INITIAL SEGMENTS OF RECURSIVE LINEAR ORDERS

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Abstract. We show that any Σ^0_2-initial segment of a recursive linear order can be presented recursively.

0. Introduction. Recursive model theory is concerned with the extent to which constructions in mathematics (typically in algebraic structures) can be made effective. A countable structure $\mathcal{M}$ is called *recursive* if it is isomorphic to a structure $\mathcal{N}$ (called its *recursive presentation*) defined on the set of natural numbers in which all the relations and functions are recursive.

A particular question here concerns the study of substructures. A substructure $\mathcal{I}$ of $\mathcal{M}$ will correspond to a particular subset of ω. Yet, because of the algebraic structure defined on $\mathcal{M}$ and thus on $\mathcal{I}$, there may exist a presentation of $\mathcal{I}$ of much lower complexity than that of $\mathcal{I}$ itself.

In this paper, we give an example of such an interaction between algebra and computability theory concerning initial segments of linear orders. Note that the complexity of an initial

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segment of a recursive linear ordering can be very high (namely, non-hyperarithmetic). For example, by a theorem of Gandy [Ga67] and Harrison [Ha68] (see also [Sa90]), there is a recursive linear ordering with an initial segment isomorphic to ω_1^{CK}, the least nonrecursive ordinal.

The study of initial segments also arises naturally in the study of automorphisms of recursive linear orders: If a recursive linear order has only countably many automorphisms then it can be analyzed as a disjoint union of intervals such that an automorphism maps intervals onto intervals in some simple way (see Raw [Ra95]).

Here, we consider initial segments of recursive linear orders of fairly low (arithmetical) complexity. Raw [Ra95] obtained two results in this connection, one in each direction:

Theorem (Raw [Ra95]).

(1) *If adjacency in a recursive linear order is recursive then any Π^0_1-initial segment is recursive.*
(2) *There is a recursive linear order with a Π^0_3-initial segment that is not recursive.*

1. The theorem and proof. In this paper, we will narrow the gap between Raw's results by showing that indeed any Σ^0_2-initial segment of any recursive linear order must be recursive.[1] It is easy to see that such results on initial segments immediately extend to general segments.

Theorem. *Given a recursive linear order $\mathcal{L} = (L, <_L)$ and a Σ^0_2-initial segment $I \subseteq L$, there is a recursive linear order $\mathcal{M} = (M, <_M)$ isomorphic to $\mathcal{I} = (I, <_L)$ (via a Δ^0_2-isomorphism).*

Proof. Without loss of generality, we may assume $L = \omega$. We also fix a recursive approximation $\{\mathcal{L}_s\}_{s\in\omega} = \{(L_s, <_{L_s})\}_{s\in\omega}$, which is simply $\mathcal{L}$ restricted to the interval $[0, s]$.

We first exhibit a cofinal sequence of elements of $\mathcal{I}$ which will serve as a skeleton for our isomorphism from $\mathcal{I}$ to $\mathcal{M}$.

Lemma. *Given a recursive linear order $\mathcal{L} = (L, <_L)$ and a Σ^0_2-initial segment $I \subseteq L$ without greatest element, there is an increasing Δ^0_2-sequence of elements of I cofinal in $\mathcal{I}$.*

Proof. Since the index set of all finite recursively enumerable sets is Σ^0_2-complete, the Σ^0_2-subset I of L can be represented as

$$I = \{x \mid W_{f(x)} \text{ finite}\}$$

for some recursive function f. At each stage, we can now define a sequence $(x_{n,s}, w_{n,s})$ as follows: Let $(x_{0,s}, w_{0,s})$ be the least pair (in some coding of all pairs of natural numbers) such that $w_{0,s}$ is the cardinality of $W_{f(x_0),s}$, and let $(x_{n+1,s}, w_{n+1,s})$ be the least pair (if any) such that $x_{n+1,s} >_L x_{n,s}$ and $w_{n+1,s}$ is the cardinality of $W_{f(x_{n+1}),s}$. Since $\mathcal{I}$ has no greatest element (and is thus, in particular, infinite), the limit $x_n = \lim_s x_{n,s}$ exists for each n and will settle down on an element of I, i.e., we have a Δ^0_2-sequence of elements x_n of I. By the choice of the pairs $(x_{n,s}, w_{n,s})$ above, the sequence x_n will be increasing and cofinal in $\mathcal{I}$ as desired. ■

