An Extended Lachlan Splitting Theorem †

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Abstract. We show that the top of any diamond with bottom $\mathbf{0}$ in the r.e. degrees is also the top of a stack of n diamonds with bottom $\mathbf{0}$.

Let $\mathbf{R}$ be the upper semilattice of the recursively enumerable degrees.

A *minimal pair* consists of two incomparable r.e. degrees with infimum equal to the recursive degree $\mathbf{0}$. An r.e. degree is *cappable* if it is one half of a minimal pair. An r.e. degree $\mathbf{a}$ is the *top of a diamond* (or *1-diamond*) if $\mathbf{a}$ is the join of a minimal pair. For any $n > 1$, $\mathbf{a}$ is the *top of an n-diamond* if there is a nontrivial splitting $\mathbf{a}_0$ and $\mathbf{a}_1$ of $\mathbf{a}$ such that the infimum of $\mathbf{a}_0$ and $\mathbf{a}_1$ exists and is the top of an $(n-1)$-diamond.

Lachlan [1966] and Yates [1966] proved that there is a minimal pair in $\mathbf{R}$. Ambos-Spies, Jockusch, Shore and Soare [1984] proved that $\mathbf{M}$, the set of all cappable r.e. degrees, is an ideal in $\mathbf{R}$, and $\mathbf{R} - \mathbf{M}$, which coincides with the class of all promptly simple r.e. degrees, is a strong filter in $\mathbf{R}$. We mention some facts about the distribution of the r.e. degrees which are tops of diamonds. Let $\mathbf{T}$ be the set of such degrees. Then $\mathbf{T}$ has no maximal or minimal elements since $\mathbf{M}$ is not a principal ideal, and given any nonrecursive r.e. degrees $\mathbf{a}_0$ and $\mathbf{a}_1$ there exist $\mathbf{0} < \mathbf{b}_0 \leq \mathbf{a}_0$ and $\mathbf{0} < \mathbf{b}_1 \leq \mathbf{a}_1$ such that $\mathbf{b}_0 \cup \mathbf{b}_1 < \mathbf{a}_0 \cup \mathbf{a}_1$. Furthermore, by a recent result of Downey, Lempp, and Shore [1993], there is a high$_2$ r.e. degree bounding only degrees in $\mathbf{T}$.

In this paper we shall modify Lachlan's construction (Lachlan [1980]) of splitting any nonrecursive r.e. degree into two r.e. degrees with infimum to show that every top of a diamond is the top of an n-diamond for every $n > 0$.

Theorem 1. *Given any nonrecursive r.e. sets A_0 and A_1, there exist r.e. sets B_0, B_1, C, C_0, and C_1 such that $A_0 \oplus A_1 \equiv_{\rm wtt} B_0 \oplus B_1 >_{\rm wtt} C \geq_{\rm wtt} C_0 \oplus C_1$; $B_0, B_1 \not\leq_{\rm T} C$; $\emptyset <_{\rm wtt} C_0 \leq_{\rm wtt} A_0$; $\emptyset <_{\rm wtt} C_1 \leq_{\rm wtt} A_1$; and $\deg_{\rm T}(C) = \deg_{\rm T}(C \oplus B_0) \cap \deg_{\rm T}(C \oplus B_1)$.*

Corollary 2. *Every top of a diamond is the top of a double diamond (i.e. a 2-diamond), and hence the top of an n-diamond for any $n \geq 1$.*

Proof. Let $\mathbf{a} = \mathbf{a}_0 \cup \mathbf{a}_1$ be the top of a diamond, where $\mathbf{a}_0$ and $\mathbf{a}_1$ form a minimal pair, and let A_0 and A_1 be sets of degree $\mathbf{a}_0$ and $\mathbf{a}_1$, respectively. Let $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{c}$, $\mathbf{c}_0$, and $\mathbf{c}_1$ be the Turing degrees of the sets in Theorem 1, respectively. Thus $\mathbf{c} = \mathbf{b}_0 \cap \mathbf{b}_1$ and $\mathbf{a} = \mathbf{b}_0 \cup \mathbf{b}_1$. Since $C \leq_{\rm wtt} A_0 \oplus A_1$ and by the distributivity of $\mathbf{R}_{\rm wtt}$, the upper semilattice of the r.e. wtt-degrees, there exist r.e. sets D_0 and D_1 such that $D_0 \leq_{\rm wtt} A_0, D_1 \leq_{\rm wtt} A_1$, and $D_0 \oplus D_1 \equiv_{\rm T} C$. Clearly D_0 and D_1 are not recursive, else C would be recursive in A_0 or A_1. As $\mathbf{a}_0 \cap \mathbf{a}_1 = \mathbf{0}, \deg_{\rm T}(D_0) \cap \deg_{\rm T}(D_1) = \mathbf{0}$, and so $\mathbf{a}$ is the top of the double diamond formed by $\mathbf{a}, \mathbf{b}_0, \mathbf{b}_1, \mathbf{c}, \deg_{\rm T}(D_0), \deg_{\rm T}(D_1)$, and $\mathbf{0}$.

