A SURVEY OF RESULTS ON THE D.C.E. AND n-C.E. DEGREES

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1. Early history

This paper gives a brief survey of work on the d.c.e. and n-c.e. degrees done over the past fifty years, with particular emphasis on work done by or in collaboration with the Kazan logic group founded and headed by Arslanov.

1.1. **Definitions.** The history of the subject dates back almost fifty years to the following

Definition 1.1 (Putnam [Pu65])**.** For $n > 0$, a set $A \subseteq \omega$ is *n-c.e.* (or *n-r.e.*) if there is a computable approximation $\{A_s\}_{s \in \omega}$ such that $A_0 = \emptyset$ and for all x,

$$A(x) = \lim_s A_s(x), \text{ and}$$
$$|\{s \mid A_s(x) \neq A_{s+1}(x)\}| \leq n.$$

So a 1-c.e. set is simply a c.e. set, and a 2-c.e. set is a difference of two c.e. sets (also called a *d.c.e. set*).

Putnam actually called the n-c.e. sets "n-trial and error predicates" (and did not require $A_0 = \emptyset$). On the other hand, Gold [Go65], in a paper published in the same volume of the journal, defined "n-r.e." to mean Σ^0_n (and is sometimes falsely credited with the above definition).

Ershov [Er68a, Er68b, Er70] expanded this definition into the transfinite, defining the *α-c.e. sets* for every computable ordinal α. He proved many of the fundamental results about this so-called *Ershov hierarchy* Σ^{-1}_α.

For $\alpha \geq \omega$, his notion depends on the ordinal notation for α. E.g., given any fixed notation for any fixed computable α, Ershov showed that the α-c.e. sets do not exhaust all the Δ^0_2-sets; on the other hand, varying over all notations for ω^2, even the ω^2-c.e. sets exhaust all the Δ^0_2-sets.

In this paper, we will concentrate on the Turing degrees of the α-c.e. sets for $\alpha \leq \omega$, and up to degree, one can define a set A to be *ω-c.e.* by replacing the second condition in the above definition by

$$|\{s \mid A_s(x) \neq A_{s+1}(x)\}| \leq f(x)$$

for some computable strictly increasing function f.

A Turing degree is called *α-c.e.* if it contains an α-c.e. set.

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1.2. **Questions and basic results.** The main focus of this paper will be the answers that have been obtained over the past fifty years to some of the following

Questions 1.2.
- What is the structure of the d.c.e. degrees? Of the n-c.e. degrees?
- Are they isomorphic, or at least elementarily equivalent, to the c.e. degrees? To the Δ^0_2-degrees? To each other?
- What is the structure of the c.e. degrees inside the d.c.e. degrees? Of the m-c.e. degrees inside the n-c.e. degrees for $m < n$? In either case, does the former form an elementary substructure in the latter? Or at least a Σ_1-elementary substructure?
- Is the first-order theory of the d.c.e. degrees decidable? Of the n-c.e. degrees? If not, which fragments are decidable?

While Ershov's work focused on the n-c.e. sets and their m-degrees, the first result on their Turing degrees is probably the following

Theorem 1.3 (Lachlan (1968, unpublished))**.** *Every nonzero d.c.e. degree* $\mathbf{d}$ *bounds a nonzero c.e. degree* $\mathbf{a} < \mathbf{d}$ *such that* $\mathbf{d}$ *is c.e. in* $\mathbf{a}$*.*

More generally, every nonzero $(n+1)$*-c.e. degree* $\mathbf{d}$ *bounds a nonzero* n*-c.e. degree* $\mathbf{a} < \mathbf{d}$ *such that* $\mathbf{d}$ *is c.e. in* $\mathbf{a}$*.*

So, in particular, the n-c.e. degrees are downward dense for all n, giving an elementary, in fact a Σ_2-elementary, difference to the Δ^0_2-degrees.

Proof (for $n = 1$): Use the Sacks Density Theorem if $\mathbf{d}$ is c.e.

Otherwise, fix a d.c.e. set $D = E - F \in \mathbf{d}$ (for c.e. sets E and F) and let $A = \{\langle x, s\rangle \mid \exists t\,(x \in E_s - F_t)\}$. □

In his thesis, Cooper first separated the c.e. and d.c.e. Turing degrees:

Theorem 1.4 (Cooper [Co71])**.** *There is a* properly d.c.e. *degree, i.e., a non-c.e. d.c.e. degree.*

In fact, Cooper stated his theorem as "There is a set btt-reducible to K whose degree is not recursively enumerable".

