COMPARING NOTIONS OF RANDOMNESS

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Abstract. It is an open problem in the area of effective (algorithmic) randomness whether Kolmogorov-Loveland randomness coincides with Martin-Löf randomness. Joe Miller and André Nies suggested some variations of Kolmogorov-Loveland randomness to approach this problem and to provide a partial solution. We show that their proposed notion of injective randomness is still weaker than Martin-Löf randomness. Since in its proof some of the ideas we use are clearer, we also show the weaker theorem that permutation randomness is weaker than Martin-Löf randomness.

1. Introduction

There are currently many competing notions of randomness, based on different intuitions of randomness. Some are based on the idea that no random real should belong to certain measure zero sets, others on the frequency interpretation of probability, and yet others on the notion of a fair betting game. Some of these notions are known to be equivalent, others are known to be not equivalent, and for yet others, it is not known whether they are equivalent. This paper is a contribution to this classification.

The notions of randomness we will be concerned with in this paper are all based on the notion of a martingale. A martingale is a formalization of the idea of a fair betting game; while betting on the outcome of a coin flip, the game would be fair if the expected value of your capital after the game is the same as before the game. That means that your win on heads is the same as your loss on tails. A martingale then describes a game consisting of simple games like that, repeated infinitely often. Part of the intuition for using martingales to define

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randomness is that if the real is random, then you should not be able to predict the bits; this means that in the game against the real (considered as infinitely many games against bits), your capital will not be unbounded.

For the different notions we study, the main differences lie in the effectiveness of the martingale, the order in which the martingale bets on bits, and the speed by which the martingale is required to succeed. One of the big open questions in the area is whether the notions of Martin-Löf randomness (a notion of monotonic randomness with a very weakly effective martingale) and Kolmogorov-Loveland randomness (a notion of nonmonotonic randomness with a somewhat more effective, but not monotonic, martingale) coincide. Joe Miller and André Nies [MN06] suggested a weakening of Kolmogorov-Loveland randomness as a way to approach this question. The weakening involves limiting the freedom of the nonmonotonic martingale in choosing the next bit to bet on.

We show that their notion of injective randomness does not coincide with Martin-Löf randomness.

We start in the next subsection with the definitions of the different types of martingales and some background. Then, in Section 2, we prove that permutation randomness does not coincide with Martin-Löf randomness. This is a weaker theorem than the theorem we show in Section 3. However, the proof for the permutation random case introduces many of the ideas in a simpler context.

1.1. **The Definitions and Background.** The space we are working in is the Cantor space 2^ω, the space of infinite sequences of zeros and ones, with the topology induced by the sets $[\sigma] = \{A \in 2^\omega \mid \sigma \subseteq A\}$ for any $\sigma \in 2^{<\omega} = \bigcup_{n\in\omega} 2^n$, that is, σ a finite sequence of zeros and ones. If $\sigma \in 2^{<\omega}$ then $\sigma \in 2^n$ for some n, and we write $|\sigma|$ for this n. Note that when convenient we will use the convention that $n = \{0, \dots, n-1\}$. For a function σ, we write $\mathrm{dom}(\sigma)$ for the domain of σ (which for $\sigma \in 2^{<\omega}$ is equal to $|\sigma|$), and $\mathrm{ran}(\sigma)$ for the range of σ. We also identify both functions and sequences with their graphs; e.g., for $\sigma \in 2^{<\omega}$, we have $\sigma = \{(n, \sigma(n)) \mid n \in \mathrm{dom}(\sigma)\}$. For $A \in 2^\omega$ and $n \in \omega$, we write $A \upharpoonright n$ for $\{(l, A(l)) \mid l < n\}$ (and analogously for $\sigma \upharpoonright n$). For $\sigma, \tau \in 2^{<\omega}$, we write $\sigma \preceq \tau$ to mean that σ is an initial segment (not necessarily proper) of τ. We write $\sigma \prec \tau$ to mean that σ is a proper initial segment of τ, and $\sigma \perp \tau$ to mean that σ and τ are incomparable (i.e., that neither $\sigma \preceq \tau$ nor $\tau \prec \sigma$ hold). For $\sigma \in 2^n$ we write $\sigma 1$ ($\sigma 0$, resp.) for the finite sequence in 2^{n+1} that agrees with σ on n and maps n to 1 (0, resp.); i.e., the sequence σ with 1 (0, resp.) concatenated. For $\Sigma \subseteq 2^{<\omega}$, we write $[\Sigma]$ for the set $\{A \in 2^\omega \mid \forall n \in \omega (A \upharpoonright n \in \Sigma)\}$. We will also write, with some abuse of notation, $[\sigma] \cap 2^{<\omega}$ for the set $\{\tau \in 2^{<\omega} \mid \sigma \preceq \tau\}$ and $[\sigma] \cap 2^k$ for the set $\{\tau \in 2^k \mid \sigma \preceq \tau\}$.

We will use the same notation for finite partial functions $\sigma : \omega \rightharpoonup 2$. If f is a function, and S a subset of its domain we write $f[S]$ for $\{f(x) \mid x \in S\}$.

A *Martin-Löf test* is a uniformly computably enumerable sequence $\langle \Sigma_n \subseteq 2^{<\omega} \mid n \in \omega \rangle$ such that $\mu([\Sigma_n]) \leq 2^{-n}$, where μ is the Lebesgue measure. A Martin-Löf test *succeeds on* a real $A \in 2^\omega$ iff $A \in \bigcap_{n\in\omega}[\Sigma_n]$. The set of reals on which a given Martin-Löf test succeeds is a null set; a Martin-Löf test is a particular notion of an effective null set. A real $A \in 2^\omega$ is *Martin-Löf random* iff no Martin-Löf test succeeds on it.

The notion of Martin-Löf randomness can also be explained using martingales. We will define our martingales in different ways. What we give in the definitions will, however, always be enough so that given a real $A \in 2^\omega$ we can compute a function $d^A : \omega \to \mathbb{R}_0^+$ (where $\mathbb{R}_0^+$ is the set of nonnegative reals). Here $d^A(n)$ gives the player's capital after n bets. We will consistently use d (with superscripts and subscripts) to denote martingales described in this way (by giving the capital after a given number of bets). The martingale formulation of Martin-Löf randomness will show a closer connection to the other notions of randomness we will define and use.

We start by describing a *martingale* as a function $f : 2^{<\omega} \to \mathbb{R}_0^+$ such that for all $\sigma \in 2^{<\omega}$ we have $f(\sigma) = \frac{f(\sigma 0)+f(\sigma 1)}{2}$. (Later on, when f is allowed to be partial, we assume this equality only when all three terms are defined.) A martingale *succeeds on* a real $A \in 2^\omega$ iff $\limsup_{n\to\infty} f(A \upharpoonright n) = \infty$. (Again, when later f is allowed to be partial, we then require for such A that $f(A \upharpoonright n)$ be defined for all n.)

To give an obviously equivalent definition more in line with the definitions given later, we can define the *capital function* $d^A : \omega \to \mathbb{R}_0^+$ by setting $d^A(n) = f(A \upharpoonright n)$, the capital after n bets on A. Then the martingale *succeeds* iff $\limsup_{n\to\infty} d^A(n) = \infty$ (this, in particular, means that $d^A(n)\downarrow$ (is defined) for all n).

For a simple but useful example of a martingale, let us define what an elementary martingale is. Given $\sigma \in 2^{<\omega}$, the *elementary martingale* $E_\sigma : 2^{<\omega} \to \mathbb{R}_0^+$ is defined as follows:

$$E_\sigma(\nu) = \begin{cases} 2^{|\sigma|} & \text{if } \sigma \preceq \nu, \\ 2^{|\nu|} & \text{if } \nu \prec \sigma, \\ 0 & \text{otherwise } (\nu \perp \sigma). \end{cases}$$

This martingale bets all its capital along σ, and bets even outside of that. Note that given $\sigma_0, \sigma_1 \in 2^{<\omega}$ the sum $E_{\sigma_0} + E_{\sigma_1}$ is also a martingale.

A martingale g is *effective* if there exists a computable function $\hat{g} : \omega \times 2^{<\omega} \to \mathbb{Q}_0^+$ (where $\mathbb{Q}_0^+$ is the set of nonnegative rationals) which is nondecreasing in the first coordinate and such that $\lim_{n\to\infty} \hat{g}(n, \sigma) = g(\sigma)$.

Theorem 1 (Schnorr [Sch71])**.** *A real is Martin-Löf random iff no effective martingale succeeds on it.*

A martingale is *(partial) computable* iff it is a (partial) computable function $f : 2^{<\omega} \to \mathbb{Q}_0^+$. A real is *(partial) computably random* iff no (partial) computable martingale succeeds on it. (So, in particular, a real is partial computable if any partial computable martingale is either bounded or partial along this real.) Clearly, any Martin-Löf random is partial computably random, and any partial computably random is computably random. However, these three notions do not coincide:

Theorem 2 (Ambos-Spies [AS98])**.** *There are reals which are computably random but not partial computably random.*

Theorem 3 (Muchnik [MSU98], Schnorr [Sch73])**.** *There are reals which are partial computably random but not Martin-Löf random.*

Theorem 3 is a combination of Theorem 9.5 in Muchnik [MSU98] and Theorem 3 in Schnorr [Sch73]. The proofs of Theorems 1, 2, and 3 can also be found in [DH] or [Nie09]. Note Wang [Wan99] proves a closely related result, he proves the separation between Schnorr and computable randomness.

All the notions of randomness above have in common that the martingale bets on all the bits of the real in order, i.e., they bet monotonically. A nonmonotonic betting strategy has the flexibility to choose which bits of the real to bet on (for instance, it might first bet on bit number 5 and then, depending on the outcome, bet on bit number 2 or 7, respectively). The exact definition (as taken essentially from [MMN$^+$06]) is as follows.

A *scan rule* is a partial function $s : (\omega \times 2)^{<\omega} \to \omega$ such that for all $w \in (\omega \times 2)^{<\omega}$, we have that $s(w) \notin \mathrm{dom}(w)$. Given the history of play—a sequence of bit locations and their values—the scan rule selects the next bit to bet on. The requirement $s(w) \notin \mathrm{dom}(w)$ corresponds to not being allowed to bet on a bit you have already seen.

