda_{j+1} = \lambda(\Lambda_j)$. So suppose $j \geq i$ or $\alpha_{j+1}^{\sigma}\widehat{\ }\langle\infty\rangle \subseteq \Lambda_{j+1}$, and (a) or (c) fails, respectively. We distinguish two cases.

If Λ_j is a node (i.e. of finite length) then $\lambda^k(\Lambda_j) = \Lambda_{j+k}$ for all $k \leq n-j$. Since $\mathrm{up}^k(\alpha_{j+1}^{\sigma}) \subset \Lambda_{j+1+k}$ for all $k < n-j$ by hypothesis, (some instance of) α_{j+1}^{σ}'s requirement would be assigned to Λ_j, contradicting (i) (or the first paragraph of this proof).

If Λ_j is an infinite path then by an application of (ii), there are infinitely many t such that for all $k < n-j$, $\lambda_{k+1}(\Lambda_j \restriction t) \supseteq \mathrm{up}^k(\alpha_{j+1}^{\sigma})$. Thus an instance of α_{j+1}^{σ} is consistent at $\Lambda \restriction t$ for infinitely many t, contradicting the failure of (a) or (c).

This concludes the proof of Lemma 1. ■

The general procedure to use this framework will then be as follows: For a theorem on r.e. objects, we define an ω-sequence of requirements of the form

$$(\rho \to \sigma)\ \&\ (\neg\rho \to \tau) \tag{7}$$

where ρ is a Σ_i-formula (or Π_i-formula), σ a Σ_{i+1}-formula (or Π_{i+1}-formula), and τ a Π_i-formula (or Σ_i-formula) if the *complexity level* i is odd (or even, respectively). We allow ρ, σ, and τ to contain free variables that will be substituted by parameters of nodes (or numbers computed from them), and we allow the innermost quantifier of ρ to be restricted to a set of "stages" $S = \{\, \mathrm{par}(\xi) \mid \xi \subset \Lambda_0 \,\&\, \mathrm{up}^n(\xi) = \alpha \,\&\, \forall j \leq n\big(\mathrm{up}^j(\xi) \subset \Lambda_j\big) \,\}$ when a T_0-strategy works for this requirement with correct guesses at the true path on all trees. In splitting (sub)requirements into subrequirements, we bound the outermost unbounded quantifier in ρ, σ, and τ by k (for the kth instance of that requirement) or by $\mathrm{par}(\beta)$ (for β working on the kth instance of that requirement), proceeding from T_n down to T_i without splitting up, and then from T_i all the way down to T_0 while splitting up. The requirements assigned to T_0-nodes then determine the *true path* Λ_0 on T_0 (where we follow $\alpha\widehat{\ }\langle\infty\rangle$ if the $i$fold instance of ρ from the requirement for $\alpha \in T_0$ is true, and we follow $\alpha\widehat{\ }\langle 0\rangle$ otherwise). We now have to define the action of an $\alpha \in T_0$ depending upon its "outcome" (∞ or 0). We thus obtain the (recursive) true path Λ_0 on T_0 by starting with $\emptyset \in T_0$ and, whenever α has been determined to be on Λ_0, letting α act and determine $\alpha\widehat{\ }\langle a\rangle$ to be on Λ_0 for α's outcome a. The construction consists of the actions of all $\alpha \subset \Lambda_0$. We finally define $\Lambda_i = \lambda^i(\Lambda_0)$ to be the *true path* on T_i (which is recursive in $\mathbf{0}^{(i)}$) and verify that all $\alpha \in \Lambda_n$ satisfy their requirements.

3. The Sample Theorems. We now present the details of the construction in our framework for four well-known theorems of recursion theory. (We assume familiarity with the traditional proofs, as e.g. in Soare [**19**].)

Friedberg-Muchnik Theorem (Muchnik [**13**], Friedberg [**3**]). *There are two r.e. sets A and B of incomparable Turing degree.*

Proof. We need to satisfy $A \neq \Phi^B$ and $B \neq \Phi^A$ for all partial recursive (p.r.) functionals Φ. By symmetry, we assume throughout that we want to satisfy the former whenever we discuss an individual requirement. To ensure such an inequality, we need to ensure it at a number x. A requirement thus takes the form

$$\begin{aligned} \exists s \in S\,\big(\Phi_s^{B_s}(x)\!\downarrow = 0\big) \to & \exists s \forall t \geq s\big(\Phi_s^{B_s}(x)\!\downarrow = 0 \;\&\; x \in A_{s+1} \;\&\\ & \qquad B_s \restriction (\varphi_s(x)+1) = B_t \restriction (\varphi_s(x)+1)\big) \qquad (8)\\ \neg\exists s \in S\big(\Phi_s^{B_s}(x)\!\downarrow = 0\big) \to & \forall s\,(x \notin A_s)\,. \end{aligned}$$

The complexity level is thus 1. We fix an effective ω-ordering $\{\mathcal{R}_e\}_{e\in\omega}$ of all requirements, assign requirement $\mathcal{R}_e$ to all nodes $\alpha \in T_1$ with $|\alpha| = e$, and substitute the x in α's requirement by $\mathrm{par}(\alpha)$ for all $\alpha \in T_1$.

The split-up requirements for $\beta \in T_0$ are obtained by restricting $s \leq \mathrm{par}(\beta)$ (where $B_{\mathrm{par}(\beta)}$ is that part of B enumerated before β acts). The action of $\beta \in T_0$ is to measure if

$$\exists s \leq \mathrm{par}(\beta)\;\big(s \in S \;\&\; \Phi_s^{B_s}(\mathrm{par}(\mathrm{up}(\beta)))\!\downarrow = 0\big)\,, \qquad (9)$$

which, since all previous instances of α's requirement before β have had outcome 0, i.e. have been answered negatively, just means

$$\Phi_{\mathrm{par}(\beta)}^{B_{\mathrm{par}(\beta)}}(\mathrm{par}(\mathrm{up}(\beta)))\!\downarrow = 0. \qquad (9')$$

If the answer is no then β has outcome 0, otherwise it has outcome ∞ and enumerates $\mathrm{par}(\mathrm{up}(\beta))$ into A.

We now verify that the construction satisfies the requirements.

We first observe a combinatorial fact about finite injury, namely that $\beta \subset \Lambda_0$ and $\mathrm{up}(\beta) \subset \Lambda_1$ implies that $\mathrm{up}(\beta)$ and $\mathrm{up}(\beta')$ are comparable for all β' with $\beta \subset \beta' \subset \Lambda_0$.

Suppose $A = \Phi^B$. Then some unique $\alpha \subset \Lambda_1$ was supposed to ensure $A(\mathrm{par}(\alpha)) \neq \Phi^B(\mathrm{par}(\alpha))$. First assume $\alpha\widehat{\ }\langle\infty\rangle \subset \Lambda_1$. Then, by Lemma 1, there are infinitely many $\beta \subset \Lambda_0$ with $\mathrm{up}(\beta) = \alpha$, and for all such β we have $\beta\widehat{\ }\langle 0\rangle \subset \Lambda_0$. Thus $\Phi_{\mathrm{par}(\beta)}^{B_{\mathrm{par}(\beta)}}(\mathrm{par}(\alpha))\!\downarrow = 0$ fails for all these β. This implies $\neg\Phi^B(\mathrm{par}(\alpha))\!\downarrow = 0$ and $\mathrm{par}(\alpha) \notin A$, a contradiction.

On the other hand, assume $\alpha\widehat{\ }\langle\beta\rangle \subset \Lambda_1$ for some $\beta \in T_0$. Thus $\beta\widehat{\ }\langle\infty\rangle \subset \Lambda_0$, $\Phi_{\mathrm{par}(\beta)}^{B_{\mathrm{par}(\beta)}}(\mathrm{par}(\alpha))\!\downarrow = 0$, and $\mathrm{par}(\alpha) \in A$. It suffices to show that no β' with $\beta \subset \beta' \subset \Lambda_0$ will enumerate a number $\leq \varphi_{\mathrm{par}(\beta)}(\mathrm{par}(\alpha))$ into B in order for us to establish that $B_{\mathrm{par}(\beta)} \restriction (\varphi_{\mathrm{par}(\beta)}(\mathrm{par}(\alpha)) + 1) = B \restriction (\varphi_{\mathrm{par}(\beta)}(\mathrm{par}(\alpha)) + 1)$. We analyze the different possibilities for the position of $\alpha' = \mathrm{up}(\beta')$ relative to α. By (3.1) and (3.3), $\alpha'\widehat{\ }\langle\beta''\rangle \subseteq \alpha$ (for some β'') is impossible. By (3.1), $\alpha\widehat{\ }\langle\infty\rangle \subseteq \alpha'$ is impossible. By $\alpha \subset \Lambda_1$, $\alpha'\widehat{\ }\langle\infty\rangle \subseteq \alpha$ implies that β' does not enumerate any number. By (3.1), $\alpha\widehat{\ }\langle\beta''\rangle \subseteq \alpha'$ (for some β'') implies $\beta = \beta''$, and thus $\mathrm{par}(\alpha') > \mathrm{par}(\beta) > \varphi_{\mathrm{par}(\beta)}(\mathrm{par}(\alpha))$ by (4). Suppose α and α' are incomparable, say they split at $\bar{\alpha}$. But then $\beta \subset \beta' \subset \Lambda_0$ is impossible by (3.1) and our observation. ■

Sacks Splitting Theorem (Sacks [**14**]). *Any nonrecursive r.e. set A can be split into two r.e. subsets A_0 and A_1 of incomparable Turing degree.*

Proof. Given an r.e. set A, we need to satisfy (for all p.r. functionals Φ and $i = 0$ or 1) the requirements

$$A = A_0 \sqcup A_1, \text{ and} \tag{10.1}$$
$$A = \Phi^{A_i} \to A \leq_{\mathrm{T}} \emptyset. \tag{10.2}$$

(Here $\sqcup$ denotes disjoint union.) We ensure $A_0 \cap A_1 = \emptyset$ simply by not enumerating the same number into both A_0 and A_1, and we ensure $A_0, A_1 \subseteq A$ simply by only enumerating elements of A into A_0 or A_1; so ensuring (10.1) is reduced to showing

$$A \subseteq A_0 \cup A_1. \tag{10.1*}$$

We try to show that A is recursive by building a partial recursive function Δ_α such that $A = \Delta_\alpha$ (for some $\alpha \in T_2$ working on (10.2)). Our requirements thus take the following form (for all x and all i and Φ):

$$\exists s \in S\,(x \in A_s) \to \exists s\big(x \in A_{0,s+1} \cup A_{1,s+1}\big), \text{ and} \tag{10.1$'$}$$
$$\forall x \exists s \in S\, \forall y \leq x\big(A_s(y) = \Phi_s^{A_{i,s}}(y)\downarrow\big) \to \forall x \exists s \forall t > s\big(A_t(x) = \Delta_{\alpha,t}(x)\downarrow\big). \tag{10.2$'$}$$

In the notation of (7), formulas τ are trivial, say $0 = 0$, so the requirements take the simpler form $\rho \to \sigma$. The complexity level of requirements (10.1$'$) and (10.2$'$) is 1 and 2, respectively. We index the requirements (10.1$'$) and (10.2$'$) by $\{\mathcal{P}_x\}_{x\in\omega}$ (for the x in (10.1$'$)) and $\{\mathcal{N}_e\}_{e\in\omega}$, respectively. We will assign requirements to nodes of T_2 in a different fashion than usual, namely as follows: We assign $\mathcal{N}_0$ to $\emptyset \in T_2$. Given assignments to all $\alpha' \subset \alpha$ for some $\alpha \in T_2$, we denote by α_0 the longest $\alpha' \subset \alpha$ to which a requirement (10.2$'$), say $\mathcal{N}_{e_0}$, has been assigned. If $\alpha_0\widehat{\ }\langle\infty\rangle \subseteq \alpha$ then no requirement is assigned to α (since α_0 has shown that A is recursive contrary to the hypothesis of the theorem). Otherwise let $\alpha_0\widehat{\ }\langle\beta\rangle \subseteq \alpha$ for some β; then $\mathcal{N}_{e_0+1}$ is assigned to α if (for all $x \leq \mathrm{par}(\beta)$) $\mathcal{P}_x$ has already been assigned to some $\alpha' \subset \alpha$; otherwise let x_0 be minimal such that $\mathcal{P}_{x_0}$ has not been assigned to some $\alpha' \subset \alpha$, and assign $\mathcal{P}_{x_0}$ to α. It is now easy to check that along any path $\Lambda \in [T_2]$, all requirements have been assigned unless $A \leq_{\mathrm{T}} \emptyset$ is shown by $\alpha\widehat{\ }\langle\infty\rangle \subset \Lambda$ for some α working on a requirement (10.2$'$).

In going from T_2 to T_1, we do not split up requirements (10.1$'$), and we split up requirements (10.2$'$) by bounding $x \leq k$ (for $\beta \in T_1$ working on the kth instance of (10.2$'$)). The split-up requirements for T_0 are simply obtained by bounding $s \leq \mathrm{par}(\gamma)$ for $\gamma \in T_0$. (Here we fix an enumeration $\{A_s\}_{s\in\omega}$ of A, and $A_{i,\mathrm{par}(\gamma)+1}$ is the part of A_i enumerated by all $\gamma' \subseteq \gamma$, and likewise for Δ_α.)