Proof of Theorem 1. Fix any r.e. sets A_0 and A_1 of degrees $\mathbf{a}_0$ and $\mathbf{a}_1$, respectively, and set $A = A_0 \oplus A_1$. We shall recursively enumerate sets B_0, B_1, C, C_0, and C_1 such that $C \leq_{\rm wtt} A_0 \oplus A_1$. For every $e, j \in \omega, i = 0, 1$

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the construction will satisfy the following requirements:

$$\begin{aligned}
J_0 &: A_0 \oplus A_1 \equiv_{\text{wtt}} B_0 \oplus B_1,\\
J_1 &: A_0 \geq_{\text{wtt}} C_0,\ A_1 \geq_{\text{wtt}} C_1,\\
R_{e,i} &: C_i \neq \omega - W_e,\\
J_2 &: C \geq_{\text{wtt}} C_0 \oplus C_1,\\
P_{e,i} &: B_i \neq \Phi_e(C \oplus B_{1-i}),\ \text{and}\\
N_j &: \{j\}^{C\oplus B_0} = \{j\}^{C\oplus B_1} = f_j \text{ total} \to f_j \leq_{\text{T}} C.
\end{aligned}$$

We use a modified Sacks splitting strategy to split A into B_0 and B_1 and satisfy J_0 and $P_{e,i}$ for every $e \in \omega, i = 0, 1$. At any stage s we define the *length of agreement* and *restraint functions*

$$\begin{aligned}
l(e,i,s) &= \max\{x : \forall y < x(B_{i,s}(y) = \Phi_{e,s}(C_s \oplus B_{1-i,s}; y))\},\\
r(e,i,s) &= \max\{u(C_s \oplus B_{1-i,s}; e, y, s) : y \leq l(e,i,s)\}.
\end{aligned}$$

At any stage s, if $n \in A_s - A_{s-1}$ then we attempt to enumerate n into B_i, where $\langle e, i\rangle$ is the least pair such that $n < r(e,i,s)$.

We code $C_0 \oplus C_1$ directly into C to satisfy J_2.

To satisfy N_j, at any stage s we define the *length of agreement* and *use functions*

$$\begin{aligned}
l(j,s) &= \max\{x : \forall y < x(\{j\}_s^{C_s\oplus B_{0,s}}(y) = \{j\}_s^{C_s\oplus B_{1,s}}(y))\},\\
p_0(j,x,s) &= u(C_s \oplus B_{0,s}; j, x, s),\ \text{and}\\
p_1(j,x,s) &= u(C_s \oplus B_{1,s}; j, x, s).
\end{aligned}$$

At any stage s, if $l(j,s) > x$ and there is an $n < p_i(j,x,s)$ enumerated into B_i then, until the B_i-side of the computations at x recovers, say until stage $t > s$, we attempt to enumerate elements of A into the same side B_i. Since we must simultaneously satisfy requirements $P_{e,i'}$, there may be an element enumerated into B_{1-i}, which allows $f_{j,s}(x) \neq f_{j,t}(x)$; in this case, we shall enumerate a certain element into C to trace such a change of $f_j(x)$.

The *priority tree* is the complete binary tree. We assign N_j to every node α of length j, and assign $P_{e,i}$ to α if $|\alpha| = 4e+i$; $R_{e,i}$ to α if $|\alpha| = 4e+i+2$. We define the string β_s (of length s) of *nodes accessible at stage* s by

$$\beta_s(j) = 0 \leftrightarrow \forall t < s(\beta_t \supseteq \beta_s\lceil j \to l(j,t) < l(j,s)).$$

We say s is γ-*expansionary* if $\gamma\hat{\ }0 \subseteq \beta_s$, and that s is a γ-*stage* if $\gamma \subset \beta_s$. We define the *length of* γ-*agreement* $l(\gamma, s)$ and γ-*restraint function* $r(\gamma, s)$ by

$$\begin{aligned}
l(\gamma,s) &= \begin{cases} l(j,s) & \text{if } \gamma\hat{\ }0 \subseteq \beta_s,\\ l(\gamma, s-1) & \text{otherwise;}\end{cases}\\
r(\gamma,s) &= \begin{cases} \max\{p_0(j,y,s), p_1(j,y,s) : y < l(j,s)\} & \text{if } \gamma\hat{\ }0 \subseteq \beta_s,\\ r(\gamma, s-1) & \text{otherwise,}\end{cases}
\end{aligned}$$

where $|\gamma| = j$. A strategy ξ is *initialized at stage* s by setting all of ξ's parameters (followers for R) to 0.