Given a real $A \in 2^\omega$, the scan rule selects a real $\tilde{A}$; however, the scan rule, and a betting strategy using it, use the full history of the play.

This means that the object of interest is $\bar{A} : \omega \to (\omega \times 2)^{<\omega}$ defined as follows:

$$\bar{A}(0) = \big(s(\emptyset), A(s(\emptyset))\big),$$
$$\bar{A}(n) = \big(s(\bar{A} \upharpoonright n), A(s(\bar{A} \upharpoonright n))\big).$$

From this, the real played, $\tilde{A}$, can be defined by $\tilde{A}(n) = \pi_1(\bar{A}(n))$. Also, if $\tau \in 2^{<\omega}$, then we can in the same way define $\bar{\tau} \upharpoonright m$ for each m such that for all $n < m$, $s(\bar{\tau} \upharpoonright n) \in |\tau|$.

A *stake function* is a partial function $q : (\omega \times 2)^{<\omega} \to [0, 2]$. The stake function gives the bet towards the next bit selected being 0. A *nonmonotonic betting strategy* is a triple (λ, s, q) where $\lambda \in \mathbb{R}^+$ is the initial capital, s is a scan rule, and q a stake function. Define the capital after play n recursively by

$$d^A_{(\lambda,s,q)}(0) = \lambda,$$
$$d^A_{(\lambda,s,q)}(n+1) = \begin{cases} d^A_{(\lambda,s,q)}(n) q(\bar{A} \upharpoonright n) & \text{if } \tilde{A}(n) = 0; \\ d^A_{(\lambda,s,q)}(n)(2 - q(\bar{A} \upharpoonright n)) & \text{if } \tilde{A}(n) = 1. \end{cases}$$

We also use $d^\sigma_{(\lambda,s,q)}(n)$ when $\sigma \in 2^{<\omega}$. This is defined as follows: $d^\sigma_{(\lambda,s,q)}(n) = d^A_{(\lambda,s,q)}(n)$ for $A \in [\sigma]$ if all the bits used are from σ, otherwise $d^\sigma_{(\lambda,s,q)}(n)$ is undefined.

The capital function now takes the following form: for all $A \in 2^\omega$, if we set $\sigma = (\bar{A})[\{0, \ldots, n-1\}]$ then

$$d^\sigma_{(\lambda,s,q)}(n) = \frac{d^{\sigma \cup \{(s(\sigma),0)\}}_{(\lambda,s,q)}(n+1) + d^{\sigma \cup \{(s(\sigma),1)\}}_{(\lambda,s,q)}(n+1)}{2}.$$

Note that obtaining σ from a real here is just a trick to not have to define the set of σ that can arise during a run of the martingale. We will use the term *nonmonotonic martingale* to refer to a capital function obtained from a nonmonotonic betting strategy. Sometimes, exclusively for emphasis, we will use the term *monotonic martingale* for martingale.

We say the betting strategy (λ, s, q) *succeeds* on A iff

$$\limsup_{n\to\infty} d^A_{(\lambda,s,q)}(n) = \infty.$$

A *partial computable nonmonotonic betting strategy* is a nonmonotonic betting strategy (λ, s, q) where s and q are partial computable. Then a real is *Kolmogorov-Loveland random* if no partial computable nonmonotonic betting strategy succeeds on it. (Here, we may assume without loss of generality that all reals involved in any partial computable betting strategy, that is λ and the outputs of q, are actually rational. This assumption makes the notion simpler.)

The following theorem about these notions is well known.

Theorem 4 (Muchnik [MSU98]). *Every Martin-Löf random real is Kolmogorov-Loveland random.*

However, the following question is a major open question in the area of randomness. It was first posed in Muchnik et al. [MSU98, Question 8.11] (the wording is different there; chaotic is the same as ML-random, and unpredictable is the same as KL-random). It is also the last remaining open question from Ambos-Spies and Kučera [ASK00, Open Problem 2.9] (the wording is different there, too; nonmonotonic computable random is the same as Kolmogorov-Loveland random, and Σ^0_1-random is the same as Martin-Löf random).

Question 5. *Do the notions of Martin-Löf randomness and Kolmogorov-Loveland randomness coincide?*

In Miller and Nies [MN06], some weakenings of Kolmogorov-Loveland randomness are suggested as a way of approaching this question. They define restrictions of nonmonotonic betting strategies by how the sequence of bits bet on is generated.

Let $h : \omega \to \omega$ be an injection. Then we can bet on bit $h(n)$ in the n^{th} round of betting: a betting strategy that uses h in the selection of bits is a betting strategy (λ, s, q) with $s(\sigma) = h(|\sigma|)$ for all $\sigma \in 2^{<\omega}$. We will write (λ, h, q) for the betting strategy in case s is computed from h in this fashion. (Thus the selection of bits no longer depends on the values of the previous bits bet on.)

Miller and Nies then use this to define several notions of randomness (where q is always a partial computable stake function): A real is *permutation random* if no partial betting strategy succeeds where h is any computable permutation of ω; and *injective random* iff no partial betting strategy succeeds where h is any computable injection. Since a betting strategy using an h that is not total does not succeed on any real, these notions stay the same if we only require h to be partial. These notions, however, would change if we required the stake function to be total (in fact, total permutation random (where both the permutation and the stake function are required to be total) is the same as computably random).

It is not clear what a partial permutation is; however, in this paper we take that to be a finite injection. With this interpretation, we have an effective enumeration of all *partial computable permutation* *betting strategies* (partial computable nonmonotonic betting strategies where the selection function is obtained from a partial computable permutation) as follows: first enumerate all triples (λ, h, q) of a rational number, a partial computable function, and partial computable stake functions, $\langle(\lambda_i, h_i, q_i) \mid i \in \omega\rangle$. Then, given a partial computable injection $f : \omega \to \omega$, define

$$\hat{f}(n) = \begin{cases} f(n) & \exists k\big(f \restriction k \text{ total injective} \wedge \{0, \ldots, n\} \subseteq \operatorname{ran}(f \restriction k)\big), \\ \uparrow & \text{otherwise.} \end{cases}$$

Note that if f is partial computable, then so is $\hat{f}$. Now $\langle(\lambda_i, \hat{h}_i, q_i) \mid i \in \omega\rangle$ is an enumeration of all partial permutation betting strategies.

To get an effective enumeration of all *partial computable injective betting strategies* (partial computable nonmonotonic betting strategies where the selection function is obtained from a partial computable injection), follow the strategy from the previous paragraph using $\check{f}$ instead of $\hat{f}$, where

$$\check{f}(n) = \begin{cases} f(n) & \exists k\big(f \restriction k \text{ total injective}\big), \\ \uparrow & \text{otherwise.} \end{cases}$$

Since it will always be enough to ensure that no martingale with initial capital 1 succeeds, we will only deal with such martingale and write (h, q) for $(1, h, q)$.

It is not hard to see that Kolmogorov-Loveland randomness implies injective randomness, which in turn implies permutation randomness. Miller and Nies now ask whether one can at least separate the latter two notions from Martin-Löf randomness. In this paper, we show that injective randomness can be separated from Martin-Löf randomness:

Theorem 6. *There is a real $A \in 2^\omega$ which is injective random but not Martin-Löf random.*

To introduce many of the ideas we will use in an easier context, we will first prove the following weaker theorem. We believe that the three main ideas in that proof (expected martingale, monotonizing, and the method of dealing with partiality) are also of independent interest.

Theorem 7. *There is a real $A \in 2^\omega$ which is permutation random but not Martin-Löf random.*

Note that the idea of an expected martingale, which is basic to all our considerations, was already used by Buhrman et al. [BvMR$^+$00].

We also observe in passing the following

Remark 8. The real A from Theorem 6 can be made of effective Hausdorff dimension 0.

2. The Proof for Permutation Randomness

We need to construct a real $A \in 2^\omega$ and a computable function $\tilde{g} : \omega \times 2^{<\omega} \to [0, \infty)$ (where we write $g_s(\nu)$ for $\tilde{g}(s, \nu)$). We need to ensure that $\tilde{g}$ is nondecreasing in the first coordinate such that $g = \lim_{s\to\infty} g_s(\sigma)$ is a martingale which succeeds on A, and that no partial permutation betting strategy succeeds on A. In fact, in our construction, for every s, $\sigma \mapsto g_s(\sigma)$ will be a martingale.

2.1. **Outline of the Construction.** In this subsection, we give a global view of the construction, which we hope will provide the reader with enough background to later fit everything in.

Roughly speaking, we construct a sequence which is permutation random but not Martin-Löf random. The proof works by diagonalization against permutation martingales, and the sequence is shown to exist only after the fact by finite extension. It is well known (cf. [MMN$^+$06, MSU98]) how to diagonalize against the class of total monotonic martingales. By an argument of Buhrman et al. [BvMR$^+$00], a total permutation martingale is equivalent to a total monotonic one. So the "extra work" comes from the fact that we have to work against possibly partial and nonmonotonic martingales. To deal with these martingales, we introduce a monotonization technique. During the construction by diagonalization, whenever a new partial permutation martingale is added, there are two possibilities. Either (1) there exists an extension of the sequence already constructed that makes the new martingale diverge without increasing the capital of the active (monotonic) martingales, in which case we pick that extension, or (2) if this is not the case, we argue that the new martingale can be made monotonic and added to the active martingales. This way, the diagonalization we perform is similar to the Muchnik-Merkle technique, and in particular is sufficient to ensure that the constructed sequence is not Martin-Löf random.

More precisely, we start with $g_0 \equiv 0$ and a global counter $c_g = 0$. The construction consists of starting a strategy $N(\emptyset, 0)$ that consists of infinitely many substrategies. Each of these substrategies searches for τ that satisfy certain conditions. If it finds such τ, it *acts* by (1) starting a strategy $N(\tau, 1, \ldots)$ where the parameters we left out here indicate the results of certain searches, and (2) defining g_{c_g+1} from g_{c_g} and incrementing c_g. This $N(\tau, 1, \ldots)$ consists of infinitely many substrategies looking for extensions ν of τ satisfying certain conditions;

when one of them succeeds in finding such a ν, it starts $N(\nu, 2, \ldots)$. This picture of substrategies starting new strategies iterates.