The action of a $\gamma \in T_0$ working on a requirement (10.1$'$) is to measure if

$$\exists s \in S\,(s \leq \mathrm{par}(\gamma)\ \&\ x \in A_s), \tag{11}$$

which just amounts to measuring if

$$x \in A_{\mathrm{par}(\gamma)}. \tag{11$'$}$$

If the answer to (11′) is no then γ has outcome 0; otherwise it has outcome ∞, and, if now $x \notin A_0 \cup A_1$, then γ enumerates x into A_{1-i_0} where $\alpha_0 \subset \mathrm{up}^2(\gamma)$ is the longest strategy working on a requirement (10.2′), and α_0 uses $i = i_0$.

The action of a $\gamma \in T_0$ working on the kth instance of a requirement (10.2′) is to measure if

$$\forall x \leq k \exists s \leq \mathrm{par}(\gamma) \forall y \leq x \left(s \in S \;\&\; A_s(y) = \Phi_s^{A_{i,s}}(y)\right)\downarrow, \tag{12}$$

which, just as for (9), simply amounts to measuring if

$$A_{\mathrm{par}(\gamma)} \upharpoonright (k+1) = \Phi_{\mathrm{par}(\gamma)}^{A_{i,\mathrm{par}(\gamma)}} \upharpoonright (k+1)\downarrow. \tag{12′}$$

If the answer is no then γ has outcome 0; otherwise it has outcome ∞ and defines $\Delta_\alpha(k) = A_{\mathrm{par}(\gamma)}(k)$ (if $\Delta_\alpha(k)$ is undefined so far where $\alpha = \mathrm{up}^2(\gamma)$).

We now verify that the construction satisfies all the requirements (up to the first one, if any, that contradicts the hypothesis of the theorem by showing $A \leq_{\mathrm{T}} \emptyset$).

First, it is easy to see that any $\alpha \subset \Lambda_2$ working on a requirement $\mathcal{P}_x$ will ensure $\mathcal{P}_x$; so if Λ_2 is an infinite path then (10.1) is satisfied.

Next, we observe that $\gamma \subset \Lambda_0$ and $\mathrm{up}^2(\gamma) \subset \Lambda_2$ implies $\mathrm{up}(\gamma) \subset \Lambda_1$. Suppose not, say $\mathrm{up}(\gamma)$ and Λ_1 split at $\bar\beta$. By $\mathrm{up}(\gamma) \subseteq \lambda(\gamma)$, we have $\bar\beta\widehat{\ }\langle\infty\rangle \subseteq \mathrm{up}(\gamma)$ and $\bar\beta\widehat{\ }\langle\bar\gamma\rangle \subset \Lambda_1$ for some $\bar\gamma$ with $\gamma \subset \bar\gamma \subset \Lambda_0$. By the way requirements are assigned to nodes of T_2, we have $\mathrm{up}(\bar\beta) \subseteq \mathrm{up}^2(\gamma)$, and by (3.2) or (3.3), even $\mathrm{up}(\bar\beta) \subset \mathrm{up}^2(\gamma)$. But $\mathrm{up}^2(\gamma) \subset \Lambda_2$. So if $\mathrm{up}(\bar\beta)$ works on some requirement (10.1′) then $\mathrm{up}^2(\gamma)$ has a correct guess on $\bar\beta$'s outcome by $\bar\beta \subset \Lambda_1$. And if $\mathrm{up}(\bar\beta)$ works on some requirement (10.2′) then, by (3.1), we have $\mathrm{up}(\bar\beta)\widehat{\ }\langle\bar\beta\rangle \subseteq \mathrm{up}^2(\gamma)$, contradicting $\bar\beta\widehat{\ }\langle\bar\gamma\rangle \subset \Lambda_1$. This establishes the observation, and furthermore that $\mathrm{up}(\gamma) \subseteq \lambda(\gamma')$ and $\mathrm{up}^2(\gamma) \subseteq \lambda^2(\gamma')$ for all γ' with $\gamma \subseteq \gamma' \subset \Lambda_0$.

Now suppose that $\alpha\widehat{\ }\langle\infty\rangle \subset \Lambda_2$ for some strategy working on a requirement (10.2′). Then $\alpha\widehat{\ }\langle\infty\rangle = \Lambda_2$ by the assignment of requirements to nodes of T_2, and $\beta\widehat{\ }\langle\gamma\rangle \subset \Lambda_1$ (for some γ) for all $\beta \subset \Lambda_1$ with $\mathrm{up}(\beta) = \alpha$. Thus, by Lemma 1(iii)(c), Δ_α is total. So suppose $\Delta_\alpha(x)\downarrow \neq A(x)$ for some (least) x. Then $\Delta_\alpha(x)$ was defined by some $\gamma \subset \Lambda_0$, and later x entered A. Since $\varphi_{\mathrm{par}(\gamma)}(x) < \mathrm{par}(\gamma)$, it suffices to show that no $\tilde\gamma$ with $\gamma \subset \tilde\gamma \subset \Lambda_0$ will put any number $\leq \mathrm{par}(\gamma)$ into A_i.

So suppose there is such a $\tilde\gamma$. We set $\tilde\alpha = \mathrm{up}^2(\tilde\gamma)$. By our observation above and $\alpha \subset \Lambda_2$, α and $\tilde\alpha$ must be comparable. If $\tilde\alpha \subset \alpha$ then $\mathrm{up}(\tilde\gamma) \subset \mathrm{up}(\gamma)$ and, by $\gamma \subset \tilde\gamma$, $\mathrm{up}(\tilde\gamma)\widehat{\ }\langle\infty\rangle \subseteq \mathrm{up}(\gamma) \subset \Lambda_1$, and so $\tilde\gamma$ will not enumerate any number.

We conclude that $\alpha \subset \tilde\alpha$. Let $\alpha_0 \subset \tilde\alpha$ be the strategy determining if $\tilde\gamma$ enumerates its $\tilde x$ into A_0 or A_1 in the construction. Obviously, $\alpha \subseteq \alpha_0$; and since y enters A_i we even have $\alpha \subset \alpha_0$, say $\alpha\widehat{\ }\langle\beta_0\rangle \subseteq \alpha_0 \subset \tilde\alpha$. Then $\tilde x > \mathrm{par}(\beta_0)$ by the way requirements are assigned to nodes of T_2. But by our observation, $\mathrm{up}(\gamma) \subset \Lambda_1$ and so $\mathrm{up}(\gamma)\widehat{\ }\langle\gamma\rangle \subseteq \beta_0$; thus $\tilde x > \mathrm{par}(\beta_0) > \mathrm{par}(\gamma)$.

We have thus shown that $A >_{\mathrm{T}} \emptyset$ forces $\alpha\widehat{\ }\langle\beta\rangle \subset \Lambda_2$ (for some $\beta \in T_1$) for all $\alpha \subset \Lambda_2$ working on a requirement (10.2′). Thus Λ_2 is infinite, and all requirements are satisfied. ■

Sacks Jump Inversion Theorem (Sacks [**15**]). *For any set $J \geq_{\mathrm{T}} \emptyset'$ r.e. in $\emptyset'$, there is an r.e. set A with $A' \equiv_{\mathrm{T}} J$.*

Proof. We need to build an r.e. set A and p.r. functionals Γ and Δ satisfying, for all x, the requirements

$$J(x) = \lim_y \Gamma^A(x,y) \tag{13}$$

(establishing $J \leq_{\mathrm{T}} A'$ by the Limit Lemma), and

$$A'(x) = \Delta^{J \oplus \Lambda_1}(x) \tag{14}$$

(where $\Lambda_1 \leq_{\mathrm{T}} \emptyset'$ is the true path on T_1, thus establishing $A' \leq_{\mathrm{T}} J \oplus \emptyset'$). Since J is a Σ_2-set there is a recursive relation R such that $x \in J$ iff $\neg\forall y \exists s R(x,y,s)$. We ensure (13) by requiring

$$\forall y \exists s R(x,y,s) \rightarrow \forall y > y_0 \exists s \in S\big(\Gamma^{A_{s+1}}_{s+1}(x,y)\downarrow = 0 \;\&\; \gamma_{s+1}(x,y) = -1\big), \text{ and} \tag{15.1}$$

$$\neg\forall y \exists s R(x,y,s) \rightarrow \exists y \, \forall z \geq y \, \forall s \geq s_z \big(s \in S \rightarrow \Gamma^{A_{s+1}}_{s+1}(x,z)\downarrow = 1 \;\& \;\gamma_{s+1}(x,z) \leq \gamma_{s_z+1}(x,z)\big). \tag{15.2}$$

(Here y_0 and the s_z will be numbers determined by $\alpha \in T_2$ and the $\beta \in T_1$ working on (15). In the above, use -1 means that the oracle string is $\emptyset$.) The complexity level of this requirement is 2.

Recall that $A' = \{\, x \mid \Phi^A_x(x)\downarrow \,\}$. We ensure (14) by requiring

$$\exists s \in S \left(\Phi^{A_s}_{x,s}(x)\downarrow\right) \rightarrow \exists s \in S \, \forall t > s \big(\Delta^{\sigma \oplus (\beta^\frown\langle\gamma\rangle)}_{s+1}(x)\downarrow = 1 \;\&\; \Phi^{A_s}_{x,s}(x)\downarrow \;\& \; A_s \restriction (\varphi_{x,s}(x)+1) = A_t \restriction (\varphi_{x,s}(x)+1)\big), \text{ and} \tag{16.1}$$

$$\neg\exists s \in S \left(\Phi^{A_s}_{x,s}(x)\downarrow\right) \rightarrow \forall s \in S\big(\Delta^{\sigma \oplus (\beta^\frown\langle\infty\rangle)}_{s+1}(x)\downarrow = 0\big) \tag{16.2}$$

where $\sigma \subset J$ and $\beta^\frown\langle\gamma\rangle \subset \Lambda_1$ (or $\beta^\frown\langle\infty\rangle \subset \Lambda_1$, respectively) are determined by $\alpha \in T_2$ and the $\beta \in T_1$ working on (16), respectively. The complexity level of this requirement is 1.

We fix an effective ω-ordering $\{\, \mathcal{R}_e \,\}_{e \in \omega}$ of all the above requirements (in ascending order of x) and assign requirement $\mathcal{R}_e$ to all nodes $\alpha \in T_2$ with $|\alpha| = e$. For α working on (16), we set

$$\sigma(i) = \begin{cases} 0, & \text{if (for some } j) \ \alpha(j)\downarrow = \infty \text{ and } \alpha \restriction j \text{ works on (15) with } x = i, \\ 1, & \text{if (for some } j) \ \alpha(j)\downarrow \neq \infty \text{ and } \alpha \restriction j \text{ works on (15) with } x = i, \\ \uparrow, & \text{otherwise.} \end{cases}$$

In going from T_2 to T_1, we split up the left-hand sides of α's requirement (15) into the kth instance for $\beta \in T_1$ by bounding $y \leq \mathrm{par}(\beta)$ in (15). (The splitting-up of the right-hand sides is a bit more complicated, there we bound y by a number computable from β.) We do not split up requirement (16) from T_2 to T_1, but we identify $\beta \in T_1$ with the β in (16) (and γ will be the unique γ' with $\beta^\frown\langle\gamma'\rangle \subset \Lambda_1$ if it exists). In going from T_1 to T_0, we just split up β's (sub)requirement (for $\beta \in T_1$) into the lth instance for $\gamma \in T_0$ by bounding $s, z \leq \mathrm{par}(\gamma)$.

The action of a $\gamma \in T_0$ working on (the lth instance of the kth instance of) (15) is to measure if

$$\forall y \leq \mathrm{par}(\mathrm{up}(\gamma))\, \exists s \leq \mathrm{par}(\gamma) R(x, y, s). \tag{17}$$

If the answer is no then γ has outcome 0 and sets $\Gamma^A(x, y) = 1$ for all $y \leq \mathrm{par}(\gamma)$ (unless now defined to a different value). If the answer is yes then γ has outcome ∞; defines β_0 to be the longest $\beta' \subset \mathrm{up}(\gamma)$ with $\mathrm{up}(\beta') = \mathrm{up}^2(\gamma)$ and γ_0 to be the longest $\gamma' \subset \gamma$ with $\mathrm{up}(\gamma') = \beta_0$ (if β_0 exists); enumerates $\mathrm{par}(\gamma^-)$ into A (if β_0 exists) where $\gamma^- \subseteq \gamma$ is minimal with $\mathrm{up}^2(\gamma^-) = \mathrm{up}^2(\gamma)$ and $\mathrm{up}(\gamma^-) \supset \beta_0$; sets $\Gamma^A(x, y) = 0$ (unless now defined to a different value) for all y with $\mathrm{par}(\gamma_0) < y \leq \mathrm{par}(\gamma)$ (if γ_0 exists); and sets $\Gamma^A(x, y) = 1$ (unless now defined to a different value) for all other $y \leq \mathrm{par}(\gamma)$. The use $\gamma(x, y)$ for setting $\Gamma^A(x, y) = 0$ here is -1 (i.e. oracle string $\emptyset$), and the use $\gamma(x, y)$ for setting $\Gamma^A(x, y) = 1$ here is $\mathrm{par}(\gamma_*)$ where $\gamma_* \subseteq \gamma$ is minimal with $\mathrm{par}(\gamma_*) \geq y$, $\mathrm{up}(\gamma_*) \subseteq \mathrm{up}(\gamma)$, and $\mathrm{up}^2(\gamma_*) = \mathrm{up}^2(\gamma)$. (When using oracle string $\emptyset$, we still get a p.r. functional if we adopt the convention that if two contradictory definitions would apply to a fixed oracle then the definition enumerated first is the one used. In our example, there will never be contradictory definitions applying to the r.e. oracle A.)

