A jump operator on the Weihrauch degrees

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Abstract

A partial order $(P, \leq)$ admits a jump operator if there is a map $j\colon P \to P$ that is strictly increasing and weakly monotone. Despite its name, the jump in the Weihrauch lattice fails to satisfy both of these properties: it is not degree-theoretic and there are functions f such that $f \equiv_{\mathrm{W}} f'$. This raises the question: is there a jump operator in the Weihrauch lattice? We answer this question positively and provide an explicit definition for an operator on partial multi-valued functions that, when lifted to the Weihrauch degrees, induces a jump operator. This new operator, called the *totalizing jump*, can be characterized in terms of the total continuation, a well-known operator on computational problems. The totalizing jump induces an injective endomorphism of the Weihrauch degrees. We study some algebraic properties of the totalizing jump and characterize its behavior on some pivotal problems in the Weihrauch lattice.

Contents

1 Introduction

Weihrauch reducibility is a notion of reducibility between computational problems that calibrates uniform computational strength. Despite growing interest in the Weihrauch degrees, their underlying structure remains relatively unexplored. Early work showed that the Weihrauch degrees form a distributive lattice with a bottom element; see [7] for an overview.

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In the context of classical computability theory, a central role is played by the Turing jump. It is therefore natural to ask whether there is an analogous operation in the Weihrauch lattice. An answer to this question requires a precise description of the desired properties of a jump operator.

Definition 1.1 ([11, Def. 1.1])**.** Let $(P, \le)$ be a partial order. A *jump operator* on P is a function $j \colon P \to P$ that is

(1) strictly increasing, i.e., for every $p \in P$, $p < j(p)$, and

(2) (weakly) monotone, i.e., for every $p, q \in P$, if $p \le q$ then $j(p) \le j(q)$.

The structure $(P, \le, j)$ is called a *jump partial order*.

This definition comes from Hinman and Slaman, who showed that every countable jump partial order is embeddable in the Turing degrees [11, Thm. 1.8]. Later, Lerman [14, Theorem 10.1.2] extended this result to every countable jump partial order with least element preserved under the embedding; and Montalbán [16, Thm. 4.17] extended this result by showing that every countable jump upper semilattice can be embedded in the Turing degrees preserving also the join operation.

Using the Axiom of Choice, it is not hard to show that every upper semilattice without maximum $(P, \le, \oplus)$ admits a jump operator: given a well-ordering $(p_\alpha)_\alpha$ of P, we can define $j(p) := p \oplus p_\alpha$, where α is least such that $p_\alpha \not\le p$. It is straightforward to check that this map is indeed a jump operator on $(P, \le, \oplus)$. However, this argument heavily uses the Axiom of Choice, and most likely, the defined jump operation will not be "natural".

In the context of Weihrauch reducibility, Brattka, Gherardi and Marcone [6] defined the *jump* of a partial multi-valued function (see Section 2 for the precise definition). While this operator (that originally was also called the *derivative*) has some connections with the Turing jump, it fails to satisfy both properties (1) and (2) in the definition of a jump operator.

In this paper, we explicitly define a jump operator on computational problems which we call the *totalizing jump*. We show that, while the explicit definition may look technical, it has a natural connection with the totalization operator T, a well-known operator on computational problems.

After recalling the necessary background notions in Section 2, we define and study the totalizing jump in Section 3. In particular, we show that the degree of the totalizing jump $\mathsf{tJ}(f)$ of a problem f is the maximum degree of $\mathsf{T}g$ for $g \equiv_{\mathrm{W}} f$ (Theorem 3.3). We also show that the map tJ is injective on the Weihrauch degrees (Theorem 3.7). This, in turn, implies that tJ is an injective (but not surjective) endomorphism of the Weihrauch degrees into themselves. As a corollary, this induces two new embeddings of the Medvedev degrees into the Weihrauch degrees. In Section 4, we explicitly characterize the totalizing jump of specific well-known problems. We make some remarks on abstract jump operators in Section 5, and finally, in Section 6, we highlight some open problems.

2 Background

In this section, we provide a short introduction to the Weihrauch degrees, focusing on what will be needed in this paper. For a more thorough presentation, the reader is referred to [7].

We let $\mathbb{N}^\mathbb{N}$ and $2^\mathbb{N}$ denote Baire and Cantor space, respectively. Let $\mathbb{N}^{<\mathbb{N}}$ and $2^{<\mathbb{N}}$ denote the sets of finite strings of natural numbers and of finite binary sequences, respectively. We write $(x_0, \dots, x_{n-1})$ for the string $\sigma := i \mapsto x_i$ of length n. The length of σ is denoted $|\sigma|$. If x is a finite or infinite string, we write $x[n]$ for the prefix of x of length n. We use $\sigma^\frown\tau$ to denote the concatenation of σ and τ, and $\sqsubseteq$ for the prefix relation.

We will use the symbol $\langle\cdot\rangle$ to denote a fixed computable bijection $\mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$. An explicit definition for $\langle\cdot\rangle$ can be found in any basic textbook on computability theory. We assume that this map has all the standard computability-theoretic properties, e.g., that $\sigma \mapsto |\sigma|$ is computable. For the sake of readability, we write $\langle n_0, \dots, n_k\rangle$ in place of $\langle(n_0, \dots, n_k)\rangle$.

Often, the symbol $\langle\cdot\rangle$ is used to denote the *join* between two (finite or infinite) strings with the same length. The meaning of $\langle\cdot\rangle$ will be clear from the context. Moreover, if $(x_i)_{i\in\mathbb{N}}$ is a sequence of infinite strings, we define $\langle x_0, x_1, \dots\rangle(\langle i,j\rangle) := x_i(j)$.

We write $f :\subseteq X \rightrightarrows Y$ for a partial multi-valued function with domain contained in X and codomain Y. For every $f, g :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$, we say that f is *Weihrauch reducible* to g, and write $f \le_{\mathrm{W}} g$, if there are two computable functionals $\Phi :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and $\Psi :\subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that, for every $p \in \mathrm{dom}(f)$,

(1) $\Phi(p) \in \mathrm{dom}(g)$, and

(2) for every $q \in g\Phi(p)$, $\Psi(p,q) \in f(p)$.

The functionals Φ and Ψ are often called the *forward functional* and the *backward functional*, respectively. Unless otherwise mentioned, we will assume that Φ is the name for the forward functional and Ψ is the name for the backward functional.

If Ψ need not have access to the original input p, we say that f is *strongly Weihrauch reducible* to g, and write $f \le_{\mathrm{sW}} g$. Formally, $f \le_{\mathrm{sW}} g$ if there are two computable functionals Φ and Ψ such that, for every $p \in \mathrm{dom}(f)$,

(1) $\Phi(p) \in \mathrm{dom}(g)$, and

(2) for every $q \in g\Phi(p)$, $\Psi(q) \in f(p)$.

Weihrauch reducibility is often formulated in the more general context of partial multi-valued functions on represented spaces, also called computational problems. However, if we are interested in the structure of the degrees, there is no loss of generality in assuming that computational problems have domain and codomain $\mathbb{N}^\mathbb{N}$. Indeed, for every computational problem on represented spaces, there is a canonical choice for a Weihrauch equivalent problem on the Baire space (see, e.g., [7, Lemma 3.8]). With a small abuse of notation, we can consider problems with other domains and codomains (e.g., $\mathbb{N}$, $2^{<\mathbb{N}}$, and $\mathbb{N}^{<\mathbb{N}}$). They can be identified with problems on $\mathbb{N}^\mathbb{N}$ using canonical representations (e.g., $n \in \mathbb{N}$ is represented by any $p \in \mathbb{N}^\mathbb{N}$ with $p(0) = n$, and a tree is represented by its characteristic function).

We say that f is a *cylinder* if for all g, $g \le_{\mathrm{W}} f$ if and only if $g \le_{\mathrm{sW}} f$. The notion of cylinder is often useful to prove separation results (as proving the non-existence of a strong Weihrauch reduction can be easier).

As mentioned in the introduction, the Weihrauch degrees form a distributive lattice, where join $\sqcup$ and meet $\sqcap$ are obtained by lifting the following operators to the degrees:

- $(f \sqcup g)(i,x) := \{i\} \times f(x)$ if $i = 0$ and $(f \sqcup g)(i,x) := \{i\} \times g(x)$ if $i = 1$;
- $(f \sqcap g)(x,z) := \{0\} \times f(x) \cup \{1\} \times g(z)$.

There is a natural bottom element, which is the (degree of the) empty function. The existence of a top element is equivalent to the failure of some relatively mild form of the Axiom of Choice (see [8, §2.1]). In this paper, we work in ZFC, so we will assume that the Weihrauch degrees do not have a maximum element.

There is a plethora of operators defined on computational problems, each of which captures a specific (natural) way to combine or modify computational problems. Most (but not all) of them lift to Weihrauch degrees. It is beyond the scope of this paper to list them all; we will instead mention the ones that are relevant to this work.

The *parallel product* $f \times g$ is defined as $(f \times g)(x,y) := f(x) \times g(y)$ and captures the idea of using f and g in parallel. Its infinite generalization is called *parallelization*, and can be defined as the problem $\widehat{f} := (x_i)_{i\in\mathbb{N}} \mapsto \{(y_i)_{i\in\mathbb{N}} : (\forall i \in \mathbb{N})(y_i \in f(x_i))\}$. In other words, given a countable sequence of f-instances, $\widehat{f}$ produces an f-solution for every f-instance.

To capture the idea of using f and g in series, we introduce the *compositional product*: Let $(\Gamma_p)_{p\in\mathbb{N}^\mathbb{N}}$ be an effective enumeration of all partial continuous functionals with G_δ domain. We define $f * g$ as the problem that takes as input an element of the set

$$\{(p,x) \in \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \,:\, x \in \mathrm{dom}(g) \text{ and } (\forall q \in g(x))\, \Gamma_p(q) \in \mathrm{dom}(f)\},$$

and produces a pair (y, w) with $w \in g(x)$ and $y \in f(\Gamma_p(w))$. Historically, the compositional product is defined as a map on a pair of computational problems or Weihrauch degrees (see [6]) that corresponds to $\max_{\leq_\mathrm{W}}\{f_0 \circ g_0 \,:\, f_0 \leq_\mathrm{W} f \text{ and } g_0 \leq_\mathrm{W} g\}$. However, it is convenient to fix a specific representative of such degree (see [18] for a short proof of the fact that $f * g$ as defined above works). Recalling that the compositional product is associative ([7, Prop. 11.5.6]), we denote by $f^{[n]}$ the n-fold compositional product of f with itself (i.e., $f^{[1]} := f$, $f^{[2]} := f * f$, and so on).

All the operators mentioned so far are degree-theoretic. We now introduce a few operators that, despite not being degree-theoretic, still play an important role in the theory.

The *jump* f' of $f :\subseteq X \rightrightarrows Y$ is the problem that takes as input a convergent sequence $(p_n)_{n\in\mathbb{N}}$ in $\mathbb{N}^\mathbb{N}$ and is defined as

$$f'((p_n)_{n\in\mathbb{N}}) := f\left(\lim_{n\to\infty} p_n\right).$$

Observe that, letting $\mathsf{lim} :\subseteq (\mathbb{N}^\mathbb{N})^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ be the computational problem that computes the limit in the Baire space, $f' \leq_\mathrm{W} f * \mathsf{lim}$. The converse reduction does not hold in general (take, e.g., a function f that only has computable outputs).

As anticipated, this jump operation fails to be a jump in the abstract sense: A simple counterexample is the constant function $c := p \mapsto 0^\mathbb{N}$ that maps every $p \in \mathbb{N}^\mathbb{N}$ to the constant 0 string. Indeed, given that the input plays no role, it is apparent that $c' \equiv_\mathrm{W} c$. This shows that the operator $'$ is not strictly increasing. At the same, letting id be the identity on the Baire space, we have

$$c' \equiv_\mathrm{W} c \equiv_\mathrm{W} \mathrm{id} <_\mathrm{W} \mathrm{id}' \equiv_\mathrm{W} \mathsf{lim},$$

where $\mathrm{id}' \equiv_\mathrm{W} \mathsf{lim}$ is straightforward from the definition (see also [6, Ex. 5.3(5)]).

One may think that the constant function is a somewhat weird exception, but this is not the case. For example, as mentioned, f' intuitively corresponds to using lim once, and then applying f to the result. For any computational problem strong enough to be closed under compositional product with lim, the jump is not strictly increasing.

As a side note, we mention that, even though the jump is not weakly monotone on the Weihrauch degrees, it is weakly monotone on the strong Weihrauch degrees. It still fails to be a jump, as it is not strictly increasing on the strong Weihrauch degrees.

In the definition of the totalizing jump, a central role is played by the *total continuation* or *totalization* operator. For every partial multi-valued $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$, its totalization is the total multi-valued function $\mathsf{T}f : \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ defined as

$$\mathsf{T}f(x) := \begin{cases} f(x) & \text{if } x \in \mathrm{dom}(f), \\ \mathbb{N}^\mathbb{N} & \text{otherwise.} \end{cases}$$

Again, the totalization is not a degree-theoretic operation: as a simple counterexample, it is enough to consider a total computable function and a partial computable function with no total computable extension.

We conclude this section by listing a few computational problems that will be useful in the rest of the paper. We already introduced the identity problem id and the problem $\mathsf{lim} :\subseteq (\mathbb{N}^\mathbb{N})^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ that maps a convergent sequence in $\mathbb{N}^\mathbb{N}$ to its limit. It is well-known that $\mathsf{lim} \equiv_\mathrm{sW} \widehat{\mathsf{LPO}}$, where $\mathsf{LPO} : \mathbb{N}^\mathbb{N} \to 2$ is defined as $\mathsf{LPO}(p) := 1$ iff $(\exists n \in \mathbb{N})\; p(n) \neq 0$. It is convenient to think of $\mathsf{LPO}(p)$ as the answer to a single $\Sigma^{0,p}_1$- (or $\Pi^{0,p}_1$-)question.

Some benchmark examples in the Weihrauch lattice are *choice problems*. The choice problem C_X can be intuitively described as the problem of finding elements of non-empty subsets of X given an enumeration of the complement of the subset. Their formal definition is usually given in the more general context of represented spaces, but for the sake of this paper, we can define them in a (strongly Weihrauch) equivalent way as problems on Baire space as follows:

- C_k: Given $p \in (k+1)^{\mathbb{N}}$ such that $(\exists n < k)\ n+1 \notin \mathrm{ran}(p)$, find $n < k$ such that $n+1 \notin \mathrm{ran}(p)$.
- $\mathsf{C}_{\mathbb{N}}$: Given $p \in \mathbb{N}^{\mathbb{N}}$ such that $(\exists n)\ n+1 \notin \mathrm{ran}(p)$, find n such that $n+1 \notin \mathrm{ran}(p)$.
- $\mathsf{C}_{2^{\mathbb{N}}}$: Given (the characteristic function of) an infinite subtree of $2^{<\mathbb{N}}$, find a path through it.
- $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$: Given (the characteristic function of) an ill-founded subtree of $\mathbb{N}^{<\mathbb{N}}$, find a path through it.

The restrictions of the choice problems to instances with a unique solution are denoted with the symbol UC_X. It is known that $\mathsf{UC}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$ ([6, Thm. 3.8]), $\mathsf{UC}_k \equiv_{\mathrm{W}} \mathsf{UC}_{2^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathrm{id}$ (where the second equivalence follows from the fact that $2^{\mathbb{N}}$ is computably compact, see, e.g., [3, Cor. 6.4]), and $\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}} <_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ ([12, Cor. 3.7]).