To aid the reader's intuition for the construction of the desired isomorphism below, let us now merely build an approximate isomorphism in the following sense: Introduce a new

[1] Coles, Downey, and Khoussainov have recently shown that this fails for Π^0_2-initial segments.

least element x_{-1} of $\mathcal{I}$. We build a recursive linear order $(M, <_M)$ and a cofinal sequence of elements y_n of M such that for any $n \geq 0$, the intervals (x_{n-1}, x_n) of $\mathcal{I}$ and (y_{n-1}, y_n) of $\mathcal{M}$ are either isomorphic to each other, or else they are both finite (i.e., we have made only a "small" number of mistakes). This sequence $\{y_n\}_{n \geq -1}$ and the model $\mathcal{M}$ can easily be built by fixing the elements y_n beforehand and then constructing the isomorphisms between the corresponding intervals, restarting every time our approximation $x_{n,s}$ to x_n changes. This will clearly make corresponding intervals isomorphic unless "we have already put more elements into the interval (y_{n-1}, y_n) than there will be eventually in the interval (x_{n-1}, x_n) before $x_{n-1,s}$ and $x_{n,s}$ settle down".

The next approximation to the construction below is then to "adjust" the isomorphism for the finitely many mistakes we will have made by mapping x_n "a bit to the left of" y_n. There are now two cases: Either almost all intervals (x_{n-1}, x_n) are finite, so there is a final segment of $\mathcal{I}$ of order type ω (and $\mathcal{I}$ is easily seen to be isomorphic to a recursive linear order); or else there will be infinitely many infinite intervals (x_{n-1}, x_n) at which we will be able to "catch up with our isomorphism". We will see below that this nonuniformity in the construction of our isomorphism is actually not even necessary.

Formally, our construction of the recursive linear order $\mathcal{M}$ and an isomorphism $f : \mathcal{I} \to \mathcal{M}$ now proceeds as follows. (Again, we will assume that $M = \omega$.) As in the lemma, fix the sequence $\{x_n\}_{n \geq -1}$ and its approximation $\{x_{n,s}\}_{n \geq -1, s \in \omega}$ ($x_{n,s}$ need not be defined for all s here). At stage $s = 0$, define $f_0(x_{-1}) = y_{-1}$, and let M_0 consist of an increasing sequence of elements $\{y_n\}_{n \geq -1}$. (We may assume that $M_0 =$ the set of even numbers, and that in going from M_s to M_{s+1}, we add some odd numbers.)

At each stage $s > 0$, define (for each $x_{n,s}$ which is defined) an image $z_{n,s} = f_s(x_{n,s})$ by induction as follows. Let $z_{-1,s} = y_{-1}$. For $n \geq 0$, let

$$z_{n,s} = \begin{cases} y_n, & \text{if } |(z_{n-1,s}, y_n] \cap M_s| \leq |(x_{n-1,s}, x_{n,s}] \cap L_s| \\ k\text{th point of } M_s \text{ past } z_{n-1,s}, & \text{otherwise, where } k = |(x_{n-1,s}, x_{n,s}] \cap L_s|. \end{cases}$$

Now define the isomorphism f_s on each interval $(x_{n-1,s}, x_{n,s}]$ by the first case which applies:

Case 1: $|(z_{n-1,s}, z_{n,s}] \cap M_s| = |(x_{n-1,s}, x_{n,s}] \cap L_s|$. Then let the image under f_s of the kth element of $(x_{n-1,s}, x_{n,s}] \cap L_s$ be the kth element of $(z_{n-1,s}, z_{n,s}] \cap M_s$.

Now note that if Case 1 fails to apply then $|(z_{n-1,s}, z_{n,s}] \cap M_s| < |(x_{n-1,s}, x_{n,s}] \cap L_s|$.