The action of a strategy is completely determined by its parameters; therefore, once we describe $N(\sigma, i, \ldots)$, we have described the complete construction. Here, we will give an outline of how one such strategy $N(\sigma, i, \ldots)$ behaves. The full details are given in Section 2.5.

We need to ensure that $\lim_{c\to\infty} g_c$ is a martingale. We will ensure this to be the case in the construction by having every g_c be a martingale, and $\lim_{c\to\infty} g_c(\emptyset) \leq 1$.

To achieve the second property, first note that $1 = \Sigma_{c=1}^{\infty} (\frac{1}{2})^c$. Then the c^{th} active strategy can use capital $(\frac{1}{2})^c$ from the root. This means that the c^{th} time a new strategy $N(\tau, i, \ldots)$ gets started, it is assigned $(\frac{1}{2})^c$. Now, if by its actions it does not increase the capital at the root by more than the capital it is assigned, we ensure this property. Note that the counter c_g exactly measures how many strategies have acted, i.e., we can use this counter to determine how much capital the next strategy gets to use.

The strategy $N(\sigma, i, \ldots)$ gets assigned some capital K, and it uses that $K = \sum_{j=1}^{\infty} K \cdot (\frac{1}{2})^j$ with the same strategy as above to provide each of the substrategies with some capital they can spend (there will be a fixed enumeration of its infinitely many substrategies, and the j^{th} substrategy gets assigned $K \cdot (\frac{1}{2})^j$).

There will be two types of substrategies: The 0^{th} strategy works under assumption (C:$\downarrow$), see page 15; all the others together achieve the desired result under assumption (C:$\uparrow$). The construction will be organized so that strategy $N(\sigma, i, \ldots)$ takes care of the i^{th} partial computable permutation in the enumeration defined on page 6.

The j^{th} substrategy of strategy $N(\sigma, i, \ldots)$ searches for an extension τ of σ satisfying certain conditions. These conditions have to ensure that there exists a sequence $\langle \tau_k \mid k \in \omega \rangle$ such that $\tau_k \prec \tau_{k+1}$ and $\bigcup_{k\in\omega} \tau_k$ is the real A we are looking for. The ideas for showing that we can find such τ_k such that A is permutation random are fairly complicated and explained below. The ideas for ensuring that the martingale we construct succeeds on A are easier. If we ensure that once this j^{th} substrategy acts at some extension τ of σ, we ensure that $g(\tau) \geq 2^i$, then the martingale wins on A, as A is a limit of the τ_k which have been acted on by $N(\tau_{k-1}, l, \ldots)$ for unbounded values of l (a strategy $N(\sigma, i, \ldots)$ starts strategies $N(\tau, i{+}1, \ldots)$, i.e., on every such extension, the corresponding value of i increases by 1).

The substrategies define g_{c_g+1} from g_{c_g} *using its capital K' from the root along* τ by setting $g_{c_g+1} := g_{c_g} + (K' \cdot E_\tau)$, that is we add an elementary martingale: for all $\nu \notin [\tau] \cup \{\eta \in 2^{<\omega} \mid \eta \prec \tau\}$, $g_{s+1}(\nu) := g_s(\nu)$; for all $\nu \prec \tau$, $g_{s+1}(\nu) := g_s(\nu) + K' 2^{|\nu|}$; and for all $\nu \in [\tau]$, $g_{s+1}(\nu) := g_s(\nu) + K' 2^{|\tau|}$.

Note that the martingale g we construct will not have capital exactly equal to 1, as then it would be closely approximable by a computable martingale, something we know cannot happen (the martingale we construct wins on a real on which no partial permutation martingale wins; therefore, certainly no computable martingale wins on it).

This can also be seen in the construction, since the substrategies (and therefore strategies) need to know how much capital they can spend in order to determine the length of the string they are searching for. Since it is undecidable which searches will succeed, the limit value of $g(\emptyset)$ cannot be determined.

In Subsection 2.2, we describe the idea that motivates the remainder of the work. It suggests that we want to consider the sum of all martingales we want to defeat. Since we can only sum monotonic martingales (i.e., betting strategies cannot be added), we show in Subsection 2.3 how to construct for a total nonmonotonic betting strategy an equivalent monotonic martingale. Then, in Subsection 2.4, we show how to extend this also to partial betting strategies. The case distinction we have to make is not quite partial versus total, however; it is "sufficiently total", case (C:↓), versus "not sufficiently total", case (C:↑). In Subsection 2.5, we put all these ideas together to describe the full solution, which we verify in Subsection 2.6.

2.2. **Combining Martingales.** The collection of martingales has a few easy but important closure properties. If f and g are monotonic martingales, then so are cf, for any $c \in \mathbb{R}^+$, and $f + g$.

If we have an enumeration of martingales $\langle f_i \mid i \in \omega \rangle$ with initial capital less than or equal to 1 and we want to find a real $A \in 2^\omega$ such that none of the martingales f_i succeeds on A, we can go about this as follows: Find suitable constants $\langle s_i \mid i \in \omega \rangle$ such that $f = \sum_{i \in \omega} s_i f_i$ is a martingale (note that the existence of these easily follows with the ideas from the previous paragraph), and ensure that f does not succeed on A. This means that none of the f_i succeeds, either. Since we certainly cannot computably consider the sum of infinitely many martingales, we observe that as long as every martingale is considered from some point onward, this idea still works.

We describe how to do that in some more detail here. First, we find a σ_0 on which f_0 does not gain too much, i.e., $f_0(\sigma_0) < 2$ (in fact, we can make sure it makes no gain at all). On σ_0, the martingale f_1 might have gained a lot, therefore set $s_1 = \frac{2 - f_0(\sigma)}{2 f_1(\sigma)}$ and notice that then

$f_0(\sigma_0) + s_1 f_1(\sigma_0) < 2$. So we can find an extension σ_1 of σ_0 where the martingale $f_0 + s_1 f_1$ does not gain too much, i.e., still has capital less than 2. Note that there, f_0 or f_1 could have increased, but not too much, since $f_0(\sigma_1) < 2$ and $f_1(\sigma_1) < \frac{2}{s_1}$. If we can iterate this construction, we have found A as required (namely, $A = \bigcup_{i\in\omega} \sigma_i$).

The difficulty is that since the betting strategies we have to beat are not monotonic, we cannot add them in this way. The way to overcome this difficulty is shown in the following two subsections.

2.3. **Monotonizing a Betting Strategy.** If (λ, s, q) is a total non-monotonic betting strategy, then we can define $f^{\text{expec}}_{(\lambda,s,q)} : 2^{<\omega} \to [0, \infty)$, the expected capital function. We will work with the case that s is obtained from an injection $h : \omega \to \omega$ and initial capital 1. Then we can define $f^{\text{expec}}_{(h,q)}$ as follows:

For $\sigma \in 2^{<\omega}$, let $n_\sigma, l_\sigma \in \omega$ be such that

$$\begin{gathered} l_\sigma > |\sigma|, \\ \operatorname{ran}(h \restriction n_\sigma) \subseteq l_\sigma, \text{ and} \\ \operatorname{ran}(h) \cap \operatorname{dom}(\sigma) = \operatorname{ran}(h \restriction n_\sigma) \cap \operatorname{dom}(\sigma). \end{gathered}$$

This means that after n_σ many bets, all bets on σ will have been placed, and to complete these bets, no bets beyond the l_σ^{th} bit are needed. Then define

$$f^{\text{expec}}_{(h,q)}(\sigma) := \sum_{\substack{\tau \in 2^{l_\sigma} \\ \sigma \prec \tau}} d^{\tau}_{(h,q)}(n_\sigma) 2^{-(l_\sigma - |\sigma|)},$$

where d^τ was defined on page 5.

Lemma 9. *$f^{\text{expec}}_{(h,q)}$ is a well-defined monotonic martingale.*

Intuitively, this lemma is clear from the probabilistic interpretation, but we give here a combinatorial proof.

Proof. In the context of this proof, we drop the subscript (h, q) from the martingales and capital functions.

Given $\sigma \in 2^{<\omega}$, we have to show

(1) that in the computation of f^{expec}, for any values n and l which satisfy the requirements, we compute the same value; and
(2) that for all σ,

$$f^{\text{expec}}(\sigma) = \frac{f^{\text{expec}}(\sigma 1) + f^{\text{expec}}(\sigma 0)}{2}.$$

For (1), first let $n, l, l' \in \omega$ be such that both pairs (n, l) and (n, l') satisfy the requirements in the definition of f^{expec} and such that $l' > l$. Then

$$
\begin{aligned}
\sum_{\sigma \prec \tau' \in 2^{l'}} d^{\tau'}(n) 2^{-(l'-|\sigma|)} &= \sum_{\sigma \prec \tau \in 2^{l}} \left(\sum_{\tau \prec \tau' \in 2^{l'}} d^{\tau'}(n) 2^{-(l'-|\sigma|)} \right) \\
&= \sum_{\sigma \prec \tau \in 2^{l}} \left(\sum_{\tau \prec \tau' \in 2^{l'}} d^{\tau}(n) 2^{-(l'-|\sigma|)} \right) \\
&= \sum_{\sigma \prec \tau \in 2^{l}} \left(d^{\tau}(n) 2^{-(l'-|\sigma|)} \sum_{\tau \prec \tau' \in 2^{l'}} 1 \right) \\
&= \sum_{\sigma \prec \tau \in 2^{l}} d^{\tau}(n) 2^{-(l'-|\sigma|)} 2^{l'-l} \\
&= \sum_{\sigma \prec \tau \in 2^{l}} d^{\tau}(n) 2^{-(l-|\sigma|)}.
\end{aligned}
$$