The action of a $\gamma \in T_0$ working on (the kth instance of) (16) is to measure if

$$\exists s \leq \mathrm{par}(\gamma) \left(s \in S \;\&\; \Phi^{A_s}_{x,s}(x)\downarrow\right), \tag{18}$$

which, as for (9), just means

$$\Phi^{A_{\mathrm{par}(\gamma)}}_{x,\mathrm{par}(\gamma)}(x)\downarrow. \tag{18$'$}$$

(Here $A_{\mathrm{par}(\gamma)}$ is the subset of A enumerated before γ's action.) If the answer is no then γ has outcome 0 and sets $\Delta^{\sigma \oplus (\beta^\frown\langle\infty\rangle)}(x) = 0$ (for its σ and β, unless previously set to 1 for a compatible initial segment of the oracle); otherwise γ has outcome ∞ and sets $\Delta^{\sigma \oplus (\beta^\frown\langle\gamma\rangle)}(x) = 1$ (for its σ and β, unless previously set to 0 for a compatible initial segment of the oracle).

We now verify that the construction satisfies all the requirements.

We first observe a combinatorial fact about infinite injury, namely that if $\gamma \subset \Lambda_0$, $\mathrm{up}(\gamma) \subset \Lambda_1$, and $\mathrm{up}^2(\gamma) \subset \Lambda_2$ then for all γ' with $\gamma \subset \gamma' \subset \Lambda_0$, either $\mathrm{up}^2(\gamma)$ and $\mathrm{up}^2(\gamma')$ are comparable, or $\mathrm{up}^2(\gamma) \supseteq \bar{\alpha}^\frown\langle\infty\rangle$ and $\mathrm{up}^2(\gamma') \supseteq \bar{\alpha}^\frown\langle\bar{\beta}\rangle$ for some $\bar{\alpha}$ and $\bar{\beta}$ with $\mathrm{lev}(\bar{\alpha}) = 2$. Suppose not, and say $\mathrm{up}^2(\gamma)$ and $\mathrm{up}^2(\gamma')$ split at $\bar{\alpha}$. First assume $\mathrm{lev}(\bar{\alpha}) = 1$. Then $\bar{\beta} \subset \mathrm{up}(\gamma)$ for some $\bar{\beta}$ with $\mathrm{up}(\bar{\beta}) = \bar{\alpha}$, and so $\Lambda_1 \restriction (|\bar{\beta}|+1) \subseteq \lambda(\gamma')$ by $\mathrm{up}(\gamma) \subset \Lambda_1$, contradicting $\mathrm{up}^2(\gamma)$ and $\mathrm{up}^2(\gamma')$ splitting at $\bar{\alpha}$. So assume $\bar{\alpha}^\frown\langle\bar{\beta}\rangle \subseteq \mathrm{up}^2(\gamma)$ for some $\bar{\beta} \in T_1$. Then $\bar{\beta}^\frown\langle\infty\rangle \subseteq \mathrm{up}(\gamma) \subset \Lambda_1$, and so again $\Lambda_1 \restriction (|\bar{\beta}| + 1) \subseteq \lambda^2(\gamma')$, contradicting $\mathrm{up}^2(\gamma)$ and $\mathrm{up}^2(\gamma')$ splitting at $\bar{\alpha}$.

Now suppose $x_0 \in J$. Then $\alpha^\frown\langle\beta\rangle \subset \Lambda_2$ (for some β) for the unique $\alpha \subset \Lambda_2$ working on (15) with this x_0. Thus $\beta^\frown\langle\infty\rangle \subset \Lambda_1$, and $\neg\forall y \leq \mathrm{par}(\beta)\exists s R(x_0, y, s)$. Let $\gamma \subset \Lambda_0$ be minimal with $\mathrm{up}(\gamma) = \beta$. Then for all γ' with $\gamma \subseteq \gamma' \subset \Lambda_0$, we have $\lambda(\gamma') \supseteq \beta$. Any such γ' with $\mathrm{up}(\gamma') \supseteq \beta$ working on (15) with x_0 will measure (17) negatively (since $\mathrm{par}(\mathrm{up}(\gamma')) \geq \mathrm{par}(\beta)$) and thus set $\Gamma^A(x_0, y) = 1$. For any $\gamma' \subset \Lambda_0$ with $\mathrm{up}(\gamma') \subset \beta$ working on (15) with this x_0, if γ' sets $\Gamma^A(x_0, y) = 0$ then $\mathrm{up}(\gamma')^\frown\langle\gamma'\rangle \subset \beta$ by $\beta \subset \Lambda_1$, and thus $\gamma' \subset \gamma$. Therefore $\Gamma^A(x_0, y) = 1$ for all $y \geq \mathrm{par}(\gamma)$. (Note that $\Gamma^A(x_0, y)$ is defined for all y since all $\gamma' \subset \Lambda_0$ with $\mathrm{up}(\gamma') \subset \Lambda_1$ and $\mathrm{up}^2(\gamma') = \alpha$ define $\Gamma^A(x_0, y)$

with the same use $\mathrm{par}(\gamma_*)$ for the fixed string γ_* that γ' uses for y if $y \le \mathrm{par}(\gamma')$ unless already set to 0 with use -1.)

On the other hand, suppose $x_0 \notin J$. Then $\alpha^\frown\langle\infty\rangle \subset \Lambda_2$ for the unique $\alpha \subset \Lambda_2$ working on (15) with this x_0. Thus $\beta^\frown\langle\gamma_\beta\rangle \subset \Lambda_1$ (for some γ_β) for all $\beta \subset \Lambda_1$ with $\mathrm{up}(\beta) = \alpha$; and for all these β (except for β_0, the least of them), γ_β will try to ensure $\Gamma^A(x_0, y) = 0$ with use -1 for all $y \in (\mathrm{par}(\gamma_{\beta^-}), \mathrm{par}(\gamma_\beta)]$ (where β^- is the maximal $\beta' \subset \beta$ with $\mathrm{up}(\beta') = \mathrm{up}(\beta)$). This will establish $\Gamma^A(x_0, y) = 0$ with use -1 for all $y > \mathrm{par}(\gamma_{\beta_0})$ (and $\Gamma^A(x_0, y)$ is defined for $y \le \mathrm{par}(\gamma_{\beta_0})$ as above).

Thus suppose $\Gamma^A(x_0, y)\downarrow = 1$ for some (least) $y > \mathrm{par}(\gamma_{\beta_0})$. Then some (minimal) γ_β would like to set it to 0. Thus $\mathrm{par}(\gamma_{\beta^-}) < y \le \mathrm{par}(\gamma_\beta)$, and some $\bar\gamma$ with $\gamma_{\beta^-} \subset \bar\gamma \subset \gamma_\beta$ set $\Gamma^A(x_0, y) = 1$. Set $\bar\alpha = \mathrm{up}^2(\bar\gamma)$ and $\bar\beta = \mathrm{up}(\bar\gamma)$.

First suppose $\alpha \ne \bar\alpha$. Since both work on the same x_0, they must be incomparable, say they split at $\tilde\alpha$.

By our observation, $\tilde\alpha^\frown\langle\infty\rangle \subseteq \alpha$ and $\tilde\alpha^\frown\langle\tilde\beta\rangle \subseteq \bar\alpha$ for some $\tilde\beta \in T_1$, and so $\lambda(\bar\gamma) \supseteq \bar\beta \supseteq \tilde\beta^\frown\langle\infty\rangle$ but $\lambda(\gamma_\beta) \not\supseteq \tilde\beta^\frown\langle\infty\rangle$. Pick $\hat\gamma$ maximal with $\bar\gamma \subset \hat\gamma \subset \gamma_\beta$ and $\lambda(\hat\gamma) \supseteq \tilde\beta^\frown\langle\infty\rangle$. Set $\hat\alpha = \mathrm{up}^2(\hat\gamma)$ and $\hat\beta = \mathrm{up}(\hat\gamma)$. So $\hat\beta \subseteq \tilde\beta$. We will show $\hat\beta = \tilde\beta$ and thus $\hat\alpha = \tilde\alpha$. First suppose $\hat\alpha \ne \tilde\alpha$. Now $\hat\alpha \subset \tilde\alpha$ is impossible by $\tilde\alpha \subset \Lambda_2$ and $\hat\beta \subseteq \tilde\beta$. Also $\tilde\alpha \subset \hat\alpha$ implies $\tilde\alpha^\frown\langle\tilde\beta\rangle \subseteq \hat\alpha$ by $\lambda(\hat\gamma) \supseteq \tilde\beta^\frown\langle\infty\rangle$, contradicting $\hat\beta \subseteq \tilde\beta$. So $\tilde\alpha$ and $\hat\alpha$ must be incomparable, say they split at $\check\alpha$. By our observation, $\check\alpha^\frown\langle\check\beta\rangle \subseteq \hat\alpha$ and $\check\alpha^\frown\langle\infty\rangle \subseteq \tilde\alpha$ for some $\check\beta \subset \hat\beta$, contradicting $\hat\beta \subseteq \tilde\beta$. Thus $\hat\alpha = \tilde\alpha$. But then $\hat\beta \subset \tilde\beta$ contradicts $\mathrm{up}(\hat\beta) = \mathrm{up}(\tilde\beta)$ (as $\tilde\beta$ changes outcome), so $\hat\beta = \tilde\beta$. By $\gamma_{\beta^-} \subset \bar\gamma$ and $\beta^- \subset \Lambda_1$, we have $\beta^- \subset \tilde\beta$. By $\tilde\alpha \subset \alpha$, there is some $\tilde\beta_0 \subset \beta^-$ with $\mathrm{up}(\tilde\beta_0) = \tilde\alpha$. Thus $\hat\gamma$ enumerates $\mathrm{par}(\hat\gamma^-)$ into A for its node $\gamma^- = \hat\gamma^-$. By $\bar\beta \supseteq \tilde\beta^\frown\langle\infty\rangle$ and $\bar\alpha \supset \tilde\alpha$, necessarily $\hat\gamma^- \subset \bar\gamma$, and so $\mathrm{par}(\hat\gamma^-) < \mathrm{par}(\bar\gamma)$. Thus $\hat\gamma^-$ would destroy the computation $\Gamma^A(x_0, y) = 1$ that was defined by $\bar\gamma$ as desired.

So α and $\bar\alpha$ must be equal. By $\beta^- \subset \Lambda_1$ and $\gamma_{\beta^-} \subset \bar\gamma$, we must have $\beta^{-\frown}\langle\gamma_{\beta^-}\rangle \subseteq \bar\beta$. But then γ_β would put a number $\le \mathrm{par}(\bar\gamma)$ into A, destroying $\bar\gamma$'s definition of $\Gamma^A(x_0, y)$, yielding the desired contradiction. Thus $\Gamma^A(x_0, y) = 0$ for all $y > \mathrm{par}(\gamma_{\beta_0})$.

As for the other type of requirements, suppose $\alpha \subset \Lambda_2$ is the unique strategy working on some fixed requirement (16). We will show first that any $\gamma \subset \Lambda_0$ with $\mathrm{up}(\gamma) \subset \Lambda_1$ and $\mathrm{up}^2(\gamma) = \alpha$ can define $\Delta^{J\oplus\Lambda_1}(x)$ for its x without being prevented from doing so by a previous definition. For the sake of a contradiction, suppose some $\gamma' \subset \Lambda_0$ sets $\Delta^{\sigma'\oplus\tau'}(x)$ to a different value for $\sigma' \oplus \tau'$ compatible with γ's intended oracle $\sigma \oplus \tau$. Since each requirement $\mathcal{R}_e$ is worked on exactly by all $\alpha' \in T_2$ with $|\alpha'| = e$, we must have $|\sigma| = |\sigma'|$ and thus $\sigma = \sigma'$. By the construction, $\beta = \mathrm{up}(\gamma) \subset \tau$ and $\beta' = \mathrm{up}(\gamma') \subset \tau'$; and by compatibility, $\beta' \subset \beta$ or $\beta \subset \beta'$. Furthermore, $|\mathrm{up}(\beta)| = |\mathrm{up}(\beta')|$, and again by the splitting-up from T_2 to T_1 and $\mathrm{lev}(\beta) = \mathrm{lev}(\beta') = 1$, $\mathrm{up}(\beta)$ and $\mathrm{up}(\beta')$ must be incomparable, say they split at $\bar\alpha$. Since β and β' are comparable, we must have $\mathrm{up}(\beta) \supseteq \bar\alpha^\frown\langle\infty\rangle$ or $\mathrm{up}(\beta') \supseteq \bar\alpha^\frown\langle\infty\rangle$. By $\sigma = \sigma'$, $\bar\alpha$ cannot work on a requirement (15), thus it works on a requirement (16). By $\mathrm{lev}(\beta) = 1$, there is a unique $\bar\beta$ such that $\mathrm{up}(\bar\beta) = \bar\alpha$, $\bar\beta \subset \beta$, and $\bar\beta \subset \beta'$. But β and β' make different predictions on the outcome of $\bar\beta$, a contradiction. This establishes that γ can define $\Delta^{J\oplus\Lambda_1}(x)$.

Now suppose first that $\alpha^\frown\langle\infty\rangle \subset \Lambda_2$ for the unique $\alpha \subset \Lambda_2$ working on a requirement (16). Then $\beta^\frown\langle\infty\rangle \subset \Lambda_1$ for the unique $\beta \subset \Lambda_1$ working on this requirement, so all $\gamma \subset \Lambda_0$ with $\mathrm{up}(\gamma) = \beta$ will measure $(18')$ negatively, i.e. $\Phi^A_x(x)\uparrow$. Furthermore, the least such γ, say γ_0, will set $\Delta^{\sigma\oplus(\beta^\frown\langle\infty\rangle)}(x) = 0$.