3 The totalizing jump

We fix a computable enumeration $(\Phi_e)_{e\in\mathbb{N}}$ of partial computable functionals from $\mathbb{N}^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$. We now introduce the following new operator on computational problems:

Definition 3.1. Let $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be a partial multi-valued function. We define the *totalizing jump* (or *tot-jump* for brevity) of f as follows: For every $e, i \in \mathbb{N}$ and every $p \in \mathbb{N}^{\mathbb{N}}$,

$$\mathsf{tJ}(f)(e,i,p) := \begin{cases} \{\Phi_i(p,q) \,:\, q \in f\Phi_e(p)\} & \text{if } \Phi_e(p) \in \mathrm{dom}(f) \text{ and} \\ & \quad (\forall q \in f\Phi_e(p))\ \Phi_i(p,q)\downarrow, \\ \mathbb{N}^{\mathbb{N}} & \text{otherwise.} \end{cases}$$

Remark 3.2. For some proofs, it may be convenient to use the following (strongly Weihrauch equivalent) definition for the tot-jump: For every partial multi-valued function $f :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ and every $x = (e,i)^\frown p \in \mathbb{N}^{\mathbb{N}}$, we define

$$\overline{\mathsf{tJ}}(f)(x) := \begin{cases} \{\Phi_i(x,q) \,:\, q \in f\Phi_e(x)\} & \text{if } \Phi_e(x) \in \mathrm{dom}(f) \text{ and} \\ & \quad (\forall q \in f\Phi_e(x))\ \Phi_i(x,q)\downarrow, \\ \mathbb{N}^{\mathbb{N}} & \text{otherwise.} \end{cases}$$

To show that $\overline{\mathsf{tJ}}(f) \equiv_{\mathrm{sW}} \mathsf{tJ}(f)$, observe that the only difference between the two problems is that in $\overline{\mathsf{tJ}}(f)$, the functionals Φ_e and Φ_i receive as input their own indices. In particular, to prove that $\overline{\mathsf{tJ}}(f) \le_{\mathrm{sW}} \mathsf{tJ}(f)$, it suffices to notice that, given e, i, p, we can uniformly compute $e', i' \in \mathbb{N}$ so that $\Phi_{e'}(p) = \Phi_e((e,i)^\frown p)$ and $\Phi_{i'}(p,q) = \Phi_i((e,i)^\frown p, q)$. The other reduction is proved analogously.

Intuitively, we can think of the tot-jump $\mathsf{tJ}(f)$ of f as a problem that "collects all possible Weihrauch reductions to f and totalizes". In particular, the name "totalizing jump" is motivated by the following characterization of the Weihrauch degree of tJ.

Theorem 3.3. *For every problem f,*

- *if $g \le_{\mathrm{W}} f$, then $\mathsf{T}g \le_{\mathrm{W}} \mathsf{tJ}(f)$;*
- *there is a $g \equiv_{\mathrm{W}} f$ such that $\mathsf{tJ}(f) = \mathsf{T}g$.*

In other words, the Weihrauch degree of $\mathsf{tJ}(f)$ *is the maximum of the Weihrauch degrees of the totalizations of the g's which are Weihrauch equivalent (equivalently, reducible) to f.*

Proof. Fix a problem f. Assume that $g \le_{\mathrm{W}} f$ via Φ_e, Φ_i. It is straightforward to see that $\mathsf{T}g \le_{\mathrm{W}} \mathsf{tJ}(f)$ is witnessed by the maps $p \mapsto (e,i,p)$ and $(p,q) \mapsto q$. Indeed, if $p \in \operatorname{dom}(g)$ then $\Phi_e(p) \in \operatorname{dom}(f)$ and, for every $q \in f\Phi_e(p)$, $\Phi_i(p,q) \in g(p)$. In particular, $\mathsf{tJ}(f)(e,i,p) \subseteq g(p)$. On the other hand, if $p \notin \operatorname{dom}(g)$ then $\mathsf{tJ}(f)(e,i,p) \subseteq \mathsf{T}g(p) = \mathbb{N}^{\mathbb{N}}$.

To conclude the proof, let us define

$$g(e,i,p) := \{\Phi_i(p,q) \,:\, q \in f\Phi_e(p)\}$$

with $\operatorname{dom}(g) := \{(e,i,p) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \,:\, \Phi_e(p) \in \operatorname{dom}(f) \text{ and } (\forall q \in f\Phi_e(p))\ \Phi_i(p,q)\downarrow\}$. It is immediate from the definition of g that $g \equiv_{\mathrm{W}} f$ and $\mathsf{tJ}(f) = \mathsf{T}g$. ☐

We now show that the tot-jump is a degree-theoretic operator that induces a jump operator on the Weihrauch degrees.

Theorem 3.4. *For every f, $f <_{\mathrm{W}} \mathsf{tJ}(f)$. Moreover, for every f, g, if $f \le_{\mathrm{W}} g$ then $\mathsf{tJ}(f) \le_{\mathrm{W}} \mathsf{tJ}(g)$.*

Proof. The reduction $f \le_{\mathrm{W}} \mathsf{tJ}(f)$ is straightforward (just map x to (e,i,x), where e,i are indices for the identity function and the projection on the second component respectively), so we only need to show that $\mathsf{tJ}(f) \not\le_{\mathrm{W}} f$.

Let $d\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the function defined as $d(p)(0) := p(0)+1$ and $d(p)(n+1) := p(n+1)$. Observe first of all that $\overline{\mathsf{tJ}}(f)$ (and hence $\mathsf{tJ}(f)$) is strongly Weihrauch equivalent to the multi-valued function f_d that, on input $x = (e,i)^\frown p$, is defined as

$$f_d(x) := \begin{cases} \{d \circ \Phi_i(x,q) \,:\, q \in f\Phi_e(x)\} & \text{if } \Phi_e(x) \in \operatorname{dom}(f) \text{ and} \\ & \quad (\forall q \in f\Phi_e(x))\ \Phi_i(x,q)\downarrow, \\ \mathbb{N}^{\mathbb{N}} & \text{otherwise.} \end{cases}$$

Indeed, the reduction $f_d \le_{\mathrm{sW}} \overline{\mathsf{tJ}}(f)$ follows from the fact that, for every $x \in \mathbb{N}^{\mathbb{N}}$ and every $q \in \overline{\mathsf{tJ}}(f)(x)$, $d(q) \in f_d(x)$. The converse reduction is witnessed by the maps id and

$$q \mapsto \begin{cases} q & \text{if } q(0) = 0, \\ d^{-1}(q) & \text{if } q(0) > 0. \end{cases}$$

It is therefore enough to show that $f_d \not\le_{\mathrm{W}} f$. Assume towards a contradiction that $f_d \le_{\mathrm{W}} f$ is witnessed by the functionals Φ_e and Φ_i. Fix $p \in \mathbb{N}^{\mathbb{N}}$ and let $y := (e,i)^\frown p$. Since f_d is total, $y \in \operatorname{dom}(f_d)$. Moreover, by definition of Weihrauch reduction, $\Phi_e(y) \in \operatorname{dom}(f)$ and for every $q \in f\Phi_e(y)$, $\Phi_i(y,q)\downarrow$.

We have now reached a contradiction, as for every non-empty $X \subseteq \mathbb{N}^{\mathbb{N}}$, $X \not\subseteq d(X)$ (consider $p \in X$ such that $p(0)$ is minimal). In particular, taking $X = \{\Phi_i(y,t) \,:\, t \in f\Phi_e(y)\} \neq \emptyset$, there is $q \in f\Phi_e(y)$ such that

$$\Phi_i(y,q) \notin d(X) = \{d \circ \Phi_i(y,t) \,:\, t \in f\Phi_e(y)\} = f_d(y),$$

contradicting the definition of Weihrauch reducibility.[1]

To prove the last part of the statement, assume that $f \le_{\mathrm{W}} g$ via the functionals Φ_e, Φ_i. Let (a,b,p) be an input for $\mathsf{tJ}(f)$. We can uniformly compute $c,d \in \mathbb{N}$ so that $\Phi_c(p) = \Phi_e(\Phi_a(p))$ and $\Phi_d(p,q) = \Phi_b(p, \Phi_i(\Phi_a(p),q))$. The reduction $\mathsf{tJ}(f) \le_{\mathrm{W}} \mathsf{tJ}(g)$ is witnessed by the functionals $(a,b,p) \mapsto (c,d,p)$ and $(p,q) \mapsto q$.

[1] Without sufficiently strong choice axioms, we cannot prove the existence of $q \in f\Phi_e(y)$ witnessing the contradiction.

Indeed, if $\Phi_a(p) \in \operatorname{dom}(f)$ and, for every $q \in f\Phi_a(p)$, $\Phi_b(p,q)\downarrow$, then $\mathsf{tJ}(f)(a,b,p) = \{\Phi_b(p,q) : q \in f\Phi_a(p)\}$. In this case, by the definition of Weihrauch reducibility, $\Phi_c(p) = \Phi_e(\Phi_a(p)) \in \operatorname{dom}(g)$ and for every $t \in g\Phi_c(p)$, $\Phi_i(\Phi_a(p), t) \in f\Phi_a(p)$. In particular, $\Phi_b(p, \Phi_i(\Phi_a(p), t)) \downarrow\in \mathsf{tJ}(f)(a,b,p)$. In other words, $\mathsf{tJ}(g)(c,d,p) \subseteq \mathsf{tJ}(f)(a,b,p)$. The other case (i.e., if $\Phi_a(p) \notin \operatorname{dom}(f)$ or if there is $q \in f\Phi_a(p)$ such that $\Phi_b(p,q)\uparrow$) is trivial as $\mathsf{tJ}(g)(c,d,p) \subseteq \mathsf{tJ}(f)(a,b,p) = \mathbb{N}^\mathbb{N}$. □

Remark 3.5. Notice that the same proof shows that tJ induces a jump operator on the strong Weihrauch degrees. Indeed, the reductions $f \le_{\mathrm{W}} \mathsf{tJ}(f)$ and $\mathsf{tJ}(f) \le_{\mathrm{W}} \mathsf{tJ}(g)$ (when $f \le_{\mathrm{W}} g$) are both strong Weihrauch reductions. Moreover, as noticed, $f_d \equiv_{\mathrm{sW}} \overline{\mathsf{tJ}}(f) \equiv_{\mathrm{sW}} \mathsf{tJ}(f)$, hence $f <_{\mathrm{sW}} \mathsf{tJ}(f)$.

Observe that the definition of $\mathsf{tJ}(f)$ is $\Delta^{1,f}_2$ in the language of third-order arithmetic, i.e., there is a Δ^1_2-formula with parameter f that says "$t \in \mathsf{tJ}(f)(e,i,p)$". Indeed,

- $\alpha_f(p) := \Phi_e(p) \downarrow\in \operatorname{dom}(f)$ is equivalent to $(\exists r, q)[\, r = \Phi_e(p) \land (r,q) \in f\,]$, which is $\Sigma^{1,f}_1$;
- $\beta_f(p) := (\forall q)[\, (\Phi_e(p), q) \in f \to \Phi_i(p,q)\downarrow\,]$ is $\Pi^{1,f}_1$;

hence the formula $t \in \mathsf{tJ}(f)(e,i,p)$ can be written as

$$(\alpha_f(p) \land \beta_f(p)) \to (\exists q \in f\Phi_e(p))\; t = \Phi_i(p,q).$$

Remark 3.6. No jump operator on partial multi-valued functions can be defined by a $\Sigma^{1,f}_1$ formula. To show this, we use the fact that $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ (essentially proved in [12, Thm. 3.11]), where $\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ is the problem of finding elements in non-empty analytic subsets of $\mathbb{N}^\mathbb{N}$. Assume that j is an operator defined by a $\Sigma^{1,f}_1$-formula. In particular, $q \in \mathsf{j}(\mathsf{C}_{\mathbb{N}^\mathbb{N}})(p)$ is a Σ^1_1-formula $\varphi(p,q)$ where $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ may appear. Since "$x \in \mathsf{C}_{\mathbb{N}^\mathbb{N}}(A)$" is arithmetic in x, A (in fact, $\mathsf{C}_{\mathbb{N}^\mathbb{N}}(A)$ is Π^0_1 relative to A), any arithmetic formula involving it is arithmetic as well. This implies that $\varphi(p,q)$ is actually Σ^1_1 uniformly in p, q and hence we can use $\boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ to pick a point in $\{q \in \mathbb{N}^\mathbb{N} : \varphi(p,q)\}$. In other words, $\mathsf{j}(\mathsf{C}_{\mathbb{N}^\mathbb{N}}) \le_{\mathrm{W}} \boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$, so j is not strictly increasing.

It is straightforward to see that the map tJ is injective. More interestingly, it is injective on the Weihrauch degrees.

Theorem 3.7. *For every f, g, if $\mathsf{tJ}(f) \le_{\mathrm{W}} \mathsf{tJ}(g)$ then $f \le_{\mathrm{W}} g$. This implies that the map tJ is an injective endomorphism on the Weihrauch degrees.*

Proof. Let f, g be two partial multi-valued functions and assume that $\mathsf{tJ}(f) \le_{\mathrm{W}} \mathsf{tJ}(g)$ via the functionals Φ and Ψ. Consider a pair (e,i) such that e is an index for id and $\Phi_i(p,q)$ is defined as follows: Let $m \in \mathbb{N}$ be the first number found such that $(\exists z \in \mathbb{N}^\mathbb{N})\; \Psi((e,i,p), z)(0) \downarrow= m$. Then

$$\Phi_i(p,q)(n) := \begin{cases} q(0) + m + 1 & \text{if } n = 0, \\ q(n) & \text{otherwise.} \end{cases}$$

To show that $f \le_{\mathrm{W}} g$, let $p \in \operatorname{dom}(f)$ and consider the input (e,i,p) for $\mathsf{tJ}(f)$. Let $(a,b,t) = \Phi(e,i,p)$ be an input for $\mathsf{tJ}(g)$. Observe that $\Phi_a(t) \in \operatorname{dom}(g)$ and for every $r \in g\Phi_a(t)$, $\Phi_b(t,r)\downarrow$. Indeed, if not, then any $z \in \mathbb{N}^\mathbb{N}$ is a valid solution for $\mathsf{tJ}(g)(a,b,t)$. In particular, we could take z so that $\Psi((e,i,p), z)(0) \downarrow= m$. This would lead to a contradiction as, by definition of Φ_i, for every $y \in \mathsf{tJ}(f)(e,i,p)$ we have $y(0) > m$. In other words,

$$\mathsf{tJ}(g)(\Phi(e,i,p)) = \{\Phi_b(t,r) : (a,b,t) = \Phi(e,i,p) \text{ and } r \in g\Phi_a(t)\}.$$

Hence, a solution for $\mathsf{tJ}(f)(e,i,p)$, and in turn for $f(p)$, can be uniformly obtained from $g(\Phi_a(t))$. □

As we will discuss extensively later, tJ is not surjective, even on the cone above $\mathsf{tJ}(\emptyset)$. The previous theorem implies that there is a proper substructure of the Weihrauch degrees that is isomorphic to the Weihrauch degrees. Note that, using this endomorphism, we obtain two (and, by iterating, infinitely many) new embeddings of the Medvedev degrees into the Weihrauch degrees (see [10] for a survey on Medvedev reducibility, and see [7, Thm. 9.1] for the two known embeddings of the Medvedev degrees into the Weihrauch degrees).

Observe that, as a corollary of Theorem 3.3, we obtain that $\mathsf{tJ}(f)$ is never a cylinder. Indeed, we can prove something slightly stronger:

Proposition 3.8. *For every $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ such that $f <_{\mathrm{sW}} \mathsf{T}f$, $\mathsf{T}f$ is not a cylinder.*

Proof. It is well-known that g is a cylinder iff $\mathrm{id} \times g \le_{\mathrm{sW}} g$ ([4, Cor. 3.6]). Assume towards a contradiction that $\mathrm{id} \times \mathsf{T}f \le_{\mathrm{sW}} \mathsf{T}f$ is witnessed by the functionals Φ, Ψ. Notice that, for some computable $p \in \mathbb{N}^\mathbb{N}$ and some $x \in \mathbb{N}^\mathbb{N}$, $\Phi(p, x) = z$ for some $z \notin \mathrm{dom}(f)$. Indeed, if this were not the case, then we would obtain $\mathsf{T}f \le_{\mathrm{sW}} f$, contradicting the hypothesis.

