THE FIRST-ORDER THEORY OF THE COMPUTABLY ENUMERABLE EQUIVALENCE RELATIONS IN THE UNCOUNTABLE SETTING

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Abstract. We generalize the analysis of Andrews, Schweber and Sorbi of the first-order theory of the partial order of degrees of c.e. equivalence relations to higher computability theory, specifically to the setting of a regular cardinal.

1. Introduction

We lift the analysis of Andrews, Schweber and Sorbi [ASS20] of the first-order theory of the partial order of degrees of c.e. equivalence relations to higher computability theory. Specifically, we work in the setting of κ-recursion for a regular cardinal κ. Andrews, Schweber, and Sorbi showed that the structure **Ceers** of degrees of c.e. equivalence relations (or *ceers*) under computable reducibility ($R \leq S$ if and only if there is a computable function f so that $x\ R\ y$ if and only if $f(x)\ S\ f(y)$) interprets $(\mathbb{N}, +, \cdot)$. In particular, the theory of **Ceers** is computably isomorphic with the theory of first-order arithmetic. We show the analogous result for the structure $\mathbf{Ceers}_\kappa$:

Theorem 1.1 (Main Theorem)**.** *The partial order $\mathbf{Ceers}_\kappa$ of κ-ceers under κ-computable reducibility (defined analogously) interprets the structure L_κ. In particular, the theory of $\mathbf{Ceers}_\kappa$ is computably isomorphic with the theory of L_κ.*

Since we are forced to pay more attention to the combinatorial principles at work, we are ultimately led to a simplification of the original argument. This simplification stems from the fact that it is much easier to code finite structures into the ceers, or κ-finite structures into the κ-ceers. In particular, every element of L_κ is contained in an admissible set which is itself κ-finite. We can therefore within $\mathbf{Ceers}_\kappa$ try to build L_κ itself by "pasting together" appropriately coded versions of smaller admissible sets. The new wrinkle in this approach is the danger of overshooting; this is handled by a general fact about interpretations between admissible sets (Lemma 5.3).

We also present a different argument for the special case $\kappa = \omega_1$; while applicable to a much narrower collection of ordinals, essentially the successor cardinals which define their predecessors in a "simple" way, it relies less on the specific nature

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of α-recursion theory and is applicable to a wider range of higher computability notions.

Convention 1.2. Throughout this paper, κ is an uncountable regular cardinal.

We refer for the basics and the background on c.e. equivalence relations (or *ceers* on ω, i.e., the classical case) to the papers by Gao and Gerdes [GG01] or Andrews, Badaev, and Sorbi [ABS17] and Andrews and Sorbi [AS19]. However, for the convenience of the reader, we recall some of the basic notions in the following section, phrased in terms of κ-ceers.

Convention 1.3. Throughout this paper, we assume for simplicity $\mathsf{V} = \mathsf{L}$. This hypothesis, however, can be removed after the fact since the theorem we prove is absolute between V and L. Indeed, the objects the Main Theorem refers to, such as κ-ceers, κ-computable reductions between κ-ceers, and L_κ can each be defined with quantifiers ranging over L as opposed to V. Thus the interpretation of L_κ in $\mathbf{Ceers}_\kappa$ which we will construct in L also gives an interpretation in V. Thus also the computable isomorphism between the first-order theories of these structures also holds in V.

The simplifying role of the assumption $\mathsf{V} = \mathsf{L}$ in our argument is purely linguistic. It lets us conflate "κ-finite" and "size $< \kappa$". In our view, this makes the flow of ideas clearer.

Lemma 1.4. *A subset of L_κ is κ-finite if and only if it has size $< \kappa$. In particular, if f is a partial function from κ to κ whose domain has size $< \kappa$, then f is κ-finite.*

To reiterate, this is *not* provable in ZFC alone. However, the results of this paper *are* theorems of ZFC since they are absolute between V and L, despite the proofs using the lemma above.

Proof. Let X be a subset of L_κ of size less than κ. Then by regularity of κ, there must be an $\alpha < \kappa$ so that $X \subseteq L_\alpha$. Since $\mathsf{V} = \mathsf{L}$, $X \in L_{\alpha+1}$, so $X \in L_\kappa$. That is, X is κ-finite.

Note that we used both the regularity of κ and the convention that $\mathsf{V} = \mathsf{L}$. For example, if we did not have regularity of κ, then a smaller cofinal subset of κ would not be κ-finite, and without $\mathsf{V} = \mathsf{L}$, a non-constructible subset of ω is small but is not in L, so certainly not κ-finite. □

We will use this, for example, to build κ-computable reductions from a κ-ceer which has $\lambda < \kappa$ many classes to any κ-ceer with $\geq \lambda$ many classes. We will be able to name a map from a representative of each class, and it's free that this map is κ-computable because it is κ-finite. We can then extend this to a κ-computable reduction of κ-ceers. There is a related topic where the hypothesis $\mathsf{V} = \mathsf{L}$ would no longer be benign. Namely, Andrews, Belin, and San Mauro [ABSMta] consider the partial order of all equivalence relations on ω of arbitrary complexity under computable reducibility. By analogy, we could fix an uncountable cardinal κ which is regular in V and consider the preorder of arbitrary equivalence relations on κ under κ-computable reductions. A priori, basic facts about this preorder might depend heavily on whether or not $\mathsf{V} = \mathsf{L}$ since the underlying set of the preorder is *not* determined by L alone. However, in this paper, we will only consider the local structure of κ-ceers.

In the next section, we cover basic properties of κ-ceers including the combinatorics of working with κ-finite κ-ceers. In section 3, we will introduce the definable collection of "almost self-full minimal κ-ceers" which have nice combinatorial properties. In section 4, we show how to code any κ-finite graph into $\mathbf{Ceers}_\kappa$. The almost self-full minimal κ-ceers will be used as the domains of these encodings, and the combinatorial properties from section 3 will be critical to carrying out the coding. In section 5, we will show that $\mathbf{Ceers}_\kappa$ interprets L_κ without parameters by "pasting together" the encodings of L_α for admissible $\alpha < \kappa$. Finally, in section 6, we give an alternate approach to the fact that $\mathbf{Ceers}_{\omega_1}$ interprets L_{ω_1}. This approach is applicable to a wider range of notions of uncountable computability.

1.1. Basic notions of κ-computability.

We recall the following definitions:

Definition 1.5. A set $B \subseteq \kappa^n$ is said to be *κ-computably enumerable* if B is Σ_1-definable over L_κ. Note that parameters from L_κ are permitted in the definition.

A partial function $f : \kappa \to \kappa$ is said to be *partial κ-computable* if its graph is a κ-computably enumerable set. If f is total on κ, then it is said to be *κ-computable.*

The *κ-finite* subsets of κ are those that belong to L_κ.

All of these notions are transferred from κ to L_κ via a natural enumeration of L_κ that maps κ onto L_κ.

We note that using our conventions that κ is regular and $\mathsf{V} = \mathsf{L}$, Lemma 1.4 tells us that κ-finiteness has a simpler description. A subset of L_κ is a member of L_κ as long as its size is less than κ.