We shall define a *restraint function* K, an *index function* I and a *trace marker function* F on any α. Let α be a strategy for some requirement $P_{e,i}$. At any stage s, if $\alpha \subset \beta_s$, and $K(\alpha)$ is undefined or $K(\alpha)$ is defined and $\max\{l(e,i,s), r(e,i,s)\} > K(\alpha)$ then define $K(\alpha)$ to be great enough to preserve $\Phi_{e,s}(C_s \oplus B_{1-i,s})\lceil l(e,i,s)$, $\{j\}_s^{C_s\oplus B_{0,s}}\lceil l(j,s)$, and $\{j\}_s^{C_s\oplus B_{1,s}}\lceil l(j,s)$ for every $j \leq 4e+i$ such that s is j-expansionary. Define $F(\alpha)$ to be the least unused number $> K(\alpha)$, and $I(\alpha) = i$.

Hence, at any stage s, let $\alpha = \beta_s\lceil j$, and

$$f_{j,s}\lceil l(\alpha,s) = \{j\}_s^{C_s\oplus B_{0,s}}\lceil l(\alpha,s) = \{j\}_s^{C_s\oplus B_{1,s}}\lceil l(\alpha,s),$$

then at stage s we define or redefine $K(\gamma)$ for any $\gamma \geq \alpha$ such that $K(\gamma) > r(\alpha, s)$ to ensure that for any $t > s$, and any $x < l(\alpha, s)$ either

$$\{j\}_t^{C_t \oplus B_{0,t}}(x) = \{j\}_s^{C_s \oplus B_{0,s}}(x)$$

or

$$\{j\}_t^{C_t \oplus B_{1,t}}(x) = \{j\}_s^{C_s \oplus B_{1,s}}(x)$$

or

$$\exists n < s_x(n \in C_t - C_s)$$

(where $\{s_x\}_{x \in \omega}$ is a C-recursive sequence). If there is an $n < r(\alpha, s)$ enumerated into A then let γ be least such that $n < K(\gamma)$, enumerate n into $B_{I(\gamma)}$ (where $I(\gamma)$ is as defined in the previous paragraph) and $F(\gamma)$ into C, and move $F(\gamma)$ to be an unused number. Until we go back to α, we shall enumerate the elements $n' < K(\gamma)$ of A into the same side $B_{I(\gamma)}$ if γ is least such that $K(\gamma) > n'$. Any $n' < K(\gamma')$ for some $\gamma' < \gamma$ enumerated into A at $s' > s$ may be enumerated into the other side $B_{1-I(\gamma)}$, because it may be the case that $I(\gamma) = 1 - I(\gamma')$. In this case, $F_s(\gamma')$ is enumerated into C to trace the injuries to $\{j\}_s^{C_s \oplus B_{0,s}}(y), \{j\}_s^{C_s \oplus B_{1,s}}(y)$ for any y such that $K(\gamma') < n < p_{I(\gamma)}(j, y, s)$ and $n' < K(\gamma') < p_{1-I(\gamma)}(j, y, s)$.

To satisfy J_1 we use a direct permitting argument. To satisfy $R_{e,i}$, for any e and i, let α be a strategy for $R_{e,i}$. At any stage s, if $\alpha \subset \beta_s$, $R_{e,i}$ is not satisfied and there is no unrealized follower, i.e., $x' \in W_e$ for every follower x' of $R_{e,i}$, then firstly we assign an unused number x to be a follower of α, and secondly we define $K(\alpha)$ such that $K(\alpha) > x$ and $I(\alpha) = i$. x is canceled at any stage $t > s$ only if α is initialized. We shall show that if α is on the true path then α is initialized only finitely often by showing that every positive requirement requires attention only finitely often if A_0 and A_1 are not recursive.