Since $\mathsf{T}f(z) = \mathbb{N}^\mathbb{N}$, for every $q \in \mathbb{N}^\mathbb{N}$ we obtain $\Psi(q) = (p, y)$ for some $y \in \mathsf{T}f(x)$. If we consider (p', x) with $p' \neq p$ and $q \in \mathsf{T}f\Phi(p', x)$, we reach a contradiction, as $\Psi(q) = (p, y) \notin (\mathrm{id} \times \mathsf{T}f)(p', x)$. □

Since, as proved in Theorem 3.3, for every f there is $g \equiv_{\mathrm{W}} f$ such that $\mathsf{tJ}(g) \equiv_{\mathrm{W}} \mathsf{tJ}(f) = \mathsf{T}g$, the previous proposition implies that $\mathsf{tJ}(f)$ is not a cylinder.

In the rest of the section, we prove several properties of the tot-jump, including various results that better describe the range of tJ. We first provide an alternative characterization of $\mathsf{tJ}(f)$. For this, we introduce the following computational problem:

Definition 3.9. Let us define $\mathsf{W}_{\Pi^0_2 \to \Pi^0_1} :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ as

$$\mathsf{W}_{\Pi^0_2 \to \Pi^0_1}(p) := \{q \in \mathbb{N}^\mathbb{N} \,:\, (\forall i)\ q(i+1) > q(i) \text{ and } p \circ q = 0^\mathbb{N}\}.$$

In other words, $\mathsf{W}_{\Pi^0_2 \to \Pi^0_1}(p)$ lists the addresses of infinitely many zeroes of p.

Notice that $\mathsf{W}_{\Pi^0_2 \to \Pi^0_1}$ is uniformly computable (it can be solved by unbounded search) and partial, as $\mathrm{dom}(\mathsf{W}_{\Pi^0_2 \to \Pi^0_1}) = \{p \in \mathbb{N}^\mathbb{N} \,:\, (\exists^\infty i)\ p(i) = 0\}$. An important property of $\mathsf{W}_{\Pi^0_2 \to \Pi^0_1}$ is that, for any given $p \in \mathrm{dom}(\mathsf{W}_{\Pi^0_2 \to \Pi^0_1})$ and any $q \in \mathbb{N}^\mathbb{N}$, it is c.e. to check if $q \notin \mathsf{W}_{\Pi^0_2 \to \Pi^0_1}(p)$. This also motivates the choice of the notation, as $\mathsf{W}_{\Pi^0_2 \to \Pi^0_1}$ is "translating a Π^0_2-question into a Π^0_1-question".[2]

Theorem 3.10. *For every f, $\mathsf{tJ}(f) \equiv_{\mathrm{W}} \mathsf{T}(\mathsf{W}_{\Pi^0_2 \to \Pi^0_1} * f * \mathsf{W}_{\Pi^0_2 \to \Pi^0_1})$.*

Proof. Since $\mathsf{W}_{\Pi^0_2 \to \Pi^0_1}$ is uniformly computable, $\mathsf{W}_{\Pi^0_2 \to \Pi^0_1} * f * \mathsf{W}_{\Pi^0_2 \to \Pi^0_1} \le_{\mathrm{W}} f$, so, in light of Theorem 3.3, we only need to show that $\mathsf{tJ}(f) \le_{\mathrm{W}} \mathsf{T}(\mathsf{W}_{\Pi^0_2 \to \Pi^0_1} * f * \mathsf{W}_{\Pi^0_2 \to \Pi^0_1})$. For every input (e, i, p) for $\mathsf{tJ}(f)$, let $t \in \mathbb{N}^\mathbb{N}$ be such that t has infinitely many zeroes iff $\Phi_e(p)\downarrow$. Let also $v \in \mathbb{N}^\mathbb{N}$ be such that $\Gamma_v(q) = \Phi_e(p)$ for every q and $w \in \mathbb{N}^\mathbb{N}$ be such that $\Gamma_w(q)$ has infinitely many zeroes iff $\Phi_i(p, q)\downarrow$. It is clear that t, v, w are uniformly computable from e, i, p. The forward functional Φ of the reduction $\mathsf{tJ}(f) \le_{\mathrm{W}} \mathsf{T}(\mathsf{W}_{\Pi^0_2 \to \Pi^0_1} * f * \mathsf{W}_{\Pi^0_2 \to \Pi^0_1})$ is the map $(e, i, p) \mapsto (w, v, t)$. We define the backward functional Ψ as follows: A solution for $\mathsf{T}(\mathsf{W}_{\Pi^0_2 \to \Pi^0_1} * f * \mathsf{W}_{\Pi^0_2 \to \Pi^0_1})$ is a string of the form $\langle y_1, y_2, y_3 \rangle$. Given (e, i, p) and $\langle y_1, y_2, y_3 \rangle$, we compute $\Phi(e, i, p) = (w, v, t)$ and do the following operations in parallel:

- check whether $y_3 \in \mathsf{W}_{\Pi^0_2 \to \Pi^0_1}(t)$;
- check whether $y_1 \in \mathsf{W}_{\Pi^0_2 \to \Pi^0_1}(\Gamma_w(y_2))$;

[2] It is probably more correct to say that $\mathsf{T}(\mathsf{W}_{\Pi^0_2 \to \Pi^0_1})$ translates a Π^0_2-question into a Π^0_1-question. We discuss this computational problem in Section 4.

- compute $\Phi_i(p, y_2)$.

Since it is c.e. to check if $y_3 \notin \mathsf{W}_{\Pi^0_2\to\Pi^0_1}(t)$ or $y_1 \notin \mathsf{W}_{\Pi^0_2\to\Pi^0_1}(\Gamma_w(y_2))$, as long as y_1 and y_3 "appear to be correct", the backward functional produces $\Phi_i(p, y_2)$. If we see that $y_3 \notin \mathsf{W}_{\Pi^0_2\to\Pi^0_1}(t)$ or $y_1 \notin \mathsf{W}_{\Pi^0_2\to\Pi^0_1}(\Gamma_w(y_2))$, we extend the partial output with $0^{\mathbb{N}}$.

Recall that, if $\Phi_e(p) \in \operatorname{dom}(f)$ and, for every $q \in f\Phi_e(p)$, $\Phi_i(p,q)\downarrow$, then $\mathsf{tJ}(f)(e,i,p) = \{\Phi_i(p,q) : q \in f\Phi_e(p)\}$. It is straightforward to check that, in this case, $t \in \operatorname{dom}(\mathsf{W}_{\Pi^0_2\to\Pi^0_1})$, $\Gamma_v(y) = \Phi_e(p) \in \operatorname{dom}(f)$ for every y, and for every $q \in f\Phi_e(p)$, $\Gamma_w(q)$ has infinitely many zeroes. In particular, every solution $\langle y_1, y_2, y_3\rangle$ of $\mathsf{T}(\mathsf{W}_{\Pi^0_2\to\Pi^0_1} * f * \mathsf{W}_{\Pi^0_2\to\Pi^0_1})(\Phi(e,i,p))$ is such that $y_1 \in \mathsf{W}_{\Pi^0_2\to\Pi^0_1}(\Gamma_w(y_2))$, $y_2 \in f\Phi_e(p)$, and $y_3 \in \mathsf{W}_{\Pi^0_2\to\Pi^0_1}(t)$. By definition, the backward functional will therefore compute $\Phi_i(p, y_2) \in \mathsf{tJ}(f)(e,i,p)$.

On the other hand, if $\Phi_e(p) \notin \operatorname{dom}(f)$ or if there is $q \in f\Phi_e(p)$ such that $\Phi_i(p,q)\uparrow$, then $\mathsf{tJ}(f)(e,i,p) = \mathbb{N}^{\mathbb{N}}$. Since $\mathsf{T}(\mathsf{W}_{\Pi^0_2\to\Pi^0_1} * f * \mathsf{W}_{\Pi^0_2\to\Pi^0_1})$ and Ψ are total, the claim follows. □

The following proposition shows that, in general, the compositions with $\mathsf{W}_{\Pi^0_2\to\Pi^0_1}$ on both sides are necessary.

Proposition 3.11. *There is f such that $\mathsf{tJ}(f) \not\le_{\mathrm{W}} \mathsf{T}(\mathsf{W}_{\Pi^0_2\to\Pi^0_1} * f)$ and $\mathsf{tJ}(f) \not\le_{\mathrm{W}} \mathsf{T}(f * \mathsf{W}_{\Pi^0_2\to\Pi^0_1})$.*

Proof. Let $A \in 2^{\mathbb{N}}$ be such that $\emptyset <_T A <_T \emptyset'$. Let f be the function with $\operatorname{dom}(f) := \{A\}$ that maps A to $\emptyset'$.

We first show that $\mathsf{tJ}(f) \not\le_{\mathrm{W}} \mathsf{T}(\mathsf{W}_{\Pi^0_2\to\Pi^0_1} * f)$. Assume towards a contradiction that the reduction is witnessed by the maps Φ, Ψ. Let i be an index for the projection on the second coordinate. Let also e be such that $\Phi_e(x)$ searches for $k \in \mathbb{N}$ such that $x = 0^k 1^\frown p$ for some $p \in \mathbb{N}^{\mathbb{N}}$ and then outputs p. Since $(e, i, 0^{\mathbb{N}}) \in \operatorname{dom}(\mathsf{tJ}(f))$ and A is not computable, there is k such that $\Phi(e,i,0^k) = \langle \sigma, \tau\rangle$ and $\tau \neq A[|\tau|]$. In particular, for every $p \in \mathbb{N}^{\mathbb{N}}$, $y := \Phi(e,i,0^{k\frown} p) \notin \operatorname{dom}(\mathsf{W}_{\Pi^0_2\to\Pi^0_1} * f)$, and therefore $0^{\mathbb{N}} \in \mathsf{T}(\mathsf{W}_{\Pi^0_2\to\Pi^0_1} * f)(y)$. On the other hand, it is immediate to check that $\mathsf{tJ}(f)(e,i,0^k 1^\frown A) = \emptyset'$. We have reached a contradiction, as $\Psi((e,i,0^k 1^\frown A), 0^{\mathbb{N}}) = \emptyset'$ would imply that $\emptyset' \le_T A$, against the hypothesis on A.

Let us now show that $\mathsf{tJ}(f) \not\le_{\mathrm{W}} \mathsf{T}(f * \mathsf{W}_{\Pi^0_2\to\Pi^0_1})$. To this end, assume towards a contradiction that the reduction is witnessed by the functionals Φ and Ψ. The idea is to diagonalize by choosing an input (e,i,x) for $\mathsf{tJ}(f)$ so that the output of Φ_i is different from any output of Ψ. Let e be an index for the identity functional and let $x = A$.

To find $i \in \mathbb{N}$ we use the recursion theorem. First we define $\Phi_i(p,q)(0)$: In parallel, we compute $\Psi((e,i,p), \langle\sigma_1,\sigma_2\rangle)$ for all possible $\sigma_1, \sigma_2 \in \mathbb{N}^{<\mathbb{N}}$ until we find a pair (σ_1, σ_2) such that

$$\Psi((e,i,p), \langle\sigma_1,\sigma_2\rangle)(0)\downarrow = m,$$

for some $m \in \mathbb{N}$ and then set $\Phi_i(p,q)(0) := m+1$. At least one such pair exists, otherwise Ψ is never defined when the first input is (e,i,p), contradicting the definition of Weihrauch reducibility.

Since we are interested in defining the behavior of $\Phi_i(p,q)$ only when $p = x = A$ and $q = \emptyset'$, we describe a procedure for computing $\Phi_i(x, \emptyset')(1)$ which on different inputs may not converge, or converge to an arbitrary string. Start from $\langle w, z\rangle = \Phi(e,i,x)$ and use $\emptyset'$ to check if there is a $\tau \in \mathbb{N}^{<\mathbb{N}}$ that satisfies all the following properties:

- $(\forall j < |\tau| - 1)(\tau(j+1) > \tau(j))$,
- $(\forall j < |\tau|)(z \circ \tau(j) = 0)$,
- $\Psi((e,i,x), \langle \emptyset'[|\tau|], \tau\rangle)(1)\downarrow$.

Intuitively, we are searching for an initial segment τ of a solution of $\mathsf{W}_{\Pi^0_2\to\Pi^0_1}(z)$. A solution for $f * \mathsf{W}_{\Pi^0_2\to\Pi^0_1}(w,z)$ is of the form (r,s), where $s \in \mathsf{W}_{\Pi^0_2\to\Pi^0_1}(z)$ and $r \in f(\Gamma_w(s))$. Since f is constant $\emptyset'$, to obtain a prefix of a solution we only need to search for a prefix τ of s. To diagonalize, we require that τ is sufficiently long so that $\Psi((e,i,x),\langle\tau,\emptyset'[|\tau|]\rangle)(1)\downarrow$. Such a τ need not exist, as we do not know if z has infinitely many zeroes. This is the reason why we use the oracle to check if such a search terminates.

If τ exists, then we can search for it and define $\Phi_i(x,\emptyset')(1) := \Psi((e,i,x),\langle\emptyset'[|\tau|],\tau\rangle)(1)+1$. Otherwise, define $\Phi_i(x,\emptyset')(1) := 0$.

For every $n>1$ and every p and q, define $\Phi_i(p,q)(n) := 0$.

Notice that $\mathsf{tJ}(f)(e,i,A) = \Phi_i(A,\emptyset')$. If the input z for $\mathsf{W}_{\Pi^0_2\to\Pi^0_1}$ has finitely many 0's or, for some $s\in\mathsf{W}_{\Pi^0_2\to\Pi^0_1}(z)$, $\Gamma_w(s)\neq A$, then $\langle\sigma_1,\sigma_2\rangle$ is the initial segment of a solution for $\mathsf{T}(f*\mathsf{W}_{\Pi^0_2\to\Pi^0_1})(w,z)$, hence $\Phi_i(A,\emptyset')(0)\neq\Psi((e,i,A),\langle\sigma_1,\sigma_2\rangle^\frown 0^{\mathbb{N}})(0)$. Otherwise, $\langle\emptyset'[|\tau|],\tau\rangle$ is the prefix of a solution $f*\mathsf{W}_{\Pi^0_2\to\Pi^0_1}(w,z)$, and therefore $\Phi_i(A,\emptyset')(1)\neq\Psi((e,i,A),\langle\tau,\emptyset'[|\tau|]\rangle)(1)$. □

The previous result shows that, in general, the use of $\mathsf{W}_{\Pi^0_2\to\Pi^0_1}$ cannot be avoided on either side of f. There are, however, many f such that $\mathsf{tJ}(f)\equiv_{\mathrm{W}}\mathsf{T}f$. We now provide a sufficient condition for this to happen.

Theorem 3.12. *Fix a problem f. If there are two total computable functions $\varphi,\psi\colon\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ such that*

- *for every e,i, $\varphi(e,i)$ and $\psi(e,i)$ are indices of total functionals and*
- *whenever $g\le_{\mathrm{W}} f$ via Φ_e,Φ_i (which might be partial), then $g\le_{\mathrm{W}} f$ via $\Phi_{\varphi(e,i)}$ and $\Phi_{\psi(e,i)}$,*

then $\mathsf{tJ}(f)\equiv_{\mathrm{W}}\mathsf{T}f$.

Proof. By Theorem 3.3, we only need to show that $\mathsf{tJ}(f)\le_{\mathrm{W}}\mathsf{T}f$. We let the forward functional of the reduction be defined by $\Phi(e,i,p)=\Phi_{\varphi(e,i)}(p)$. Similarly, we let the backward functional be defined by $\Psi((e,i,p),q)=\Phi_{\psi(e,i)}(p,q)$. The proof is then straightforward: Notice indeed that if (e,i,p) is an input for $\mathsf{tJ}(f)$ such that $\Phi_e(p)\in\operatorname{dom}(f)$ and for every $q\in f\Phi_e(p)$, $\Phi_i(p,q)\downarrow$, then the functionals Φ_e,Φ_i are witnessing the reduction $g\le_{\mathrm{W}} f$ for some problem g (e.g., we can take g to be the problem that maps p to the set $\{\Phi_i(p,q) : q\in f\Phi_e(p)\}$). In particular, the second item in the hypotheses implies that $\Phi(e,i,p)=\Phi_{\varphi(e,i)}(p)\in\operatorname{dom}(f)$ and, for every $q\in f\Phi(e,i,p)$, $\Psi(p,q)=\Phi_{\psi(e,i)}(p,q)\in g(p)=\mathsf{tJ}(f)(e,i,p)$.