Next, let $n, l \in \omega$ be such that the pair (n, l) satisfies the requirement in the definition of f^{expec} and such that $h(n+1) = i < l$. If $\tau \in 2^l$, then let $\tilde{\tau} \in 2^l$ be such that $\tilde{\tau}(i) = 1 - \tau(i)$, and such that for all $j < l$ with $j \neq i$, $\tilde{\tau}(j) = \tau(j)$. Then (remembering the definition of $\bar{\tau}$ on page 5)

$$
\begin{aligned}
&\sum_{\sigma \prec \tau \in 2^{l}} d^{\tau}(n) 2^{-(l-|\sigma|)} = \\
&\quad = \frac{1}{2} \sum_{\sigma \prec \tau \in 2^{l}} \left(q(\bar{\tau} \restriction n) + (2 - q(\bar{\tau} \restriction n))\right) d^{\tau}(n) 2^{-(l-|\sigma|)} \\
&\quad = \frac{1}{2} \sum_{\sigma \prec \tau \in 2^{l}} \big((q(\bar{\tau} \restriction n) d^{\tau}(n) 2^{-(l-|\sigma|)}) + (2 - q(\bar{\tau} \restriction n)) d^{\tau}(n) 2^{-(l-|\sigma|)}\big) \\
&\quad = \frac{1}{2} \sum_{\sigma \prec \tau \in 2^{l}} \big(d^{\tau}(n+1) 2^{-(l-|\sigma|)} + d^{\tilde{\tau}}(n+1) 2^{-(l-|\sigma|)}\big) \\
&\quad = \frac{1}{2} \left(\sum_{\sigma \prec \tau \in 2^{l}} d^{\tau}(n) 2^{-(l-|\sigma|)} + \sum_{\sigma \prec \tau \in 2^{l}} d^{\tilde{\tau}}(n) 2^{-(l-|\sigma|)} \right) \\
&\quad = \sum_{\sigma \prec \tau \in 2^{l}} d^{\tau}(n+1) 2^{-(l-|\sigma|)}.
\end{aligned}
$$

Note that in the third equality, the terms might be reordered (depending on whether $\tau(i) = 0$).

This shows that in the definition of f^{expec}, the exact values of n and l are irrelevant as long as they are big enough. It remains to show (2), i.e., that f^{expec} satisfies the martingale equation. So let both n and l be large enough, then

$$\begin{aligned} f^{\text{expec}}(\sigma) &= \sum_{\sigma \prec \tau \in 2^l} d^\tau(n) 2^{-(l-|\sigma|)} \\ &= \sum_{\substack{\sigma \prec \tau \in 2^l \\ \tau(|\sigma|)=0}} d^\tau(n) 2^{-(l-|\sigma|)} + \sum_{\substack{\sigma \prec \tau \in 2^l \\ \tau(|\sigma|)=1}} d^\tau(n) 2^{-(l-|\sigma|)} \\ &= \frac{1}{2} f^{\text{expec}}(\sigma 0) + \frac{1}{2} f^{\text{expec}}(\sigma 1). \end{aligned}$$

□

There are now two problems to overcome. Firstly, we need to see that we can use f^{expec} to beat the original nonmonotonic betting strategy, and secondly, that we can have a sufficiently computable version of it.

The problem with seeing that f^{expec} succeeds on the same reals that the nonmonotonic betting strategy (s, q) succeeds on is simplified by taking the slowly-but-surely winning version of (s, q) (also known as the saving version of (s, q)). The problem solved by this is that f^{expec} does not reflect all fluctuations that appear in the capital history of (s, q).

Lemma 10. *Let (s, q) be a partial computable nonmonotonic betting strategy. Then there exists a partial computable nonmonotonic betting strategy $(\bar{s}, \bar{q})$ that succeeds on the same reals as (s, q) where the capital function $\bar{d}$ of $(\bar{s}, \bar{q})$ satisfies for all reals A*

$$\forall n \, \forall m \, (\bar{d}^A(n+m) > \bar{d}^A(n) - 2, \text{ and}$$
$$\forall n \, (\bar{d}^A(n) < 2(n+1)).$$

A version of this well known lemma can be found as [BvMR+00, Lemma 2.3, p. 579], and the computations in their proof immediately generalize to this context, so we will not repeat them here. The intuition is that in the betting of the betting strategy, every time your capital increases to more than 2, you take 2 from your capital and "keep it in the bank" and only continue betting with the remaining little bit of capital. This way, your capital can never decrease by more than 2. If the original betting strategy succeeds, then infinitely often the little bit of capital you are betting with will increase above 2 so that this betting strategy succeeds as well.

Now, for a nonmonotonic betting strategy (s, q), define $f^{\mathrm{ss-expec}}$ to be the expected capital function computed for the slowly-but-surely winning version $(\bar{s}, \bar{q})$ of (s, q).

Lemma 11. *$f^{\mathrm{ss-expec}}$ is a monotonic martingale which succeeds on every real on which (s, q) succeeds.*

Proof. Let $A \in 2^\omega$ be a real on which (s, q) succeeds. Then $(\bar{s}, \bar{q})$ also succeeds on A. We need to see that for every L, there is an $n \in \omega$ such that $f^{\mathrm{ss-expec}}(A \upharpoonright n)$ is greater than L.

Let k be such that $\bar{d}^A(k) > L + 2$, and let $\sigma \prec A$ be such that all bets that are made in the computation of $\bar{d}(k)$ are made on σ. This means that by the slowly-but-surely winning condition for all τ extending σ and all $l \geq k$, we have $\bar{d}^\tau(l) > L$. This in turn implies that $f^{\mathrm{ss-expec}}(\sigma) > L$, as was to be proven. □

The final bit of analysis of $f^{\mathrm{ss-expec}}$ that is needed is to come up with a usable condition under which we can compute it. Existence of n_σ and l_σ for any σ is clear, but it is not clear under what conditions they can be found computably. We next give such conditions.

Say we have already decided on σ_0 as the initial segment of the real A we are constructing, and that we now want to extend it to length $k > |\sigma_0|$. Then we want to be able to compute $f^{\mathrm{ss-expec}}$ on $[\sigma_0] \cap 2^k$.

Lemma 12. *For a total nonmonotonic betting strategy (s, q) where q is obtained from an injective $h : \omega \to \omega$ and σ_0 and k are as in the paragraph above, we can compute $f^{\mathrm{ss-expec}}$ on $[\sigma_0] \cap 2^k$ from $|\operatorname{ran}(h) \cap k|$.*

Proof. From $|\operatorname{ran}(h) \cap k|$, we can compute an n such that $\operatorname{ran}(h \upharpoonright n) \cap k = \operatorname{ran}(h) \cap k$, i.e., after n many bets, all bets on every τ of length k will have been made. This means that $f^{\mathrm{ss-expec}}$ can be computed on any τ using $n_\tau = n$ and $l_\tau = \max\{\operatorname{ran}(h \upharpoonright n)\}$. □

Note that the hypothesis needed for $d^{\mathrm{ss-expec}}$ to be computable is obviously satisfied for permutation betting strategies as then always $|\operatorname{ran}(h) \cap k| = k$.

2.4. **Partiality.** Initially, dealing also with partial permutation betting strategies seems straightforward. Once you have decided on σ_0 as an initial segment for the real and (h, q) is the next permutation betting strategy to consider, the strategy working under the assumption that (h, q) is partial should just look for an extension τ of σ_0 where (h, q) diverges. The problem with this is that the earlier betting strategies already considered, of which all the ones which are "sufficiently total" have been combined into a single monotonic martingale f_{mon} (the other ones have been completely defeated already), might make a large gain somewhere along each such τ. This would exclude τ from being an initial segment of the real we are trying to construct, as we need to make the capital function bounded on it, and we have chosen 2 as the bound to maintain (so we must ensure that on no initial segment, the capital function goes above 2).

The situation we have is $\sigma \in 2^{<\omega}$, a monotonic martingale f_{mon} which is total on $[\sigma] \cap 2^{<\omega}$ and for which $\max\{f_{\mathsf{mon}}(\sigma') \mid \sigma' \preceq \sigma\} < 2$, and a partial permutation betting strategy (h, q). We have to either find a $\tau \succ \sigma$ such that $d_{(h,q)}$ diverges on τ and $\max\{f_{\mathsf{mon}}(\tau') \mid \tau' \preceq \tau\} < 2$, or we have a method of adding $d_{(h,q)}$ to f_{mon}.

We need to be explicit about what we mean when $d_{(h,q)}$ diverges on τ. For this, we define $d^{\tau}_{(h,q)} \uparrow^*$ by

$$\exists i \left(\operatorname{ran}(h \upharpoonright i) \subsetneq |\tau| \wedge h(i)\uparrow\right) \vee \exists i \left(\operatorname{ran}(h \upharpoonright i) \subseteq |\tau| \wedge \exists n < i \,(q(\bar{\tau} \upharpoonright n)\uparrow)\right),$$

and $d^{\tau}_{(h,q)} \downarrow^*$ as its negation (note that it will often be the case that for $\tau \prec \tau'$ we have that $d^{\tau}_{(h,q)} \downarrow^*$ and $d^{\tau'}_{(h,q)} \uparrow^*$; this happens for instance when $h(0) \in |\tau'| \setminus |\tau|$ and $h(1)$ diverges).

The case distinction which needs to be made is the following: Either

$$\text{(C:}\uparrow\text{)} \qquad \exists \tau \succ \sigma \left(\max\{f_{\mathsf{mon}}(\tau') \mid \tau' \preceq \tau\} < 2 \;\wedge\; d^{\tau}_{(h,q)} \uparrow^* \right),$$

or

$$\text{(C:}\downarrow\text{)} \qquad \forall \tau \succ \sigma \left(\max\{f_{\mathsf{mon}}(\tau') \mid \tau' \preceq \tau\} < 2 \rightarrow d^{\tau}_{(h,q)} \downarrow^* \right).$$

Note that when the permutation is partial (C:$\uparrow$) holds.

The strategy is then as follows: In case (C:$\uparrow$), we search for such a τ. That is, at stage s, we assume that any computation which does not converge within s steps diverges, and we look for the length-lexicographically first τ which satisfies (C:$\uparrow$).