2. Basic Results about κ-ceers

Definition 2.1.
1. A *κ-ceer* is a κ-c.e. equivalence relation on the set κ.
2. A κ-ceer R is *reducible* to a κ-ceer S (denoted $R \leq S$) if there is a κ-computable function $f : \kappa \to \kappa$ such that for all $\alpha, \beta < \kappa$, $\alpha \mathrel{R} \beta$ if and only if $f(\alpha) \mathrel{S} f(\beta)$. κ-ceers R and S are *equivalent* (denoted $R \equiv S$) if $R \leq S$ and $S \leq R$. The *degree* of a κ-ceer is its $\equiv$-equivalence class.
3. The structure of the degrees of all κ-ceers in the language of $\leq$ will be denoted by $\mathbf{Ceers}_\kappa$.
4. The *uniform join* of κ-ceers R and S is $R \oplus S$, defined by $(\alpha + m)\ (R \oplus S)\ (\beta + n)$ (for non-successor ordinals $\alpha, \beta < \kappa$ and $m, n \in \omega$) if and only if $(\alpha + \frac{m}{2})\ R\ (\beta + \frac{n}{2})$ (and m and n are both even) or $(\alpha + \frac{m-1}{2})\ S\ (\beta + \frac{n-1}{2})$ (and m and n are both odd). We extend this definition to arbitrarily large κ-finite sums: For a sequence $(R_\alpha)_{\alpha<\lambda}$ for some $\lambda < \kappa$, we can define E to be an equivalence relation on $\kappa \times \lambda$ given by $(\alpha, \beta)\ E\ (\gamma, \delta)$ if and only if $\beta = \delta$ and $\alpha\ R_\beta\ \gamma$. Finally, using a uniform sequence of κ-computable bijections between κ and ordinals $\kappa \times \lambda$ for $\lambda < \kappa$, we translate E to an equivalence relation $\oplus_\alpha R_\alpha$ on κ. (Note that $\oplus$ is well-defined on degrees even though $\deg(R \oplus S)$ need *not* be the least upper bound of the degrees of R and S.)
5. For any (possibly finite) nonzero cardinal $\lambda < \kappa$, the κ-ceer Id_λ is defined by $\alpha\ \mathrm{Id}_\lambda\ \beta$ (for $\alpha \leq \beta$) if and only if $\lambda \cdot \gamma + \alpha = \beta$ for some γ. The κ-ceer Id is given by equality, i.e., $\alpha\ \mathrm{Id}\ \beta$ if and only if $\alpha = \beta$.
6. A κ-ceer R is *κ-finite* if $R \equiv \mathrm{Id}_\lambda$ for some $\lambda < \kappa$; otherwise R is *κ-infinite.* We will show immediately below that κ-finiteness is equivalent to having

fewer than κ many classes. A κ-ceer R is *light* if $\mathrm{Id} \leq R$; and it is *dark* if it is neither κ-finite nor light.

(7) A κ-ceer R is *self-full* if $R \oplus \mathrm{Id}_1 \not\leq R$.

(8) If A is any κ-ceer and W is a κ-infinite κ-c.e. set, then $A \mid W$ is the κ-ceer obtained by taking a κ-computable bijection g of κ with W and defining $a\ A \mid W\ b$ if and only if $g(a)\ A\ g(b)$. This is well-defined up to degree, so we do not need to specify the bijection g we use. If W is κ-finite, we write $A \mid W$ for Id_λ, where λ is the cardinality of the set of A-equivalence classes of members of W.

(9) We say $A \leq_{\mathbf{Fin}_\kappa} B$ if there is a κ-finite κ-ceer F so that $A \leq B \oplus F$. This gives the notion of the "mod-κ-finite" κ-ceer-degrees.

We note that it might be more natural to define κ-ceers to have domain any κ-c.e. subset of κ. We would then demand a reduction to be a κ-computable function defined on the whole domain. In this case, (8) above has the more natural definition of $A \mid W$ being literally given by restriction. Of course, this intuition gave rise to the notation. We do not take this approach, though it may be more elegant, in order to match the approach taken in the classical setting of ω-ceers. We would also define Id_λ to be given by equality on the domain λ. Again, this clearly motivates the notation.

Lemma 2.2. *We note the following basic facts:*

(1) If f is a reduction of A to B and the range of f intersects every B-class, then $A \equiv B$.

(2) If R is a κ-ceer with exactly λ many classes for some $\lambda < \kappa$, then $R \equiv \mathrm{Id}_\lambda$. So, a κ-ceer is κ-finite if and only if it has fewer than κ many classes.

(3) If f is a reduction of A to B, then $A \equiv B \mid \mathrm{ran}(f)$.

(4) If $X \leq Y \oplus Z$, then $X \equiv Y_0 \oplus Z_0$ for some $Y_0 \leq Y$ and $Z_0 \leq Z$.

(5) If R is not self-full, then $\mathrm{Id}_\omega \leq R$. (In the classical setting, where $\mathrm{Id} = \mathrm{Id}_\omega$, this is the result that dark degrees are self-full.)

Proof. (1) Given an element $x \in \kappa$, we can κ-computably search for some $y \in \kappa$ so that $x\ B\ y$ and y is in the range of f, say, $y = f(a)$. Then the map $x \mapsto a$ gives a reduction of B to A.

(2) Fix an enumeration of the classes of R in order type λ. Then the map that sends the γth class to γ is a κ-computable reduction of R to Id_λ, and it is onto the classes of Id_λ, which shows that $R \equiv \mathrm{Id}_\lambda$ by (1). Note that we are using Lemma 1.4 here, which relies on the regularity of κ, and our convention that $\mathsf{V} = \mathsf{L}$. Using Lemma 1.4 allows us to act analogously to what we do in the classical setting of defining a computable function by first non-uniformly fixing its values on finitely many inputs.

(3) In the case when $\mathrm{ran}(f)$ is κ-infinite, this follows by (1). In the case when $\mathrm{ran}(f)$ is κ-finite, this follows by (2).

(4) If $X \leq Y \oplus Z$ via the map f, then let $Y_0 = Y \mid \{x \mid 2x \in \mathrm{ran}(f)\}$ and $Z_0 = Z \mid \{x \mid 2x+1 \in \mathrm{ran}(f)\}$. Note that $y \in Y_0$ if and only if the image of y in the reduction of Y to $Y \oplus Z$ is in the range of f (similarly for $z \in Z_0$). Then we can easily tweak f to give a reduction of X to $Y_0 \oplus Z_0$ which is onto, and so $X \equiv Y_0 \oplus Z_0$ by (1).

(5) Say $R \oplus \mathrm{Id}_1 \leq R$ via a map f. Then $g = f(2x)$ is a reduction from R to itself whose range is disjoint from some R-class (namely, the class that Id_1 was

sent to by f). Let a be in this class. Then $\{g^{(n)}(a) \mid n \in \omega\}$ is an infinite set of pairwise R-inequivalent elements, since $g^{(n)}(a)$ cannot be R-equivalent to a as the class of a does not intersect the range of g; but since g is a reduction of R to itself, this implies that $g^{(n+k)}(a)$ is not R-equivalent to $g^{(k)}(a)$ for any $n, k \in \omega$. □

We will need the following theorem whose proof is a straightforward adaptation of [AS19, Theorem 7.2]. Here we give only a proof sketch. For a careful exposition of the priority argument and how each parameter is chosen, see [AS19].

Theorem 2.3 (Exact Pair Theorem). *Let $(A_i)_{i<\kappa}$ be a uniformly κ-c.e. sequence of κ-ceers. Then there exist two κ-ceers X and Y above $\bigoplus_{i<\alpha} A_i$ for every $\alpha < \kappa$ so that any κ-ceer Z which is reducible to both X and Y is below $\bigoplus_{i<\alpha} A_i$ for some $\alpha < \kappa$.*

Proof. We construct the κ-ceers X and Y to meet requirements:

Q_α: There is some column of X and Y which codes A_α.

$P_{j,k}$: If Z is reducible to X via φ_j and reducible to Y via φ_k, then $Z \leq \bigoplus_{i<\alpha} A_i$ for some $\alpha < \kappa$.

For the sake of a Q-requirement, we "restrain a column" of each of X and Y to prevent collapse aside from those used for coding on that column. For the sake of P-requirements, we collapse potentially many columns all together to the class of 0 in either X or Y in order to perform a diagonalization. If we cannot perform the diagonalization, this will be because the range of φ_j or φ_k is contained entirely in the κ-finitely many columns restrained by higher-priority Q-requirements. The argument is put together as a standard κ-finite-injury priority construction. □

2.1. **Working with κ-finite κ-ceers.** In this section, we establish facts about working with κ-finite κ-ceers. In particular, we will characterize the relation $\leq_{\mathbf{Fin}_\kappa}$. We begin with a *negative* observation, pointing out a key difference between κ-ceers and ω-ceers.