On the other hand, if $\Phi_e(p)\notin\operatorname{dom}(f)$ or if there is $q\in f\Phi_e(p)$ such that $\Phi_i(p,q)\uparrow$, then $\mathsf{tJ}(f)(e,i,p)=\mathbb{N}^{\mathbb{N}}$, hence the claim follows by the totality of Φ,Ψ (guaranteed by the first item in the hypotheses). □

Intuitively, the hypotheses of the previous result require that any reduction $g\le_{\mathrm{W}} f$ witnessed by Φ_e,Φ_i is, in fact, witnessed by total functionals, and the indices for such functionals can be found uniformly in e,i. Theorem 3.12 can be rephrased as follows:

Corollary 3.13. *Fix a partial multi-valued function f. Assume there are total computable functionals Φ and Ψ such that for every $e,i\in\mathbb{N}$ and every $p\in\mathbb{N}^{\mathbb{N}}$, if $\Phi_e(p)\in\operatorname{dom}(f)$ and for every $q\in f\Phi_e(p)$, $\Phi_i(p,q)\downarrow$, then we have $\Phi(e,i,p)\in\operatorname{dom}(f)$ and $\Psi((e,i,p),f\Phi(e,i,p))\subseteq\{\Phi_i(p,q) : q\in f\Phi_e(p)\}$.*

Then $\mathsf{tJ}(f)\equiv_{\mathrm{W}}\mathsf{T}f$.

This rewording, on top of showing a closer connection with the definition of the tot-jump, allows us to draw a connection with the notion of *transparent* functions, introduced in [6]. More precisely, a function $H:\subseteq\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$ is called *transparent* if for every computable $F:\subseteq\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$ there is a computable $G:\subseteq\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$ such that $F\circ H=H\circ G$.

This notion can be naturally generalized to computational problems by requiring that given two (indices for) functionals Φ_e,Φ_i witnessing $g\le_{\mathrm{W}} f$, we can uniformly compute the index of a total function $\Phi\colon\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$ such that the reduction $g\le_{\mathrm{W}} f$ is witnessed by the functionals Φ and $(p,q)\mapsto q$.

This assumption implies that $\mathsf{tJ}(f) \equiv_{\mathrm{W}} \mathsf{T}f$. Indeed, these conditions are equivalent to requiring that there is a total computable Φ such that, for every $e, i \in \mathbb{N}$ and every $p \in \mathbb{N}^{\mathbb{N}}$, if $\Phi_e(p) \in \operatorname{dom}(f)$ and for every $q \in f\Phi_e(p)$, $\Phi_i(p,q)\downarrow$, then

$$f\Phi(e,i,p) \neq \emptyset \quad \text{and} \quad f\Phi(e,i,p) \subseteq \{\Phi_i(p,q) \,:\, q \in f\Phi_e(p)\}.$$

While many problems (including lim and its iterations) satisfy the above conditions, the assumption that the backward functional is exactly the projection on the second component is unnecessarily strong. In fact, any total computable functional would serve the same purpose. Thus we have the hypotheses of Corollary 3.13.

In Section 4, we will use Corollary 3.13 to describe the tot-jump of some natural problems.

Lemma 3.14. *There are $\bar{e}, \bar{\imath} \in \mathbb{N}$ such that for every f and every $k \in \mathbb{N}$, $\mathsf{tJ}(f)$ is Weihrauch equivalent to its restriction to $X_k := \{(\bar{e},\bar{\imath})^\frown 0^{k\frown} x \,:\, x \in \mathbb{N}^{\mathbb{N}}\}$.*

Proof. We let $\bar{e}, \bar{\imath}$ be the indices of two universal functionals such that $\Phi_{\bar{e}}(p) = \Phi_n(p)$ and $\Phi_{\bar{\imath}}(p,q) = \Phi_m(p,q)$ where $\langle n,m\rangle$ is least such that $p(\langle n,m\rangle) = 1$. It is obvious that any restriction of $\mathsf{tJ}(f)$ is Weihrauch reducible to $\mathsf{tJ}(f)$. To prove that $\mathsf{tJ}(f) \le_{\mathrm{W}} \mathsf{tJ}(f)|_{X_k}$, we let the forward functional be the map $\Phi(e,i,p) := (\bar{e},\bar{\imath})^\frown 0^{\langle n,m\rangle} 1^\frown p$, where $n, m > k$ are such that, for every t,

$$\Phi_n(0^t 1^\frown p) = \Phi_e(p), \text{ and}$$
$$\Phi_m(0^t 1^\frown p, q) = \Phi_i(p,q).$$

Clearly such n, m can be uniformly computed from e, i, k. The backward functional is the projection on the second coordinate. Note that $\Phi_{\bar{e}}(0^{\langle n,m\rangle} 1^\frown p) = \Phi_n(0^{\langle n,m\rangle} 1^\frown p) = \Phi_e(p)$, and similarly $\Phi_{\bar{\imath}}(0^{\langle n,m\rangle} 1^\frown p, q) = \Phi_m(0^{\langle n,m\rangle} 1^\frown p, q) = \Phi_i(p,q)$. Thus $\mathsf{tJ}(f)(e,i,p) = \mathsf{tJ}(f)(\bar{e},\bar{\imath},0^{\langle n,m\rangle} 1^\frown p)$, which concludes the proof. □

Theorem 3.15. *For every f, $\mathsf{tJ}(f)$ is total and join-irreducible.*

Proof. The fact that $\mathsf{tJ}(f)$ is total is apparent by definition. We show that if $\mathsf{tJ}(f) \equiv_{\mathrm{W}} g_0 \sqcup g_1$ then there is $b < 2$ such that $\mathsf{tJ}(f) \le_{\mathrm{W}} g_b$, and thus $g_{1-b} \le_{\mathrm{W}} g_b$. Assume that the reduction $\mathsf{tJ}(f) \le_{\mathrm{W}} g_0 \sqcup g_1$ is witnessed by the functionals Φ and Ψ and let $\bar{e}$ and $\bar{\imath}$ be as in Lemma 3.14. By the continuity of the forward functional, there exist k and $b < 2$ such that $\Phi(\bar{e},\bar{\imath},0^k)(0)\downarrow = b$. In particular, for every $x \sqsupseteq 0^k$, the functional Φ produces an input for g_b. Since, by Lemma 3.14, $\mathsf{tJ}(f)$ is equivalent to its restriction to $\{(\bar{e},\bar{\imath})^\frown 0^{k\frown} x \,:\, x \in \mathbb{N}^{\mathbb{N}}\}$, the claim follows. □

We will show in Corollary 4.9 that the range of tJ is a proper subset of the total, join-irreducible degrees.

Corollary 3.16. *For every f, g, $\mathsf{tJ}(f) \sqcup \mathsf{tJ}(g) \le_{\mathrm{W}} \mathsf{tJ}(f \sqcup g)$. Moreover, the reduction is strict iff $f \mid_{\mathrm{W}} g$.*

Proof. The fact that $\mathsf{tJ}(f) \sqcup \mathsf{tJ}(g) \le_{\mathrm{W}} \mathsf{tJ}(f \sqcup g)$ follows by the monotonicity of tJ and the fact that $\sqcup$ is the join in the Weihrauch degrees. Clearly, if $f \le_{\mathrm{W}} g$ then $f \sqcup g \equiv_{\mathrm{W}} g$, hence $\mathsf{tJ}(f) \sqcup \mathsf{tJ}(g) \equiv_{\mathrm{W}} \mathsf{tJ}(g) \equiv_{\mathrm{W}} \mathsf{tJ}(f \sqcup g)$. Conversely, if $f \mid_{\mathrm{W}} g$ then $\mathsf{tJ}(f) \mid_{\mathrm{W}} \mathsf{tJ}(g)$ by Theorem 3.7, and hence $\mathsf{tJ}(f \sqcup g) \not\equiv_{\mathrm{W}} \mathsf{tJ}(f) \sqcup \mathsf{tJ}(g)$ (by Theorem 3.15), which implies that $\mathsf{tJ}(f) \sqcup \mathsf{tJ}(g) <_{\mathrm{W}} \mathsf{tJ}(f \sqcup g)$. □

Theorem 3.17. *For every f, there is h such that $f <_{\mathrm{W}} h <_{\mathrm{W}} \mathsf{tJ}(f)$.*

Proof. We distinguish three cases: If $\mathrm{id} \le_{\mathrm{W}} f$, then the claim immediately follows from the fact that the Weihrauch degrees are dense above id (see [13]).

If $\mathrm{id} \mid_{\mathrm{W}} f$, then $f <_{\mathrm{W}} f \sqcup \mathrm{id} =: h$. Moreover, $\mathrm{id} \le_{\mathrm{W}} \mathsf{tJ}(f)$ (as $\mathsf{tJ}(f)$ is total), hence $f \sqcup \mathrm{id} \le_{\mathrm{W}} \mathsf{tJ}(f)$. The fact that the reduction is strict follows from the fact that $\mathsf{tJ}(f)$ is join-irreducible (Theorem 3.15).

If $\emptyset <_W f <_W \mathrm{id}$, then we have $f <_W \mathrm{id} <_W \mathsf{tJ}(f)$ by $\mathsf{tJ}(\emptyset) \equiv_W \mathrm{id}$ (Theorem 4.1 below) and the injectivity of the tot-jump proved in Theorem 3.7. Finally, if $f = \emptyset$ then $\mathsf{tJ}(f) \equiv_W \mathrm{id}$ by Theorem 4.1, and the existence of h with $\emptyset <_W h <_W \mathrm{id}$ is well-known. □

Theorem 3.18. *The range of* tJ *is not dense, i.e., there exist* f *and* g *such that* $f <_W g$*, the set* $X := \{h : \mathsf{tJ}(f) <_W h <_W \mathsf{tJ}(g)\}$ *is non-empty and, for every* $h \in X$, $h \notin \mathrm{ran}(\mathsf{tJ})$.

Proof. The claim follows immediately from the fact that the Weihrauch degrees are dense above id and that there are (strong) minimal covers in the Weihrauch lattice (see [13]). Indeed, it is enough to choose f and g such that g is a minimal cover of f. Since the tot-jump is always total, $\mathrm{ran}(\mathsf{tJ})$ is contained in the cone above id and there exists h such that $\mathsf{tJ}(f) <_W h <_W \mathsf{tJ}(g)$. By Theorem 3.7, if $h \equiv_W \mathsf{tJ}(h_0)$ then $f <_W h_0 <_W g$, contradicting the fact that g is a minimal cover of f. □

Proposition 3.19. *For every* f, g, $\mathsf{tJ}(f \sqcap g) \le_W \mathsf{tJ}(f) \sqcap \mathsf{tJ}(g)$. *There exist* f *and* g *such that* $\mathsf{tJ}(f \sqcap g) <_W \mathsf{tJ}(f) \sqcap \mathsf{tJ}(g)$.

Proof. The first part of the statement is straightforward using the fact that tJ is monotone and that $\sqcap$ is the meet in the Weihrauch lattice.

To prove the second part of the statement, let $X, Y \subseteq \mathbb{N}^\mathbb{N}$ be two incomparable Turing degrees (i.e., for some Turing-incomparable $x, y \subseteq \mathbb{N}$, X and Y are, respectively, the equivalence classes of x and y under $\equiv_T$), and let $f := \mathrm{id}|_X$ and $g := \mathrm{id}|_Y$. Assume towards a contradiction that the reduction $\mathsf{tJ}(f) \sqcap \mathsf{tJ}(g) \le_W \mathsf{tJ}(f \sqcap g)$ is witnessed by the functionals Φ and Ψ. We claim that one of $\mathsf{tJ}(f)$ or $\mathsf{tJ}(g)$ is uniformly computable. This contradicts Theorem 4.1 below, which implies that $\mathsf{tJ}(\emptyset)$ is the only uniformly computable tot-jump.

Let $\bar{e}, \bar{\imath}$ be as in Lemma 3.14. Observe that, for every k and every $x \in X$,

$$0^\mathbb{N} \in \mathsf{tJ}(f \sqcap g)(\Phi((\bar{e}, \bar{\imath}, 0^k \frown x), (\bar{e}, \bar{\imath}, 0^k \frown x))).$$

Indeed, if we let (e, i, z) be the input for $\mathsf{tJ}(f \sqcap g)$ given by the value $\Phi((\bar{e}, \bar{\imath}, 0^k \frown x), (\bar{e}, \bar{\imath}, 0^k \frown x))$, then $\Phi_e(z) \notin \mathrm{dom}(f \sqcap g)$ as, by hypothesis, X and Y are Turing-incomparable (in particular x does not compute any $y \in Y$). Analogously, by swapping the roles of X and Y in the above argument, for every $k \in \mathbb{N}$ and $y \in Y$, $0^\mathbb{N} \in \mathsf{tJ}(f \sqcap g)(\Phi((\bar{e}, \bar{\imath}, 0^k \frown y), (\bar{e}, \bar{\imath}, 0^k \frown y)))$.

By continuity, there is $k \in \mathbb{N}$ such that $\Psi(((\bar{e}, \bar{\imath}, 0^k), (\bar{e}, \bar{\imath}, 0^k)), 0^\mathbb{N})(0)$ commits to some $b < 2$. Without loss of generality, we can assume that $b = 0$, i.e., the backward functional commits to producing a solution for $\mathsf{tJ}(f)$. We therefore obtain that

$$\Psi(((\bar{e}, \bar{\imath}, 0^k \frown x), (\bar{e}, \bar{\imath}, 0^k \frown x)), 0^\mathbb{N}) \in \mathsf{tJ}(f)(\bar{e}, \bar{\imath}, 0^k \frown x),$$

i.e., we can uniformly compute a solution for $\mathsf{tJ}(f)(\bar{e}, \bar{\imath}, 0^k \frown x)$ which, in turn, implies that $\mathsf{tJ}(f)$ is uniformly computable. □

The previous proof can be generalized by letting X, Y be two incomparable Muchnik degrees (see [10] for the definition of Muchnik reducibility).

Definition 3.20 ([5, Def. 4.13])**.** We call a partial multi-valued function f *co-total* if, for every problem g,

$$f \le_W \mathsf{T}g \iff f \le_W g.$$

We observe that a problem is co-total exactly when the tot-jump "cannot help" to solve it.

Theorem 3.21. *For every* f, f *is co-total iff for every* g,

$$f \le_W \mathsf{tJ}(g) \iff f \le_W g.$$

In particular, if f *is co-total, then* $f \notin \mathrm{ran}(\mathsf{tJ})$.

Proof. The proof is straightforward using Theorem 3.3. Indeed, assume that f is co-total and let g be such that $f \le_W \mathsf{tJ}(g)$. Since $\mathsf{tJ}(g) \equiv_W \mathsf{T}g_0$ for some $g_0 \equiv_W g$, we obtain $f \le_W \mathsf{T}g_0$ and hence $f \le_W g_0 \equiv_W g$.

The converse implication is similar: Assume that for every g, $f \le_W \mathsf{tJ}(g)$ implies $f \le_W g$, and let h be such that $f \le_W \mathsf{T}h$. Since $\mathsf{T}h \le_W \mathsf{tJ}(h)$, we immediately obtain $f \le_W h$.

Finally, if f is co-total and $f \equiv_W \mathsf{tJ}(h)$, for some h, then $f \le_W h$, contradicting $\mathsf{tJ}(h) \not\le_W h$. □

In [5], several problems are proved to be co-total, including $\mathsf{C}_{\mathbb{N}}$ and $\mathsf{C}_{2^{\mathbb{N}}}$, hence we immediately have the following result:

Corollary 3.22. *The problems* $\mathsf{C}_{\mathbb{N}}$ *and* $\mathsf{C}_{2^{\mathbb{N}}}$ *are not in* $\operatorname{ran}(\mathsf{tJ})$. □

The next theorem leads to a sufficient condition for a function to be co-total. Let $\mathsf{U} :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be a fixed universal Turing functional. Let $\mathsf{DIS}\colon \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be the problem defined as

$$\mathsf{DIS}(p) := \{q \in \mathbb{N}^{\mathbb{N}} \,:\, \mathsf{U}(p) \neq q\}.$$

It is immediate from the definition that DIS is total. In fact, for every $p \in \mathbb{N}^{\mathbb{N}}$, $\mathsf{DIS}(p)$ is either $\mathbb{N}^{\mathbb{N}}$ (if $\mathsf{U}(p)\uparrow$) or $\mathbb{N}^{\mathbb{N}} \setminus \{\mathsf{U}(p)\}$. The problem DIS was studied extensively in [2] and is one of the weakest discontinuous problems.[3] In particular, $\mathsf{DIS} \le_W \mathsf{C}_2$ (see, e.g., [1, Prop. 5.10]).