Case 2: $x_{i,s} \neq x_{i,s-1}$ for some $i \leq n$. Let $k_0 = |(z_{n-1,s}, z_{n,s}] \cap M_s|$. Then for each $k \in [1, k_0]$, let the image under f_s of the kth element of $(x_{n-1,s}, x_{n,s}] \cap L_s$ be the kth element of $(z_{n-1,s}, z_{n,s}] \cap M_s$. For each element of $(x_{n-1,s}, x_{n,s}] \cap L_s$ past the k_0th element, add a new element to the interval $(z_{n-1,s}, z_{n,s}] \cap M_s$, and extend f_s on these elements according to their ordering in L_s.

Case 3: Otherwise. Then we define $f_s(x) = f_{s-1}(x)$ for each element $x \in (x_{n-1,s}, x_{n,s}] \cap L_{s-1}$, and for each element of $(x_{n-1,s}, x_{n,s}] \cap (L_s - L_{s-1})$, add a new element to the interval $(z_{n-1,s}, z_{n,s}] \cap M_s$, and extend f_s on these elements according to their ordering in L_s.

This concludes the description of the construction of $\mathcal{M}$. We begin the verification that our construction works as desired in the following

Claim. *For each $n \geq -1$, $z_n = f(x_n) = \lim_s f_s(x_{n,s})$ exists, and for $n \geq 0$, the pointwise limit f of f_s on (x_{n-1}, x_n) exists and is an isomorphism onto (z_{n-1}, z_n).*

Proof. We proceed by induction on n. Our claim is trivial for $n = -1$. For $n \geq 0$, fix the least stage s_0 such that $z_{n-1} = f_s(x_{n-1,s})$ and $x_{i,s_0} = x_{i,s}$ for all $i \leq n$ and all $s \geq s_0$. We distinguish two cases:

Case 1: $|(z_{n-1,s_0}, z_{n,s_0}] \cap M_{s_0}| = |(x_{n-1}, x_n] \cap L|$. Then, in particular, this cardinality is finite, so once $(x_{n-1,s}, x_{n,s}] \cap L_s$ is stable, f will be an order-preserving map from $(x_{n-1}, x_n] \cap L$ onto $(z_{n-1,s_0}, z_{n,s_0}] \cap M_{s_0} = (z_{n-1}, z_n] \cap M$.

Case 2: Otherwise. Then $|(z_{n-1,s_0}, z_{n,s_0}] \cap M_{s_0}| < |(x_{n-1}, x_n] \cap L|$. Let $s_1 \geq s_0$ be least such that $|(z_{n-1,s_0}, z_{n,s_0}] \cap M_{s_0}| \geq |(x_{n-1,s_1}, x_{n,s_1}] \cap L_{s_1}|$. At that stage, our construction will determine the final images $f(x)$ for each $x \in (x_{n-1}, x_n] \cap L_{s_1}$. From then on, the construction will merely make new (and final) definitions of $f(x)$ for each $x \in (x_{n-1}, x_n] \cap (L - L_{s_1})$ when such x enters L.

This establishes the claim. ■

We now finish the proof of our theorem by distinguishing two cases:

Case I: There are infinitely many n such that $z_n = y_n$. Then clearly f gives an isomorphism from $\mathcal{I}$ onto $\mathcal{M}$.

Case II: Otherwise. Fix $n \geq -1$ maximal such that $z_n = y_n$. We now observe that f gives an isomorphism between $(x_{-1}, x_n]$ and $(z_{-1}, z_n]$ as in Case I; and also an isomorphism between the intervals $(x_n, \infty) \cap I$ and (z_n, ∞), since both of these have order type ω. ■

2. Concluding remarks. Our paper leaves open the case of a Π^0_2-initial segment of a recursive linear ordering. It is interesting to note that neither our technique for Σ^0_2-initial segments nor Raw's technique for Π^0_3-initial segments can work for Π^0_2-initial segments: On the one hand, for Π^0_2-initial segments, the cofinal Δ^0_2-sequence of elements of the initial segment need not exist as can easily be verified. On the other hand, for Π^0_2-initial segments, no coding in the spirit of Raw's proof for Π^0_3-initial segments can exist, since under very mild assumptions on the recursive linear ordering, any first-order definable fact about the initial segment (defined using $\mathcal{I}$ as a unary predicate) can actually be defined in $\mathcal{L}$ by a formula of the same complexity without using the unary predicate for $\mathcal{I}$.

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