We start with the following

Fact 2.4. *Let A and B be κ-ceers. Then we cannot "cancel κ-finite κ-ceers" in the following sense: The fact that $A \oplus F \equiv B \oplus F$ for a κ-finite κ-ceer F does not imply that $A \equiv B$. For example,* $\mathrm{Id}_1 \oplus \mathrm{Id}_\omega \equiv \mathrm{Id}_2 \oplus \mathrm{Id}_\omega$, *but* $\mathrm{Id}_1 \not\equiv \mathrm{Id}_2$. □

This does not happen in the setting of ω-ceers. In fact, even for κ-ceers A and B, if F is truly finite (i.e., has $< \omega$ equivalence classes), then $A \oplus F \equiv B \oplus F$ implies $A \equiv B$.

That said, much of what we want for finite ceers does carry through. For example, κ-infinite κ-ceers bound all κ-finite κ-ceers.

Lemma 2.5. *If X is a κ-infinite κ-ceer, then every κ-finite κ-ceer is reducible to X.*

Proof. Fix $\lambda < \kappa$. Let $(a_i)_{i\in\lambda}$ be a sequence of X-inequivalent elements of κ. Then the map that sends $\lambda \cdot \gamma + \alpha$ to a_α gives a κ-computable reduction of Id_λ to X. Note that we are again using Lemma 1.4. □

Also, taking the uniform join of a degree with the degree of a κ-finite ceer does not move the degree very far in $\mathbf{Ceers}_\kappa$:

Lemma 2.6. *If $A \leq B$ and F is κ-finite, then either $B \in [A, A \oplus F]$ or $A \oplus F < B$.*

Proof. Let $F \equiv \mathrm{Id}_\gamma$ for some $\gamma < \kappa$, and let g be the reduction from A to B. We consider two cases: Either there are γ many classes in B which do not intersect the range of g or not. If there are, then we can reduce $A \oplus F$ to B by taking the union of g with a map sending the classes of Id_γ to these unused classes. If not, then let $\beta < \gamma$ be so that there are precisely β many B-classes which do not intersect the range of g. Then we can build a reduction g' from $A \oplus \mathrm{Id}_\beta$ to B which is onto the classes of B. Thus $B \equiv A \oplus \mathrm{Id}_\beta$. In both cases, we used Lemma 1.4 to κ-computably send the fewer than κ many classes to the right images. $\square$

Next we characterize the relation $\leq_{\mathbf{Fin}_\kappa}$ in terms of taking uniform joins with κ-finite κ-ceers.

Lemma 2.7. *If $X \leq_{\mathbf{Fin}_\kappa} Y$, then either $X \leq Y$ or there is some $\gamma < \kappa$ so that $X \equiv Y \oplus \mathrm{Id}_\gamma$.*

Proof. We have $X \leq Y \oplus \mathrm{Id}_\gamma$ for some $\gamma < \kappa$ via a reduction f. Then $X \equiv Y_0 \oplus \mathrm{Id}_\beta$ with $Y_0 \leq Y$ given by the image of the reduction f (since the only κ-ceers below Id_γ are equivalent to Id_β for some $\beta \leq \gamma$ by 2.2(2)). We consider two cases: If Y contains β many classes that do not intersect the range of f, then we can send the classes in X to these instead of Id_β, so we obtain a reduction witnessing $X \leq Y$. In the second case, there are not β many classes in Y which do not intersect the range of f. In this case, we have that $Y_0 \oplus \mathrm{Id}_\alpha \equiv Y$ for some $\alpha < \beta$. By sending α many of the elements in X which were sent into Id_β instead to these classes, we can build a map from X to $Y \oplus \mathrm{Id}_\beta$ which is onto the classes of Y in $Y \oplus \mathrm{Id}_\beta$. It follows that $X \equiv Y \oplus \mathrm{Id}_\delta$ for some $\delta \leq \beta$. $\square$

Corollary 2.8. *If $X < Z < X \oplus \mathrm{Id}_\gamma$, then $Z \equiv X \oplus \mathrm{Id}_\beta$ for some $\beta < \gamma$.* $\square$

Corollary 2.9. *If $X \equiv_{\mathbf{Fin}_\kappa} Y$, then either $X \equiv Y$ or there is a $\gamma < \kappa$ so that $X \equiv Y \oplus \mathrm{Id}_\gamma$ or $Y \equiv X \oplus \mathrm{Id}_\gamma$.* $\square$

From our characterization of $\leq_{\mathbf{Fin}_\kappa}$, we derive that the relation $\leq_{\mathbf{Fin}_\kappa}$ is definable in $\mathbf{Ceers}_\kappa$.

Lemma 2.10. *The set of pairs of degrees $\mathbf{a}, \mathbf{b}$ so that $\mathbf{a} \leq_{\mathbf{Fin}_\kappa} \mathbf{b}$ is definable in $\mathbf{Ceers}_\kappa$.*

Proof. This definition is similar to [AS19, Obs. 9.7]. By Lemma 2.7, $X \leq_{\mathbf{Fin}_\kappa} Y$ is equivalent to either $X \leq Y$ or $X > Y \wedge X \equiv Y \oplus \mathrm{Id}_\gamma$ for some $\gamma < \kappa$. The former is clearly definable, so we need only define the latter condition. We will show that for $Y > X$, $Y \equiv X \oplus \mathrm{Id}_\gamma$ for some $\gamma < \kappa$ is equivalent to $[X, Y]$ being linearly ordered and $(\forall Z)[X \leq Z \to (Y < Z \vee Z \in [X, Y])]$.

Suppose that $Y > X$ and $Y \equiv X \oplus \mathrm{Id}_\gamma$. Then Corollary 2.8 and Lemma 2.6 show that $[X, Y]$ is linearly ordered and $(\forall Z)[X \leq Z \to (Y < Z \vee Z \in [X, Y])]$.

Conversely, suppose that $Y > X$, $[X, Y]$ is linearly ordered, and $(\forall Z)[X \leq Z \to (Y < Z \vee Z \in [X, Y])]$. From the Exact Pair theorem (Theorem 2.3) applied to the sequence $(A_i)_{i \in \kappa}$ defined by $A_0 = X$ and $A_\gamma = \mathrm{Id}_1$ for $\gamma \neq 0$, we see that there are κ-ceers Z and W which are incomparable and form an exact pair for the set of degrees which are $\leq X \oplus \mathrm{Id}_\gamma$ for some $\gamma < \kappa$. By the condition, Y is comparable with each of Z and W. Since they are incomparable, Y is either above both or below both. If Y were above both Z and W, then $[X, Y]$ would not be linearly ordered, contradicting the condition. So Y must be $\leq Z, W$. Thus Y must be $\leq X \oplus \mathrm{Id}_\gamma$ for some $\gamma < \kappa$. Then $Y \equiv X \oplus \mathrm{Id}_\gamma$ for some $\gamma < \kappa$ by Corollary 2.8. $\square$

3. Almost self-full κ-ceers

In the ω-setting, the entire coding machinery from [ASS20] revolved around the good combinatorial behavior of dark minimal ceers, which were used as the domains for coded graphs. This in turn relied on defining the collection of dark minimal ceers, which leaned heavily on true finiteness, and in particular on the fact that classically, dark minimality is equivalent to self-full minimality.

Self-fullness is a very useful property in the study of ceers. In the setting of ω-ceers, if A is non-self-full, then for any finite ceer F, $A \oplus F \equiv A$. But in the κ-ceers, a non-self-full κ-ceer A can satisfy $A \oplus \mathrm{Id}_\lambda \not\equiv A$ for some $\lambda \geq \omega$. Consider for example the case with $\kappa > \aleph_1$ and $A = B \oplus \mathrm{Id}_\omega$ where B is self-full. A is clearly non-self-full, but $A \oplus \mathrm{Id}_{\omega_1} > A$.

The shift from ω to κ forces us back to the drawing board here. We now introduce the notion of a κ-ceer being almost self-full, and in this section, we show that the class of almost self-full κ-ceers is definable in $\mathbf{Ceers}_\kappa$ and that we can recover some of the nice combinatorial properties of self-full ceers for the almost self-full κ-ceers.

Definition 3.1. We say that a κ-ceer A is *almost self-full* if there is some κ-finite κ-ceer F so that $A \oplus F \not\leq A$.

A κ-infinite κ-ceer A is *minimal* if every $X < A$ is κ-finite.

A degree is *almost self-full* (or *minimal*) if some, or equivalently every, κ-ceer in the degree is almost self-full (or minimal, respectively).