In case (C:$\downarrow$), we will find a total (on $[\sigma]$) permutation betting strategy $(\tilde{h}, \tilde{q})$ such that $d_{(\tilde{h},\tilde{q})}(\tau)$ is equal to $d_{(h,q)}(\tau)$ for all τ for which $\max\{f_{\mathsf{mon}}(\tau') \mid \tau' \preceq \tau\}$ is less than 2. The idea is that in order to compute $\tilde{d}^{\tau}_{(h,q)}(n)$, we find the largest part of τ where $\max\{f_{\mathsf{mon}}(\tau') \mid \tau' \preceq \tau\}$ is less than 2, and give $\tilde{d}$ on τ the value of $d_{(h,q)}$ on that part. Essentially this means as soon as we lose the guarantee — from the assumption that we are in case (C:$\downarrow$) — that (h, q) converges, we always bet even (i.e., the stake function will then have value 1).

We need to be somewhat careful with what "largest part" means, since we do need the property that if ν_0 and ν_1 are such that $\nu_0 \upharpoonright$

$\mathrm{ran}(h \restriction j) = \nu_1 \restriction \mathrm{ran}(h \restriction j)$ (that is, ν_0 and ν_1 agree on the bit values of the first j places inspected) then for all $i < j$, $\tilde{d}^{\nu_0}_{(h,q)}(i) = \tilde{d}^{\nu_1}_{(h,q)}(i)$.

The betting strategy $(\tilde{h}, \tilde{q})$ is now defined as follows: Since in this case h is total we can set $\tilde{h} = h$. $\tilde{q}(w)$ where $w \in (\omega \times 2)^{<\omega}$ is computed by searching for the maximum $j \leq \mathrm{lh}(w)$ for which there exists a $\tau \succ \sigma$ such that for all $i \leq j$ we have $\tau(\pi_0(w(i))) = \pi_1(w(i))$ and $\max\{f_{\mathsf{mon}}(\tau') \mid \tau' \preceq \tau\} < 2$ (if no such j exists, then w is not consistent with σ and therefore is not relevant to the behavior of the capital function on $[\sigma]$, in this case just diverge). Now if $j = \mathrm{lh}(w)$ set $\tilde{q}(w) = q(w)$, otherwise (if $j < \mathrm{lh}(w)$) set $\tilde{q}(w) = 1$. Note that τ guarantees (since we are in case (C:$\downarrow$)) that $q(w)$ converges if $j = \mathrm{lh}(w)$, since $d^{\tau}_{(h,q)} \downarrow^*$.

Note that our restriction $\max\{f_{\mathsf{mon}}(\tau') \mid \tau' \preceq \tau\} < 2$ ensures that this is a computable procedure. We only need to consider τ that are long enough to have initial segments of the betting as determined by w be performed. For these, either this restriction is satisfied, or for all extensions, it is not satisfied. We will not need this additional information in the verification.

2.5. **Putting it all together.** Let $\langle (h_i, q_i) \mid i \in \omega \rangle$ be the enumeration of all partial computable permutation betting strategies with initial capital 1 defined on page 6 (i.e., we dropped the hat from h_i). We will now complete the construction described in Subsection 2.1 by describing the action of the strategies $N(i, \sigma, \ldots)$ exactly.

The strategy $N(i, \sigma, \epsilon_0, \ldots, \epsilon_{i-1})$ (where, from now on, we will write $\bar{\epsilon}$ for $\epsilon_0, \ldots, \epsilon_{i-1}$) is the strategy with the following parameters

- i, the number of martingales supposedly already dealt with on σ,
- σ, indicating the cone, $[\sigma]$, on which this strategy will act, and
- $\epsilon_j \in \{0, 1\}$ (for $j < i$), denoting whether the previous strategies were able to find a τ where (h_j, q_j) diverges and $\max\{f_j^{\bar{\epsilon}\restriction j+1}(\tau') \mid \tau' \preceq \tau\} < 2$ (if $\epsilon_j = 0$) or where (h_j, q_j) converges on every extension τ of σ where $\max\{f_j^{\bar{\epsilon}\restriction j+1}(\tau') \mid \tau' \preceq \tau\} < 2$ (if $\epsilon_j = 1$). Here and below, $f_j^{\bar{\epsilon}\restriction j+1}$ is one of the monotonic martingales constructed by the strategy $N(j, \sigma', \bar{\epsilon} \restriction j)$ where $\sigma' \prec \sigma$.

This strategy performs three kinds of actions:

- it will start new strategies $N(i+1, \tau, \bar{\epsilon}\epsilon)$ where $\sigma \prec \tau$ and $\epsilon \in \{0, 1\}$, and
- it will at various times define g_{c_g+1} from g_{c_g}, and increment the counter c_g.

- it will (attempt to) define $f_i^{\bar{\epsilon}\epsilon}$ for $\epsilon \in \{0, 1\}$.

In fact, whenever a new strategy is started, a new g_{c_g+1} will also be defined. It is important to note here that c_g is not determined by this strategy, but by the overall results of all strategies active so far. In all cases, c_g will be the maximum value such that g_{c_g} is defined. If we think of defining g_{c_g+1} from g_{c_g} as modifying the martingale g, then c_g is the number of times we have already modified g before (this is the terminology we will use below).

If $N(i, \sigma, \bar{\epsilon})$ is the c_g-th strategy that is started, then the strategy as a whole can use capital $C := (\frac{1}{2})^{c_g}$ from the root (as explained in Section 2.1). Since this strategy will possibly infinitely often modify g, it partitions $C = \sum_{l \in \omega} k_l$ (where all $k_l > 0$) and uses these parts from C as appropriate (k_0 is used for the case (C:$\downarrow$), and k_{l+1} is used for the l-th time the actions for the case (C:$\uparrow$) modify g).

Define $\hat{f}_i^{\bar{\epsilon}\restriction i}$ to be the monotonic martingale $\tilde{d}^{\mathrm{ss-expec}}_{(h_i, q_i)}$ determined from $d_{(h_i, q_i)}$ and the monotonic martingale $f_{i-1}^{\bar{\epsilon}\restriction i}$ as in the previous section (where $f_{-1}^{\emptyset} \equiv 0$).

In case $\forall \nu \succ \sigma\, (f_{i-1}^{\bar{\epsilon}}(\nu) < 2 \to d^{\nu}_{(h_i, q_i)} \downarrow^*)$, i.e., in case (C:$\downarrow$), $N(i, \sigma, \bar{\epsilon})$ has to find a long enough extension τ of σ such that $f_i^{\bar{\epsilon}1}(\tau) := f_{i-1}^{\bar{\epsilon}}(\tau) + \lambda_i \hat{f}_i^{\bar{\epsilon}}(\tau) < 2$, where λ_i is determined below in the second substrategy. In the other case, (C:$\uparrow$), it has to find a long enough extension τ of σ such that $f_i^{\bar{\epsilon}0}(\tau) := f_{i-1}^{\bar{\epsilon}}(\tau) < 2$ and $d^{\tau}_{(h_i, q_i)} \uparrow^*$, which then exists. The next paragraph explains what long enough means.

Since we need to ensure that g wins on the real A we construct, and the τ are approximations to this A, the strategy $N(i, \sigma, \bar{\epsilon})$ ensures that $g(\tau) \geq 2^i$. So, for case (C:$\downarrow$), it looks for a τ such that $k_0 2^{|\tau|} \geq 2^i$, and in case (C:$\uparrow$), if active for the l-th time, we look for a τ such that $k_{l+1} 2^{|\tau|} \geq 2^i$.

At stage s in the construction, this strategy computes everything for at most s steps.

The first substrategy has two local variables, a (initially 0) and ν (initially $\emptyset$). a counts the number of times this strategy has acted, and ν the string it acted with. It searches for the length-lexicographically least $\tau' \succ \sigma$ such that $d^{\tau'}_{(h_i, q_i)} \uparrow^*$ and $f_{i-1}^{\bar{\epsilon}}(\tau') < 2$, and then for the least $\tau \succ \tau'$ such that $f_{i-1}^{\bar{\epsilon}}(\tau) < 2$ and $k_{a+1} 2^{|\tau|} > 2^i$. If $\tau' = \nu$ it does nothing. If $\tau' \neq \nu$ (which means that the computation on ν has converged), then it sets $\nu = \tau'$, increases a by one, stops the previous strategies $N(i{+}1, \tau'', \bar{\epsilon}0)$ this strategy started, and starts $N(i{+}1, \tau, \bar{\epsilon}0)$. Also, we define g_{c_g+1} from g_{c_g} using capital k_{a+1} from the root along τ (as explained in Section 2.1), and increase c_g by one.

It is clear that in case (C:↑) (where $\sigma = \sigma$, $f_{\mathsf{mon}} = f^{\bar{\epsilon}}_{i-1}$, and $(h,q) = (h_i,q_i)$) and where $\bar{\epsilon}$ is correct, this strategy succeeds; it will find a pair (τ,τ') which permanently satisfies the requirement.

Simultaneously, for the second substrategy, we wait for a stage at which $\hat{f}^{\bar{\epsilon}}_i$ and $f^{\bar{\epsilon}}_{i-1}$ converge on σ. Then we set $\lambda_i := \frac{1}{2}\frac{2-f^{\bar{\epsilon}}_{i-1}(\sigma)}{\hat{f}^{\bar{\epsilon}}_i(\sigma)}$ (to ensure $f^{\bar{\epsilon}1}_i(\sigma) < 2$). After this stage, we search for the length-lexicographically least $\tau \succ \sigma$ such that $f^{\bar{\epsilon}1}_i(\tau) < 2$ and $k_0 2^{|\tau|} > 2^i$. If we find such a τ, we start the strategy $N(i+1,\tau,\bar{\epsilon}1)$ and define g_{s+1} from g_s using capital k_0 from the root along τ (as explained in Section 2.1), where s is the number of times we have already modified the martingale g before.

It is clear that in case (C:↓) (where $\sigma = \sigma$, $f_{\mathsf{mon}} = f^{\bar{\epsilon}}_{i-1}$, and $(h,q) = (h_i,q_i)$) and where $\bar{\epsilon}$ is correct, this strategy succeeds.

We start the construction by starting $N(0,\emptyset,\emptyset)$.

2.6. **Verification.** Recursively (but certainly not computably!), define σ_i and ϵ_i (where $\sigma_{-1} = \emptyset$) as follows.