On the other hand, suppose $\alpha^\frown\langle\gamma\rangle \subset \Lambda_2$ (for some $\gamma \in T_0$) where α works on (16). Then $\beta^\frown\langle\gamma\rangle \subset \Lambda_1$ for the unique $\beta \subset \Lambda_1$ with $\mathrm{up}(\beta) = \alpha$, and $\gamma^\frown\langle\infty\rangle \subset \Lambda_0$. Thus γ measures $(18')$ positively and will set $\Delta^{\sigma\oplus(\beta^\frown\langle\gamma\rangle)}(x) = 1$. So we need to show $\Phi^A_x(x)\downarrow$.

It suffices to show that $A_{\mathrm{par}(\gamma)} \upharpoonright (\varphi_{x,\mathrm{par}(\gamma)}(x){+}1) = A \upharpoonright (\varphi_{x,\mathrm{par}(\gamma)}(x){+}1)$. Note that $\mathrm{par}(\gamma) > \varphi_{x,\mathrm{par}(\gamma)}(x)$. For the sake of a contradiction, suppose some $\bar\gamma$ with $\gamma \subset \bar\gamma \subset \Lambda_0$ puts some $y \leq \mathrm{par}(\gamma)$ into A where $y = \mathrm{par}(\bar\gamma^-)$ for $\bar\gamma$'s node $\gamma^- = \bar\gamma^-$ from the construction. We set $\bar\beta^- = \mathrm{up}(\bar\gamma^-)$. By $\mathrm{par}(\bar\gamma^-) \leq \mathrm{par}(\gamma)$, we have $\bar\gamma^- \subset \gamma$. We set $\bar\alpha = \mathrm{up}^2(\bar\gamma)$ and $\bar\beta = \mathrm{up}(\bar\gamma)$. We denote by $\bar\beta_0$ and $\bar\gamma_0$ the nodes β_0 and γ_0 of $\bar\gamma$ from the construction, respectively. Then $\bar\beta_0{}^\frown\langle\bar\gamma_0\rangle \subseteq \bar\beta$. We proceed by comparing the positions of α and $\bar\alpha$ on T_2. As $\beta \subset \Lambda_1$, $\bar\beta_0 \subset \beta \subset \bar\beta$.

First, $\alpha \subset \bar\alpha$ is impossible since α changes outcome at γ, and thus $\lambda^2(\bar\gamma_0) \supseteq \alpha^\frown\langle\infty\rangle$ and $\lambda^2(\bar\gamma) \supseteq \alpha^\frown\langle\gamma\rangle$, contradicting $\bar\alpha \subseteq \lambda^2(\bar\gamma_0), \lambda^2(\bar\gamma)$.

Next, suppose α and $\bar\alpha$ are incomparable, say they split at $\tilde\alpha$. By our observation, $\alpha \supseteq \tilde\alpha^\frown\langle\infty\rangle$ and $\bar\alpha \supseteq \tilde\alpha^\frown\langle\tilde\beta\rangle$ for some $\tilde\beta \in T_1$, thus $\lambda(\bar\gamma_0) \supseteq \tilde\beta^\frown\langle\infty\rangle$ and $\lambda(\gamma) \not\supseteq \tilde\beta^\frown\langle\infty\rangle$, so $\lambda(\bar\gamma) \supseteq \tilde\beta^\frown\langle\infty\rangle$ is impossible by $\bar\gamma_0 \subset \gamma \subset \bar\gamma$.

Next, $\bar\alpha^\frown\langle\tilde\beta\rangle \subseteq \alpha$ (for some $\tilde\beta \in T_1$) is impossible since then $\tilde\beta^\frown\langle\infty\rangle \subseteq \beta$; thus, by $\gamma \subset \bar\gamma$ and $\lambda(\bar\gamma) \supseteq \tilde\beta^\frown\langle\infty\rangle$, we have $\mathrm{up}(\bar\gamma) = \tilde\beta$, and so $\bar\gamma$ will not enumerate any number by $\mathrm{up}(\bar\gamma)^\frown\langle\infty\rangle \subset \Lambda_1$.

Finally, assume $\bar\alpha^\frown\langle\infty\rangle \subseteq \alpha$. By $\bar\gamma_0{}^\frown\langle\infty\rangle \subseteq \bar\gamma^- \subset \gamma$, we must have $\bar\beta_0{}^\frown\langle\bar\gamma_0\rangle \subseteq \bar\beta^-$ and $\bar\beta_0{}^\frown\langle\bar\gamma_0\rangle \subseteq \beta$. By $\beta = \mathrm{up}(\gamma) \subset \Lambda_1$, we must have $\lambda(\gamma') \supseteq \beta$ for all γ' with $\gamma \subseteq \gamma' \subset \Lambda_0$; thus β and $\bar\beta$ must be comparable. If $\bar\beta \subset \beta$ then by $\beta \subset \Lambda_1$, we have $\bar\beta^\frown\langle\bar\gamma\rangle \subseteq \beta$, contradicting $\gamma \subset \bar\gamma$. Thus $\beta^\frown\langle\gamma\rangle \subseteq \bar\beta$ by $\gamma^\frown\langle\infty\rangle \subseteq \bar\gamma$. By our assumption on $\bar\gamma$ and $\bar\gamma_0$, there are no β' with $\bar\beta_0 \subset \beta' \subset \beta$ and $\mathrm{up}(\beta') = \mathrm{up}(\bar\beta_0)$, and by $\bar\alpha^\frown\langle\infty\rangle \subseteq \alpha$, there can be no $\gamma' \subset \gamma$ with $\mathrm{up}(\gamma') = \bar\beta$; so β and $\bar\beta^-$ must be incomparable, say they split at $\tilde\beta$. Note that $\tilde\beta \supseteq \bar\beta_0{}^\frown\langle\bar\gamma_0\rangle$ since $\beta, \bar\beta^- \supseteq \bar\beta_0{}^\frown\langle\bar\gamma_0\rangle$; and that $\bar\beta^- \supseteq \tilde\beta^\frown\langle\infty\rangle$ since $\bar\gamma^- \subset \gamma$.

We now set $\tilde\alpha = \mathrm{up}(\tilde\beta)$ and compare the positions of $\bar\alpha$ and $\tilde\alpha$ on T_2 (still under the assumption $\bar\alpha^\frown\langle\infty\rangle \subseteq \alpha$).

If $\tilde\alpha \subset \bar\alpha$ then, by our observation, $\mathrm{lev}(\alpha) = 2$ and $\tilde\alpha^\frown\langle\infty\rangle \subseteq \bar\alpha$ by $\bar\beta_0 \subset \tilde\beta$ and $\bar\alpha \subset \Lambda_2$. But this is impossible by $\bar\beta^- \supseteq \tilde\beta^\frown\langle\infty\rangle$.

If $\bar\alpha \subset \tilde\alpha$ then, by $\tilde\beta \subset \bar\beta$, we must have $\mathrm{lev}(\tilde\alpha) = 2$ and $\bar\alpha^\frown\langle\infty\rangle \subseteq \tilde\alpha$, contradicting $\bar\beta \supseteq \tilde\beta^\frown\langle\infty\rangle$.

If $\bar\alpha$ and $\tilde\alpha$ are incomparable, say they split at $\check\alpha$, then, by our observation, we must have $\bar\alpha \supseteq \check\alpha^\frown\langle\infty\rangle$ and $\tilde\alpha \supseteq \check\alpha^\frown\langle\check\beta\rangle$ for some $\check\beta \in T_1$, which in turn contradicts $\tilde\beta \subset \bar\beta$ and $\bar\alpha = \mathrm{up}(\bar\beta)$.

We have now established that the existence of a $\bar\gamma \supset \gamma$ destroying $\Phi^A_x(x)$ is impossible, so $\Phi^A_x(x)\downarrow$ as desired. ■

Sacks Density Theorem (Sacks [**16**]). *For any r.e. set $D <_{\mathrm{T}} C$ there is an r.e. set A such that $D <_{\mathrm{T}} A <_{\mathrm{T}} C$.*

Proof. We will actually build an r.e. set A such that if $D <_{\mathrm{T}} C \oplus D$ then $D <_{\mathrm{T}} A \oplus D <_{\mathrm{T}} C \oplus D$. So we need to build A satisfying

$$A \leq_{\mathrm{T}} C \oplus D \tag{19}$$

and, for all p.r. functionals Φ and Ψ,

$$C = \Phi^{A \oplus D} \to C \leq_{\mathrm{T}} D, \text{ and} \tag{20}$$

$$A = \Psi^{D} \to C \leq_{\mathrm{T}} D. \tag{21}$$

We could ensure (19) by building an explicit reduction. Instead, we will show at the end that $A \leq_{\mathrm{T}} C \oplus D$ essentially by showing that Λ_2 is co-r.e. in $C \oplus D$. We ensure (20) and (21) by strategies $\alpha \in T_3$ threatening to build a reduction Γ_α or Δ_α, respectively, showing $C \leq_T D$. (Of course, once we have established $C \leq_{\mathrm{T}} D$ for some requirement, we do not have to ensure the lower-priority requirements since the hypothesis of the theorem fails.) Requirements (20) and (21) then take the form

$$\begin{aligned} &\forall x \exists s \forall y \leq x \forall t \geq s \big(C_s(y) = \Phi_s^{A_s \oplus D_s}(y)\downarrow \;\&\; (A_s \oplus D_s) \upharpoonright (\varphi_s(y)+1) = \\ &\qquad (A_t \oplus D_t) \upharpoonright (\varphi_s(y)+1)\big) \to \\ &\forall x > x_0 \exists s \forall y \forall t > s \big(x_0 < y \leq x \to C_s(y) = \Gamma_{\alpha,s+1}^{D_s}(y)\downarrow \;\&\; \\ &\qquad D_s \upharpoonright (\gamma_{\alpha,s}(y)+1) = D_t \upharpoonright (\gamma_{\alpha,s}(y)+1)\big), \end{aligned} \tag{22}$$

and

$$\begin{aligned} &\forall x \exists s \forall y \leq x \forall t \geq s \big(A_s(y) = \Psi_s^{D_s}(y)\downarrow \;\&\; \\ &\qquad D_s \upharpoonright (\psi_s(y)+1) = D_t \upharpoonright (\psi_s(y)+1)\big) \to \\ &\forall x \exists s \forall y \leq x \forall t > s \big(C_s(y) = \Delta_{\alpha,s+1}^{D_s}(y)\downarrow \;\&\; \\ &\qquad D_s \upharpoonright (\delta_{\alpha,s}(y)+1) = D_t \upharpoonright (\delta_{\alpha,s}(y)+1)\big). \end{aligned} \tag{23}$$

(Here x_0 will be a number determined by $\alpha \in T_3$. Note that (22) and (23) both only use $\neg\rho \to \tau$ in (7) whereas $\rho \to \sigma$ is vacuous.)

We fix an effective ω-ordering of the above requirements (22) and (23) and assign requirement $\mathcal{R}_e$ to all nodes in $\{\, \alpha \in T_3 \mid |\alpha| = e \;\&\; \forall i < e(\alpha(i) \neq \infty) \,\}$ since outcome ∞ of any requirement corresponds to showing $C \leq_{\mathrm{T}} D$.

In going from T_3 to T_2, we split up α's requirement into the kth instance for $\beta \in T_2$ by bounding $x \leq k$, except in the left-hand side of (23) where we bound $x \leq \mathrm{par}(\beta)$.

In going from T_2 to T_1, and from T_1 to T_0, we split up a subrequirement into its lth, or mth, instance for $\gamma \in T_1$, or $\delta \in T_0$, by bounding $s \leq l$, or $t \leq \mathrm{par}(\delta)$, respectively.

The action of a $\delta \in T_0$ working on (the mth instance of the lth instance of the kth instance of) (22) is to measure if

$$\begin{aligned} &\forall x \leq k \exists s \leq l \, \forall y \leq x \forall t \in [s, \mathrm{par}(\delta)] \big(C_s(y) = \Phi_s^{A_s \oplus D_s}(y)\downarrow \;\&\; \\ &\qquad (A_s \oplus D_s) \upharpoonright (\varphi_s(y)+1) = (A_t \oplus D_t) \upharpoonright (\varphi_s(y)+1)\big), \end{aligned} \tag{24}$$

which just means

$$\begin{aligned} C_l \restriction (k+1) = \Phi_l^{A_l \oplus D_l} \restriction (k+1)\downarrow \ \& \\ (A_l \oplus D_l) \restriction (\varphi_l(k)+1) = (A_{\mathrm{par}(\delta)} \oplus D_{\mathrm{par}(\delta)}) \restriction (\varphi_l(k)+1). \end{aligned} \tag{24'}$$

(This uses the tacit assumption that use functions are increasing in the argument and nondecreasing in the stage. Here and in (25), $A_{\mathrm{par}(\delta)}$ and $D_{\mathrm{par}(\delta)}$ are the sets A and D enumerated before δ's action.) If the answer is yes then δ has outcome 0 (recall that (24′) is an instance of $\neg\rho$ in (7)) and defines $\Gamma_\alpha^D(k) = C_l(k)$ with use $\gamma_\alpha(k) = \varphi_l(k)$ (unless already defined to a different value) where $\alpha = \mathrm{up}^3(\delta)$; otherwise δ has outcome ∞.