Note that in the ω-ceers, being almost self-full is the same as being self-full. In the ω-ceers, the following formula defines the self-full ceers. In the κ-ceers, it defines the collection of almost self-full κ-ceers.

Lemma 3.2. *A κ-ceer A is almost self-full if and only if there exists a κ-ceer $B > A$ so that $\forall X\ (X > A \to X \geq B)$*

Proof. First suppose that A is almost self-full. Then for some $\gamma < \kappa$, we have that $A \oplus \mathrm{Id}_\gamma > A$. Fix a minimal such γ and let $B = A \oplus \mathrm{Id}_\gamma$. Then Lemma 2.6 shows that $X > A$ implies that either $X \geq B$ or X is strictly between A and B. The minimality of γ and Corollary 2.8 rule out the latter possibility.

Now suppose that A is not almost self-full. That is, $A \oplus F \equiv A$ for every κ-finite F. There are two κ-ceers X and Y which are incomparable so that $Z < X, Y$ implies $Z \leq A$. This follows directly from the Exact Pair Theorem (Theorem 2.3) applied to the sequence $(A_i)_{i\in\kappa}$ defined by $A_0 = A$ and $A_\gamma = \mathrm{Id}_1$ for $\gamma \neq 0$. Thus there can be no such B, which would have to be below both X and Y. □

Corollary 3.3. *The class of almost self-full κ-ceers is definable in* $\mathbf{Ceers}_\kappa$. □

The class that we will use in place of the dark minimal ceers will be the almost self-full minimal κ-ceers. That is, a degree is almost self-full minimal iff it is almost self-full and does not strictly bound any degree of an infinite κ-ceer.

3.1. Properties of reductions of minimal κ-ceers.

In this subsection, we see two nice combinatorial properties regarding reductions of minimal κ-ceers. Firstly, they are "atomic" with regard to uniform joins.

Lemma 3.4. *If A is minimal and $A \leq B \oplus C$, then $A \leq B$ or $A \leq C$.*

Proof. If $A \leq B \oplus C$, then we know $A \equiv B_0 \oplus C_0$ for some $B_0 \leq B$ and $C_0 \leq C$. Then B_0 is reducible to A as well, so it is either equivalent to A or κ-finite.

Similarly C_0 is reducible to A as well, so it is either equivalent to A or κ-finite. Both cannot be κ-finite since A is not κ-finite. If $B_0 \equiv A$, then $A \leq B$, and if $C_0 \equiv A$, then $A \leq C$. □

This fact extends even to κ-finite uniform joins.

Lemma 3.5. *Suppose that A is minimal and $A \leq \bigoplus_{i<\lambda} B_i$ for $\lambda < \kappa$. Then for some $i < \lambda$, $A \leq B_i$.*

Proof. By the same argument as in Lemma 2.2(4), we see that $A \equiv \bigoplus_{i<\lambda} B_i^0$, where $B_i^0 \leq B_i$. If any of the B_i^0 is infinite, then $A \equiv B_i^0$ and so $A \leq B_i$. If all of the B_i^0 were to be κ-finite, then the regularity of κ would imply that A is κ-finite, but A is minimal, so κ-infinite. □

Secondly, incomparability extends to $\leq_{\mathbf{Fin}_\kappa}$-incomparability.

Lemma 3.6. *Suppose that R and S are $\leq$-incomparable minimal κ-ceers. Then they are also $\leq_{\mathbf{Fin}_\kappa}$-incomparable. Namely, for no κ-finite F is $R \leq S \oplus F$.*

Proof. By Lemma 3.4, if R were to reduce to $S \oplus F$ for some κ-finite F, then R would reduce to either S or F. The first is assumed to be false and the second is impossible since R is κ-infinite. □

3.2. **Existence and combinatorial properties of almost self-full minimal κ-ceers.** In order to do the necessary coding using almost self-full minimal κ-ceers, we will need to show that there are κ many almost self-full minimal κ-ceers. The easiest way to do this is to show that there are κ many self-full minimal κ-ceers. We will use the following combinatorial characterization of the self-full minimal κ-ceers, which is the direct analog of the combinatorial characterization used in [ASS20].

Lemma 3.7. *A κ-ceer A is self-full and minimal if and only if it is κ-infinite and for every κ-c.e. set W, if W intersects κ many A-classes, then W intersects every A-class.*

Proof. Suppose that A is self-full and minimal. Let W be a κ-c.e. set which intersects κ many A-classes. Then $A \mid W$ is κ-infinite and $\leq A$. By minimality of A, it must be equivalent to A itself. Thus we have a reduction of A to $A \mid W$. If W were not to intersect every A-class, then we could extend this to a reduction of $A \oplus \mathrm{Id}_1$ to A, but A is assumed to be self-full.

Suppose that A is κ-infinite and for every κ-c.e. set W, if W intersects κ many A-classes, then W intersects every A-class. Let $X \leq A$ via f. If X is κ-infinite, then $\mathrm{ran}(f)$ is a κ-c.e. set which intersects κ many A-classes. Thus it intersects every A-class. Thus $X \equiv A$. So, A is minimal. If it were true that $A \oplus \mathrm{Id}_1$ reduced to A via some f, then the range of the even ordinals under f would be a κ-c.e. set which intersects κ many A-classes but would have to be disjoint from one class, namely, the class of $f(1)$. This is impossible by the condition, so A is self-full. □

It is immediate that a self-full minimal κ-ceer is almost self-full minimal.

Theorem 3.8. *There are κ many pairwise incomparable self-full minimal (and thus almost self-full minimal) κ-ceers.*

Proof. For simplicity, we build a pair of incomparable self-full minimal κ-ceers; the construction of κ many is similar.

We wish to build κ-ceers A_0, A_1 which are self-full minimal and incomparable. Towards this end, we must meet the following requirements for each $i < 2$ and $x, y \in \kappa$:

- $R^i_{x,y}$: if W_x intersects κ many A_i-classes then W_x intersects $[y]_{A_i}$.
- S^i_x: φ_x is not a reduction of A_i to A_{1-i}.

At every stage s, we enforce reflexivity, symmetry and transitivity; and we must describe when we collapse elements in A_i. The possible actions will be to make two elements of κ equivalent in A_0 or A_1 or to lay down (or maintain or remove) κ-finitely many restraints of the form "x and y remain A_i-inequivalent." Conversely, each such requirement will be faced with κ-finitely many such restraints, and if there is some stage after which a given requirement is never injured, then that requirement will act at most once after that stage and will be satisfied. Consequently, the κ-ceers A_i will have the desired properties. The strategies we use to meet the R- and S-requirements are the following:

- To meet $R^i_{x,y}$, we simply follow a greedy algorithm: If at a given stage, there is some element of W_x which we can A_i-collapse to y, we do so; otherwise, we wait. The only way we can be prevented from collapsing y to a given element u is if u itself is collapsed to an element involved in one of the restraints set by a higher-priority requirement. These restraints at the end of the day will only apply to κ-finitely many A_i-classes. So if in fact W_x meets κ many A_i-classes, then at some stage, there must have been an element enumerated into W_x which we were free to A_i-collapse to y.
- To meet S^i_x, we pick fresh distinct a and b on the A_i-side and restrain lower-priority requirements from causing A_i-collapse involving either of their classes; we then wait for $\varphi_x(a)$ and $\varphi_x(b)$ to halt. If this never happens, then the requirement is vacuously satisfied; otherwise, if $\varphi_x(a)\downarrow = c$ and $\varphi_x(b)\downarrow = d$, say, then we check to see whether $cA_{1-i}d$. If so, then we maintain our restraint; otherwise, we restrain c and d from becoming A_{1-i}-equivalent and A_i-connect a and b. Since we restrained lower-priority requirements from causing collapse involving a and b or c and d, the only way this strategy does not succeed is if a higher-priority requirement acts after it begins. In that case, we reinitialize the strategy, choosing fresh a and b. Since this can only happen fewer than κ many times, the strategy eventually succeeds. □

Finally, we note that almost self-full minimal κ-ceers satisfy something similar to the combinatorial condition for self-full minimality.