If there is an $n \in A_{i,s} - A_{i,s-1}$ and a realized follower x of α such that $n < x$, and α is least such that $n < K(\alpha)$ then enumerate x into C_i,n into $B_{I(\alpha)}$ and $F(\alpha)$ into C, and $R_{e,i}$ is satisfied. Hence, if α is on the true path and $R_{e,i}$ is not satisfied then there are infinitely many uncanceled followers x of α such that $K(\alpha)$ is reset infinitely often and no element $< K(\alpha)$ is enumerated into A_i after x is realized. Therefore, either $R_{e,i}$ is eventually satisfied or A_i is recursive.

We say that α *requires attention at s* if $\alpha \subset \beta_s$, and

(1) α is a strategy for $P_{e,i}$,; and $K(\alpha)$ is undefined, or it is defined and $\max\{r(e, i, s), l(e, i, s)\} > K(\alpha)$;

or

(2) α is a strategy for $R_{e,i}$, every follower x of α is *realized* (i.e., $x \in W_{e,s}$), and $R_{e,i}$ is not satisfied.

Construction:

Stage 0: Initialize every node α.

Stage $s > 0$: Find the least $\alpha \subset \beta_s$ requiring attention. If α requires attention via (2) then assign an unused number $x > s$ to be a follower of α, set $I(\alpha) = i$ (as defined via the requirement requiring attention),

$$K(\alpha) = \max\{F(\gamma), x : \gamma < \alpha\},$$

and set $F(\alpha) > K(\alpha)$ to be an unused number. Initialize every $\gamma > \alpha$.

If α requires attention via (1) then define $I(\alpha) = i$ (again defined via the requirement requiring attention),

$$K(\alpha) = \max\{F(\gamma), s + 1 : \gamma < \alpha\},$$

and set $F(\alpha) > K(\alpha)$ to be an unused number. Initialize every $\gamma > \alpha$.

Let $n \in A_s - A_{s-1}$. Let α be least such that $n < K(\alpha)$. (If α fails to exist then enumerate n into B_0 and initialize every $\gamma > \beta_s$.) Enumerate $F(\alpha)$ into C and n into $B_{I(\alpha)}$. If α is a strategy for some $R_{e,i}$, x is realized at s, $n \in A_{i,s} - A_{i,s-1}$, and $x > n$, where x is currently the largest follower of α, then enumerate x into C_i and $R_{e,i}$ is satisfied. Move $F(\alpha)$ equal to the first unused number $> K(\alpha)$, and initialize every $\gamma > \alpha$.

This ends the description of the construction.

Let

$$\beta = \liminf_s \beta_s$$

be the true path.

Lemma 3. *Assume that A_0 and A_1 are not recursive. Let $\alpha \subset \beta$. Then (i) $K(\alpha)$ and $F(\alpha)$ are eventually constant; (ii) the positive requirement assigned to α is satisfied; (iii) α requires attention at most finitely often; and (iv) any $\gamma > \alpha$ is initialized at most finitely often.*

Proof. Assume that the lemma holds for any $\gamma \subset \alpha$. Then α is initialized only finitely often, and hence, $K(\alpha)$ becomes defined eventually.

(i) If $K(\alpha)$ is reset infinitely often then $K(\alpha)$ is reset at a stage s only if α is initialized at a stage $s' < s$, or $L(\alpha, s)$ increases, where

$$L(\alpha, s) = \begin{cases} \max\{r(e,i,s), l(e,i,s)\} & \text{if } \alpha \text{ is a strategy for some } P_{e,i}, \\ x & \text{otherwise,} \end{cases}$$

where x is currently the largest follower of α. Hence, if $K(\alpha)$ is reset infinitely often then $L(\alpha, s)$ tends to infinity, and so does $K(\alpha)$. Eventually any number $< K(\alpha)$ entering A is enumerated into $B_{I(\alpha)}$, and no number $< K(\alpha)$ enumerated in $B_{1-I(\alpha)} \oplus C$. Hence $B_{1-I(\alpha)} \oplus C$ is recursive. If α is a strategy for some $R_{e,i}$ then no number $< L(\alpha, s)$ enters $A_{I(\alpha)}$ after $L(\alpha, s)$ is realized, hence $A_{I(\alpha)}$ is recursive. If α is a strategy for some $P_{e,i}$ then, by a similar argument, $B_{I(\alpha)}$ is recursive, contradicting requirement J_0, which is obviously satisfied.

(ii) (iii) Let s_0 be the least stage such that $\alpha \leq \beta_s$ for all $s > s_0$, and such that no γ requires attention and no set changes below $K(\gamma)$ for any $\gamma < \alpha$.