The action of a $\delta \in T_0$ working on (the mth instance of the lth instance of the kth instance of) (23) is to measure if

$$\begin{aligned} \forall x \le \mathrm{par}(\beta) \exists s \le l\, \forall y \le x \forall t \in [s, \mathrm{par}(\delta)] \big(A_s(y) = \Psi_s^{D_s}(y)\downarrow \ \& \\ D_s \restriction (\psi_s(y)+1) = D_t \restriction (\psi_s(y)+1) \big), \end{aligned} \tag{25}$$

which just means

$$\begin{aligned} A_l \restriction (\mathrm{par}(\beta)+1) = \Psi_l^{D_l} \restriction (\mathrm{par}(\beta)+1)\downarrow \ \& \\ D_l \restriction (\psi_l(\mathrm{par}(\beta))+1) = D_{\mathrm{par}(\delta)} \restriction (\psi_l(\mathrm{par}(\beta))+1). \end{aligned} \tag{25'}$$

Independent of the answer to (25′), δ enumerates $\mathrm{par}(\beta)$ into A if now $\Delta_\alpha^D(k)\downarrow \neq C(k)$. If the answer to (25′) is yes then δ has outcome 0 (again recall that (25′) is an instance of $\neg\rho$ in (7)) and defines $\Delta_\alpha^D(k) = C_l(k)$ (unless already defined to a different value) with use $\delta_\alpha(k) = \psi_l(\mathrm{par}(\beta))$. (Here $\alpha = \mathrm{up}^3(\delta)$ and $\beta = \mathrm{up}^2(\delta)$.) If the answer to (25′) is no then δ has outcome ∞.

We now verify that the construction satisfies all requirements (22) and (23) (up to the highest-priority one showing $C \le_{\mathrm{T}} D$, if any), and that $A \le_{\mathrm{T}} C \oplus D$.

We observe first that on T_3, $\alpha\widehat{\ }\langle\infty\rangle \subseteq \alpha'$ implies that no requirement is assigned to α'; therefore on T_2, $\beta \subseteq \beta'$ (if subrequirements are assigned to β and β') implies $\mathrm{up}(\beta) \subseteq \mathrm{up}(\beta')$ as follows: If $\mathrm{up}(\beta) \not\subseteq \mathrm{up}(\beta')$, say $\mathrm{up}(\beta) \supseteq \alpha_0\widehat{\ }\langle\beta_0\rangle$ and $\mathrm{up}(\beta') \not\supseteq \alpha_0\widehat{\ }\langle\beta_0\rangle$ for some minimal α_0 and some β_0, then $\beta \supseteq \beta_0\widehat{\ }\langle\infty\rangle$ but $\beta' \not\supseteq \beta_0\widehat{\ }\langle\infty\rangle$, contradicting $\beta \subseteq \beta'$. Furthermore, if β_1 and β_2 are incomparable on T_2 and $\mathrm{up}(\beta_1) \subseteq \mathrm{up}(\beta_2)$ then β_1 and β_2 split at a $\beta \in T_2$ with $\mathrm{up}(\beta) = \mathrm{up}(\beta_1)$. This is because otherwise $\mathrm{up}(\beta) \subset \mathrm{up}(\beta_1)$, say $\mathrm{up}(\beta)\widehat{\ }\langle\bar\beta\rangle \subseteq \mathrm{up}(\beta_1)$, and so $\bar\beta\widehat{\ }\langle\infty\rangle \subseteq \beta_1, \beta_2$ and $\beta \subseteq \bar\beta$ by (3.3), contradicting β_1 and β_2 splitting at β.

First suppose that $C = \Phi^{A\oplus D}$ (and none of the higher-priority requirements has shown $C \le_{\mathrm{T}} D$). Then $\alpha\widehat{\ }\langle\infty\rangle = \Lambda_3$ for the unique $\alpha \subset \Lambda_3$ working on this requirement. Thus $\beta_k\widehat{\ }\langle\gamma_k\rangle \subset \Lambda_2$ (for some $\gamma_k \in T_1$) for all k where β_k is the unique $\beta \subset \Lambda_2$ working on the kth instance of α's requirement. Since $\gamma_k\widehat{\ }\langle\infty\rangle \subset \Lambda_1$, all $\delta \subset \Lambda_0$ with $\mathrm{up}(\delta) = \gamma_k$ answer (24′) positively, and so one of them defines $\Gamma_\alpha^D(k)$ D-correctly (unless some other strategy does so). So Γ_α^D is total. Suppose $\Gamma_\alpha^D(k) \neq C(k)$ for some k. Then some $\delta \subset \Lambda_0$ last defines $\Gamma_\alpha^D(k) = 0$ D-correctly, but $k \in C$ and some (least) $\bar\delta \supset \delta$ enumerates some $y \le \frac{1}{2}\varphi_{\mathrm{par}(\delta)}(k)$ into A (since $\Phi^{A\oplus D}(k) = C(k)$). We set $\beta = \mathrm{up}^2(\delta)$, $\gamma = \mathrm{up}(\delta)$, $\bar\alpha = \mathrm{up}^3(\bar\delta)$, $\bar\beta = \mathrm{up}^2(\bar\delta)$, and $\bar\gamma = \mathrm{up}(\bar\delta)$. Then $y = \mathrm{par}(\bar\beta)$. Notice that $\varphi_l(k) < l \le \mathrm{par}(\gamma)$ (where l bounds s for γ above) and $\varphi_{\mathrm{par}(\delta)}(k) < \mathrm{par}(\delta)$ by the usual convention of stage bounding use. Below, we will typically show $\mathrm{par}(\gamma) < \mathrm{par}(\bar{\beta})$ or $\mathrm{par}(\delta) < \mathrm{par}(\bar{\beta})$ for a contradiction.

We argue by cases, comparing the positions of α and $\bar{\alpha}$ on T_3. First assume that α and $\bar{\alpha}$ are incomparable, say $\alpha \supseteq \alpha_0{}^\frown\langle\beta_0\rangle$ and $\bar{\alpha} \supseteq \alpha_0{}^\frown\langle\bar{\beta}_0\rangle$ for $\beta_0 \neq \bar{\beta}_0$. (Recall that there is no outcome ∞ on T_3.) Thus $\beta \supseteq \beta_0{}^\frown\langle\infty\rangle$ and $\bar{\beta} \supseteq \bar{\beta}_0{}^\frown\langle\infty\rangle$. We distinguish subcases, comparing the positions of β_0 and $\bar{\beta}_0$.

First assume $\beta_0 \subset \bar{\beta}_0$. Then $\beta_0{}^\frown\langle\gamma_0\rangle \subseteq \bar{\beta}_0$ for some γ_0 since $\mathrm{up}(\beta_0) = \mathrm{up}(\bar{\beta}_0)$. Thus $\bar{\gamma} \supseteq \gamma_0{}^\frown\langle\infty\rangle$ but $\gamma \not\supseteq \gamma_0{}^\frown\langle\infty\rangle$. If $\gamma \subseteq \gamma_0$ then $\mathrm{par}(\gamma) \leq \mathrm{par}(\gamma_0) < \mathrm{par}(\bar{\beta}_0) < \mathrm{par}(\bar{\beta})$, a contradiction. Thus γ and $\gamma_0{}^\frown\langle\infty\rangle$ are incomparable. Since $\delta \subset \bar{\delta}$ we must have $\gamma \supseteq \tilde{\gamma}{}^\frown\langle\infty\rangle$ and $\bar{\gamma} \supseteq \gamma_0{}^\frown\langle\infty\rangle \supseteq \tilde{\gamma}{}^\frown\langle\tilde{\delta}\rangle$ for some $\tilde{\gamma}$ and some $\tilde{\delta} \supset \delta$, again a contradiction by $\mathrm{par}(\bar{\beta}) > \mathrm{par}(\bar{\beta}_0) > \mathrm{par}(\gamma_0) > \mathrm{par}(\tilde{\delta}) > \mathrm{par}(\delta)$.

Next assume $\bar{\beta}_0 \subset \beta_0$. Then $\bar{\beta}_0{}^\frown\langle\bar{\gamma}_0\rangle \subseteq \beta_0$ for some $\bar{\gamma}_0$ since $\mathrm{up}(\bar{\beta}_0) = \mathrm{up}(\beta_0)$. Thus $\gamma \supseteq \bar{\gamma}_0{}^\frown\langle\infty\rangle$ but $\bar{\gamma} \not\supseteq \bar{\gamma}_0{}^\frown\langle\infty\rangle$. Now $\alpha_0{}^\frown\langle\beta_0\rangle \subseteq \alpha \subset \Lambda_3$, so $\bar{\beta}_0{}^\frown\langle\bar{\gamma}_0\rangle \subseteq \beta_0 \subset \Lambda_2$ and $\bar{\gamma}_0{}^\frown\langle\infty\rangle \subset \Lambda_1$. Thus $\bar{\gamma} \subseteq \bar{\gamma}_0$ is impossible by $\bar{\delta} \supset \delta$. Also $\bar{\gamma}_0{}^\frown\langle\infty\rangle \subseteq \gamma \subseteq \lambda(\delta)$, so $\bar{\gamma}_0{}^\frown\langle\infty\rangle \subseteq \lambda(\delta')$ for all δ' with $\delta \subseteq \delta' \subset \Lambda_0$, contradicting $\bar{\gamma} \subseteq \lambda(\bar{\delta})$ and $\delta \subset \bar{\delta} \subset \Lambda_0$.

Finally assume that β_0 and $\bar{\beta}_0$ are incomparable, say they split at β_1. By our observation above, $\mathrm{up}(\beta_1) = \alpha_0$, and there are distinct γ_1 and $\bar{\gamma}_1$ such that $\beta_1{}^\frown\langle\gamma_1\rangle \subseteq \beta_0$ and $\beta_1{}^\frown\langle\bar{\gamma}_1\rangle \subseteq \bar{\beta}_0$. So $\gamma_1{}^\frown\langle\infty\rangle \subseteq \gamma$ and $\bar{\gamma}_1{}^\frown\langle\infty\rangle \subseteq \bar{\gamma}$. Thus γ and $\bar{\gamma}_1{}^\frown\langle\infty\rangle$ are incomparable. Since $\delta \subset \bar{\delta}$ we must have $\gamma \supseteq \tilde{\gamma}{}^\frown\langle\infty\rangle$ and $\bar{\gamma} \supseteq \bar{\gamma}_1{}^\frown\langle\infty\rangle \supseteq \tilde{\gamma}{}^\frown\langle\tilde{\delta}\rangle$ for some $\tilde{\gamma}$ and some $\tilde{\delta} \supset \delta$, again a contradiction by $\mathrm{par}(\bar{\beta}) > \mathrm{par}(\bar{\beta}_0) > \mathrm{par}(\bar{\gamma}_1) > \mathrm{par}(\tilde{\delta}) > \mathrm{par}(\delta)$.

We have thus established that α and $\bar{\alpha}$ must be comparable. So assume next that $\alpha \subset \bar{\alpha}$, say $\alpha{}^\frown\langle\beta_0\rangle \subseteq \bar{\alpha}$ for some β_0. Then $\beta_0{}^\frown\langle\infty\rangle \subseteq \bar{\beta}$. We distinguish subcases, comparing the positions of β and β_0.

First assume $\beta \subset \beta_0$. Then $\beta{}^\frown\langle\gamma_0\rangle \subseteq \beta_0$ for some γ_0 since $\mathrm{up}(\beta) = \mathrm{up}(\beta_0)$, and so $\bar{\gamma} \supseteq \gamma_0{}^\frown\langle\infty\rangle$ but $\gamma \not\supseteq \gamma_0{}^\frown\langle\infty\rangle$. If $\gamma \subseteq \gamma_0$ then $\mathrm{par}(\gamma) \leq \mathrm{par}(\gamma_0) < \mathrm{par}(\beta_0) < \mathrm{par}(\bar{\beta})$, a contradiction. Thus γ and $\gamma_0{}^\frown\langle\infty\rangle$ are incomparable. Since $\delta \subset \bar{\delta}$ we must have $\gamma \supseteq \tilde{\gamma}{}^\frown\langle\infty\rangle$ and $\bar{\gamma} \supseteq \gamma_0{}^\frown\langle\infty\rangle \supseteq \tilde{\gamma}{}^\frown\langle\tilde{\delta}\rangle$ for some $\tilde{\gamma}$ and some $\tilde{\delta} \supset \delta$, again a contradiction by $\mathrm{par}(\bar{\beta}) > \mathrm{par}(\beta_0) > \mathrm{par}(\gamma_0) > \mathrm{par}(\tilde{\delta}) > \mathrm{par}(\delta)$.