Theorem 3.23 (with Arno Pauly)**.** *If* $\mathsf{DIS} \times g \le_W \mathsf{tJ}(f)$*, then* $g \le_W f$*.*

Proof. Let Φ and Ψ witness the reduction $\mathsf{DIS} \times g \le_W \mathsf{tJ}(f)$. Let $U :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ be the universal computable functional used in the definition of DIS. By the recursion theorem, there is a computable $p \in \mathbb{N}^{\mathbb{N}}$ such that $U(\langle p, x\rangle)$ is the first component of $\Psi((\langle p, x\rangle, x), 0^{\mathbb{N}})$. Consider the reduction $g \le_W \mathsf{tJ}(f)$ where the forward functional is given by $x \mapsto \Phi(\langle p, x\rangle, x)$ and the backward functional maps (x, y) to the second component of $\Psi((\langle p, x\rangle, x), y)$.

We claim that if $x \in \operatorname{dom}(g)$, then $\mathsf{tJ}(f)(\Phi(\langle p, x\rangle, x)) \neq \mathbb{N}^{\mathbb{N}}$ and so it does not fall in the "otherwise" case. This implies that our reduction of g to $\mathsf{tJ}(f)$ actually is a Weihrauch reduction of g to f. To prove the claim, note that if $x \in \operatorname{dom}(g)$, then $(\langle p, x\rangle, x) \in \operatorname{dom}(\mathsf{DIS} \times g)$. If $0^{\mathbb{N}} \in \mathsf{tJ}(f)(\Phi(\langle p, x\rangle, x))$, then $\Psi((\langle p, x\rangle, x), 0^{\mathbb{N}})$ must converge to an element $(q, z) \in \mathsf{DIS}(\langle p, x\rangle) \times g(x)$. But $\mathsf{DIS}(\langle p, x\rangle)$ must be different from q, the first component of $\Psi((\langle p, x\rangle, x), 0^{\mathbb{N}})$. □

Corollary 3.24. *If* $\mathsf{DIS} \times g \le_W g$*, then* g *is co-total and hence it is not in the range of* tJ. □

As mentioned, DIS is quite weak and hence being closed under parallel product with DIS is a rather weak condition satisfied by many natural problems, like $\mathsf{C}_{\mathbb{N}}$, $\mathsf{C}_{2^{\mathbb{N}}}$, lim, $\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$, and $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$.

Proposition 3.25. DIS *is not co-total.*

Proof. Observe that $\mathsf{DIS} = \mathsf{T}(\mathsf{DIS}|_X)$, where $X := \{p \in \mathbb{N}^{\mathbb{N}} \,:\, \mathsf{U}(p)\downarrow\}$. To show that DIS is not co-total, it is enough to show that $\mathsf{DIS} \not\le_W \mathsf{DIS}|_X$. This follows from the fact that $\mathsf{DIS}|_X \le_W \mathrm{id}$: indeed, for every p such that $\mathsf{U}(p)\downarrow$, it is enough to consider $q := n \mapsto \mathsf{U}(p)(n) + 1$. □

Combining this result with Corollary 3.24, we immediately obtain:

Corollary 3.26. $\mathsf{DIS} \times \mathsf{DIS} \not\le_W \mathsf{DIS}$. □

Proposition 3.27. DIS *is join-irreducible.*

[3] In [2], it was shown that, under $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$, DIS is a strong minimal cover of id in the topological version of Weihrauch reducibility. Such a result cannot be transferred to the (plain) Weihrauch degrees, as the cone above id is dense in the Weihrauch degrees under ZFC [13].

Proof. Without loss of generality, we can assume that $\mathsf{U}((e)^\frown p) = \Phi_e(p)$. Let e be such that $\Phi_e(0^j 1^\frown p) = \Phi_j(p)$ and $\Phi_e(0^{\mathbb{N}})\uparrow$. Notice that $\mathsf{DIS} \le_{\mathrm{W}} \mathsf{DIS}|_X$, where $X := \{x \in \mathbb{N}^{\mathbb{N}} : x(0) = e\}$ (it is enough to map p to $(e)^\frown 0^i 1^\frown p$, where i is an index of U).

We show that if $\mathsf{DIS}|_X \le_{\mathrm{W}} f_0 \sqcup f_1$, then $\mathsf{DIS}|_X \le_{\mathrm{W}} f_0$ or $\mathsf{DIS}|_X \le_{\mathrm{W}} f_1$. If Φ, Ψ witness the reduction $\mathsf{DIS}|_X \le_{\mathrm{W}} f_0 \sqcup f_1$ (in particular, for every input q for DIS, $\Phi(q)$ produces a pair (b, x) with $x \in \mathrm{dom}(f_b)$), then by continuity, there is k such that $\Phi((e)^\frown 0^k)(0) = b < 2$. Observe that we can uniformly map any $x \in X$ of the form $(e)^\frown 0^j 1^\frown p$ to $(e)^\frown 0^h 1^\frown p$ for some $h > k$ such that $\Phi_j(p) = \Phi_h(p)$. The reduction $\mathsf{DIS}|_X \le_{\mathrm{W}} f_b$ is therefore witnessed by the maps $(e)^\frown 0^j 1^\frown p \mapsto \Phi((e)^\frown 0^h 1^\frown p)(1)$ and Ψ. □

While being join-irreducible and not being co-total are necessary conditions for a Weihrauch degree to be in the range of tJ, we will show in Corollary 4.8 that they are not sufficient, as DIS is not equivalent to the tot-jump of any problem. In other words, the range of the tot-jump is a proper subset of the set of total, non-co-total, join-irreducible degrees.

4 The jump of specific problems

In this section, we explicitly characterize the tot-jump of various well-known problems. Let us start with a straightforward example.

Theorem 4.1. $\mathsf{tJ}(\emptyset) \equiv_{\mathrm{W}} \mathrm{id}$.

Proof. The proof is trivial as for every e, i, p, $\mathsf{tJ}(\emptyset)(e, i, p) = \mathbb{N}^{\mathbb{N}}$, as there are no e, p such that $\Phi_e(p) \in \mathrm{dom}(\emptyset)$. Therefore, $\mathsf{tJ}(\emptyset)$ is total and uniformly computable, and hence equivalent to id. □

To characterize $\mathsf{tJ}(\mathrm{id})$, we first introduce the following problem.

Definition 4.2. Let us define $\chi_{\Pi^0_2 \to \Pi^0_1} : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ as

$$\chi_{\Pi^0_2 \to \Pi^0_1}(p) := \{q \in \mathbb{N}^{\mathbb{N}} : (\exists^\infty n)(p(n) = 0) \iff (\forall n)(q(n) = 0)\}.$$

Intuitively, $\chi_{\Pi^0_2 \to \Pi^0_1}$ transforms a Π^0_2-question into a Π^0_1-question. An alternate form of $\chi_{\Pi^0_2 \to \Pi^0_1}$ was introduced by Neumann and Pauly [17], who defined

$$\mathsf{isFinite}_{\mathbb{S}}(p) := \{q \in \mathbb{N}^{\mathbb{N}} : (\exists^\infty n)(p(n) = 1) \iff (\forall n)(q(n) = 0)\}.$$

It is immediate that $\chi_{\Pi^0_2 \to \Pi^0_1} \equiv_{\mathrm{W}} \mathsf{isFinite}_{\mathbb{S}}$.

Proposition 4.3. $\chi_{\Pi^0_2 \to \Pi^0_1} \equiv_{\mathrm{W}} \mathsf{T}(\mathsf{W}_{\Pi^0_2 \to \Pi^0_1})$.

Proof. For the left-to-right reduction, recall that, given $p, q \in \mathbb{N}^{\mathbb{N}}$, it is c.e. to check whether $q \notin \mathsf{W}_{\Pi^0_2 \to \Pi^0_1}(p)$. In particular, given p and $q \in \mathsf{T}(\mathsf{W}_{\Pi^0_2 \to \Pi^0_1})(p)$, we can uniformly compute $t \in \mathbb{N}^{\mathbb{N}}$ defined as $t(n) := 0$ if $q(n+1) > q(n)$ and $p \circ q(n) = 0$, and $t(n) := 1$ otherwise. It is apparent that $t \in \chi_{\Pi^0_2 \to \Pi^0_1}(p)$.

Similarly, for the right-to-left reduction, given $p \in \mathbb{N}^{\mathbb{N}}$ and $q \in \chi_{\Pi^0_2 \to \Pi^0_1}(p)$, we can compute a solution for $\mathsf{T}(\mathsf{W}_{\Pi^0_2 \to \Pi^0_1})(p)$ as follows: Let $\sigma_0 := ()$. For every n, if $q(i) = 0$ for all $i < n$ we check if $p(n) = 0$. If it is, we define $\sigma_{n+1} := \sigma_n{}^\frown(n)$, otherwise we let $\sigma_{n+1} := \sigma_n$. If instead $q(i) > 0$ for some $i < n$, we let $\sigma_{n+1} := \sigma_n{}^\frown(0)$. It is straightforward to check that $r := \bigcup_n \sigma_n \in \mathbb{N}^{\mathbb{N}}$ is uniformly computable from p and q and that $r \in \mathsf{T}(\mathsf{W}_{\Pi^0_2 \to \Pi^0_1})(p)$. □

Theorem 4.4. $\mathsf{tJ}(\mathrm{id}) \equiv_{\mathrm{W}} \chi_{\Pi^0_2 \to \Pi^0_1}$.

Proof. For the left-to-right reduction, observe that $\mathsf{tJ}(\mathrm{id})$ can be written as follows:

$$\mathsf{tJ}(\mathrm{id})(e,i,x) = \begin{cases} \Phi_i(x,\Phi_e(x)) & \text{if } \Phi_e(x)\downarrow \wedge \Phi_i(x,\Phi_e(x))\downarrow, \\ \mathbb{N}^\mathbb{N} & \text{otherwise.} \end{cases}$$

Since the domain of a computable functional is a Π^0_2-set, we can uniformly compute $p \in \mathbb{N}^\mathbb{N}$ such that

$$(\exists^\infty n)\, p(n) = 0 \iff \Phi_e(x)\downarrow \wedge \Phi_i(x,\Phi_e(x))\downarrow .$$

Given $q \in \chi_{\Pi^0_2 \to \Pi^0_1}(p)$, we are able to uniformly compute a solution for $\mathsf{tJ}(f)(e,i,x)$ as follows: In parallel, run $\Phi_i(x,\Phi_e(x))$ and check whether there is n such that $q(n) \neq 0$. If no such n is found then we are producing $\Phi_i(x,\Phi_e(x))$, which is the correct solution for $\mathsf{tJ}(f)(e,i,x)$. Otherwise, it means that $\mathsf{tJ}(f)(e,i,x) = \mathbb{N}^\mathbb{N}$, hence we can stop simulating $\Phi_i(x,\Phi_e(x))$ and continue the output with $0^\mathbb{N}$.

For the right-to-left reduction, note that $\mathsf{W}_{\Pi^0_2 \to \Pi^0_1} \le_\mathrm{W} \mathrm{id}$, so Theorem 3.3 and Proposition 4.3 give us $\chi_{\Pi^0_2 \to \Pi^0_1} \equiv_\mathrm{W} \mathsf{T}(\mathsf{W}_{\Pi^0_2 \to \Pi^0_1}) \le_\mathrm{W} \mathsf{tJ}(\mathrm{id})$. ☐

With a more careful analysis, we can characterize the n-th tot-jump of id. Intuitively, we can think of $\mathsf{tJ}^n(\mathrm{id})$ as a problem capturing the following: You are allowed to ask n many Σ^0_2-questions serially. For every $j < n$, you can see in finite time if the answer to the j-th question is "yes" and then you can ask the next question. However, if the j-th answer is "no", then the procedure hangs and it is impossible to see the answers of the remaining questions.

Theorem 4.5. *For every $p \in \mathbb{N}^\mathbb{N}$, let $A^p_n := \{k < n : (\exists^\infty j)\ p(j) = k\}$. For every $n > 0$, let $g_n \colon \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ be defined as*

$$g_n(p) := \begin{cases} \left\{0^{\sigma(0)}1^{\sigma(1)}\dots(m-1)^{\sigma(m-1)}m^\mathbb{N} : \sigma \in \mathbb{N}^m\right\} & \text{if } A^p_n \neq \emptyset \text{ and } m := \min A^p_n, \\ \left\{0^{\sigma(0)}1^{\sigma(1)}\dots(n-1)^{\sigma(n-1)}n^\mathbb{N} : \sigma \in \mathbb{N}^n\right\} & \text{if } A^p_n = \emptyset. \end{cases}$$

Then $\mathsf{tJ}^n(\mathrm{id}) \equiv_\mathrm{W} g_n$.

Proof. By induction on n: The base case $n = 1$ is Theorem 4.4, as it is straightforward to see that $g_1 \equiv_\mathrm{W} \chi_{\Pi^0_2 \to \Pi^0_1}$. For the induction step, it suffices to show that $\mathsf{tJ}(g_n) \equiv_\mathrm{W} g_{n+1}$.

Let us first prove that $\mathsf{tJ}(g_n) \le_\mathrm{W} g_{n+1}$. Let (e,i,x) be an input for $\mathsf{tJ}(g_n)$. For every $k < n$, we can uniformly compute y_k so that $\mathrm{ran}(y_k) \subseteq \{k, n+1\}$ and k occurs infinitely many times in y_k if and only if

$$\Phi_e(x)\downarrow \wedge\, k \in A^{\Phi_e(x)}_n \wedge (\forall \sigma \in \mathbb{N}^k)\ \Phi_i(x, 0^{\sigma(0)}1^{\sigma(1)}\dots(k-1)^{\sigma(k-1)}k^\mathbb{N})\downarrow .$$

This can be done because the displayed formula is uniformly Π^0_2 in (e,i,x). Similarly, we can uniformly compute y_n so that $\mathrm{ran}(y_n) \subseteq \{n, n+1\}$ and n occurs infinitely many times in y_n if and only if

$$\Phi_e(x)\downarrow \wedge (\exists^\infty j)(\Phi_e(x)(j) \ge n) \wedge (\forall \sigma \in \mathbb{N}^n)\ \Phi_i(x, 0^{\sigma(0)}1^{\sigma(1)}\dots(n-1)^{\sigma(n-1)}n^\mathbb{N})\downarrow .$$

Let $y := \langle y_0,\dots,y_n\rangle$. We now claim that a solution for $\mathsf{tJ}(g_n)(e,i,x)$ can be uniformly computed from any $z \in g_{n+1}(y)$ as follows: We compute $\Phi_i(x,z)$ as long as $z(j) \le n$. If $z(j) = n+1$ for some j, we stop the computation and continue the output with $0^\mathbb{N}$.

Let us now show that this procedure correctly produces a solution for $\mathsf{tJ}(g_n)(e,i,x)$.

- If $\Phi_e(x)\uparrow$, then for every $k \le n$, y_k only has finitely many k's. This implies that $(\forall^\infty j)\ y(j) = n+1$, hence $n+1 \in \mathrm{ran}(z)$. The procedure produces an eventually null string, which is clearly a valid solution as $\mathsf{tJ}(g_n)(e,i,x) = \mathbb{N}^\mathbb{N}$.