We set $\epsilon_i = 1$ if (C:↓) is true for $\sigma = \sigma_{i-1}$, $f_{\mathsf{mon}} = f^{\bar{\epsilon}}_{i-1}$, and $(h,q) = (h_i,q_i)$. In that case, we set $\sigma_i = \tau$ where τ is found by the strategy $N(i,\sigma_{i-1},\bar{\epsilon}\restriction i)$ by its second substrategy. Note that then $N(i+1,\sigma_i,\bar{\epsilon}\restriction i,1)$ is started.

We set $\epsilon_i = 0$ if (C:↑) is true for $\sigma = \sigma_{i-1}$, $f_{\mathsf{mon}} = f^{\bar{\epsilon}}_{i-1}$, and $(h,q) = (h_i,q_i)$. In that case, we let $\tau' \in 2^{<\omega}$ be the length-lexicographically least element of $[\sigma_{i-1}] \cap 2^{<\omega}$ for which $d^{\tau'}_{(h_i,q_i)} \uparrow^*$, $f^{\bar{\epsilon}}_{i-1}(\tau') < 2$, and s is a stage at which for all ν length-lexicographically before τ', $d^{\nu}_{(h_i,q_i)}$ converges in fewer than s steps if $f^{\bar{\epsilon}}_{i-1}(\nu) < 2$. Then, at stage s, the first substrategy of $N(i,\sigma_{i-1},\bar{\epsilon}\restriction i)$ will pick τ' as well as an extension τ of τ' (if it hadn't already done so at an earlier stage), and we will never again find a new pair (τ',τ). Then set $\sigma_i = \tau$. Note that then $N(i+1,\sigma_i,\bar{\epsilon}\restriction i,0)$ is started and never stopped thereafter.

Define $A := \bigcup_{i\in\omega}\sigma_i$. Note that clearly g succeeds on A since $g(\sigma_i) \geq 2^i$ (as a result of the modification of g done by $N(i,\sigma_{i-1},\bar{\epsilon}\restriction i)$ when $N(i+1,\sigma_i,\bar{\epsilon}\restriction i+1)$ was started). Note that A is Δ^0_3, since

$$\begin{aligned}\sigma \prec A \Leftrightarrow\ & \exists\bar{\epsilon}(\bar{\epsilon}\text{ is correct} \wedge \sigma\text{ is obtained when running}\\ &\qquad\qquad\text{the construction above knowing }\bar{\epsilon})\\ \Leftrightarrow\ & \forall\bar{\epsilon}(\bar{\epsilon}\text{ is correct } \rightarrow \sigma\text{ is obtained when running}\\ &\qquad\qquad\text{the construction above knowing }\bar{\epsilon}).\end{aligned}$$

Here the statement "$\bar{\epsilon}$ is correct" is Δ^0_3, and with that information determining the outcome of the construction is Σ^0_2.

We need to see that none of the partial computable permutation martingales (h_i, q_i) succeeds on A. Suppose that (h_i, q_i) succeeds on A. We will derive a contradiction to the fact that for all $i \in \omega$, $f_i^{\bar\epsilon\restriction i+1}(\sigma_i) < 2$.

We know that if (h_i, q_i) succeeds on A, then $\hat f_i^{\bar\epsilon\restriction i}$ also succeeds on A (note that this, in particular, implies that (C:↓) is the correct case for strategy $N(i, \sigma_{i-1}, \bar\epsilon \restriction i)$, i.e., $\epsilon_i = 1$). Pick γ such that $\sigma_i \prec \gamma \prec A$ and $\hat f_i^{\bar\epsilon\restriction i}(\gamma) > \frac{2}{s_i} + 2$. Since $\hat f_i^{\bar\epsilon\restriction i}$ is a slow-but-sure martingale, this means that for all $\nu \succ \gamma$, we have $\hat f_i^{\bar\epsilon\restriction i}(\nu) > \frac{2}{s_i}$. This holds, in particular, for any j such that $\sigma_j \succ \gamma$ (which implies $j \geq i$). But this in turn implies that

$$f_j^{\bar\epsilon\restriction j+1}(\sigma_j) = \sum_{\substack{l \leq j \\ \epsilon_l = 1}} s_l \hat f_l^{\bar\epsilon\restriction l}(\sigma_j) > s_i \hat f_i^{\bar\epsilon\restriction i}(\sigma_j) > 2,$$

which contradicts the choice of σ_j, showing that (h_i, q_i) does not succeed on A.

3. The Proof for Injective Randomness

We again need to construct a real $A \in 2^\omega$ and a computable function $\tilde g : \omega \times 2^{<\omega} \to [0, \infty)$ which is nondecreasing in the first coordinate and such that $g = \lim_{s\to\infty} g_s$ is a martingale which succeeds on A (where we write $g_s(\nu)$ for $\tilde g(s, \nu)$). This time no partial injective martingale can succeed on A.

When dealing with partial injective betting strategies we have no hope of producing a martingale with the same good properties the expected martingale had for partial permutation betting strategies. Lemma 12 cannot be used, since for an injective function we cannot compute $|\operatorname{ran}(h) \cap k|$, and there are too many possibilities to guess (exponential in k). The type of guess we need to make is one where when we guess correctly, we can use the guess to find arbitrarily long sequences with it (to ensure that given σ, i and k, we can find an extension τ of σ, using the guess, such that $k \cdot 2^{|\tau|} \geq 2^i$. (Note here that each "guess" really corresponds to a substrategy based on that guess, which the substrategy uses as a parameter.)

We were able to come up with a sufficient approximation, however. The idea is to ignore most bits, and to only look for bits of a given type. If this type is chosen, or rather guessed, correctly, then we can use this approximation, the average operator, to construct a sequence of clopen sets of decreasing measure (a Martin-Löf test). We can then prove that in the intersection of these clopen sets, there is a real on which none of the partial injective betting strategies wins. Since we do not know the correct guesses, we construct a Martin-Löf martingale that combines all our different attempts, and, in particular, will include all the correct guesses.

To bring this construction more in line with the previous construction, instead of constructing the Martin-Löf test, we use a notion similar to elementary martingales defined from clopen sets. So let C be a clopen subset of 2^ω and define E_C as follows:

$$E_C(\nu) = \frac{\mu([\nu] \cap C)}{\mu([\nu]) \cdot \mu(C)}.$$

Here μ is the Lebesgue measure. Note that for $\sigma \in 2^{<\omega}$ we have $E_\sigma = E_{[\sigma]}$, so that this is indeed a generalization of the notion of elementary martingale. Also we will only use it for basic open sets, i.e., those of the form $\{A \in 2^\omega \mid \forall i \in \mathrm{dom}(\sigma)\ A(i) = \sigma(i)\}$ where $\sigma : \omega \rightharpoonup 2$ is a finite partial map.

Let $\langle (h_i, q_i) \mid i \in \omega \rangle$ be an enumeration of all partial injective betting strategies (with initial capital 1); without loss of generality we can assume that all these martingales are slow-but-sure.

3.1. **Types of Bits.** The main difficulty to solve in this proof is that not all betting strategies bet on all bits. This means that during the construction, you never know whether the betting on a bit is done (unless you happen to stumble upon a bit on which all betting strategies you are considering make a bet). The solution is to make guesses as to which betting strategies will bet on a bit. This does not work in a completely straightforward manner—we need a method to bring the number of guesses we have to make down to a manageable number.

Let us assume we are in the part of the construction where we are considering the first $n+1$ betting strategies, $(h_0, q_0), \dots, (h_n, q_n)$. Let $A \in 2^\omega$ be a real and $k \in \omega$ a bit location. If $T \subseteq \{0, \dots, n\}$ is such that during the betting, the betting strategies (h_i, q_i) bet on location k iff $i \in T$, then we know when the betting on location k is done (that is when all (h_i, q_i) for $i \in T$ have bet on bit k). Not knowing what the appropriate T is, we need to guess for it. If our guess T is correct for infinitely many bit locations then we could hope to use it in our construction.

However, the locations for which T is correct cannot be recognized during the construction, since there might also be infinitely many bits for which the correct guess is a proper superset of T. If then we see all martingales in T bet on a location and we act on this, we might still act inappropriately since more betting strategies might bet on this location. The solution is to not just guess for such T but to guess for maximal such T. We work this out in detail below.

Definition 13. We say betting strategy (h_i, q_i) bets on bit k iff there is an $n \in \omega$ such that $h_i(n) = k$.

Note that since the locations the betting strategy (h_i, q_i) bets on are determined by the injection h_i, they do not depend on the real, i.e., if (h_i, q_i) bets along $A \in 2^\omega$ bets on location k, then for all $B \in 2^\omega$ (h_i, q_i), (h_i, q_i) bets on location k when betting on B. This shows that the notion in the next definition is well-defined. Also note that even when (h_i, q_i) bets on a bit k, it might still be the case that the partial function q_i does not sufficiently converge along a certain real to compute the value of the martingale.

Definition 14. The set $\mathsf{type}_n(k) = \mathsf{type}(n, k) \subseteq \{0, \dots, n\}$ is defined by $i \in \mathsf{type}_n(k)$ iff $i \leq n$ and (h_i, q_i) bets on bit k. We will say k is *of n-type T* iff $\mathsf{type}_n(k) = T$.

We need the following observations:

- For all $n \in \omega$,
$$\omega = \coprod_{T \subseteq \{0,\dots,n\}} \{k \in \omega \mid \mathsf{type}_n(k) = T\}$$
(here $\coprod$ denotes disjoint union).
- If k is of n-type T, then when all (h_i, q_i) for $i \in T$ have bet on k, no more bets on k will be made. More precisely, if t is such that for all $i \in T$, we have $k \in \operatorname{ran}(h_i \restriction (t + 1))$, then for all $t' > t$ and all $i \leq n$, we have that $h_i(t') \neq k$.
- Let $K \in \omega$. If T is $\subseteq$-maximal in
$$\{T \subseteq \{0, \dots, n\} \mid \exists k > K \ \mathsf{type}(n, k) = T\},$$
and we find $k > K$ and t such that for all $j \in T$, martingale (h_i, q_i) bets on k before or on bet t ($k \in \operatorname{ran}(h_i \restriction (t+1))$), then $\mathsf{type}(n, k) = T$ and no more bets will be made on k.
- If T is $\subseteq$-maximal in
$$\{T \subseteq \{0, \dots, n\} \mid \exists^\infty k \ \mathsf{type}(n, k) = T\},$$
then there is a $K \in \omega$ such that T is $\subseteq$-maximal in
$$\{T \subseteq \{0, \dots, n\} \mid \exists k > K \ \mathsf{type}(n, k) = T\}.$$

The last observation motivates the guesses we will make; our guesses will be of the form (K, T). And this represents the guess that T is a maximal type appearing infinitely often, and K is big enough so that any bits whose type is a proper superset of T appear below level K.