1.3. **Related results.** We digress to highlight some connections between n-c.e. degrees and relative enumerability:

Theorem 1.5 (Cooper and Yi [CY71])**.**
(1) *There is an* isolated d.c.e. degree $\mathbf{d}$*, i.e., there is a c.e. degree* $\mathbf{a} < \mathbf{d}$ *such that any c.e. degree* $\mathbf{b} \leq \mathbf{d}$ *is actually* $\leq \mathbf{a}$*. (So, by Lachlan,* $\mathbf{d}$ *is c.e. in* $\mathbf{a}$*.)*
(2) *There is a* nonisolated d.c.e. degree $\mathbf{d}$*, i.e., one for which the c.e. degrees* $\leq \mathbf{d}$ *have no maximal element.*

This theorem was later strengthened as follows:

Theorem 1.6 (Ishmukhametov [Is99])**.**
(1) *There is a d.c.e. degree* $\mathbf{d}$ *for which there is a unique c.e. degree* $\mathbf{a} < \mathbf{d}$ *such that* $\mathbf{d}$ *is c.e. in* $\mathbf{a}$*.*
(2) *There is a d.c.e. degree* $\mathbf{d}$ *for which there are c.e. degrees* $\mathbf{a} < \mathbf{b} < \mathbf{d}$ *such that* $\mathbf{d}$ *is c.e. exactly in the c.e. degrees in the interval* $[\mathbf{a}, \mathbf{b}]$*.*

The following theorem gives an early hint at a difference between d.c.e. degrees and n-c.e. degrees for $n > 2$:

Theorem 1.7 (Arslanov, LaForte and Slaman [ALS98])**.** (1) *Every ω-c.e. degree $\mathbf{d}$ which is c.e. in a c.e. degree $\mathbf{a} < \mathbf{d}$ is actually 2-c.e.*
(2) *On the other hand, there is a d.c.e. degree $\mathbf{a}$ such that for every $n > 2$, there is a degree $\mathbf{d} > \mathbf{a}$ c.e. in $\mathbf{a}$ which is $(n+1)$-c.e. but not n-c.e.*

2. A survey of results and open questions

We now return to the main theme of this paper, answers and partial answers to the questions posed above:

2.1. **Elementary differences.** The first elementary difference between the c.e. and the d.c.e. Turing degrees was found by Arslanov:

Theorem 2.1 (Arslanov [Ar85, Ar88])**.** *Every nonzero d.c.e. degree* $\mathbf{d}$ cups to $\mathbf{0}'$ *in the d.c.e. degrees, i.e., there is an incomplete d.c.e. degree* $\mathbf{e}$ *such that* $\mathbf{d} \cup \mathbf{e} = \mathbf{0}'$.
(The same holds for the n-c.e. degrees for all $n \geq 2$.)

By Yates (1973, unpublished), the above fails for the c.e. degrees, so there is a Σ_3-elementary difference between the c.e. degrees on the one hand, and the n-c.e. degrees (for $n \geq 2$) on the other hand. A sharper elementary difference was given by Downey soon afterwards:

Theorem 2.2 (Downey [Do89])**.** *There are nonzero d.c.e. degrees* $\mathbf{d}$ *and* $\mathbf{e}$ *such that* $\mathbf{d} \cap \mathbf{e} = \mathbf{0}$ *and* $\mathbf{d} \cup \mathbf{e} = \mathbf{0}'$.

Of course, the Lachlan Nondiamond Theorem shows that the above fails for the c.e. degrees, which gives a Σ_2-elementary difference between the c.e. degrees on the one hand, and the n-c.e. degrees (for $n \geq 2$) on the other hand.

A third elementary difference was given by the following

D.C.E. Nondensity Theorem (Cooper, Harrington, Lachlan, Lempp and Soare [CHLLS91]**).** There is a maximal incomplete d.c.e. degree $\mathbf{d}$, so the d.c.e. degrees are not dense.

(In fact, $\mathbf{d}$ is also maximal in the α-c.e. degrees for all $\alpha \in [2, \omega]$, so the α-c.e. degrees are not dense.)