- If $\Phi_e(x)\downarrow$ and $A_n^{\Phi_e(x)}=\emptyset$ then for every $k<n$ and for almost all j, $y_k(j)=n+1$; moreover, n occurs infinitely many times in y_n if and only if, for all $\sigma\in\mathbb{N}^n$, $\Phi_i(x,0^{\sigma(0)}1^{\sigma(1)}\dots(n-1)^{\sigma(n-1)}n^{\mathbb{N}})\downarrow$. This, in turn, implies that if $z\in g_{n+1}(x)$ and $n+1\notin\operatorname{ran}(z)$ then $\Phi_i(x,z)\downarrow\in \mathsf{tJ}(g_n)(e,i,x)$, while otherwise $\mathsf{tJ}(g_n)(e,i,x)=\mathbb{N}^{\mathbb{N}}$ and the procedure computes an eventually null string.

- Finally, assume that $\Phi_e(x)\downarrow$ and $A_n^{\Phi_e(x)}\neq\emptyset$. Let $k:=\min A_n^{\Phi_e(x)}$ and notice that for every $k'<k$ we have $(\forall^\infty j)\ y_{k'}(j)=n+1$. Moreover, y_k has infinitely many k if and only if, for all $\sigma\in\mathbb{N}^n$, $\Phi_i(x,0^{\sigma(0)}1^{\sigma(1)}\dots(k-1)^{\sigma(k-1)}k^{\mathbb{N}})\downarrow$. If y_k has infinitely many k, then we can just run the computation $\Phi_i(x,z)$ for any $z\in g_{n+1}(y)$. Otherwise, $\mathsf{tJ}(g_n)(e,i,x)=\mathbb{N}^{\mathbb{N}}$ and the described procedure is guaranteed to produce an infinite string (and therefore a valid solution).

Let us now show $g_{n+1}\le_{\mathrm{W}}\mathsf{tJ}(g_n)$. Intuitively, the reduction works as follows: The forward functional maps an input p for g_{n+1} to (e,i,p), where e is an index for the identity functional and i is an index for the functional that, given (p,q), tries to produce a list of positions witnessing the fact that $q\in g_n(p)$. More precisely, if q is of the form $0^{\sigma(0)}1^{\sigma(1)}\dots(k-1)^{\sigma(k-1)}k^{\mathbb{N}}$ for some $\sigma\in\mathbb{N}^{<\mathbb{N}}$ and some $k\le n$, then $\Phi_i(p,q)$ produces a strictly increasing string such that for some strictly increasing sequence $(v_j)_{j\in\mathbb{N}}$ and for every j,

$$p(\Phi_i(p,q)(j))=q(v_j).$$

This can be done iteratively as follows: At stage 0, search for u_0,v_0 such that $p(u_0)=q(v_0)$. At stage $j+1$, we search for $u_{j+1}>u_j$ and $v_{j+1}>v_j$ such that $p(u_{j+1})=q(v_{j+1})$. The sequence $(u_j)_{j\in\mathbb{N}}$ is the output of $\Phi_i(p,q)$.

If instead q is not of the form $0^{\sigma(0)}1^{\sigma(1)}\dots(k-1)^{\sigma(k-1)}k^{\mathbb{N}}$ for any $\sigma\in\mathbb{N}^{<\mathbb{N}}$ and any $k\le n$, we let $\Phi_i(p,q)\uparrow$.

Given $z\in\mathsf{tJ}(g_n)(e,i,p)$, the backward functional Ψ is defined as $\Psi(p,z)(0):=\min\{p(z(0)),n\}$ and $\Psi(p,z)(j+1):=p(z(j+1))$ if $p(z(j))\le p(z(j+1))\le n$ and $z(j+1)>z(j)$, and $\Psi(p,z)(j+1):=n+1$ otherwise.

To conclude the proof we show that, for every $z\in\mathsf{tJ}(g_n)(e,i,p)$, $\Psi(p,z)$ correctly produces a solution for $g_{n+1}(p)$. Observe that if $A_{n+1}^p\neq\emptyset$, then $g_{n+1}(p)=g_n(p)$. In particular, for every j,

$$\Psi(p,z)(j)=p(z(j))=p(\Phi_i(p,q)(j))=q(v_j)$$

for some $q\in g_n(p)$ and some strictly increasing $(v_j)_{j\in\mathbb{N}}$, and hence $\Psi(p,z)$ is a correct solution for $g_{n+1}(p)$. On the other hand, if $A_{n+1}^p=\emptyset$ then, for every $t\in g_n(p)$, $\Phi_i(p,t)\uparrow$. This implies that $\Psi(p,z)$ is of the form $0^{\sigma(0)}1^{\sigma(1)}\dots n^{\sigma(n)}(n+1)^{\mathbb{N}}$ for some σ, and therefore is a valid solution for $g_{n+1}(p)$. □

We now show that the set $\{\mathsf{tJ}(f)\,:\,f\le_{\mathrm{W}}\mathrm{id}\}$ is a proper subset of $\{h\,:\,\mathsf{tJ}(\emptyset)\le_{\mathrm{W}}h\le_{\mathrm{W}}\mathsf{tJ}(\mathrm{id})\}$.

Theorem 4.6. *Let $f<_{\mathrm{W}}\mathsf{tJ}(\mathrm{id})$ and let $X:=\{p\in\operatorname{dom}(f)\,:\,p$ is computable$\}$. If $\mathrm{id}<_{\mathrm{W}}f|_X$, then there is no h such that $f\equiv_{\mathrm{W}}\mathsf{tJ}(h)$.*

Proof. By Theorem 3.7, if $\mathsf{tJ}(h)\equiv_{\mathrm{W}}f<_{\mathrm{W}}\mathsf{tJ}(\mathrm{id})$ then $h<_{\mathrm{W}}\mathrm{id}$, i.e., $h\equiv_{\mathrm{W}}\mathrm{id}|_A$ for some $A\subseteq\mathbb{N}^{\mathbb{N}}$ (as the lower cone of id is isomorphic to the dual of the Medvedev degrees, see, e.g., [9, Sec. 5]). Notice that A (and therefore $\operatorname{dom}(h)$) does not have any computable point, as otherwise $h\equiv_{\mathrm{W}}\mathrm{id}$.

Assume $f|_X\le_{\mathrm{W}}\mathsf{tJ}(h)$ via Φ,Ψ. Let $p\in X$ be a computable input for f and let $\Phi(p)=(e,i,x)$. Observe that, since A has no computable point, $\Phi_e(x)\notin A$, hence $\mathsf{tJ}(h)(e,i,x)=\mathbb{N}^{\mathbb{N}}$. This implies that a reduction $f|_X\le_{\mathrm{W}}\mathsf{tJ}(h)$ would yield a reduction of $f|_X$ to the (uniformly computable) constant map $p\mapsto\mathbb{N}^{\mathbb{N}}$, contradicting $f|_X\not\le_{\mathrm{W}}\mathrm{id}$. □

Theorem 4.7. $\mathsf{LPO}<_{\mathrm{W}}\mathsf{tJ}(\mathrm{id})<_{\mathrm{W}}\widehat{\mathsf{tJ}(\mathrm{id})}\equiv_{\mathrm{W}}\mathsf{lim}$.

Proof. Let $p\in\mathbb{N}^{\mathbb{N}}$ be an input for LPO. By Theorem 4.4, we can use $\mathsf{tJ}(\mathrm{id})$ to compute $q\in\mathbb{N}^{\mathbb{N}}$ such that $(\forall n)\ q(n)=0$ if and only if $(\exists n)\ p(n)>0$. It is then straightforward to see that we can find an answer for $\mathsf{LPO}(p)$ by unbounded search (either there is a non-zero element in q or a non-zero element in p). The fact that the reduction is strict follows from Theorem 4.6.

The second reduction is immediate and the reduction $\mathsf{lim} \le_{\mathrm{W}} \widehat{\mathsf{tJ}(\mathrm{id})}$ follows from the fact that $\widehat{\mathsf{LPO}} \equiv_{\mathrm{W}} \mathsf{lim}$.

Given that lim is parallelizable, to prove that $\widehat{\mathsf{tJ}(\mathrm{id})} \le_{\mathrm{W}} \mathsf{lim}$ it suffices to show that $\mathsf{tJ}(\mathrm{id}) \le_{\mathrm{W}} \mathsf{lim}$. To this end, we prove that $\chi_{\Pi^0_2 \to \Pi^0_1} \le_{\mathrm{W}} \mathsf{lim}$, and the claim will follow from Theorem 4.4. Given $p \in \mathbb{N}^\mathbb{N}$, we can uniformly compute the converging sequence $(q_n)_{n\in\mathbb{N}}$ defined as $q_n(m) := 0$ if there is j such that $m \le j \le n$ such that $p(j) = 0$, and $q_n(m) := 1$ otherwise. Clearly, for each m, $\lim_{n\to\infty} q_n(m) = 0$ if and only if there is some 0 in p after position m. Therefore, $\lim_{n\to\infty} q_n \in \chi_{\Pi^0_2 \to \Pi^0_1}(p)$.

Finally the fact that $\mathsf{tJ}(\mathrm{id}) <_{\mathrm{W}} \widehat{\mathsf{tJ}(\mathrm{id})}$ follows from the fact that every computable input for $\mathsf{tJ}(\mathrm{id}) \equiv_{\mathrm{W}} \mathsf{T}(\mathsf{W}_{\Pi^0_2 \to \Pi^0_1})$ has a computable solution, while this is not the case for lim. □

Given that $\mathsf{DIS} \le_{\mathrm{W}} \mathsf{LPO}$, combining Theorem 4.6 and Theorem 4.7 we obtain:

Corollary 4.8. *The problems* DIS *and* LPO *are not Weihrauch-equivalent to any problem in the range of* tJ. □

Since we showed that DIS is total, non-co-total, and join-irreducible, we also obtain the following corollary:

Corollary 4.9. *The map* tJ *does not induce a surjective operator onto the total, non-co-total, join-irreducible degrees.* □

Next we show that no lower cone is closed under tot-jump, i.e., there are no tot-jump principal ideals in the Weihrauch lattice.

Theorem 4.10. *For every $g \neq \emptyset$, there exists an $f <_{\mathrm{W}} g$ such that $\mathsf{tJ}(f) \not\le_{\mathrm{W}} g$.*

Proof. If $\mathrm{id} \not\le_{\mathrm{W}} g$, then $f := \emptyset$ has the desired properties, while if $g \equiv_{\mathrm{W}} \mathrm{id}$ we can set $f := \mathrm{id}|_X$ for any X without computable elements (in this case, $\emptyset <_{\mathrm{W}} f <_{\mathrm{W}} \mathrm{id}$ and we can use Theorem 3.7). We can now assume that $\mathrm{id} <_{\mathrm{W}} g$ and distinguish two cases.

The first one is when there exists g_0 with finite domain such that $g \equiv_{\mathrm{W}} g_0$. In this case, we claim that $\mathsf{LPO} \not\le_{\mathrm{W}} g$. Granting the claim, $f := \mathrm{id}$ has the desired property by Theorem 4.7. To prove the claim, assume that Φ and Ψ witness $\mathsf{LPO} \le_{\mathrm{W}} g_0$. Then, since every point in $\mathrm{dom}(g_0)$ is isolated, there are $z \in \mathrm{dom}(g_0)$ and k_0 such that for every $x \sqsupseteq 0^{k_0}$, $\Phi(x) = z$. Besides, there is $k_1 \ge k_0$ such that for some $y \in g_0(z)$, $\Psi(0^{k_1}, y)(0) \downarrow= 0$. In particular, the string $0^{k_1}1^\mathbb{N}$ witnesses the fact that Φ and Ψ do not witness the Weihrauch reduction.

Assume now that g is not Weihrauch equivalent to any problem with finite domain. We want to define f such that $f <_{\mathrm{W}} g$ and $\mathsf{tJ}(f) \not\le_{\mathrm{W}} g$. To this end, we define a *scrambling function* $\xi :\subseteq \mathrm{dom}(g) \to \mathbb{N}$. The desired f will be defined as $f(x, \xi(x)) := g(x)$, with $\mathrm{dom}(f) := \{(x,n) \in \mathrm{dom}(g) \times \mathbb{N} : \xi(x) = n\}$. Notice that, no matter which ξ we choose, $f \le_{\mathrm{W}} g$.

To define ξ we define a sequence $(\xi_s)_{s\in\mathbb{N}}$ of functions with finite domain, starting with $\xi_0 := \emptyset$.

At stage $s+1 = 2\langle e, i\rangle + 1$, we satisfy the requirement "$g \not\le_{\mathrm{W}} f$ via Φ_e, Φ_i". To do so, we choose (in a noneffective way) some $x \in \mathrm{dom}(g)$ such that one of the following conditions holds:

- $\Phi_e(x)\uparrow$;
- $\Phi_e(x)$ produces the pair (y, n) with $y \notin \mathrm{dom}(g)$ or $\xi_s(y) \downarrow\neq n$;
- $\Phi_e(x)$ produces the pair (y, n) and there is $q \in g(y)$ such that $\Phi_i(x, q) \notin g(x)$;
- $\Phi_e(x)$ produces the pair (y, n) with $\xi_s(y)\uparrow$.

Such an x must exist, as otherwise Φ_e and Φ_i witness $g \equiv_W f_s$, where $f_s(x, \xi(x)) := g(x)$ with $\mathrm{dom}(f_s) := \{(x, n) : \xi_s(x) \downarrow= n\}$, contradicting the fact that g is not Weihrauch equivalent to any problem with finite domain. In the first three cases, there is nothing to do, and we just define $\xi_{s+1} := \xi_s$. In the last case, we let $\xi_{s+1} := \xi_s \cup \{(y, n+1)\}$.

At stage $s + 1 = 2\langle e, i\rangle + 2$, we satisfy the requirement "$\mathsf{tJ}(f) \not\leq_W g$ via Φ_e, Φ_i". Let $\tilde{\xi}$ be an arbitrary computable extension of ξ_s and let $\tilde{f} := (x, \tilde{\xi}(x)) \mapsto g(x)$. Clearly $\tilde{f} \equiv_W g$, hence $\mathsf{tJ}(\tilde{f}) \equiv_W \mathsf{tJ}(g) \not\leq_W g$. In particular, there are $\tilde{e}, \tilde{\imath}, \tilde{x}$ witnessing $\mathsf{tJ}(\tilde{f}) \not\leq_W g$ via Φ_e, Φ_i, i.e., if $\Phi_{\tilde{e}}(\tilde{x}) \downarrow\in \mathrm{dom}(\tilde{f})$ and, letting $(\tilde{y}, \tilde{n})$ be the pair produced by $\Phi_{\tilde{e}}(\tilde{x})$, for every $q \in \tilde{f}(\tilde{y}, \tilde{n}) = g(\tilde{y})$, $\Phi_{\tilde{\imath}}(\tilde{x}, q) \downarrow$, then there is $q \in \tilde{f}(\tilde{y}, \tilde{n})$ such that $\Phi_e((\tilde{e}, \tilde{\imath}, \tilde{x}), q) \notin \mathsf{tJ}(\tilde{f})$. Since $\tilde{\xi}$ is an extension of ξ_s, if $\xi_s(\tilde{x}) \downarrow$ then $\xi_s(\tilde{x}) = \tilde{\xi}(\tilde{x}) = \tilde{n}$. In this case, or if $\Phi_{\tilde{e}}(\tilde{x}) \uparrow$, then there is nothing to do, and we set $\xi_{s+1} := \xi_s$. Otherwise, let $\xi_{s+1} := \xi_s \cup \{(\tilde{x}, \tilde{n})\}$. Observe that indeed, for every extension $\bar{\xi}$ of ξ_s such that $\bar{\xi}(\tilde{x}) = \tilde{n}$, the input $(\tilde{e}, \tilde{\imath}, \tilde{x})$ witnesses $\mathsf{tJ}((x, \bar{\xi}(x)) \mapsto g(x)) \not\leq_W g$ via Φ_e, Φ_i.