First assume that α is a strategy for some $R_{e,i}$. At any stage $s > s_0$, if s is an α-stage and any uncanceled follower of α is realized then an unused number x is assigned to α, and $K(\alpha)$ is defined such that $K(\alpha) > x$ and $I(\alpha) = i$. α is initialized at any stage $t > s$ only if $\beta_t < \alpha$ or there is a $\gamma < \alpha$ requiring attention at t. By the choice of s_0, neither case ever occurs. If $R_{e,i}$ is not satisfied then there are infinitely many α-stages s such that an unused number x is assigned to $R_{e,i}$, $K(\alpha)$ is reset at s such that $K(\alpha) > x$, $I(\alpha) = i$, and there is no element $< x$ to be enumerated into A_i after x is realized, otherwise, x would be enumerated into C_i and $R_{e,i}$ is satisfied. Since x tends to infinity, A_i is recursive, a contradiction. Hence, $R_{e,i}$ is satisfied, and α requires attention at most finitely often.

Now assume that α is a strategy for some $P_{e,i}$. If $l(e,i,s)$ is unbounded for α-stages s then there exists an α-stage $s > s_0$ at which α requires attention. By the choice of s_0, $K(\alpha)$ is reset at any $t > s$ only if $L(\alpha, t) > L(\alpha, s)$ and $\beta_t \supset \alpha$. Now $I(\alpha) = i$, and $K(\alpha)$ is reset to preserve $\Phi_e^{C \oplus B_{1-i}}$ on elements $< r(e,i,t) \leq K(\alpha)$ by directing elements into $B_{I(\alpha)}$. So if $L(\alpha, t)$ tends to infinity then B_{1-i} and C are recursive, so is $B_i \leq_{\mathrm{T}} B_{1-i} \oplus C$. Hence, A is recursive, a contradiction. And again α requires attention at most finitely often.

(iv) This is obvious since α requires attention at most finitely often, so α initializes any γ only finitely often and eventually no number $< K(\alpha)$ enters any set.

Lemma 4. *Let $\alpha \subset \beta$. Then $N_{|\alpha|}$ is satisfied.*

Proof. Let $\alpha \subset \beta$ such that $|\alpha| = j$. We assume that $f_j = \{j\}^{C \oplus B_0} = \{j\}^{C \oplus B_1}$ is total. Let s^* be the least stage after which no $\gamma \leq \alpha$ requires attention and such that no $F(\gamma)$ for any $\gamma \leq \alpha$ is reset at any $s > s^*$. To C-recursively compute $f_j(x)$ for any given x, find an α-expansionary stage $s_x > s^*$ and a $\gamma > \alpha$ such that $l(\alpha, s_x) > x$, $K_{s_x}(\gamma)$ is being defined or redefined at s_x, no number $\leq F(\gamma)$ is ever enumerated into C, and $\{j\}^{C \oplus B_0}(x)[s_x] = \{j\}^{C \oplus B_1}(x)[s_x]$ are C-correct. We claim that $f_j(x) = f_{j,s_x}(x)$.

We now claim that at any stage $s > s_x$, at least one of the two computations holds. (This obviously finishes the argument.) Our proof very closely follows Lachlan's original argument [1980].

For the sake of a contradiction, suppose this fails at some least stage $s > s_x$, say, via a number n entering B_i or C and destroying the remaining computation $\{j\}^{C \oplus B_i}(x)$.

We distinguish cases as to how n enters C or B_i:

Case 1: n enters C via an $R_{e,i}$-strategy γ': Then n equals some witness, which by our hypothesis on s_x must have been picked after stage s_x. By the construction and cancellation of markers, we must have $\gamma' \leq \gamma$ and that n was picked at an α-expansionary stage $s' > s_x$, say. But then $n > s' > p_i(j, x, s')$, and the latter use cannot have increased unless the witness n is canceled.

Case 2: n enters C as a marker $F(\gamma')$: Then some number $n' < K(\gamma')$ must enter A at the same stage. Again by our hypothesis on s_x, $F(\gamma')$ must have been picked after stage s_x, and we reach a contradiction as in Case 1.

Case 3: n enters B_i: Then n enters A at the same stage. By the arguments of Cases 1 and 2, the computation $\{j\}^{C \oplus B_{1-i}}(x)$ on the "other" side must have been destroyed by a number n' entering B_{1-i} since the most recent α-expansionary stage s', say. By the cancellation of markers, $n < n'$, and some number $< n'$ must enter C at stage s, leading to a contradiction as in Case 2.

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