Next assume $\beta \supseteq \beta_0$, so $\beta \not\supseteq \beta_0{}^\frown\langle\infty\rangle$ but $\bar{\beta} \supseteq \beta_0{}^\frown\langle\infty\rangle$. By $\lambda^2(\bar{\delta}) \supseteq \bar{\beta}$, $\lambda(\bar{\delta}) \supseteq \gamma{}^\frown\langle\infty\rangle$ is impossible; so let $\tilde{\delta} \subset \bar{\delta}$ be maximal such that for all δ' with $\delta \subset \delta' \subseteq \tilde{\delta}$, $\lambda(\delta') \supseteq \gamma{}^\frown\langle\infty\rangle$. Then $\tilde{\gamma} = \mathrm{up}(\tilde{\delta}) \subseteq \gamma$, $\tilde{\gamma}{}^\frown\langle\infty\rangle \subseteq \gamma{}^\frown\langle\infty\rangle$, and $\tilde{\gamma}{}^\frown\langle\tilde{\delta}\rangle \subseteq \lambda(\tilde{\delta}{}^\frown\langle\infty\rangle)$. We set $\tilde{\beta} = \mathrm{up}(\tilde{\gamma})$ and $\tilde{\alpha} = \mathrm{up}^2(\tilde{\gamma})$. We will show $\tilde{\alpha} = \alpha$ and $\tilde{\beta} \subseteq \beta$. For the sake of a contradiction, first suppose $\alpha \not\subseteq \tilde{\alpha}$, say $\alpha_1{}^\frown\langle\beta_1\rangle \subseteq \alpha$ but $\alpha_1{}^\frown\langle\beta_1\rangle \not\subseteq \tilde{\alpha}$ for minimal such α_1 and some β_1. Thus $\beta_1{}^\frown\langle\infty\rangle \subseteq \beta$ but $\beta_1{}^\frown\langle\infty\rangle \not\subseteq \tilde{\beta}$. Assume $\beta_1 \not\subseteq \tilde{\beta}$, say $\hat{\beta} \subseteq \tilde{\beta}$ is maximal with $\hat{\beta} \subset \beta_1$. Then, by our observation, $\mathrm{up}(\hat{\beta}) = \alpha_1$, and so $\hat{\beta}{}^\frown\langle\hat{\gamma}\rangle \subseteq \beta_1$ for some $\hat{\gamma}$. By $\alpha \subset \Lambda_3$ we have $\beta_1{}^\frown\langle\infty\rangle \subset \Lambda_2$ and so $\hat{\gamma}{}^\frown\langle\infty\rangle \subset \Lambda_1$. Now $\delta \subset \tilde{\delta} \subset \Lambda_0$ implies $\hat{\gamma}{}^\frown\langle\infty\rangle \subseteq \lambda(\tilde{\delta})$ and thus $\hat{\beta}{}^\frown\langle\hat{\gamma}\rangle \subseteq \lambda^2(\tilde{\delta})$ by $\hat{\beta} \subseteq \lambda^2(\tilde{\delta})$ and $\hat{\beta} \subseteq \tilde{\beta}$. By $\tilde{\beta} \not\supseteq \hat{\beta}{}^\frown\langle\hat{\gamma}\rangle$ and $\tilde{\beta} \subseteq \lambda^2(\tilde{\delta})$, we have $\tilde{\beta} = \hat{\beta}$, and thus $\tilde{\gamma} = \hat{\gamma}$ by $\hat{\gamma}{}^\frown\langle\infty\rangle \subseteq \lambda(\tilde{\delta})$. But we have, $\tilde{\gamma}{}^\frown\langle\infty\rangle \subset \Lambda_1$, contradicting the choice of $\tilde{\gamma} = \hat{\gamma}$. Thus $\beta_1 \subseteq \tilde{\beta}$. Now $\beta_1{}^\frown\langle\gamma_1\rangle \subseteq \tilde{\beta}$ for some γ_1 is impossible by $\tilde{\gamma} \subseteq \gamma$ and $\beta_1{}^\frown\langle\infty\rangle \subseteq \beta$; so $\tilde{\beta} = \beta_1$ and thus $\tilde{\beta}{}^\frown\langle\infty\rangle \subseteq \beta$, contradicting $\tilde{\gamma}{}^\frown\langle\infty\rangle \subseteq \gamma$. Thus $\alpha \subseteq \tilde{\alpha}$, so suppose, again for the sake of a contradiction, $\alpha \subset \tilde{\alpha}$, say $\alpha{}^\frown\langle\beta_1\rangle \subseteq \tilde{\alpha}$, and so $\tilde{\beta} \supseteq \beta_1{}^\frown\langle\infty\rangle$. Now $\beta_1 \not\subseteq \beta$ is impossible since then

$\beta_2\widehat{\ }\langle\gamma_2\rangle \subseteq \beta_1$ but $\beta_2\widehat{\ }\langle\gamma_2\rangle \not\subseteq \beta$ for minimal such β_2 and some γ_2 (by $\mathrm{up}(\beta_1) = \alpha$ and our observation), and so $\tilde{\gamma} \supseteq \gamma_2\widehat{\ }\langle\infty\rangle$ but $\gamma \not\supseteq \gamma_2\widehat{\ }\langle\infty\rangle$, contradicting $\tilde{\gamma} \subseteq \gamma$. Thus $\beta_1 \subseteq \beta$. Now $\lambda(\tilde{\delta}) \supseteq \gamma\widehat{\ }\langle\infty\rangle$, and so $\lambda^2(\tilde{\delta}) \supseteq \beta\widehat{\ }\langle\gamma\rangle$ or $\lambda^2(\tilde{\delta}) \supseteq \beta_2\widehat{\ }\langle\gamma_2\rangle$ for some $\beta_2 \subset \beta$ and some $\gamma_2 \supseteq \gamma\widehat{\ }\langle\infty\rangle$. The latter implies $\tilde{\gamma} \supseteq \gamma_2\widehat{\ }\langle\infty\rangle$ by $\alpha \subset \tilde{\alpha}$, contradicting $\tilde{\gamma} \subseteq \gamma$. Thus $\lambda^2(\tilde{\delta}) \supseteq \beta\widehat{\ }\langle\gamma\rangle$, contradicting $\lambda^2(\tilde{\delta}) \supseteq \tilde{\beta}$. Thus $\alpha = \tilde{\alpha}$. We next show that $\tilde{\beta} \subseteq \beta$. Suppose not; then $\tilde{\beta} \supseteq \beta_1\widehat{\ }\langle\gamma_1\rangle$ but $\beta \not\supseteq \beta_1\widehat{\ }\langle\gamma_1\rangle$ for minimal such β_1 and some γ_1. Then $\tilde{\gamma} \supseteq \gamma_1\widehat{\ }\langle\infty\rangle$, contradicting $\tilde{\gamma} \subseteq \gamma$. We have thus established $\tilde{\alpha} = \alpha$ and $\tilde{\beta} \subseteq \beta$, and so $D_{\tilde{l}} \restriction (\varphi_{\tilde{l}}(\tilde{k}) + 1) \neq D_{\mathrm{par}(\tilde{\delta})} \restriction (\varphi_{\tilde{l}}(\tilde{k}) + 1)$ for $\tilde{\delta}$'s numbers $\tilde{k}$ and $\tilde{l}$.

We will now show that this implies $D_l \restriction (\varphi_l(k)+1) \neq D \restriction (\varphi_l(k)+1)$, contradicting the D-correctness of δ's definition of $\Gamma^D_\alpha(k)$. First of all, there is some maximal $\tilde{\delta}_0 \subset \delta$ with $\mathrm{up}(\tilde{\delta}_0) = \tilde{\gamma}$ since $\tilde{\gamma} \subseteq \gamma$. Furthermore, $\tilde{\delta}_0\widehat{\ }\langle 0\rangle \subseteq \delta$, and so $D_{\tilde{l}} \restriction (\varphi_{\tilde{l}}(\tilde{k}) + 1) = D_{\mathrm{par}(\tilde{\delta}_0)} \restriction (\varphi_{\tilde{l}}(\tilde{k}) + 1)$. By $\mathrm{up}^i(\tilde{\delta}_0) \subseteq \mathrm{up}^i(\delta)$ for all $i \in \{0, 1, 2, 3\}$ and by (5), we must have $\tilde{m}_0 \geq m + \mathrm{par}(\gamma)$ (where $\tilde{\delta}_0$ and δ work on the $\tilde{m}_0$th and the mth instance of $\tilde{\gamma}$'s and γ's subrequirement, respectively) since otherwise the $(\tilde{m}_0 + 1)$th instance of $\tilde{\gamma}$'s subrequirement would have been assigned to δ rather than the mth instance of γ's subrequirement. Thus $\mathrm{par}(\tilde{\delta}_0) \geq \tilde{m}_0 \geq \mathrm{par}(\gamma) \geq l$, and so $D_{\tilde{l}} \restriction (\varphi_{\tilde{l}}(\tilde{k}) + 1) = D_l \restriction (\varphi_{\tilde{l}}(\tilde{k}) + 1)$ and $\varphi_{\tilde{l}}(\tilde{k}) = \varphi_l(\tilde{k})$. Therefore $D_l \restriction (\varphi_{\tilde{l}}(\tilde{k}) + 1) \neq D \restriction (\varphi_{\tilde{l}}(\tilde{k}) + 1)$, which gives the desired implication above, using $\tilde{k} \leq k$ and $\varphi_{\tilde{l}}(\tilde{k}) = \varphi_l(\tilde{k}) \leq \varphi_l(k)$.

Finally assume β and β_0 are incomparable, say they split at β'. Then, by our observation, $\mathrm{up}(\beta') = \alpha$, and there are distinct γ_1 and γ_2 such that $\beta'\widehat{\ }\langle\gamma_1\rangle \subseteq \beta$ and $\beta'\widehat{\ }\langle\gamma_2\rangle \subseteq \beta_0$. So $\gamma_1\widehat{\ }\langle\infty\rangle \subseteq \gamma$ and $\gamma_2\widehat{\ }\langle\infty\rangle \subseteq \bar{\gamma}$. Thus γ and $\gamma_2\widehat{\ }\langle\infty\rangle$ are incomparable. Since $\delta \subset \bar{\delta}$ we must have $\gamma \supseteq \tilde{\gamma}\widehat{\ }\langle\infty\rangle$ and $\bar{\gamma} \supseteq \gamma_2\widehat{\ }\langle\infty\rangle \supseteq \tilde{\gamma}\widehat{\ }\langle\tilde{\delta}\rangle$ for some $\tilde{\gamma}$ and some $\tilde{\delta} \supset \delta$, again a contradiction by $\mathrm{par}(\bar{\beta}) > \mathrm{par}(\beta_0) > \mathrm{par}(\gamma_2) > \mathrm{par}(\tilde{\delta}) > \mathrm{par}(\delta)$.

We have thus established that $\bar{\alpha} \subseteq \alpha$, in fact $\bar{\alpha} \subset \alpha$ since they work on different requirements, say $\bar{\alpha}\widehat{\ }\langle\bar{\beta}_0\rangle \subseteq \alpha$. Then $\bar{\beta}_0\widehat{\ }\langle\infty\rangle \subseteq \beta$, and $\bar{\beta}_0\widehat{\ }\langle\infty\rangle \subset \Lambda_2$ by $\alpha \subset \Lambda_3$. We distinguish subcases, comparing the positions of $\bar{\beta}_0$ and $\bar{\beta}$. First assume $\bar{\beta} \subset \bar{\beta}_0$, say $\bar{\beta}\widehat{\ }\langle\bar{\gamma}_0\rangle \subseteq \bar{\beta}_0$ for some $\bar{\gamma}_0$. Now $\bar{\beta}\widehat{\ }\langle\bar{\gamma}_0\rangle \subseteq \bar{\beta}_0 \subset \beta$ and so $\bar{\gamma}_0\widehat{\ }\langle\infty\rangle \subseteq \gamma$; also $\bar{\beta}\widehat{\ }\langle\bar{\gamma}_0\rangle \subset \Lambda_2$ and so $\bar{\gamma}_0\widehat{\ }\langle\infty\rangle \subset \Lambda_1$. Thus $\lambda(\delta') \supseteq \bar{\gamma}_0\widehat{\ }\langle\infty\rangle$ for all δ' with $\delta \subset \delta' \subset \Lambda_0$. This implies $\mathrm{up}(\bar{\delta}) = \bar{\gamma}_0$, by (3.3), contradicting $\bar{\delta}\widehat{\ }\langle\infty\rangle \subset \Lambda_0$.

Next assume $\bar{\beta} \not\subseteq \bar{\beta}_0$, say $\tilde{\beta} \subset \bar{\beta}$ is maximal with $\tilde{\beta} \subseteq \bar{\beta}_0$. By our observation, $\mathrm{up}(\tilde{\beta}) = \bar{\alpha}$, and so $\tilde{\beta}\widehat{\ }\langle\tilde{\gamma}\rangle \subseteq \bar{\beta}$ for some $\tilde{\gamma}$ but $\tilde{\beta}\widehat{\ }\langle\tilde{\gamma}\rangle \not\subseteq \beta$. Thus $\tilde{\gamma}\widehat{\ }\langle\infty\rangle \subseteq \bar{\gamma}$ but $\tilde{\gamma}\widehat{\ }\langle\infty\rangle \not\subseteq \gamma$. If γ and $\tilde{\gamma}$ are incomparable, say they split at $\hat{\gamma}$, then $\hat{\gamma}\widehat{\ }\langle\infty\rangle \subseteq \gamma$ and $\hat{\gamma}\widehat{\ }\langle\hat{\delta}\rangle \subseteq \tilde{\gamma}$ for some $\hat{\delta} \supset \delta$ by $\delta \subset \bar{\delta}$. But then $\mathrm{par}(\bar{\beta}) > \mathrm{par}(\tilde{\gamma}) > \mathrm{par}(\hat{\delta}) > \mathrm{par}(\delta)$, a contradiction. So necessarily $\gamma \subset \tilde{\gamma}$, and thus $\mathrm{par}(\gamma) < \mathrm{par}(\tilde{\gamma}) < \mathrm{par}(\bar{\beta})$, again a contradiction.