The desired scrambling function is the map $\xi := \bigcup_{s\in\mathbb{N}} \xi_s$. Since all the requirements are satisfied, the above-defined function f satisfies $f <_W g$ and $\mathsf{tJ}(f) \not\leq_W g$, which concludes the proof. ☐

In light of Theorem 4.7, a natural question is how $\mathsf{tJ}(\mathrm{id})$ compares with $\mathsf{C}_\mathbb{N}$ (as $\mathsf{lim} \equiv_W \widehat{\mathsf{C}_\mathbb{N}}$). By Theorem 3.21, as $\mathsf{C}_\mathbb{N}$ is co-total, $\mathsf{C}_\mathbb{N} \leq_W \mathsf{tJ}(\mathrm{id})$ would imply that $\mathsf{C}_\mathbb{N} \leq_W \mathrm{id}$, which is a contradiction. On the other hand, $\mathsf{tJ}(\mathrm{id}) \not\leq_W \mathsf{C}_\mathbb{N}$. In fact, we have a slightly stronger result (as $\mathsf{C}_\mathbb{N} <_W \mathsf{TC}_\mathbb{N}$ by [17, Prop. 24]).

Proposition 4.11. $\mathsf{tJ}(\mathrm{id}) \not\leq_W \mathsf{TC}_\mathbb{N}$.

Proof. We use the fact that $\mathsf{tJ}(\mathrm{id}) \equiv_W \chi_{\Pi^0_2 \to \Pi^0_1}$ (Theorem 4.4). As mentioned above, $\chi_{\Pi^0_2 \to \Pi^0_1} \equiv_W \mathsf{isFinite}_\mathbb{S}$. But Neumann and Pauly proved that $\mathsf{isFinite}_\mathbb{S} \not\leq_W \mathsf{TC}_\mathbb{N}$ [17, Prop. 24(5)]. ☐

The following result is folklore.

Lemma 4.12. $\mathsf{TC}_\mathbb{N}$ *is a* fractal*, i.e., for every* $\sigma \in \mathbb{N}^{<\mathbb{N}}$*,* $\mathsf{TC}_\mathbb{N}$ *is Weihrauch equivalent to its restriction to* $X_\sigma := \{p \in \mathbb{N}^\mathbb{N} : \sigma \sqsubset p\}$*.*

Proof. Fix $\sigma \in \mathbb{N}^{<\mathbb{N}}$. To prove that $\mathsf{TC}_\mathbb{N} \leq_W \mathsf{TC}_\mathbb{N}|_{X_\sigma}$, let $m := \max \mathrm{ran}(\sigma)$. We can uniformly map $p \in \mathbb{N}^\mathbb{N}$ to

$$q := \sigma^\frown(1, \dots, m)^\frown(p(0) + m, p(1) + m, \dots).$$

Clearly, $q \in X_\sigma$ and for every $n \in \mathsf{TC}_\mathbb{N}(q)$, $\max\{n - m, 0\} \in \mathsf{TC}_\mathbb{N}(p)$. ☐

Theorem 4.13. *There is no* f *such that* $\mathsf{TC}_\mathbb{N} \equiv_W \mathsf{tJ}(f)$*.*

Proof. Observe first of all that $\mathsf{TC}_\mathbb{N} <_W \mathsf{tJ}(\mathsf{C}_\mathbb{N})$. Indeed, the reduction follows by Theorem 3.3, while the fact that the reduction is strict follows from the fact that $\mathsf{tJ}(\mathrm{id}) \not\leq_W \mathsf{TC}_\mathbb{N}$ (Proposition 4.11) whereas $\mathsf{tJ}(\mathrm{id}) \leq_W \mathsf{tJ}(\mathsf{C}_\mathbb{N})$ (by the monotonicity of the tot-jump).

This also implies that if $\mathsf{tJ}(f) \leq_W \mathsf{TC}_\mathbb{N}$ then $f <_W \mathsf{C}_\mathbb{N}$. To conclude the proof, it is enough to show that if $\mathsf{TC}_\mathbb{N} \leq_W \mathsf{tJ}(f)$ then $\mathsf{C}_\mathbb{N} \leq_W f$.

Assume now that the reduction $\mathsf{TC}_\mathbb{N} \leq_W \mathsf{tJ}(f)$ is witnessed by the functionals Φ, Ψ. Consider the input $0^\mathbb{N}$ for $\mathsf{TC}_\mathbb{N}$. By continuity, there are $k \in \mathbb{N}$ and $\sigma \in \mathbb{N}^k$ such that, for some $m \in \mathbb{N}$, $\Psi(0^k, \sigma)(0) \downarrow= m$. Let $q \in \mathbb{N}^\mathbb{N}$ be an input for $\mathsf{C}_\mathbb{N}$ such that $0^{k\frown}(m+1) \sqsubset q$. Let also $\Phi(q) = (e, i, x)$, which is an input of $\mathsf{tJ}(f)$. By definition of Weihrauch reduction, $\Phi_e(x) \in \mathrm{dom}(f)$ and, for every $y \in f(\Phi_e(x))$, $\Phi_i(x, y) \downarrow\in \mathsf{tJ}(f)(\Phi(q))$ (as otherwise σ would be the initial segment of a solution for $\mathsf{tJ}(f)(\Phi(q))$, hence Ψ would not produce a valid answer for $\mathsf{TC}_\mathbb{N}(q)$). This implies that

$$\{\Psi(q, \Phi_i(x, y)) : y \in f(\Phi_e(x))\} \subseteq \mathsf{TC}_\mathbb{N}(q) = \mathsf{C}_\mathbb{N}(q),$$

where (e, i, x) are uniformly computable from q. Since $\mathsf{C}_\mathbb{N}$ is a fractal (by the same proof as Lemma 4.12 or see [6, Fact 3.2(1)]), $\mathsf{C}_\mathbb{N}$ is Weihrauch equivalent to its restriction to $\{p \in \mathbb{N}^\mathbb{N} : 0^{k\frown}(m+1) \sqsubset p\}$.

In other words, the reduction $\mathsf{TC}_\mathbb{N} \le_\mathrm{W} \mathsf{tJ}(f)$ yields a reduction $\mathsf{C}_\mathbb{N} \le_\mathrm{W} f$, which concludes the proof. $\square$

Notice that the above argument does not necessarily yield a reduction $\mathsf{TC}_\mathbb{N} \le_\mathrm{W} f$, as $\mathsf{tJ}(f)$ is still allowed to go in the "otherwise" case when every $n > 0$ is in $\operatorname{ran}(q)$.

Remark 4.14. Notice that $\mathsf{TC}_\mathbb{N}$ is another witness for Corollary 4.9. Indeed, it is total (trivially), a fractal (by Lemma 4.12), and not in the range of tJ (by Theorem 4.13). Every fractal is join-irreducible (see [7, Prop. 4.11]), so all that is left is to show that $\mathsf{TC}_\mathbb{N}$ is not co-total. If it were, then $\mathsf{TC}_\mathbb{N} \le_\mathrm{W} \mathsf{TC}_\mathbb{N}$ would imply that $\mathsf{TC}_\mathbb{N} \le_\mathrm{W} \mathsf{C}_\mathbb{N}$, but we have already noted that $\mathsf{C}_\mathbb{N} <_\mathrm{W} \mathsf{TC}_\mathbb{N}$.

Theorem 4.15. $\mathsf{tJ}(\mathsf{C}_\mathbb{N}) \equiv_\mathrm{W} \mathsf{T}(\mathsf{C}_\mathbb{N} \times \mathsf{W}_{\Pi^0_2 \to \Pi^0_1})$.

Proof. In light of Theorem 3.3 and of the uniform computability of $\mathsf{W}_{\Pi^0_2 \to \Pi^0_1}$, it suffices to show that $\mathsf{tJ}(\mathsf{C}_\mathbb{N}) \le_\mathrm{W} \mathsf{T}(\mathsf{C}_\mathbb{N} \times \mathsf{W}_{\Pi^0_2 \to \Pi^0_1})$. Given the input (e, i, p) for $\mathsf{tJ}(\mathsf{C}_\mathbb{N})$, let $q \in \mathbb{N}^\mathbb{N}$ be such that q has infinitely many zeroes iff

$$(\forall n)[\ (\forall j)(\Phi_e(p)(j) \neq n+1) \to \Phi_i(p,n)\downarrow\ ].$$

In other words, q has infinitely many zeroes iff $\Phi_i(p,n)\downarrow$ whenever n is not enumerated by $\Phi_e(p)$. Moreover, let $\bar{e}$ be such that $\Phi_{\bar{e}}(p)$ works by simulating $\Phi_e(p)$ and padding the output with zeroes (this ensures that $\Phi_{\bar{e}}$ is total and $\mathsf{C}_\mathbb{N}(\Phi_e(p)) = \mathsf{C}_\mathbb{N}(\Phi_{\bar{e}}(p))$ whenever $\Phi_e(p)\downarrow$).

The forward functional of the reduction $\mathsf{tJ}(\mathsf{C}_\mathbb{N}) \le_\mathrm{W} \mathsf{T}(\mathsf{C}_\mathbb{N} \times \mathsf{W}_{\Pi^0_2 \to \Pi^0_1})$ is then given by the map $(e,i,p) \mapsto (\Phi_{\bar{e}}(p), q)$.

Given $\langle n, t\rangle \in \mathsf{T}(\mathsf{C}_\mathbb{N} \times \mathsf{W}_{\Pi^0_2 \to \Pi^0_1})(\Phi_{\bar{e}}(p), q)$, we uniformly compute a solution for $\mathsf{tJ}(\mathsf{C}_\mathbb{N})(e,i,p)$ as follows: We run $\Phi_i(p,n)$ until we either see that $n+1 \in \operatorname{ran}(\Phi_e(p))$ or we see that $t \notin \mathsf{W}_{\Pi^0_2 \to \Pi^0_1}(q)$ (both conditions are c.e.). If this never happens, we know that n is a valid solution for $\mathsf{C}_\mathbb{N}(\Phi_{\bar{e}}(p))$ and $\Phi_i(p,n)\downarrow$. Otherwise, it means that $\mathsf{tJ}(\mathsf{C}_\mathbb{N})(e,i,p) = \mathbb{N}^\mathbb{N}$, hence we can just continue the output with $0^\mathbb{N}$. $\square$

Unlike $\mathsf{C}_\mathbb{N}$, the fact that $2^\mathbb{N}$ is computably compact implies that $\mathsf{C}_{2^\mathbb{N}} \equiv_\mathrm{W} \mathsf{TC}_{2^\mathbb{N}}$ (see, e.g., [5, Prop. 6.1]). However, a characterization similar to the one for the tot-jump of $\mathsf{C}_\mathbb{N}$ holds for $\mathsf{C}_{2^\mathbb{N}}$.

Theorem 4.16. $\mathsf{tJ}(\mathsf{C}_{2^\mathbb{N}}) \equiv_\mathrm{W} \mathsf{T}(\mathsf{C}_{2^\mathbb{N}} \times \mathsf{W}_{\Pi^0_2 \to \Pi^0_1})$.

Proof. As above, it suffices to show that $\mathsf{tJ}(\mathsf{C}_{2^\mathbb{N}}) \le_\mathrm{W} \mathsf{T}(\mathsf{C}_{2^\mathbb{N}} \times \mathsf{W}_{\Pi^0_2 \to \Pi^0_1})$. Let (e,i,p) be an input for $\mathsf{tJ}(\mathsf{C}_{2^\mathbb{N}})$. Let $S \subseteq 2^{<\mathbb{N}}$ be a tree such that

$$[S] = \{x \in 2^\mathbb{N} \ :\ (\forall n)[\ \Phi_e(p)(\langle x[n]\rangle)\downarrow \to \Phi_e(p)(\langle x[n]\rangle) = 1\ ]\}.$$

Observe that a tree S as above can be uniformly computed from e, p (as the formula defining $[S]$ is Π^0_1 with parameters e,p) and that, if $\Phi_e(p)$ is the characteristic function of a subtree of $2^{<\mathbb{N}}$, then $[S] = [\Phi_e(p)]$. Moreover, S is well-defined, even if $\Phi_e(p)$ does not converge or it is not the characteristic function of a tree. To compute an input $q \in \mathbb{N}^\mathbb{N}$ for $\mathsf{W}_{\Pi^0_2 \to \Pi^0_1}$, we first observe that the set

$$\begin{aligned} A_{e,i,p} := \{y \in 2^\mathbb{N} \ :\ & \Phi_e(p)\uparrow \\ & \lor\ (\exists \tau \in 2^{<\omega})(\exists \sigma \sqsubseteq \tau)[\ \Phi_e(p)(\langle\tau\rangle) = 1 \land \Phi_e(p)(\langle\sigma\rangle) = 0\ ] \\ & \lor\ [\ (\forall n)[\ \Phi_e(p)(\langle y[n]\rangle)\downarrow \to \Phi_e(p)(\langle y[n]\rangle) = 1\ ] \\ & \qquad \land (\exists n)\ \Phi_i(p,y)(n)\uparrow\ ]\} \end{aligned}$$

is defined by a Σ^0_2-formula with parameters e,i,p. Intuitively, the first two rows of the definition capture "$\Phi_e(p)$ is not the characteristic function of a subtree of $2^{<\mathbb{N}}$", while the last two rows can be read as "$y \in [\Phi_e(p)]$ and $\Phi_i(p,y)\uparrow$". The third row could have been written as "$(\forall n)(\Phi_e(p)(\langle y[n]\rangle) =$

1)". This is (in general) a Π^0_2 statement, but it can be equivalently rewritten in a Π^0_1-way in light of the first row.

Since the projection of a Σ^0_2-set over a computably compact set is Σ^0_2 (see, e.g., [15, Lemma 3.9]) and that checking if a subtree of $2^{<\mathbb{N}}$ is ill-founded is a Π^0_1-problem, this implies that the statement

$$\text{"}\Phi_e(p) \text{ is an ill-founded subtree of } 2^{<\mathbb{N}} \text{ and } \neg(\exists y \in 2^{\mathbb{N}})\ y \in A_{e,i,p}.\text{"}$$

is Π^0_2. Therefore, we can uniformly compute a string $q \in \mathbb{N}^{\mathbb{N}}$ such that q has infinitely many zeroes iff the above formula holds.

The forward functional of the reduction is the map that sends (e,i,p) to (S,q). The backward functional Ψ is the map that works as follows: given e,i,p and a solution $\langle z,t\rangle$ for $\mathsf{T}(\mathsf{C}_{2^{\mathbb{N}}} \times \mathsf{W}_{\Pi^0_2\to\Pi^0_1})(S,q)$, Ψ outputs $\Phi_i(p,z)$ as long as t appears to belong to $\mathsf{W}_{\Pi^0_2\to\Pi^0_1}(q)$. As soon as an error is found, Ψ extends the partial output with $0^{\mathbb{N}}$.

It is immediate from the definition of q that an error is found only if $\Phi_e(p)$ does not define an ill-founded subtree of $2^{<\mathbb{N}}$ or if there is $y \in [\Phi_e(p)]$ such that $\Phi_i(p,y)\uparrow$. In this case, $\mathsf{tJ}(\mathsf{C}_{2^{\mathbb{N}}})(e,i,p) = \mathbb{N}^{\mathbb{N}}$, hence the computed string is a valid solution. On the other hand, if an error is never found, then $z \in [S] = [\Phi_e(p)]$ and $\Psi((e,i,p),\langle z,t\rangle) = \Phi_i(p,z) \in \mathsf{tJ}(\mathsf{C}_{2^{\mathbb{N}}})(e,i,p)$, which concludes the proof. □

We conclude this section by showing that, as a consequence of Corollary 3.13, the tot-jump of each of the problems lim, $\mathsf{lim}^{[n]}$, $\mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$, and $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ is the respective total continuation.