This shows that the c.e. degrees on the one hand, and the α-c.e. degrees (for $\alpha \in [2, \omega]$) on the other hand, are not Σ_2-elementarily equivalent, whereas they are clearly Σ_1-elementarily equivalent (since any finite partial order can be embedded in any of them).

Downey [Do89] conjectured, more as a challenge to the research community than as a firm belief, that the 2-c.e. degrees and the n-c.e. degrees are elementarily equivalent for all $n > 2$. However:

"3-Bubble" Theorem (Arslanov, Kalimullin and Lempp [AKL10]**).** The following holds in the 3-c.e. degrees but not in the 2-c.e. degrees:

There are degrees $\mathbf{c} > \mathbf{b} > \mathbf{a} > \mathbf{0}$ such that any degree $\mathbf{x} \leq \mathbf{c}$ is comparable to both $\mathbf{a}$ and $\mathbf{b}$.

(The above statement is actually a slight improvement of the original statement due to Wu and Yamaleev [WY12].)

This leaves open the following

Conjecture 2.3. For any distinct $m, n > 2$, the m-c.e. degrees and the n-c.e. degrees are not elementarily equivalent.

(We suspect that an "n-Bubble" Theorem holds.)

2.2. **Non-elementary substructures.** After studying elementary differences, we next turn to the study of elementary substructures:

Theorem 2.4 (Slaman (1983, unpublished)). *There is a* Slaman triple*, i.e., there are c.e. degrees* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *such that*

- *there is a nonzero* Δ^0_2*-degree* $\mathbf{x} \leq \mathbf{a}$ *such that* $\mathbf{c} \not\leq \mathbf{b} \cup \mathbf{x}$*, and*
- *there is no such c.e. degree* $\mathbf{x}$*.*

By Lachlan's result, this implies that the n-c.e. degrees do not form a Σ_1-elementary substructure of the Δ^0_2-degrees for any n.

Theorem 2.5 (Yang and Yu [YY06] for $n = 1$; Cai, Shore and Slaman [CSS12]). *There are n-c.e. degrees* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}$ *such that*

- *there is a nonzero $(n+1)$-c.e. degree* $\mathbf{x} \leq \mathbf{a}$ *such that* $\mathbf{x} \not\leq \mathbf{e}$ *and* $\mathbf{c} \not\leq \mathbf{b} \cup \mathbf{x}$*, and*
- *there is no such n-c.e. degree* $\mathbf{x}$*.*

This implies that the m-c.e. degrees do not form a Σ_1-elementary substructure of the n-c.e. degrees whenever $1 \leq m < n$. (For $m = 2$, this result was also claimed by Arslanov and Yamaleev (unpublished).)

2.3. **Undecidability.** Finally, degree structures are very complicated from an algebraic point of view, so it is natural to make this precise:

Theorem 2.6 (Cai, Shore, Slaman [CSS12]). *Given n and a computable partial order $(\omega, \preceq)$, there are c.e. degrees* $\mathbf{b}, \mathbf{c}$ *and (uniformly) n-c.e. degrees* $\mathbf{a}$ *and* $\mathbf{d}_i$ *(for $i \in \omega$) such that*

- *each* $\mathbf{d}_i$ *is maximal in the n-c.e. degrees with the property that* $\mathbf{d}_i \leq \cup_{j \in \omega} \mathbf{d}_j$ *and* $\mathbf{c} \not\leq \mathbf{b} \cup \mathbf{d}_i$*, and*
- $\mathbf{d}_i \leq \mathbf{d}_j \cup \mathbf{a}$ *iff $i \preceq j$.*

By Taitslin's result [Ta62] *that the theory of partial orders and the complement of the theory of finite partial orders are effectively inseparable, this implies that the first-order theory of the n-c.e. degrees is undecidable for all n.*

Undecidability had been established before by Harrington and Shelah [HS82] for the c.e. degrees, and by Epstein [Ep79] and Lerman [Le83] for the Δ^0_2-degrees.

This leads to the final

Conjecture 2.7. For all n, the first-order theory of the n-c.e. degrees is as complicated as true first-order arithmetic.

The above was shown by Harrington, Slaman and Woodin (1980's) for the c.e. degrees, and by Shore (1981) for the Δ^0_2-degrees.

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