(Recall here that each “guess” really corresponds to a substrategy based on that guess, which the substrategy uses as a parameter.)

The key idea making this work is that, given a correct guess, we can continue to use it as long as needed. This allows us to shrink the clopen set to be arbitrarily small before starting to consider the next betting strategy.

3.2. **The Average Operator.** In this subsection, we work under the assumption that the betting strategies involved are total. In the next subsection, we show how to deal with partiality.

Let $\sigma : \omega \rightharpoonup 2$ be finite (i.e., σ is a finite partial map from ω to 2) and t a number such that for all $k \in \mathrm{dom}(\sigma)$, if $j \in \mathsf{type}(n, k)$, then (h_j, q_j) has bet on k before its $(t+1)^{\mathrm{st}}$ bet, that is, t is so large that after t bets, all bets that will be made on σ have been made. Now let l be such that all bets by betting strategy (h_j, q_j) $(j \leq n)$ that are made before the $(t+1)^{\mathrm{st}}$ bet are made on bits $k < l$ (namely, $\mathrm{ran}(h_j \restriction (t+1)) \subseteq \{0, \dots, l-1\}$).

If (t, l) satisfies the requirements in the previous paragraph, then we say (t, l) is *sufficiently out there (for σ and n).*

For (t, l) sufficiently out there, define

$$\mathrm{Av}_n^{(t,l)}(\sigma) := \sum_{\substack{\tau \in 2^l \\ \sigma \prec \tau}} 2^{-(l-|\sigma|)} \Big(\sum_{j \leq n} d_j^\tau(t)\Big).$$

Remember that $d_j^\tau(t)$ is the capital that results from running the betting strategy (h_j, q_j) for t bets along τ (this is only defined if τ is long enough for all the bets that are made).

The following two lemmas show some essential properties of Av.

Lemma 15. *If both (t_0, l_0) and (t_1, l_1) are sufficiently out there for σ and n, then $\mathrm{Av}_n^{(t_0,l_0)}(\sigma) = \mathrm{Av}_n^{(t_1,l_1)}(\sigma)$.*

This follows immediately from the martingale property of the (h_j, q_j) $(j \leq n)$. It shows that we can write $\mathrm{Av}(\sigma)$ for $\mathrm{Av}^{(t,l)}(\sigma)$ where (t, l) is any pair that is sufficiently out there.

Lemma 16. *If $k \notin \mathrm{dom}(\sigma)$, $\sigma_0 = \sigma \cup \{(k, 0)\}$, and $\sigma_1 = \sigma \cup \{(k, 1)\}$, then*

$$\mathrm{Av}_n(\sigma) = \frac{\mathrm{Av}_n(\sigma_0) + \mathrm{Av}_n(\sigma_1)}{2}.$$

This is proved exactly as Lemma 9 after choosing (t, l) sufficiently out there for all three computations.

Definition 17. If $T \subseteq \{0, \ldots, n\}$, we define the *restricted* Av *operator*, Av_T, as follows: Let (t, l) be sufficiently out there for n and σ. Then

$$\mathrm{Av}_T(\sigma) = \sum_{\substack{\tau \in 2^l \\ \sigma \prec \tau}} 2^{-(l-|\sigma|)} (\sum_{j \in T} d_j^\tau(t)).$$

Clearly, the analogues of Lemma 15 and Lemma 16 hold for Av_T. Note also that there is a weaker notion of (t, l) being T-*sufficiently out there* for σ, which just requires (t, l) to be such that $\mathrm{dom}(\sigma) \cap \mathrm{ran}(h_i \restriction t) = \mathrm{dom}(\sigma) \cap \mathrm{ran}(h_i)$ for $i \in T$ and $\max(\mathrm{ran}(h_i \restriction t)) \leq l$. Then, just like in Lemma 15, the exact value of (t, l) does not influence the value of Av_T computed using it.

Lemma 18. *Let $k \notin \mathrm{dom}(\sigma)$, let bit k be of type T, and $T \subseteq S \subseteq \{0, \ldots, n\}$. If $\mathrm{Av}_T(\sigma \cup \{(k, i)\}) \leq \mathrm{Av}_T(\sigma \cup \{(k, 1-i)\})$, then $\mathrm{Av}_S(\sigma \cup \{(k, i)\}) \leq \mathrm{Av}_S(\sigma \cup \{(k, 1-i)\})$.*

We see this is true by the following computation, where σ_i denotes $\sigma \cup \{(k, i)\}$ and (t, l) is sufficiently out there:

$$\begin{aligned}
\mathrm{Av}_S(\sigma_i) &= \sum_{\substack{\tau \in 2^l \\ \sigma_i \prec \tau}} 2^{-(l-|\sigma_i|)} (\sum_{j \in S} d_j^\tau(t)) \\
&= \sum_{\substack{\tau \in 2^l \\ \sigma_i \prec \tau}} 2^{-(l-|\sigma_i|)} (\sum_{j \in S \setminus T} d_j^\tau(t)) + \sum_{\substack{\tau \in 2^l \\ \sigma_i \prec \tau}} 2^{-(l-|\sigma_i|)} (\sum_{j \in T} d_j^\tau(t)) \\
&\stackrel{*}{=} \sum_{\substack{\tau \in 2^l \\ \sigma_{1-i} \prec \tau}} 2^{-(l-|\sigma_{1-i}|)} (\sum_{j \in S \setminus T} d_j^\tau(t)) + \mathrm{Av}_T(\sigma_i) \\
&\leq \sum_{\substack{\tau \in 2^l \\ \sigma_{1-i} \prec \tau}} 2^{-(l-|\sigma_{1-i}|)} (\sum_{j \in S \setminus T} d_j^\tau(t)) + \mathrm{Av}_T(\sigma_{1-i}) \\
&= \sum_{\substack{\tau \in 2^l \\ \sigma_{1-i} \prec \tau}} 2^{-(l-|\sigma_{1-i}|)} (\sum_{j \in S \setminus T} d_j^\tau(t)) + \sum_{\substack{\tau \in 2^l \\ \sigma_{1-i} \prec \tau}} 2^{-(l-|\sigma_{1-i}|)} (\sum_{j \in T} d_j^\tau(t)) \\
&= \sum_{\substack{\tau \in 2^l \\ \sigma_{1-i} \prec \tau}} 2^{-(l-|\sigma_{1-i}|)} (\sum_{j \in S} d_j^\tau(t)) \\
&= \mathrm{Av}_S(\sigma_{1-i}).
\end{aligned}$$

Equality $(*)$ follows since $|\sigma_i| = |\sigma_{1-i}|$ and for $j \in S \setminus T$, the betting strategy (h_j, q_j) does not bet on bit k.

What this all achieves is that when we have a correct guess for the type, we can computably find which of zero or one does not increase the average value.

3.3. **Partiality.** We are going to use a similar strategy to the case of partial permutation betting strategies to deal with partiality. Note that we cannot define monotone martingales associated to partial injective betting strategies, so the details will look different.

Let σ denote the partial string (i.e., the partial function $\omega \rightharpoonup \{0,1\}$) that has already been determined, P a set of indices for which earlier strategies have determined that the associated betting strategies are partial, $n \in \omega$ the index of the next betting strategy to consider, and for $j \in \{0,\dots,n-1\} \setminus P$, write $(\tilde{h}_j, \tilde{q}_j)$ for the total version of (h_j, q_j). We write $\widetilde{\mathrm{Av}}$ for Av computed using the $(\tilde{h}_j, \tilde{q}_j)$ (actually from $(\lambda_j, \tilde{h}_j, \tilde{q}_j)$ where λ_j is an initial capital determined during the construction). The two cases to consider are then the following.

$$\text{(CI:}\uparrow\text{)}\quad \exists \tau \succ \sigma \left(\max\{\widetilde{\mathrm{Av}}_{(\{0,\dots,n-1\}\setminus P)}(\tau') \mid \tau' \prec \tau\} < 2 \wedge d^{\tau}_{(h_n,q_n)} \uparrow^* \right),$$

or

$$\text{(CI:}\downarrow\text{)}\quad \forall \tau \succ \sigma \left(\max\{\widetilde{\mathrm{Av}}_{(\{0,\dots,n-1\}\setminus P)}(\tau') \mid \tau' \prec \tau\} < 2 \rightarrow d^{\tau}_{(h_n,q_n)} \downarrow^* \right).$$

The strategy is then as follows: In case (CI:↑), we search for such a τ. That is, at stage s, we assume that any computation diverges if it does not converge within s steps, and we look for the length-lexicographically first τ which satisfies (CI:↑).

In case (CI:↓), we find a total injective betting strategy $(\tilde{h}_n, \tilde{q}_n)$ which is equal to (h_n, q_n) everywhere where the average of the previous martingales is less than 2. The definition is essentially the same as in the permutation case, but now instead of searching for $\tau \in 2^{<\omega}$, we search for a finite injective $\tau : \omega \rightharpoonup 2$.