We are thus left with only one subcase, namely $\bar{\beta} = \bar{\beta}_0$. Since $\bar{\alpha}\widehat{\ }\langle\bar{\beta}\rangle \subseteq \alpha$, this subcase can generate at most one injury for each $\bar{\alpha} \subset \alpha$. We have thus established $C =^* \Gamma^D_\alpha$ as desired. (We remark that this finite injury to the Γ's can be eliminated if we rephrase requirement (21) as

$$A = \Psi^D \to \bar{C} \text{ r.e. in } D. \qquad (21')$$

Now the witness $\bar{\beta}_0 \subset \Lambda_2$ for $(21')$ will not be the least $\beta \subset \Lambda_2$ working on $(21')$ with $A \upharpoonright (\mathrm{par}(\beta)+1) \neq \Psi^D \upharpoonright (\mathrm{par}(\beta)+1)$, but the least such for which also $k_\beta \notin C$ (unless C is cofinite in which case certainly $C \leq_{\mathrm{T}} D$). The trick is that for this $\bar{\beta}_0$, $\mathrm{par}(\bar{\beta}_0)$ will not be enumerated into A (which causes the finite injury in the above construction). This also implies that $A \leq_{\mathrm{T}} \Lambda_1 \leq_{\mathrm{T}} C \oplus D$, giving an easier proof of (19).)

Next suppose $A = \Psi^D$ (and none of the higher-priority requirements has shown $C \leq_{\mathrm{T}} D$). Then $\alpha\hat{\ }\langle\infty\rangle = \Lambda_3$ for the unique $\alpha \subset \Lambda_3$ working on this requirement. Thus $\beta_k\hat{\ }\langle\gamma_k\rangle \subset \Lambda_2$ (for some $\gamma_k \subset \Lambda_1$) for all k where β_k is the unique $\beta \subset \Lambda_2$ working on the kth instance of α's requirement. Since $\gamma_k\hat{\ }\langle\infty\rangle \subset \Lambda_1$, all $\delta \subset \Lambda_0$ with $\mathrm{up}(\delta) = \gamma_k$ answer $(25')$ positively, and one of these δ defines $\Delta^D_\alpha(k)$ D-correctly (unless some other strategy does so). So Δ^D_α is total. Suppose $\Delta^D_\alpha(k) \neq C(k)$ for some k. Then one of the $\delta \subset \Lambda_0$ with $\mathrm{up}(\delta) = \gamma_k$, say $\bar{\delta}$, will enumerate $\mathrm{par}(\beta_k)$ into A, trying to destroy the D-correct computation $\Delta^D_\alpha(k)$ defined by some $\hat{\delta} \subset \bar{\delta}$. We set $\hat{\beta} = \mathrm{up}^2(\hat{\delta})$ and $\hat{\gamma} = \mathrm{up}(\hat{\delta})$. (Note $\alpha = \mathrm{up}^3(\hat{\delta})$.) We will show $\hat{\beta} = \beta_k$, so $D_l \upharpoonright (\psi_l(\mathrm{par}(\beta_k))+1) \neq D \upharpoonright (\psi_l(\mathrm{par}(\beta_k))+1)$ (since $A(\mathrm{par}(\beta_k)) = \Psi^D(\mathrm{par}(\beta_k))$), and $\hat{\delta}$ did therefore not define $\Delta^D_\alpha(k)$ D-correctly by $\delta_\alpha(k) = \psi_l(\mathrm{par}(\beta_k))$, yielding the desired contradiction. Now since β_k and $\hat{\beta}$ work on the same k, $\beta_k \neq \hat{\beta}$ would imply that they are incomparable, say they split at β_0. Then, by our observation, we have $\mathrm{up}(\beta_0) = \alpha$; and so $\beta_0\hat{\ }\langle\gamma_0\rangle \subseteq \beta_k$ and $\beta_0\hat{\ }\langle\hat{\gamma}_0\rangle \subseteq \hat{\beta}$ for some γ_0 and $\hat{\gamma}_0$. Thus $\gamma_0\hat{\ }\langle\infty\rangle \subseteq \gamma_k$ and $\hat{\gamma}_0\hat{\ }\langle\infty\rangle \subseteq \hat{\gamma}$, and so $\gamma_0\hat{\ }\langle\infty\rangle$ and $\hat{\gamma}_0\hat{\ }\langle\infty\rangle$ must be incomparable. Therefore there is some δ' with $\hat{\delta} \subset \delta' \subseteq \bar{\delta}$ such that $\lambda(\delta') \not\supseteq \hat{\gamma}_0\hat{\ }\langle\infty\rangle$. Let $\tilde{\delta} \subset \bar{\delta}$ be maximal such that for all δ' with $\hat{\delta} \subseteq \delta' \subseteq \tilde{\delta}$, $\lambda(\delta') \supseteq \hat{\gamma}_0\hat{\ }\langle\infty\rangle$. Then $\tilde{\gamma} = \mathrm{up}(\tilde{\delta}) \subseteq \hat{\gamma}$, $\tilde{\gamma}\hat{\ }\langle\infty\rangle \subseteq \hat{\gamma}\hat{\ }\langle\infty\rangle$, and $\tilde{\gamma}\hat{\ }\langle\tilde{\delta}\rangle \subseteq \lambda(\tilde{\delta}\hat{\ }\langle\infty\rangle)$. We set $\tilde{\beta} = \mathrm{up}(\tilde{\gamma})$ and $\tilde{\alpha} = \mathrm{up}^2(\tilde{\gamma})$. We will show $\tilde{\alpha} = \alpha$ and $\tilde{\beta} \subseteq \beta_0$. For the sake of a contradiction, first suppose $\alpha \not\subseteq \tilde{\alpha}$, say $\alpha_1\hat{\ }\langle\beta_1\rangle \subseteq \alpha$ but $\alpha_1\hat{\ }\langle\beta_1\rangle \not\subseteq \tilde{\alpha}$ for minimal such α_1 and some β_1. Thus $\beta_1\hat{\ }\langle\infty\rangle \subseteq \hat{\beta}$ but $\beta_1\hat{\ }\langle\infty\rangle \not\subseteq \tilde{\beta}$. Assume $\beta_1 \not\subseteq \tilde{\beta}$, say $\check{\beta} \subseteq \tilde{\beta}$ is maximal with $\check{\beta} \subset \beta_1$. Then, by our observation, $\mathrm{up}(\check{\beta}) = \alpha_1$, and so $\check{\beta}\hat{\ }\langle\check{\gamma}\rangle \subseteq \beta_1$ for some $\check{\gamma}$. By $\alpha \subset \Lambda_3$ we have $\beta_1\hat{\ }\langle\infty\rangle \subset \Lambda_2$ and so $\check{\gamma}\hat{\ }\langle\infty\rangle \subset \Lambda_1$. Now $\hat{\delta} \subset \tilde{\delta} \subset \Lambda_0$ implies $\check{\gamma}\hat{\ }\langle\infty\rangle \subseteq \lambda(\tilde{\delta})$ and thus $\check{\beta}\hat{\ }\langle\check{\gamma}\rangle \subseteq \lambda^2(\tilde{\delta})$ by $\tilde{\beta} \subseteq \lambda^2(\tilde{\delta})$ and $\check{\beta} \subseteq \tilde{\beta}$. By $\tilde{\beta} \not\supseteq \check{\beta}\hat{\ }\langle\check{\gamma}\rangle$ we have $\tilde{\beta} = \check{\beta}$, and thus $\tilde{\gamma} = \check{\gamma}$ by $\check{\gamma}\hat{\ }\langle\infty\rangle \subseteq \lambda(\tilde{\delta})$. But we have $\tilde{\gamma}\hat{\ }\langle\infty\rangle \subset \Lambda_1$, contradicting the choice of $\tilde{\gamma} = \check{\gamma}$. Thus $\beta_1 \subseteq \tilde{\beta}$. Now $\beta_1\hat{\ }\langle\gamma_1\rangle \subseteq \tilde{\beta}$ for some γ_1 is impossible by $\tilde{\gamma} \subseteq \hat{\gamma}$; so $\tilde{\beta} = \beta_1$ and thus $\tilde{\beta}\hat{\ }\langle\infty\rangle \subseteq \hat{\beta}$, contradicting $\tilde{\gamma}\hat{\ }\langle\infty\rangle \subseteq \hat{\gamma}_0$. Thus $\alpha \subseteq \tilde{\alpha}$, so suppose, again for the sake of a contradiction, $\alpha \subset \tilde{\alpha}$, say $\alpha\hat{\ }\langle\beta_1\rangle \subseteq \tilde{\alpha}$ for some β_1, and so $\beta_1\hat{\ }\langle\infty\rangle \subseteq \tilde{\beta}$. Now $\beta_1 \not\subseteq \beta_0$ is impossible since then $\beta_2\hat{\ }\langle\gamma_2\rangle \subseteq \beta_1$ but $\beta_2\hat{\ }\langle\gamma_2\rangle \not\subseteq \beta_0$ for minimal such β_2 and some γ_2 (by $\mathrm{up}(\beta_1) = \alpha$ and by our observation above), and so $\tilde{\gamma} \supseteq \gamma_2\hat{\ }\langle\infty\rangle$ but $\hat{\gamma} \not\supseteq \gamma_2\hat{\ }\langle\infty\rangle$, contradicting $\tilde{\gamma} \subseteq \hat{\gamma}_0$. Thus $\beta_1 \subseteq \beta_0$. Now $\lambda(\tilde{\delta}) \supseteq \hat{\gamma}_0\hat{\ }\langle\infty\rangle$, and so $\lambda^2(\tilde{\delta}) \supseteq \beta_0\hat{\ }\langle\hat{\gamma}_0\rangle$ or $\lambda^2(\tilde{\delta}) \supseteq \beta_2\hat{\ }\langle\gamma_2\rangle$ for some $\beta_2 \subset \beta_0$ and some $\gamma_2 \supseteq \hat{\gamma}_0\hat{\ }\langle\infty\rangle$. The latter implies $\tilde{\gamma} \supseteq \gamma_2\hat{\ }\langle\infty\rangle$ by $\alpha \subset \tilde{\alpha}$, contradicting $\tilde{\gamma} \subseteq \hat{\gamma}_0$. Thus $\lambda^2(\tilde{\delta}) \supseteq \beta_0\hat{\ }\langle\hat{\gamma}_0\rangle$, contradicting $\lambda^2(\tilde{\delta}) \supseteq \tilde{\beta}$. Thus $\tilde{\alpha} = \alpha$. We next show $\tilde{\beta} \subseteq \beta_0$. Suppose not; then $\tilde{\beta} \supseteq \beta_1\hat{\ }\langle\gamma_1\rangle$ but $\beta_0 \not\supseteq \beta_1\hat{\ }\langle\gamma_1\rangle$ for minimal such β_1 and some γ_1. Then $\tilde{\gamma} \supseteq \gamma_1\hat{\ }\langle\infty\rangle$ but $\hat{\gamma}_0 \not\supseteq \gamma_1\hat{\ }\langle\infty\rangle$, contradicting $\tilde{\gamma} \subseteq \hat{\gamma}_0$. We have thus established $\tilde{\alpha} = \alpha$ and $\tilde{\beta} \subseteq \beta_0$. Therefore, $D_{\tilde{l}} \upharpoonright (\psi_{\tilde{l}}(\mathrm{par}(\tilde{\beta}))+1) \neq D_{\mathrm{par}(\tilde{\delta})} \upharpoonright (\psi_{\tilde{l}}(\mathrm{par}(\tilde{\beta}))+1)$ for $\tilde{\delta}$'s number $l = \tilde{l}$.

We will show that this implies $D_{\hat{l}} \upharpoonright (\psi_{\hat{l}}(\mathrm{par}(\hat{\beta}))+1) \neq D_{\mathrm{par}(\bar{\delta})} \upharpoonright (\psi_{\hat{l}}(\mathrm{par}(\hat{\beta}))+1)$ for $\hat{\delta}$'s number $l = \hat{l}$, contradicting the D-correctness of $\hat{\delta}$'s definition of $\Delta^A_\alpha(k)$. First of all, there is some maximal $\tilde{\delta}_0 \subset \hat{\delta}$ with $\mathrm{up}(\tilde{\delta}_0) = \tilde{\gamma}$ since $\tilde{\gamma} \subset \hat{\gamma}$. Also $\tilde{\delta}_0{}^\frown\langle 0\rangle \subseteq \hat{\delta}$ and so $D_{\tilde{l}} \upharpoonright (\psi_{\tilde{l}}(\mathrm{par}(\tilde{\beta}))+1) = D_{\mathrm{par}(\tilde{\delta}_0)} \upharpoonright (\psi_{\tilde{l}}(\mathrm{par}(\tilde{\beta}))+1)$. By $\mathrm{up}^i(\tilde{\delta}_0) \subseteq \mathrm{up}^i(\hat{\delta})$ for all $i \in \{0,1,2,3\}$ and by (5), we must have $\tilde{m}_0 \geq \hat{m} + \mathrm{par}(\hat{\gamma})$ (where $\tilde{\delta}_0$ and $\hat{\delta}$ work on the $\tilde{m}_0$th and the $\hat{m}$th instance of $\tilde{\gamma}$'s and $\hat{\gamma}$'s subrequirement, respectively) since otherwise the $(\tilde{m}_0+1)$th instance of $\tilde{\gamma}$'s subrequirement would have been assigned to $\hat{\delta}$ rather than the $\hat{m}$th instance of $\hat{\gamma}$'s subrequirement. Thus $\mathrm{par}(\tilde{\delta}_0) \geq \tilde{m}_0 \geq \mathrm{par}(\hat{\gamma}) \geq \hat{l}$, and so $D_{\tilde{l}} \upharpoonright (\psi_{\tilde{l}}(\mathrm{par}(\tilde{\beta}))+1) = D_{\hat{l}} \upharpoonright (\psi_{\tilde{l}}(\mathrm{par}(\tilde{\beta}))+1)$ and $\psi_{\tilde{l}}(\mathrm{par}(\tilde{\beta})) = \psi_{\hat{l}}(\mathrm{par}(\tilde{\beta}))$. Therefore $D_{\hat{l}} \upharpoonright (\psi_{\hat{l}}(\mathrm{par}(\tilde{\beta}))+1) \neq D_{\mathrm{par}(\tilde{\delta})} \upharpoonright (\psi_{\hat{l}}(\mathrm{par}(\tilde{\beta}))+1)$, which gives the desired implication above, using $\mathrm{par}(\tilde{\beta}) < \mathrm{par}(\hat{\beta})$ and $\mathrm{par}(\tilde{\delta}) < \mathrm{par}(\bar{\delta})$. Thus $\hat{\beta} = \beta_k$ as desired.