Theorem 4.17. *For every $n \geq 1$, $\mathsf{tJ}(\mathsf{lim}^{[n]}) \equiv_{\mathrm{W}} \mathsf{T}(\mathsf{lim}^{[n]})$.*

Proof. Let us first prove the theorem for $n = 1$. Let $\mathcal{P}$ be the family of all problems f such that there is a total computable Φ such that, for every $e,i \in \mathbb{N}$ and every $p \in \mathbb{N}^{\mathbb{N}}$, if $\Phi_e(p) \in \mathrm{dom}(f)$ and $(\forall q \in f\Phi_e(p))\ \Phi_i(p,q)\downarrow$, then

$$\Phi(e,i,p) \in \mathrm{dom}(f) \quad \text{and} \quad f\Phi(e,i,p) \subseteq \{\Phi_i(p,q) : q \in f\Phi_e(p)\}.$$

By Corollary 3.13 (with Ψ the projection on the second coordinate), for every $f \in \mathcal{P}$, $\mathsf{tJ}(f) \equiv_{\mathrm{W}} \mathsf{T}f$, so it is enough to show that $\mathsf{lim} \in \mathcal{P}$. Let Φ be the map that, upon input e,i,p, computes the sequence $(x_n)_{n\in\mathbb{N}}$ defined as follows: We read the output of $\Phi_e(p)$ as (an initial segment of) the join of countably many strings $\langle q_0, q_1, \ldots\rangle$. Formally, if σ_n is the string produced by $\Phi_e(p)$ in n steps, we define $k_n := \max\{j : \langle j,0\rangle < |\sigma_n|\}$ and, for every $j \leq k_n$, define $\tau_{n,j}(m) := \sigma_n(\langle j,m\rangle)$ for all m with $\langle j,m\rangle < |\sigma_n|$. With this definition, for every $j \leq k_n$, $\tau_{n,j} \sqsubseteq q_j$. We can uniformly compute the string $y \in \mathbb{N}^{\mathbb{N}}$ defined as $y_n(m) := \tau_{n,j}(m)$ for the largest j such that $\langle j,m\rangle < |\sigma_n|$, and $y_n(m) := 0$ if no such j exists. Observe that, if $(q_n)_{n\in\mathbb{N}}$ converges, then so does $(y_n)_{n\in\mathbb{N}}$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} q_n$. We define

$$x_n := \Phi_{i,n}(p,y_n)^\frown 0^{\mathbb{N}},$$

where $\Phi_{i,n}(p,t)$ denotes the output produced by $\Phi_i(p,t)$ in n steps. Clearly, the map Φ is total and computable.

Assume that $\Phi_e(p)$ produces the sequence $(q_j)_{j\in\mathbb{N}} \in \mathrm{dom}(\mathsf{lim})$ and that $\Phi_i(p,q)\downarrow$, where $q := \lim_{n\to\infty} q_n$. By the continuity of Φ_i, we immediately obtain

$$\lim_{n\to\infty} x_n = \Phi_i\left(p, \lim_{n\to\infty} y_n\right) = \Phi_i\left(p, \lim_{n\to\infty} q_n\right) = \Phi_i(p, \mathsf{lim}\ \Phi_e(p)),$$

which shows that $\mathsf{lim} \in \mathcal{P}$.

The general case follows by induction on n, as the class $\mathcal{P}$ is closed under composition and $\mathsf{lim} \circ \mathsf{lim} \equiv_{\mathrm{sW}} \mathsf{lim} * \mathsf{lim}$ (by [6, Example 4.4(1)] and the fact that lim is a cylinder). To prove that $\mathcal{P}$ is closed under composition, we can use $f \in \mathcal{P}$ first and $g \in \mathcal{P}$ later, to conclude that $fg \in \mathcal{P}$.

More precisely, let $f,g \in \mathcal{P}$ and let F and G be the total computable witnesses. Fix $e,i \in \mathbb{N}$ and $p \in \mathbb{N}^{\mathbb{N}}$ such that $\Phi_e(p) \in \mathrm{dom}(fg)$ (i.e. $\Phi_e(p) \in \mathrm{dom}(g)$ and $g(\Phi_e(p)) \subseteq \mathrm{dom}(f)$) and $(\forall q \in$

$fg\Phi_e(p))\ \Phi_i(p,q)\downarrow$. Let $\bar e, \bar\imath \in \mathbb{N}$ be such that $\Phi_{\bar e}(\langle a,b\rangle) = b$ and $\Phi_{\bar\imath}(\langle a,b\rangle, c) = \Phi_i(a,c)$. Observe that

$$\{\Phi_i(p,q) : q \in fg\,\Phi_e(p)\} = \bigcup_{t\in g\Phi_e(p)} \{\Phi_{\bar\imath}(\langle p,t\rangle, q) : q \in f\Phi_{\bar e}(\langle p,t\rangle)\}.$$

By the choice of F, for every $t \in g\,\Phi_e(p) \subseteq \operatorname{dom}(f)$ (which implies that $\Phi_{\bar e}(\langle p,t\rangle) \in \operatorname{dom}(f)$ and $(\forall q \in f\Phi_{\bar e}(\langle p,t\rangle))\ \Phi_{\bar\imath}(\langle p,t\rangle,q)\downarrow$),

$$\begin{aligned}\emptyset \neq fF(\bar e, \bar\imath, \langle p,t\rangle) &\subseteq \{\Phi_{\bar\imath}(\langle p,t\rangle, q) : q \in f\Phi_{\bar e}(\langle p,t\rangle)\}\\ &= \{\Phi_i(p,q) : q \in f(t)\}.\end{aligned}$$

In other words, letting j be such that $\Phi_j(p,t) := F(\bar e, \bar\imath, \langle p,t\rangle)$, we obtain

$$f(\{\Phi_j(p,t) : t \in g\Phi_e(p)\}) \subseteq \{\Phi_i(p,q) : q \in fg\Phi_e(p)\}.$$

To conclude the proof, note that by the choice of G, we have

$$\emptyset \neq g\,G(e,j,p) \subseteq \{\Phi_j(p,t) : t \in g\,\Phi_e(p)\}.$$

In particular, the map $(e,i,p) \mapsto G(e,j,p)$ witnesses the fact that $f \circ g \in \mathcal{P}$. ☐

Theorem 4.18. $\mathsf{tJ}(\mathsf{UC}_{\mathbb{N}^\mathbb{N}}) \equiv_{\mathrm{W}} \mathsf{TUC}_{\mathbb{N}^\mathbb{N}}$ *and* $\mathsf{tJ}(\mathsf{C}_{\mathbb{N}^\mathbb{N}}) \equiv_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$.

Proof. We only show that $\mathsf{tJ}(\mathsf{C}_{\mathbb{N}^\mathbb{N}}) \equiv_{\mathrm{W}} \mathsf{TC}_{\mathbb{N}^\mathbb{N}}$, as (with minor modifications) the same proof works for $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$.

As in the previous proof, we show that $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ satisfies the assumptions of Corollary 3.13, namely, there are two total computable functionals Φ and Ψ such that, for every $e,i \in \mathbb{N}$ and every $p \in \mathbb{N}^\mathbb{N}$, if $\Phi_e(p) \in \operatorname{dom}(\mathsf{C}_{\mathbb{N}^\mathbb{N}})$ and $(\forall x \in [\Phi_e(p)])\ \Phi_i(p,x)\downarrow$, then $\Phi(e,i,p) \in \operatorname{dom}(\mathsf{C}_{\mathbb{N}^\mathbb{N}})$ and $\Psi((e,i,p),[\Phi(e,i,p)]) \subseteq \{\Phi_i(p,x) : x \in [\Phi_e(p)]\}$. Let Φ be the map that sends (e,i,p) to (the characteristic function of) a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ such that

$$[T] = \{\langle x,y\rangle : x \in [\Phi_e(p)] \text{ and } \Phi_i(p,x) \sqsubseteq y\}.$$

As the set above is uniformly Π^0_1 in (e,i,p), a tree T as above can be uniformly computed from (e,i,p). Let $\Psi := ((e,i,p),\langle x,y\rangle) \mapsto y$.

To conclude the proof, notice that if $\Phi_e(p) \in \operatorname{dom}(\mathsf{C}_{\mathbb{N}^\mathbb{N}})$ and $\Phi_i(p,x)\downarrow$ for every $x \in [\Phi_e(p)]$, then T is ill-founded and, for every $\langle x,y\rangle \in [T]$, $\Psi((e,i,p),\langle x,y\rangle) = y \in \{\Phi_i(p,q) : q \in f\Phi_e(p)\}$. ☐

5 Remarks on abstract jump operators

In this paper, we introduce and study a natural jump operator on the Weihrauch lattice, a natural partial order. The remarks in this section address a much more abstract question: under what conditions do arbitrary partial orders admit a jump operator in the sense of Definition 1.1? We show that, without additional structure, admitting a jump is not a first-order property.

We mentioned in the introduction that the existence of an abstract jump operator is easy to see in any upper semilattice without maximum (using the Axiom of Choice). This result can be extended to countable upper directed partial orders (i.e., partial orders such that any finite number of elements have a common upper bound) without maximum. Indeed, any such partial order $(P,\leq)$ admits a strictly increasing cofinal chain $(q_n)_{n\in\mathbb{N}}$. This can be easily shown by letting $(p_n)_{n\in\mathbb{N}}$ be an enumeration of P and defining $q_0 := p_0$ and $q_{n+1} := p_m$ where m is least such that $q_n < p_m$. The existence of a jump operator follows from the fact that every partial order with a strictly increasing cofinal chain admits a jump operator (it is enough to map every element of the poset to the first element in the sequence that is strictly above it).

We mention that the same strategy cannot be used to obtain the existence of a jump operator on the Weihrauch degrees. Indeed, the third and fifth author will show in an upcoming paper that no chain in the Weihrauch degrees can be cofinal.

Remark 5.1. There is an upper directed partial order with no maximum and size $\aleph_1$ that does not admit a jump operator, which implies that the above observation cannot be generalized to larger partial orders. To show this, let $(Q, \leq_Q)$ be ω_1 ordered as an antichain (each element is only comparable with itself). Let also $(R, \leq_R)$ be the partial order of all non-empty finite subsets of ω_1 ordered by inclusion. Let P be the union of Q and R, where $\leq_P$ is defined as the transitive closure of

$$\leq_Q \cup \leq_R \cup \{(\alpha, \{\gamma\}) \,:\, \alpha \leq \gamma\}.$$

Assume towards a contradiction that P admits a jump operator j. Observe now that, for a fixed α, $j(\alpha)$ is a non-empty finite set of ordinals such that at least one is $\geq \alpha$. Let $\beta \in j(\alpha)$ be such that $\alpha \leq \beta$. In particular, for every $\gamma \geq \alpha$, we have $\alpha <_P \{\gamma\}$ and hence $j(\alpha) \leq_P j(\{\gamma\})$, i.e., $\beta \in j(\alpha) \subseteq j(\{\gamma\})$.

Define an increasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ of countable ordinals as follows: $\alpha_0 := 0$ and, for every n, $\alpha_{n+1} := \max j(\alpha_n) + 1$. For every n, let $\beta_n \in j(\alpha_n)$ be such that $\alpha_n \leq \beta_n$. In particular, we obtain

$$\alpha_0 \leq \beta_0 < \alpha_1 \leq \beta_1 \ldots,$$

which implies that all the β_n are distinct.

By the above observation, for every $\gamma \geq \sup_{n\in\mathbb{N}} \alpha_n$ and every n, $\beta_n \in j(\{\gamma\})$, hence $j(\{\gamma\})$ is infinite, which is a contradiction with $j(\{\gamma\}) \in P$.

Finally, we consider the case of arbitrary countable partial orders (without a maximum). We show that the existence of a jump operator is as complicated as its naive definition suggests: it is Σ^1_1-complete.

Theorem 5.2. *There is a computable map $F\colon \mathrm{LO} \to \mathrm{PO}_0$ from the family of countable linear orders to the family of countable partial orders without maximum (where we represent a linear/partial order using its characteristic function) such that*

$$L \text{ is ill-founded } \iff F(L) \text{ admits a jump operator.}$$

Proof. To show this, let $(X, \leq_X)$ be the partial order defined as $X := 2 \times \omega$ and $(i,n) \leq_X (j,m)$ iff $i = j$ and $n \leq m$. Intuitively, X consists of two incomparable copies of ω. We define $F(L) := 1+\sum_{x\in L^*} X$, where L^* is L with the order reversed. We order $F(L)$ as expected: in particular, every element of a given term is less than every element of the later terms. For the sake of readability, we write $X_x := \{(i,n)_x \,:\, i \in 2 \text{ and } n \in \omega\}$ for the x-th copy of X in $F(L)$. Let also $\bot$ be the minimum of $F(L)$.

Assume that L is ill-founded and let $(x_k)_{k\in\mathbb{N}}$ be an infinite descending sequence (i.e., an infinite ascending sequence in L^*). We can define a jump function j on $F(L)$ as follows:

- $j(\bot) := (0,0)_{x_0}$;
- for every $(i,n)_x$ such that $x <_{L^*} x_0$, let $j((i,n)_x) := (0,0)_{x_0}$;
- for every $(i,n)_x$ such that $x_k \leq_{L^*} x <_{L^*} x_{k+1}$, let $j((i,n)_x) := (0,0)_{x_{k+1}}$;
- for every $(i,n)_x$ such that $x_k <_{L^*} x$ for every k, let $j((i,n)_x) := (i,n+1)_x$.

It is immediate from the definition that j is strictly increasing. Proving that j is weakly monotone is also easy.

Conversely, if $F(L)$ admits a jump operator j, then we can define an ascending sequence in L^* (i.e., a descending sequence in L witnessing the fact that L is ill-founded) as follows: We let x_0 be such that $j(\bot) = (i_0, n_0)_{x_0}$ for some i_0, n_0. To define x_1, observe that, by the weak monotonicity of j, we have $j((1-i_0,n_0)_{x_0}) \geq_{F(L)} j(\bot) = (i_0,n_0)_{x_0}$. This, in combination with $j((1-i_0,n_0)_{x_0}) >_{F(L)} (1-i_0,n_0)_{x_0}$, implies that $j((1-i_0,n_0)_{x_0}) = (i_1,n_1)_{x_1}$ for some i_1, n_1 with $x_1 >_{L^*} x_0$. We can iterate this argument to obtain the desired strictly increasing sequence in L^*. ☐

This shows that the set of partial orders without maximum admitting a jump operator is a non-Borel Σ^1_1-subset of the set of countable partial orders. In particular, the existence of a jump operator for countable partial orders without maximum cannot be characterized by an arithmetic formula. It would be interesting to obtain a similar result for non-countable partial orders. This would require a detour into the realm of generalized descriptive set theory, and it is possible that additional set-theoretic axioms (for example, on the size of the continuum) would be needed.

6 Open Questions

We mentioned in Section 3 that the tot-jump of f can be defined via a Δ^1_2-formula using f as a parameter. Moreover, we showed that no jump operator on computational problems can be defined using a $\Sigma^{1,f}_1$-formula (Remark 3.6). This leaves a gap, and therefore it is natural to ask the following question:

Open Question 6.1. Is there a $\Pi^{1,f}_1$-definition for the tot-jump? More generally, is it possible to define a jump operator on the Weihrauch degrees using a $\Pi^{1,f}_1$-formula?

Despite our efforts, we could not obtain a satisfactory characterization for $\mathrm{ran}(\mathsf{tJ})$. A better characterization is especially desirable in light of Theorem 3.7, as that would give us a description of a sublattice of the Weihrauch degrees which is isomorphic to the full structure.

Open Question 6.2. Find a "natural" characterization for $\mathrm{ran}(\mathsf{tJ})$. Is tJ definable in the Weihrauch degrees?

While the "natural" condition is of course vague and informal, a satisfactory answer would allow us to promptly tell whether a given f is in the range of the tot-jump. To this end, a powerful result is provided by Theorem 3.23, and especially by Corollary 3.24. As proved, closure under product with DIS is a sufficient condition for co-totality, and we showed that a co-total problem f can be Weihrauch-reducible to $\mathsf{tJ}(g)$ only in the trivial case $f \leq_{\mathrm{W}} g$. This raises the following question:

Open Question 6.3. Does closure under product with DIS characterize co-totality?

We also showed that the range of tJ is a (proper) subset of the join-irreducible degrees (Theorem 3.15 and Corollary 4.9). This allowed us to show that the tot-jump of f and g distributes over the join of f and g only when f and g are comparable. On the other hand, we do not know whether the same holds for the meet: While Proposition 3.19 shows that, in general, the jump does not distribute over the meet, we do not know whether this is always the case when f and g are not comparable.

Open Question 6.4. Are there f and g such that $f \mid_{\mathrm{W}} g$ but $\mathsf{tJ}(f) \sqcap \mathsf{tJ}(g) \equiv_{\mathrm{W}} \mathsf{tJ}(f \sqcap g)$?

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