Lemma 3.9. *If A is almost self-full minimal and W is a κ-c.e. set which intersects κ-infinitely many A-classes, then it intersects κ-cofinitely many A-classes.*

In fact, if A is minimal, $A \oplus \mathrm{Id}_\lambda > A$, and W is a κ-c.e. set which intersects κ-infinitely many A-classes, then there are strictly fewer than λ A-classes which do not intersect W.

Proof. Suppose A is minimal, $A \oplus \mathrm{Id}_\lambda > A$, and W is a κ-c.e. set which intersects κ-infinitely many A-classes. Then $A \mid W$ is an infinite κ-ceer. By minimality of A, we must have $A \mid W \equiv A$. If W were to be disjoint from λ many A-classes, then we could reduce $A \mid W \oplus \mathrm{Id}_\lambda \equiv A \oplus \mathrm{Id}_\lambda$ to A. But since $A \oplus \mathrm{Id}_\lambda > A$, this is impossible. □

4. Coding structures with κ-ceers

We now build up the machinery to code graphs into $\mathbf{Ceers}_\kappa$. We use the almost self-full minimal κ-ceers as the domains of our graphs. We introduce the notion of a sharp cover of a pair of κ-ceers, which will be used to code edges in our graphs.

Definition 4.1. A *sharp cover* of a pair of $\leq_{\mathbf{Fin}_\kappa}$-incomparable minimal κ-ceers R_0 and R_1 is a κ-ceer S such that for all X, we have

$$X <_{\mathbf{Fin}_\kappa} S \Leftrightarrow X \leq_{\mathbf{Fin}_\kappa} R_0 \text{ or } X \leq_{\mathbf{Fin}_\kappa} R_1.$$

We now introduce the encoding of a graph by a single κ-ceer degree, and show that the graph encoded is uniformly definable in the parameter.

Definition 4.2. To each κ-ceer degree $\mathbf{c}$, we assign a graph $\mathrm{Graph}_\mathbf{c}$ as follows:

- The vertices of $\mathrm{Graph}_\mathbf{c}$ are the almost self-full minimal degrees $\leq \mathbf{c}$.
- For distinct (and hence $\leq_{\mathbf{Fin}_\kappa}$-incomparable) $\mathbf{a}, \mathbf{b} \in \mathrm{Vert}(\mathrm{Graph}_\mathbf{c})$, we set $\langle \mathbf{a}, \mathbf{b} \rangle \in \mathrm{Edge}(\mathrm{Graph}_\mathbf{c})$ if and only if there are $\leq_{\mathbf{Fin}_\kappa}$-incomparable $\mathbf{u}, \mathbf{v} \leq \mathbf{c}$ each of which is a sharp cover of $\mathbf{a}$ and $\mathbf{b}$.

Lemma 4.3. *Both the vertex set and the edge set of* $\mathrm{Graph}_\mathbf{c}$ *are definable in* $\mathbf{Ceers}_\kappa$ *uniformly in the parameter* $\mathbf{c}$.

Proof. The relation $\leq_{\mathbf{Fin}_\kappa}$ is definable by Lemma 2.10. Note that a κ-ceer A is κ-finite if and only if $A \leq_{\mathbf{Fin}_\kappa} \mathrm{Id}_1$, and the degree of Id_1 is definable as it is the least degree in $\mathbf{Ceers}_\kappa$. It follows that the κ-finite degrees, thus also the minimal degrees, and thus also the almost self-full minimal degrees (by Corollary 3.3), form definable classes. So we have shown that all of the notions used in the definition of $\mathrm{Graph}_\mathbf{c}$ are definable in $\mathbf{Ceers}_\kappa$. □

We next show how to build sharp covers of pairs of almost self-full minimal κ-ceers. This is necessary in order to show that every κ-finite graph on a set of almost self-full minimal κ-ceers is equal to $\mathrm{Graph}_\mathbf{c}$ for some $\mathbf{c}$.

Definition 4.4. For two κ-ceers A and B and two sequences $a = (a_i)_{i<\lambda}$ and $b = (b_j)_{j<\lambda}$, we let $A \oplus_{/a,b} B$ be the κ-ceer generated from $A \oplus B$ by connecting $2a_i$ and $2b_i + 1$ (that is, by connecting the A-class of a_i and the B-class of b_i) for every $i < \lambda$.

Lemma 4.5. *Let A and B be incomparable almost self-full minimal κ-ceers and $\omega \leq \lambda < \kappa$ be so that $A \oplus \mathrm{Id}_\lambda > A$ and $B \oplus \mathrm{Id}_\lambda > B$. Let a be a sequence of length λ in distinct A-classes, and let b be a sequence of length λ in distinct B-classes. Then $A \oplus B$ and $A \oplus_{/a,b} B$ are $\leq_{\mathbf{Fin}_\kappa}$-incomparable sharp covers of A and B.*

Proof. We trivially have $A, B \leq A \oplus B, A \oplus_{/a,b} B$. Note that A and B are $\mathbf{Fin}_\kappa$-incomparable by Lemma 3.6.

To see that $A \oplus B$ is a sharp cover of A and B, suppose that $X \leq A \oplus B \oplus \mathrm{Id}_\gamma$ for some $\gamma < \kappa$. Then $X \equiv A_0 \oplus B_0 \oplus \mathrm{Id}_\beta$ for some $A_0 \leq A$, $B_0 \leq B$ and $\beta \leq \gamma$ by Lemma 2.2(4). By minimality of A, A_0 is either κ-finite or equivalent to A. Similarly, B_0 is either κ-finite or equivalent to B. If both $A_0 \equiv A$ and $B_0 \equiv B$, then $X \equiv_{\mathbf{Fin}_\kappa} A \oplus B$. If both are κ-finite, then X is κ-finite. If $A_0 \equiv A$ and B_0 is κ-finite then $X \equiv_{\mathbf{Fin}_\kappa} A$, and if $B_0 \equiv B$ and A_0 is κ-finite then $X \equiv_{\mathbf{Fin}_\kappa} B$.

We now argue that $A \oplus_{/a,b} B$ is a sharp cover of A and B as well. Suppose that X reduces via f to $(A \oplus_{/a,b} B) \oplus \mathrm{Id}_\delta$ for some $\delta < \kappa$. Let A_0 be $A \mid \{x \mid 4x \in \mathrm{ran}(f)\}$.

Then A_0 is the trace of f on the copy of A in $(A \oplus_{/a,b} B) \oplus \mathrm{Id}_\delta$. Similarly, let B_0 be $B \mid \{x \mid 4x+2 \in \mathrm{ran}(f)\}$, which is the trace of f on the copy of B in $(A \oplus_{/a,b} B) \oplus \mathrm{Id}_\delta$. If A_0 (or B_0) is κ-finite, then $X \leq_{\mathbf{Fin}_\kappa} B$ (or $X \leq_{\mathbf{Fin}_\kappa} A$, respectively). On the other hand, if both A_0 and B_0 are κ-infinite, then the image of f can omit at most λ classes in $A \oplus_{/a,b} B$ as A_0 omits at most λ A-classes and B_0 omits at most λ B-classes by Lemma 3.9, so $X \equiv_{\mathbf{Fin}_\kappa} A \oplus_{/a,b} B$.