I.e., the betting strategy is defined as follows: We can set $\tilde{h}_n = h_n$, since if h_n is not total we are in case (CI:↑). $\tilde{q}(w)$ where $w \in (\omega \times 2)^{<\omega}$ is computed by searching for the maximum $j \leq \mathrm{lh}(w)$ for which there exists a $\tau \succ \sigma$ such that for all $i \leq j$ we have $\tau(\pi_0(w(i))) = \pi_1(w(i))$ and $\max\{\widetilde{\mathrm{Av}}_{(\{0,\dots,n-1\}\setminus P)}(\tau') \mid \tau' \prec \tau\} < 2$ (if no such j exists, then w is not consistent with σ and therefore is not relevant to the behavior of the capital function on $[\sigma]$, in this case just diverge). Now, if $j = \mathrm{lh}(w)$, then set $\tilde{q}(w) = q(w)$, otherwise (if $j < \mathrm{lh}(w)$), set $\tilde{q}(w) = 1$. Note that τ guarantees (since we are in case (CI:$\downarrow$)) that $q(w)$ converges if $j = \mathrm{lh}(w)$, since $d^{\tau}_{(h,q)} \downarrow^*$.

Note again that the maximum in the requirement ensures that we have a computable search, it will not be used in the verification section.

3.4. **The Strategy and the Construction.** We now describe the overall strategy, i.e., the construction of our Martin-Löf martingale. We use a similar idea as before, starting many different substrategies with associated capital. If they succeed at stage s, they construct g_{s+1} from g_s by adding a scalar multiple of the computable martingale E_C to g_s, where C is a clopen set (in fact, a basic open set) determined by the strategy.

A substrategy will have as its inputs a finite partial function $\sigma : \omega \rightharpoonup 2$, $n, K, j \in \omega$, disjoint finite sets $P, T \subseteq \{0, \ldots, n\}$, and $(t, l) \in \omega \times \omega$. It will be denoted by $\mathsf{Strat}(\sigma, n, P, j, (T, K), (t, l))$. The interpretation of these parameters is as follows:

- σ determines the clopen set inside which we will work;
- n denotes the index of the next betting strategy to be considered;
- P denotes the set of betting strategies for which earlier strategies have determined they are partial, and possibly also n, if this substrategy will work with the assumption (CI:$\uparrow$);
- if $n \in P$, then j indicates the smallest number of computation steps this strategy believes need to be taken before it believes divergence;
- if $n \in T$, then j indicates after how many computation steps (h_n, q_n) is done betting on σ;
- T is the n-type this strategy will use;
- K is an upper bound for the exceptions to the type T (i.e., above K there are no bits with type a proper superset of T); and
- (t, l) is a pair sufficiently out there to compute $\widetilde{\mathrm{Av}}_{T\setminus\{n\}}(\sigma)$.

We have scalars c_m such that $\max\{\widetilde{\mathrm{Av}}_{(T\setminus\{n\})}(\sigma') < 2 \mid \sigma' \prec \sigma\}$ when $\widetilde{\mathrm{Av}}$ is determined using the total versions $(c_m, \tilde{h}_m, \tilde{q}_m)$, $m \in T \setminus \{n\}$ (note that the actual indices for these scalars determined below are more complicated since there are possibly many different strategies determining scalars for (h_m, q_m) in different cones and after different amounts of previous work).

The substrategy will also have assigned to it an initial capital I. It will find an extension τ of σ such that $I \cdot \mu([\sigma])/\mu([\tau]) > 2^n$, and then start all consistent strategies $\mathsf{Strat}(\tau, n+1, P', j', (T', K'), (t', l'))$

with initial capitals such that the total used initial capital stays strictly below 1. A consistent strategy here means that $P \subseteq P' \subseteq \{0, \dots, n+1\}$, $T \subseteq T' \subseteq \{0, \dots, n+1\}$, $P' \cap T' = \emptyset$, both j', K' greater than the current stage of the computation, and (t', l') as determined for the computation of $\mathrm{Av}_T(\tau)$.

Now we describe the action of $\mathsf{Strat}(\sigma, n, P, j, (T, K), (t, l))$ towards finding $\tau \succ \sigma$. This is done in cases:

(1) $n \in P$: find τ' as in (CI:↑) believing that any computation that does not converge in j steps does in fact not converge.

(2) $n \in T$: In this step (and the next) only compute everything for s steps, where s is the stage of the construction.

Since, by assumption, $\max\{\widetilde{\mathrm{Av}}_{(T \setminus \{n\})}(\sigma') \mid \sigma' \prec \sigma\} < 2$, we can find a coefficient $c_{\sigma,\dots,(t,l)}$ such that if we use $(c_{\sigma,\dots,(t,l)}, \tilde{h}_n, \tilde{q}_n)$ instead of $(\tilde{h}_n, \tilde{q}_n)$, we have that $\max\{\widetilde{\mathrm{Av}}_T(\sigma') \mid \sigma' \prec \sigma\} < 2$, and in fact using j, we can find this coefficient effectively.

Find a $k > \max\{\max\{\mathrm{dom}(\sigma)\}, K\}$ such that k is of type T. In searching for this, you find (t', l') that allows you to do the computation to check that k is of this type T (that is, l' is greater than all bets made, and t' is how many computation steps it took for all betting strategies $(\tilde{h}_m, \tilde{q}_m)$, $m \in T$, to bet on bit k. Then, with the input to this strategy, we believe $t'' = \max(t, t', j)$ is sufficiently large to compute both $\widetilde{\mathrm{Av}}_T(\sigma \cup \{(k, 0)\})$ and $\widetilde{\mathrm{Av}}_T(\sigma \cup \{(k, 1)\})$, and from t'', we can compute an appropriate l'' — i.e., (t'', l'') is sufficiently out there to compute both these Av_T.

Then set τ' to be $\sigma \cup \{(k, i)\}$ for whichever $i \in \{0, 1\}$ gives the least value for $\mathrm{Av}_T(\sigma \cup \{(k, i)\})$.

(3) $n \notin T \cup P$: Find a $k > \max\{\max\{\mathrm{dom}(\sigma)\}, K\}$ such that k is of type T. In searching for this, you find (t', l') that allows you to do the computation to check that k is of this type T. Then with the input to this strategy, we believe $t'' = \max(t, t')$ and $l'' = \max(l, l')$ is sufficiently out there to compute both $\widetilde{\mathrm{Av}}_T(\sigma \cup \{(k, 0)\})$ and $\widetilde{\mathrm{Av}}_T(\sigma \cup \{(k, 1)\})$.

Now, set τ' to be $\sigma \cup \{(k, i)\}$ for whichever $i \in \{0, 1\}$ gives the least value for $\mathrm{Av}_T(\sigma \cup \{(k, i)\})$.

Finally, in all cases, extend τ' to a long enough τ by iterating the construction as in step (3).

3.5. **Verification.** Recursively (but not computably), determine the sequence

$$\langle (\sigma_i, P_i, j_i, (T_i, K_i), (t_i, l_i)) \mid i \in \omega \rangle$$

of correct parameters, that is, where all the assumptions as indicated in the previous section are in fact correct. Then it is clear from the construction that the martingale constructed in the previous section succeeds on all reals in $S = \bigcap_{i\in\omega}[\sigma_i]$.

It now remains to show that in S there is an injective random real. We see by induction that for all i, we have that $\widetilde{\mathrm{Av}}_{T_i}(\sigma_i) < 2$, and that in fact $\widetilde{\mathrm{Av}}_{\{0,\dots,n\}}(\sigma_i) = \widetilde{\mathrm{Av}}_{T_i}(\sigma_i)$.

Suppose that no real in S is injective random. Then

$$\mathcal{O} := \bigcup_{i\in\omega}\{[\sigma] \mid \exists k\,\forall\sigma' \succ \sigma\,\forall l \geq k\, d_i^{\sigma'}(l) > \frac{2}{c_{\sigma_i,\dots,(t_i,l_i)}}\}$$

is an open cover of S: Let $A \in S$. Then there is an injective betting strategy (h_i, q_i) that wins on A. This means, in particular, that there is an n such that

$$d_i^A(n) > \frac{2}{c_{\sigma_i,\dots,(t_i,l_i)}} + 2.$$

This in turn means

$$d_i^A(k) > \frac{2}{c_{\sigma_i,\dots,(t_i,l_i)}}$$

for all $k \geq n$, since (h_i, q_i) was assumed to be slow-but-sure. By now choosing m large enough, $\sigma = A \restriction m$ is as desired. Note that if $[\sigma] \in \mathcal{O}$, then for any extension $\sigma' \succ \sigma$, we have $[\sigma'] \in \mathcal{O}$ as well.

Since S is compact, we can find a finite subcover $[\nu_1], \dots, [\nu_n]$ of $\mathcal{O}$. Let b be such that for all $i \leq n$, there is a $j < b$ such that

$$\exists k\,\forall\sigma' \succ \sigma\,\forall l \geq k\, d_j^{\sigma'}(l) > \frac{2}{c_{\sigma_j,\dots,(t_j,l_j)}}.$$

By replacing some ν_i by $\nu_{i_0}, \dots, \nu_{i_k}$ such that $[\nu_i] = \cup_{j\leq k}[\nu_{i_j}]$, we can assume all ν_i have the same length and that this length is $|\sigma_h|$ for some $h > b$ big enough to satisfy

$$\forall\sigma' \succ \sigma\,\forall l \geq h\, d_j^{\sigma'}(l) > \frac{2}{c_{\sigma_j,\dots,(t_j,l_j)}}$$

(which ensures that in the computation of the average on σ_h, at least the h required bets are made). Now, for every $i \leq n$, there exists a $j < b$ such that either $c_{\sigma_j,\dots,(t_j,l_j)} \cdot d_j^h(\nu_i) = \tilde{d}_j^h(\nu_i) > 2$ or $\widetilde{\mathrm{Av}}_{0,\dots,j-1}(\nu_a) > 2$. Now we have reached the desired contradiction: On the one hand, we know $\mathrm{Av}_{\{0,\dots,b\}}(\sigma_h) \leq \mathrm{Av}_{\{0,\dots,h\}}(\sigma_h) < 2$; on the other hand, we have found an antichain covering $[\sigma_h]$ where on each element in the antichain for some $j \leq b$ either some $\tilde{d}_j$ is greater than 2 or $\widetilde{\mathrm{Av}}_{\{0,\dots,j-1\}}$ is greater than 2. This implies that the average $\widetilde{\mathrm{Av}}_{\{0,\dots,b\}}(\sigma_h) > 2$, and therefore $\widetilde{\mathrm{Av}}_{\{0,\dots,h\}}(\sigma_h) > 2$, which is a contradiction.

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