This establishes the satisfaction of requirements (20) and (21) if $C \not\leq_{\mathrm{T}} D$ since then clearly Λ_3 must be an infinite path. It remains to prove $A \leq_{\mathrm{T}} C \oplus D$. We first exhibit a $(C \oplus D)$-recursive set $M \subseteq T_2 \times \omega$ and a $(C \oplus D)$-r.e. set $M_0 \subseteq T_2$ such that

$$M_0 = \{\, \beta \mid \exists t\,(\langle \beta, t\rangle \in M)\,\} = T_2 - \Lambda_2, \text{ and} \tag{26.1}$$

$$\forall \langle \beta, t\rangle \in M\, \forall t' \geq t \big(\beta \not\subseteq \lambda^2(\Lambda_0 \upharpoonright t')\big). \tag{26.2}$$

We enumerate M as follows: The pair $\langle \beta^\frown\langle\infty\rangle, t\rangle$ is enumerated into M iff

$$\langle \beta, t\rangle \in M, \text{ or} \tag{27.1}$$

$$\begin{aligned} \exists\gamma_0\exists\delta_0\Big(&\mathrm{up}(\delta_0) = \gamma_0 \,\&\, \mathrm{up}(\gamma_0) = \beta \,\&\, \delta_0{}^\frown\langle 0\rangle \subset \Lambda_0 \,\&\, |\delta_0{}^\frown\langle 0\rangle| = t \,\&\\ &\delta_0\text{'s condition } (24') \text{ or } (25') \text{ is } D\text{-correct } \&\\ &\forall \bar{\beta} \subseteq \beta\big(k_{\bar{\beta}} \in C \to k_{\bar{\beta}} \in C_t \,\&\, \neg\big(\Delta^{D_t}_{\mathrm{up}(\bar{\beta}),t}(k_{\bar{\beta}})\!\downarrow = 0\big)\big)\Big). \end{aligned} \tag{27.2}$$

The pair $\langle \beta^\frown\langle\gamma\rangle, t\rangle$ is enumerated into M iff

$$\langle \beta, t\rangle \in M, \text{ or} \tag{28.1}$$

$$\mathrm{up}(\gamma) \neq \beta, \text{ or} \tag{28.2}$$

$$\langle \beta^\frown\langle\infty\rangle, t\rangle \in M \,\&\, \gamma^\frown\langle\infty\rangle \not\subseteq \lambda(\Lambda_0 \upharpoonright t), \text{ or} \tag{28.3}$$

$$\exists\delta_0\,\big(\mathrm{up}^2(\delta_0) = \beta \,\&\, \delta_0{}^\frown\langle\infty\rangle \subset \Lambda_0 \,\&\, |\delta_0{}^\frown\langle\infty\rangle| = t\big) \tag{28.4}$$

We first establish (26.2). By induction, and by inspection (for (28.4)), we have to verify (26.2) only for pairs enumerated into M through (27.2) or (28.3). So suppose $\langle \beta^\frown\langle\infty\rangle, t\rangle \in M$ via (27.2), or $\langle \beta^\frown\langle\gamma\rangle, t\rangle \in M$ via (28.3) (where the latter implies the former). We will show

$$\forall t' \geq t\,\big(\beta \not\subseteq \lambda^2(\Lambda_0 \upharpoonright t') \text{ or } \beta^\frown\langle\gamma_0\rangle \subseteq \lambda^2(\Lambda_0 \upharpoonright t')\big) \tag{29}$$

for the γ_0 in (27.2). Suppose this fails for some (least) $t' \geq t$. Then $\beta \subseteq \lambda^2(\Lambda_0 \upharpoonright t')$ but $\gamma_0{}^\frown\langle\infty\rangle \not\subseteq \lambda(\Lambda_0 \upharpoonright t')$. Let $\bar{\delta}$ be maximal with $\lambda(\bar{\delta}) \supseteq \gamma_0{}^\frown\langle\infty\rangle$ and $\delta \subset \bar{\delta} \subset \Lambda_0$.

Thus $\bar{\gamma}^\frown\langle\infty\rangle \subseteq \gamma_0{}^\frown\langle\infty\rangle$ and $\bar{\gamma}^\frown\langle\bar{\delta}\rangle = \lambda(\bar{\delta}^\frown\langle\infty\rangle)$ for $\bar{\gamma} = \mathrm{up}(\bar{\delta})$. We distinguish cases, comparing the positions of β and $\bar{\beta} = \mathrm{up}(\bar{\gamma})$. If β and $\bar{\beta}$ are incomparable then by $\bar{\gamma} \subset \gamma_0$, $\lambda(\gamma_0) \supseteq \beta$ and $\lambda(\bar{\gamma}) \supseteq \bar{\beta}$, we have $\hat{\beta}^\frown\langle\infty\rangle \subseteq \bar{\beta}$ and $\hat{\beta}^\frown\langle\hat{\gamma}\rangle \subseteq \beta$ for some $\hat{\beta}$ and $\hat{\gamma}$, which implies $\bar{\gamma}^\frown\langle\infty\rangle \subset \hat{\gamma}^\frown\langle\infty\rangle \subseteq \gamma_0$, contradicting $\lambda(\bar{\delta}) \supseteq \gamma_0$ and $\lambda^2(\bar{\delta}) \supseteq \bar{\beta}$. If $\bar{\beta} \subset \beta$ then $\bar{\gamma}^\frown\langle\infty\rangle \not\subseteq \lambda(\delta')$ and so $\beta \not\subseteq \lambda^2(\delta')$ for all δ' with $\bar{\delta} \subset \delta' \subset \Lambda_0$, a contradiction. If $\beta \subset \bar{\beta}$ then $\beta^\frown\langle\gamma_0\rangle \subseteq \bar{\beta}$ by (29) and $\bar{\beta} \subseteq \lambda^2(\bar{\delta})$, contradicting $\bar{\gamma} \subset \gamma_0$. So $\beta = \bar{\beta}$, and, by (3.3) and $\bar{\gamma}^\frown\langle\infty\rangle \subseteq \gamma_0{}^\frown\langle\infty\rangle$, we must have $\bar{\gamma} = \gamma_0$. Now $\bar{\delta}$'s condition (24′) or (25′) was D-correct by (27.2). Thus $\bar{\delta}$ works on a requirement (22), and $(A_l \oplus D_l) \restriction (\varphi_l(k)+1) \neq (A_{\mathrm{par}(\bar{\delta})} \oplus D_{\mathrm{par}(\bar{\delta})}) \restriction (\varphi_l(k)+1)$ by an A-change.

The y at which A changed was enumerated into A by some $\hat{\delta}$ with $\delta_0 \subset \hat{\delta} \subset \bar{\delta}$. We set $\hat{\beta} = \mathrm{up}^2(\hat{\delta})$ and $\hat{\gamma} = \mathrm{up}(\hat{\delta})$, so $y = \mathrm{par}(\hat{\beta})$. We show that such a $\hat{\delta}$ cannot exist, distinguishing subcases by comparing the positions of β and $\hat{\beta}$. If $\beta \subset \hat{\beta}$ then, by $\hat{\beta} \subseteq \lambda^2(\hat{\delta})$ and $\hat{\delta} \subset \bar{\delta}$, we have $\beta^\frown\langle\gamma_0\rangle \subseteq \hat{\beta}$; so $y = \mathrm{par}(\hat{\beta}) > \mathrm{par}(\gamma_0) \geq l > \varphi_l(k)$, a contradiction. Also $\hat{\beta} \subseteq \beta$ is impossible by the last conjunct of (27.2). So β and $\hat{\beta}$ must be incomparable, say they split at $\tilde{\beta}$. If $\tilde{\beta}^\frown\langle\tilde{\gamma}\rangle \subseteq \beta$ for some $\tilde{\gamma}$ then, by $\tilde{\beta}^\frown\langle\tilde{\gamma}\rangle \not\subseteq \hat{\beta}$, $\tilde{\gamma}^\frown\langle\infty\rangle \not\subseteq \lambda(\delta')$ for any δ' with $\hat{\delta} \subseteq \delta' \subset \Lambda_0$, contradicting $|\hat{\delta}| < t'$. So $\tilde{\beta}^\frown\langle\infty\rangle \subseteq \beta$ and $\tilde{\beta}^\frown\langle\tilde{\gamma}\rangle \subseteq \hat{\beta}$ for some $\tilde{\gamma}$, and thus $\tilde{\gamma}^\frown\langle\infty\rangle \not\subseteq \gamma_0$ but $\tilde{\gamma}^\frown\langle\infty\rangle \subseteq \hat{\gamma}$. If $\gamma_0 \subset \tilde{\gamma}$ then $\mathrm{par}(\hat{\beta}) > \mathrm{par}(\tilde{\gamma}) > \mathrm{par}(\gamma_0) \geq l > \varphi_l(k)$, a contradiction. So γ_0 and $\tilde{\gamma}$ must be incomparable, say they split at $\check{\gamma}$. By $\delta_0 \subset \hat{\delta}$ and by $\gamma_0 \subseteq \lambda(\delta_0)$, we must have $\check{\gamma}^\frown\langle\infty\rangle \subseteq \gamma_0$ and $\check{\gamma}^\frown\langle\check{\delta}\rangle \subseteq \tilde{\gamma}$ for some $\check{\delta} \supset \delta_0$. Thus $\mathrm{par}(\hat{\beta}) > \mathrm{par}(\tilde{\gamma}) > \mathrm{par}(\check{\delta}) > \mathrm{par}(\delta_0) \geq l > \varphi_l(k)$, again a contradiction. This shows that no A-change $\leq \varphi_l(k)$ can have occurred; thus $\bar{\delta}$ cannot exist, establishing (29) and (26.2).

By Lemma 1(ii) and (26.2), we can conclude $M_0 \cap \Lambda_2 = \emptyset$. We establish (26.1) by showing that $T_2 - M_0$ does not contain incomparable nodes. (Note that $T_2 - M_0$ is a tree by (27.1) and (28.1).) For the sake of a contradiction, suppose β_1 and β_2 are incomparable nodes in $T_2 - M_0$ of minimal length. By minimality, any node in $T_2 - M_0$ of length $< |\beta_1|$ must be on Λ_2; also $|\beta_1| = |\beta_2|$ and $\beta_1 \restriction (|\beta_1|-1) = \beta_2 \restriction (|\beta_1|-1) = \beta$, say. By (26.2), we can assume $\beta_1 \subset \Lambda_2$.

First assume $\beta^\frown\langle\gamma\rangle = \beta_1$ for some γ. Then $\gamma^\frown\langle\infty\rangle \subset \Lambda_1$, and for all $\delta \subset \Lambda_0$ with $\mathrm{up}(\delta) = \gamma$, the first five conjuncts of (27.2) hold true for $\gamma = \gamma_0$ and $\delta = \delta_0$. By $\mathrm{up}^j(\gamma) \subset \Lambda_{1+j}$ for $j \in \{0, 1, 2\}$, there infinitely many $\delta \subset \Lambda_0$ with $\mathrm{up}(\delta) = \gamma$, and by the correctness of the functionals $\Gamma_{\bar{\alpha}}$ and $\Delta_{\bar{\alpha}}$ (for $\bar{\alpha} \subseteq \mathrm{up}(\beta)$), all conjuncts of (27.2) must hold for some $\delta \subset \Lambda_0$ with $\mathrm{up}(\delta) = \gamma$. Thus β_2 must eventually enter M_0, a contradiction.

So assume $\beta^\frown\langle\infty\rangle = \beta_1$. Then $\mathrm{up}(\beta) \subset \Lambda_3$, and so any $\beta^\frown\langle\gamma\rangle$ must eventually enter M_1 via (28.2) or (28.4), again a contradiction. We have thus established (26.1) and (26.2). It is now easy to see why $A \leq_{\mathrm{T}} C \oplus D$ as follows: Fix x. By the surjectivity of par, determine if $x = \mathrm{par}(\beta)$ for some $\beta \in T_2$. If not then $x \notin A$. Otherwise check if β works on a requirement (23) and if $k_\beta \in C$. If not then $x \notin A$. Otherwise simultaneously enumerate (recursively in $C \oplus D$) the sets M and A. If $\langle\beta, t\rangle \in M$ for some t then $x \in A$ iff $x \in A_t$ by (26.2). Otherwise $k_\beta \in C_s$, say, so the first $\delta \subset \Lambda_0$ with $\mathrm{up}^2(\delta) = \beta$ and $s < \mathrm{par}(\delta)$ will enumerate $\mathrm{par}(\beta)$ into A if then $\Delta^D_{\mathrm{up}(\beta)}(k_\beta)\!\downarrow = 0$, or else $\mathrm{par}(\beta) \notin A$. This establishes $A \leq_{\mathrm{T}} C \oplus D$ and thus concludes the proof of the Sacks Density Theorem. ■

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