We now just need to establish $\leq_{\mathbf{Fin}_\kappa}$-incomparability of $A \oplus B$ and $A \oplus_{/a,b} B$. As they are both sharp covers of A and B, it suffices to show that $A \oplus B \not\leq (A \oplus_{/a,b} B) \oplus \mathrm{Id}_\gamma$ for any $\gamma < \kappa$. Suppose that f is such a reduction. We look at the reduction g of A to $(A \oplus_{/a,b} B) \oplus \mathrm{Id}_\gamma$ given by the composition of the obvious reduction of A to $A \oplus B$ and of f. Consider the trace of g on the κ-ceer B: That is, $B \mid W$ where W is the set of x so that $4x + 2$ is in the range of g. This κ-ceer $B \mid W$ is both reducible to B and A (via sending x to the first c found so that $g(c) = 4x+2$). Thus $B \mid W$ is κ-finite since A and B are incomparable and minimal. It follows that only κ-finitely many A-classes contain an element c so that $g(c)$ is 2 mod 4. Similarly, only κ-finitely many A-classes contain an element c so that $g(c)$ is odd. Thus, κ-infinitely many A-classes are sent entirely via g to elements that are 0 mod 4, that is, into the copy of A in $(A \oplus_{/a,b} B) \oplus \mathrm{Id}_\gamma$. Thus the set of x so that $4x$ is in the range of g must intersect κ-infinitely many A-classes. By Lemma 3.9, it misses fewer than λ many A-classes. The same argument shows that the reduction h of B to $(A \oplus_{/a,b} B) \oplus \mathrm{Id}_\gamma$ intersects all but fewer than λ many B-classes. But in the definition of $A \oplus_{/a,b} B$, we collapsed λ many A-classes with B-classes. By the pigeonhole principle, there is a pair c, d so that $g(c)\ ((A \oplus_{/a,b} B) \oplus \mathrm{Id}_\gamma)\ h(d)$. But the reductions g and h go through $A \oplus B$, where the images are not equivalent, a contradiction. □

We are finally ready for the key theorem of this section:

Theorem 4.6. *For every κ-finite undirected graph G with vertex set consisting of almost self-full minimal degrees in $\mathbf{Ceers}_\kappa$, there is a κ-ceer degree $\mathbf{c}$ with $\mathrm{Graph}_{\mathbf{c}} = G$.*

Remark 4.7. Note the equality, rather than mere isomorphism, between G and $\mathrm{Graph}_{\mathbf{c}}$: The vertices of G are precisely the vertices of $\mathrm{Graph}_{\mathbf{c}}$, and two vertices are connected in G if and only if they are connected in $\mathrm{Graph}_{\mathbf{c}}$. Note that $\mathbf{c}$ does provide extra information not present in G, namely, specific "names" for edges in the graph (the appropriate pairs of sharp covers), but this extra information isn't present in $\mathrm{Graph}_{\mathbf{c}}$ which really is just G itself. In particular, the construction $\mathbf{c} \mapsto \mathrm{Graph}_{\mathbf{c}}$ is not injective.

Proof of Theorem 4.6. Let $\mathrm{Vert}(G) = \{\deg(R_i) \mid i < \lambda\}$ for some $\lambda < \kappa$ be the vertex set of G. Each R_i is almost self-full. For each pair R_i, R_j with an edge in G, let $R_i \# R_j$ be $R_i \oplus_{/a,b} R_j$ for sequences a and b as in Lemma 4.5. Let C be the κ-ceer

$$(\bigoplus_{i<\lambda} R_i) \oplus \left(\bigoplus_{\langle i,j\rangle \in \mathrm{Edge}(G),\, i<j} R_i \# R_j \right),$$

and let $\mathbf{c} = \deg(C)$. For brevity, let $H = \mathrm{Graph}_{\mathbf{c}}$; we claim that $H = G$.

First, we observe that trivially $\mathrm{Vert}(H) \supseteq \{\deg(R_i) \mid i < \lambda\}$. Next, we check that H has no unwanted vertices. To see this, suppose that S is almost self-full minimal and that $S \leq C$. Then, by Lemma 3.5, we have that S must be reducible to one of the summands of C, so either to some R_i or to some $R_i \# R_j$. In the former case, we have $S \equiv R_i$, since R_i is minimal, and in the latter case, since $R_i \# R_j$ is a sharp cover of R_i and R_j, by Lemma 4.5, we have that $S \leq_{\mathbf{Fin}_\kappa} R_i$ or $S \leq_{\mathbf{Fin}_\kappa} R_j$. But incomparable almost self-full minimal κ-ceers are mod-finite incomparable by Lemma 3.6, so $S \equiv R_i$ or $S \equiv R_j$.

Now we turn to the edges. Note that $R_i \oplus R_j \leq C$ for all $i < j < \lambda$. So whenever G has an edge between $\deg(R_i)$ and $\deg(R_j)$, we have in H an edge between $\deg(R_i)$ and $\deg(R_j)$ as witnessed by $R_i \oplus R_j \leq C$ and $R_i \# R_j \leq C$. So it only remains to check that H has no unwanted edges.

Suppose that $i < j$ and that K is a sharp cover of R_i and R_j with $K \leq C$. Then K is of the form

$$\bigoplus_{i<\lambda} A_i \oplus \left(\bigoplus_{\langle i,j\rangle \in \mathrm{Edge}(G),\, i<j} B_{ij} \right)$$

where $A_i \leq R_i$ and $B_{ij} \leq R_i \# R_j$. Each A_i is either κ-finite or $\equiv R_i$ by minimality of R_i. Similarly, each B_{ij} is either κ-finite, $\equiv_{\mathbf{Fin}_\kappa} R_i \# R_j$, $\equiv_{\mathbf{Fin}_\kappa} R_i$ or $\equiv_{\mathbf{Fin}_\kappa} R_j$. Since K is a sharp cover of R_i and R_j, it cannot be $\geq R_k$ for any $k \neq i, j$. By Lemma 3.5, R_i and R_j each reduce to a single term in this expression. If they reduce to two different terms, then we have $R_i \oplus R_j \leq K$ and thus $R_i \oplus R_j \equiv_{\mathbf{Fin}_\kappa} K$ since K is a sharp cover of R_i and R_j. If they do not reduce to two different terms, then the only possible term they both reduce to is B_{ij}, In particular, G has an edge between $\deg(R_i)$ and $\deg(R_j)$. Thus, if H has an edge between $\deg(R_i)$ and $\deg(R_j)$, then there are two $\leq_{\mathbf{Fin}_\kappa}$-incomparable sharp covers of R_i and R_j, showing that G has an edge between $\deg(R_i)$ and $\deg(R_j)$. □

We have described how to code κ-finite graphs, but this lets us talk about coding any κ-finite structure.

Convention 4.8. It is well-known that undirected graphs are "universal" for structures of a bounded cardinality (for example, all structures of cardinality $\leq \kappa$) in any finite language. From now on, for simplicity, we will speak of (a degree of) a κ-ceer coding such a structure, rather than an undirected graph per se.

Note that the above proof gave literal equality on the vertex sets instead of mere isomorphism. This gives us additionally the ability to code *maps between* coded structures:

Corollary 4.9. *Suppose that A and B are disjoint κ-finite sets of almost self-full minimal degrees, and that $R \subseteq A \times B$ is some relation (e.g., the graph of a function from A to B). Then there is a κ-ceer coding the undirected graph with vertex set $A \sqcup B$ and with an edge between a and b if and only if either $(a, b) \in R$ or $(b, a) \in R$.*

This lets us talk about isomorphisms between coded structures, embeddings between coded structures, etc.

Corollary 4.10. *There is a first-order formula $\varphi(x, y)$ (without parameters) such that whenever $\mathbf{a}$ and $\mathbf{b}$ are κ-ceers coding κ-finite structures A and B, we have that $\mathbf{Ceers}_\kappa \models \varphi(\mathbf{a}, \mathbf{b})$ if and only if $A \cong B$.*

Proof. Though the result is stated for any encoded structures in a fixed language, it is equivalent to show it for graphs. Using Lemma 4.3, it is definable to say "There exist κ-ceers coding an isomorphism between $\mathrm{Graph}_{\mathbf{a}}$ and $\mathrm{Graph}_{\mathbf{b}}$."

We will now spell this out more formally. Let $\psi(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e})$ be the formula stating:

- The vertex set of $\mathrm{Graph}_{\mathbf{e}}$ is disjoint from the vertex sets of $\mathrm{Graph}_{\mathbf{a}}$ and $\mathrm{Graph}_{\mathbf{b}}$.
- The edges of $\mathrm{Graph}_{\mathbf{c}}$ give a function f from $\mathrm{Vert}(\mathrm{Graph}_{\mathbf{a}})$ to $\mathrm{Vert}(\mathrm{Graph}_{\mathbf{e}})$.
- The edges of $\mathrm{Graph}_{\mathbf{d}}$ give a function g from $\mathrm{Vert}(\mathrm{Graph}_{\mathbf{e}})$ to $\mathrm{Vert}(\mathrm{Graph}_{\mathbf{b}})$.
- Each of f and g are isomorphisms.

Each reference in ψ to the edges or vertex set of the graph coded by a degree is defined by a formula by Lemma 4.3. Finally, let $\varphi(\mathbf{a},\mathbf{b})$ be the formula $\exists\,\mathbf{c}\exists\,\mathbf{d}\exists\,\mathbf{e}\,\psi(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\mathbf{e})$. The formula $\varphi(\mathbf{a},\mathbf{b})$ clearly implies that the graphs encoded by $\mathbf{a}$ and $\mathbf{b}$ are isomorphic. While in general it is not clear that this is equivalent to the graphs encoded by $\mathbf{a}$ and $\mathbf{b}$ being isomorphic, it is equivalent if $\mathbf{a}$ and $\mathbf{b}$ code κ-finite structures since then the required graphs $\mathrm{Graph}_{\mathbf{c}}$, $\mathrm{Graph}_{\mathbf{d}}$, $\mathrm{Graph}_{\mathbf{e}}$ are also κ-finite objects and are thus encoded by κ-ceers by Theorem 4.6. □

5. Interpreting L_κ in $\mathbf{Ceers}_\kappa$

In this section, we will prove that $\mathbf{Ceers}_\kappa$ interprets L_κ. Consequently, L_κ and $\mathbf{Ceers}_\kappa$ are mutually interpretable. At this point, we have shown that $\mathbf{Ceers}_\kappa$ uniformly interprets all κ-finite graphs, which, like in the ω-setting is not enough to yield the result.

In this section, we show that the problem of interpreting L_κ for uncountable regular cardinals κ is different from the problem of interpreting arithmetic in the ω-case. In the κ-setting, we will show that interpreting κ-finite graphs along with embeddings between them and a mild second order quantifier suffices, whereas this would not suffice in the ω-setting. More formally: If $\mathcal{A}$ is any structure which is interpretable in L_κ and uniformly defines a collection of graphs which includes all κ-finite graphs, uniformly defines a collection of embeddings which includes all embeddings between the κ-finite graphs, and uniformly defines a collection of sets which includes all countable sets, then $\mathcal{A}$ interprets L_κ.

First, we show that a broad class of κ-ceer degrees is definable:

Lemma 5.1. *The set of $\mathbf{c} \in \mathbf{Ceers}_\kappa$ such that $\mathrm{Graph}_{\mathbf{c}}$ is a (representing undirected graph of a) structure isomorphic to some admissible level of L is definable in $\mathbf{Ceers}_\kappa$.*

Proof. A structure $\mathcal{S}$ is isomorphic to L_α for some admissible α if and only if $\mathcal{S}$ is well-founded and satisfies $\mathsf{KP}^- + \mathsf{V} = \mathsf{L}$ where KP^- is the theory obtained from the usual Kripke-Platek set theory KP by removing the foundation scheme. To see this, note that well-founded models of KP^- satisfy full KP, so it is enough to show that the admissible sets satisfying $\mathsf{V} = \mathsf{L}$ are exactly the admissible levels of the L-hierarchy. This follows from [Sa90, VII Lemma 2.5], which shows that a transitive set satisfies $\mathsf{V} = \mathsf{L}$ if and only if it is a limit level of the L-hierarchy.

The theory $\mathsf{KP}^- + \mathsf{V} = \mathsf{L}$ is axiomatizable by a single sentence, so the set of κ-ceer degrees coding structures satisfying it is definable in $\mathbf{Ceers}_\kappa$. Thus we just need to show that well-foundedness is appropriately definable.

To do this, note that since κ is an uncountable cardinal, for any ill-founded relation on κ at all, there is a κ-finite subset with no least element. Since every κ-finite graph is coded by a κ-ceer degree, this means that we can characterize the

κ-ceer degrees coding well-founded structures as those which code structures for which no other κ-ceer degree codes a counterexample to well-foundedness. $\square$

Our strategy will be to "build up" L_κ out of codes for L_λ with $\lambda < \kappa$. We have to make sure that we do not accidentally build L_γ for any $\gamma > \kappa$. This will be guaranteed by the following lemma, which is a general result about interpretability and admissible sets.

Definition 5.2. For an ordinal α, we let α^{ad} be the least admissible ordinal $> \alpha$. For a real r, we let $\omega_1^{CK}(r)$ be the least ordinal without an r-hyperarithmetic (or equivalently, per Spector, r-computable) copy.[1]

Lemma 5.3. *Suppose α is admissible. Then L_α is first-order interpretable (with parameters) in L_κ if and only if $\alpha \leq \kappa$.*

Proof. Since $L_\alpha \in L_\kappa$ when $\alpha < \kappa$, the right-to-left direction is immediate. For the other direction, by the same reasoning and since L_η interprets η, it is enough to show that κ^{ad} is not interpretable in L_κ.

We will use a forcing argument to reduce this to a problem about reals. While this is not strictly necessary, it may have the benefit of making the argument more concrete for those more used to classical computability theory.

Consider the forcing partial order $\mathrm{Coll}(\omega, \kappa)$ consisting of all finite partial functions from ω into κ ordered by reverse extension. We will think of this as building a generic copy G of κ with domain ω. The forcing $\mathrm{Coll}(\omega, \kappa)$ is a set in $L_{\kappa^{ad}}$, hence letting G be $\mathrm{Coll}(\kappa, \omega)$-generic over $L_{\kappa^{ad}}$, we have that $M := L_{\kappa^{ad}}[G]$ is itself admissible.

We now calculate $\omega_1^{CK}(G)$. Since G is literally a copy of κ, we have $\kappa < \omega_1^{CK}(G)$. On the other hand, the ordinal $\omega_1^{CK}(G)$ is the height of the smallest admissible set with G as an element, and so $\omega_1^{CK}(G) \leq \kappa^{ad}$. So, in fact, $\omega_1^{CK}(G) = \kappa^{ad}$, as by definition, there are no admissible ordinals between κ and κ^{ad}.

Now suppose, for the sake of a contradiction, that κ^{ad} is first-order interpretable (with parameters) in L_κ. Since G codes a copy of κ with domain ω, by arithmetic transfinite recursion along G, we have that κ^{ad} is hyperarithmetic in G. But this contradicts the fact that $\omega_1^{CK}(G) = \kappa^{ad}$. (Note that, strictly speaking, we merely needed $\omega_1^{CK}(G) \leq \kappa^{ad}$, but it is good practice to calculate it exactly.) $\square$

Now we are ready to interpret L_κ in $\mathbf{Ceers}_\kappa$.

Theorem 5.4. *$\mathbf{Ceers}_\kappa$ interprets L_κ. Consequently, L_κ and $\mathbf{Ceers}_\kappa$ are mutually interpretable.*

Proof. We define a definable collection of *objects* which we will use to glue together to build our copy of L_κ.

Definition 5.5. An *object* is a pair $\mathfrak{x} = \langle l(\mathfrak{x}), r(\mathfrak{x})\rangle$ of κ-ceers such that

- $l(\mathfrak{x})$ codes an admissible level of L, and
- $r(\mathfrak{x})$ is an element of the structure coded by $l(\mathfrak{x})$. (Recall that we are appropriately conflating structures and undirected graphs, and the vertices of the graph given by $l(\mathfrak{x})$ are themselves κ-ceers.)

[1] The standard notations for these are "α^+" and "ω_1^r", respectively, but these clash with the set-theoretic notations for the successor cardinal of α and for $(\omega_1)^{L[r]}$, respectively; since this may cause confusion in our context, we use nonstandard notation here.

We define an equivalence relation $\sim$ on objects by setting $\mathfrak{x} \sim \mathfrak{y}$ if there is an object $\mathfrak{z}$ such that there are κ-ceers coding initial-segment embeddings of $l(\mathfrak{x})$ and $l(\mathfrak{y})$ into $l(\mathfrak{z})$ which send $r(\mathfrak{x})$ and $r(\mathfrak{y})$ to the same element.

We define an *element relation* ε on objects by setting $\mathfrak{x}\varepsilon\mathfrak{y}$ if there is an object $\mathfrak{z}$ such that there are κ-ceers coding initial-segment embeddings f and g of $l(\mathfrak{x})$ and $l(\mathfrak{y})$, respectively, into $l(\mathfrak{z})$ such that $f(r(\mathfrak{x})) \in g(r(\mathfrak{y}))$ (in $l(\mathfrak{z})$).

In particular, every set in L_κ is contained in some L_λ for $\lambda < \kappa$, which is coded by an object. By Lemma 5.3, we cannot overshoot L_κ, so the $\sim$-classes of objects, equipped with ε, form a copy of L_κ. As in Lemma 4.10, the existence of initial segment embeddings, as used in Definition 5.5, is definable in $\mathbf{Ceers}_\kappa$. Thus we have given an interpretation of L_κ in $\mathbf{Ceers}_\kappa$ without parameters. $\square$

As an immediate corollary, this lets us calculate the logical complexity of $\mathbf{Ceers}_\kappa$:

Corollary 5.6. *The first-order theories of L_κ and $\mathbf{Ceers}_\kappa$ are classically-computably isomorphic, as are their nth-order theories for every truly-finite n.*

The $\mathcal{L}_{\kappa,\kappa}$-theories of L_κ and $\mathbf{Ceers}_\kappa$ are κ-computably isomorphic, as are their $\mathcal{L}_{\kappa,\omega}$-theories.

If $\kappa = \omega_1$, then there are total continuous functions f and g on Baire space with κ-computable codes such that f and g send codes for $\mathcal{L}_{\omega_1,\omega}$-sentences to codes for $\mathcal{L}_{\omega_1,\omega}$-sentences and (conflating codes with sentences appropriately), we have

$$L_\kappa \models \varphi \iff \mathbf{Ceers}_\kappa \models f(\varphi); \text{ and } L_\kappa \models g(\psi) \iff \mathbf{Ceers}_\kappa \models \psi.$$

(An analogous result applies to all κ for which generalized descriptive set theory is appropriate.) $\square$

Having proved our theorem for arbitrary uncountable regular cardinals, it's now natural to ask how deeply it extends into the admissible ordinals in general. There are two main obstacles to pushing further.

First, the argument above relied on a basic characterization of the κ-finite κ-ceers, and in particular that if a κ-ceer is κ-finite, then it is comparable with every other κ-ceer. We used this to define the collection of almost self-full minimal κ-ceers, which we used for coding. However, this breaks down once κ is not a regular cardinal. Consider $\kappa = \aleph_\omega$, let A be the equivalence relation where $x \mathrel{A} y \iff x = y$ or $x, y \geq \omega$, and let B be the equivalence relation where $x \mathrel{B} y$ if and only if $x, y < \omega$ or $|x| = |y|$. Both A and B are κ-ceers and A has a κ-finite transversal (which appears to be the right notion of "κ-finite" once κ is no longer a regular cardinal), but there can be no reduction from one to the other: From such a reduction, we could extract a computable set $S \subseteq \kappa$ of cardinals. Since (by a Skolem hull + Mostowski collapse argument) $L_\eta \prec_{\Pi_1} L_\lambda$ whenever η, λ are cardinals with $\eta < \lambda$, any such S would compute the κ-halting problem.

This might be avoidable by an appropriate hack, since what we really need is a definable κ-sized set of κ-ceer degrees with the right analogue of self-fullness and some form of minimality over some definable ideal. However, even ignoring this, there is a second issue. The coding idea we used relied on two closure properties of L_κ when κ is a regular cardinal, namely, that every element of L_κ is contained in an admissible set in L_κ and that every ill-founded relation in L_κ has a descending sequence in L_κ. These properties are equivalent to κ being *recursively inaccessible*, that is, to κ being an admissible limit of admissible ordinals (see [Ba75, Chapter V, Def. 6.7] or [Hi78, Section 8.6, Def. 6.1 and Theorem 6.5]).

In particular, even if we came up with another way to encode graphs that got around the issues regarding κ-finite κ-ceers, we would be unable to answer the following question.

Question 5.7. How complicated is the theory of the ω_1^{CK}-ceers?

Per the above, resolving this question would seem to require a genuinely new idea.

Andrews, Schweber and Sorbi [ASS20] also considered related degree structures and showed that they all interpret first-order arithmetic. We ask:

Question 5.8. Do the $\equiv_{\mathbf{Fin}_\kappa}$-degrees of κ-ceers also interpret L_κ? What about the set of light degrees? What about the set of dark degrees? What about the set of light $\equiv_{\mathbf{Fin}_\kappa}$-degrees or dark $\equiv_{\mathbf{Fin}_\kappa}$-degrees?

6. An alternate perspective on ω_1

In this section, we present an alternate proof that $\mathbf{Ceers}_\kappa$ and L_κ are mutually interpretable when $\kappa = \omega_1$ (or more generally, when κ is a successor cardinal whose predecessor is "easily locatable"). Within the confines of α-recursion theory, this argument is strictly less general than that above; however, it has a new degree of flexibility with respect to applicability in alternate higher computability theories on ω_1.

Proposition 6.1. *L_{ω_1} is interpretable in $\mathbf{Ceers}_{\omega_1}$ (by a proof not using admissible sets).*

Proof. Just as above, we define the coding apparatus $\mathbf{c} \mapsto \mathrm{Graph}_\mathbf{c}$, conflate structures and undirected graphs, and show that every countable structure is coded by some ω_1-ceer. Rather than using ω_1-finite structures to build L_{ω_1}, however, we start by identifying a particular structure up to isomorphism:

Lemma 6.2. *In $\mathbf{Ceers}_{\omega_1}$, the set of ω_1-ceers coding a copy of the standard model of arithmetic $\mathbb{N}$ is first-order definable without parameters.*

Proof. $\mathbb{N}$ is the unique model of Robinson arithmetic Q which has no proper submodel of Q. Now Q is finitely axiomatizable and all countable structures are represented by ceers; so if $\mathbf{c}$ codes a model of Q with a proper submodel of Q and hence a proper *countable* submodel of Q, then some ceer $\mathbf{d}$ codes that submodel. This definition is expressible in $\mathbf{Ceers}_{\omega_1}$. $\square$

We now observe that $\mathsf{HC} = L_{\omega_1}$ is bi-interpretable with the full "powerset structure" of $\mathbb{N}$, that is,
$$\mathbb{N}_2 = (\omega \sqcup \mathcal{P}(\omega); +, \cdot, <, \in).$$
This is because we can code elements of L_{ω_1} by well-founded relations on $\mathbb{N}$, and the corresponding equality and element relations are appropriately definable.

We are now ready to interpret $\mathbb{N}_2$ inside $\mathbf{Ceers}_{\omega_1}$ as follows. A *real-object* will be a pair $\langle \mathbf{c}, \mathbf{d} \rangle$, where $\mathbf{c}$ codes a copy of $\mathbb{N}$ and $\mathrm{Vert}(\mathrm{Graph}_\mathbf{d}) \subseteq \mathrm{Vert}(\mathrm{Graph}_\mathbf{c})$; we consider two real-objects to be equivalent if there is an isomorphism of their $\mathbf{c}$-parts which yields a bijection on $\mathbf{d}$-part-vertices. Meanwhile, a *number-object* will be a pair $\langle \mathbf{c}, \mathbf{n} \rangle$, where $\mathbf{c}$ codes a copy of $\mathbb{N}$ and $\mathbf{n} \in \mathrm{Vert}(\mathrm{Graph}_\mathbf{c})$. Again, we obtain a notion of equality of number-objects by asking about the existence of an appropriate isomorphism of $\mathbf{c}$-parts. All the extra-logical structure on equivalence classes of real-objects and number-objects (that is, $+, \cdot, <$, and $\in$) is straightforward to define, and so we obtain an interpretation of $\mathbb{N}_2$ inside $\mathbf{Ceers}_{\omega_1}$. □

We mention this argument because it has one noticeable advantage: It is relatively disentangled from the particular context of the L-hierarchy, and it applies to a much wider range of computability theories. Of course, these theories still need to be sufficiently well-structured so that they permit the basic finite-injury argument needed to show that there are ω_1 many minimal ω_1-ceers, but that is a fairly mild requirement; see Stoltenberg-Hansen [SH79] for details on what exactly is required. So the $\mathbb{N}_2$-based approach appears to be non-redundant if we are interested in higher computability theories besides α-recursion theory.

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