Embedding finite lattices into the Σ_2^0 enumeration degrees

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Abstract

We show that every finite lattice is embeddable into the Σ_2^0 enumeration degrees via a lattice-theoretic embedding which preserves 0 and 1.

1 Introduction

Informally, a set A is enumeration reducible to a set B if there is some effective procedure for enumerating A, given any enumeration of B. This informal notion of reducibility can be formalized using the notion of enumeration operator. Let $\{W_i\}_{i\in\omega}$ be the standard listing of the computably enumerable (c.e.) sets. With every c.e. set W_i, one can associate a mapping $\Phi_i : P(\omega) \to P(\omega)$ (where $P(\omega)$ is the power set of the set of natural numbers ω) by letting, for every B,

$$\Phi_i^B = \{x : (\exists u)[\langle x, u\rangle \in W_i \ \&\ D_u \subseteq B]\}$$

(where $\langle\cdot,\cdot\rangle$ is the usual pairing function, providing a computable one-one bijection of ω^2 onto ω; and D_u is the finite set with canonical index u, i.e. D_u denotes the finite set D for which $u = \sum_{x\in D} 2^x$; see e.g. [Soa87]. In the following, finite sets will be often identified with their canonical indices). A mapping $\Phi : P(\omega) \to P(\omega)$ is called an *enumeration operator* (or simply an *e-operator*) if $\Phi = \Phi_i$ for some i.

Given sets of numbers A and B, we say that A is *enumeration reducible* (or simply *e-reducible*) to B if $A = \Phi^B$ for some e-operator Φ. This reducibility is

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easily seen to be a partial preordering relation, which will be denoted by the symbol $\leq_e$.

The degree structure induced by $\leq_e$ is the structure of the *enumeration degrees* (simply *e-degrees*), denoted by $\mathfrak{D}_e$. The e-degree of a set X will be denoted by $\deg_e(X)$. $\mathfrak{D}_e$ is in fact an upper semilattice with least element $\mathbf{0}_e$, with $\mathbf{0}_e = \deg_e(W)$ where W is any c.e. set. It is known (Gutteridge, see also [Coo82]) that $\mathfrak{D}_e$ does not have minimal elements (although the structure is not dense, see [Coo87]; Calhoun and Slaman, [CS96]), have shown that there exist Π^0_2 e-degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ is a minimal cover of $\mathbf{a}$). An important substructure of $\mathfrak{D}_e$ is given by the Σ^0_2 e-degrees, i.e. the e-degrees of the Σ^0_2 sets. Let $\mathfrak{G}$ denote the structure of the e-degrees of the Σ^0_2 sets. Cooper ([Coo84]) shows that $\mathfrak{G} = \mathfrak{D}_e(\leq_e \mathbf{0}'_e)$ where $\mathbf{0}'_e = \deg_e(\overline{K})$, $\overline{K}$ being the complement of the halting set (for a definition of the jump operation on the e-degrees, see [Coo84] and [MC85]). Cooper ([Coo84]) shows that $\mathfrak{G}$ is dense.

An interesting feature of the e-degrees is that they provide a wider context for the Turing degrees. Indeed, let $\mathfrak{D}_T$ denote the structure of the Turing degrees. One can define an embedding $\iota : \mathfrak{D}_T \longrightarrow \mathfrak{D}_e$ which is 0-, $\cup$- and jump-preserving (simply define $\iota(\deg_T(A)) = \deg_e(c_A)$ where $\deg_T(A)$ and c_A denote the Turing degree and the characteristic function of A, respectively). It is interesting to notice that the c.e. Turing degrees (whose structure we denote by $\mathfrak{R}$) are isomorphic, under ι, to the Π^0_1 e-degrees, as one can prove (see for instance [Coo84]) that for every e-degree $\mathbf{a}$,

$$\mathbf{a} \in \text{range}(\iota \upharpoonright \mathfrak{R}) \Leftrightarrow \mathbf{a} \in \Pi^0_1.$$

Say that an e-degree $\mathbf{a}$ is *low* if $\mathbf{a}' \leq \mathbf{0}'_e$. Cooper and McEvoy (see [MC85]) give the following useful characterization of the low e-degrees.

Theorem 1.1 ([MC85]) *Given an e-degree* $\mathbf{a}$*, the following are equivalent:*

1. $\mathbf{a}$ *is low;*
2. *for every set* B*,* $\deg_e(B) \leq \mathbf{a} \Rightarrow B \in \Delta^0_2$*;*
3. *there exists a set* $A \in \mathbf{a}$*, together with a* Σ^0_2 *approximation* $\{A^s\}_{s\in\omega}$ *to* A *(i.e. a uniformly computable sequence of computable sets such that* $A = \{x : (\exists t)(\forall s \geq t)[x \in A^s]\}$*) such that, for every* e, x*,* $\lim_s \Phi^{A^s}_{e,s}(x)$ *exists.*

The main sources for notation and terminology are [Odi89], [Rog67], [Soa87]. If $\mathcal{A}$ is an expression and s is a stage then (as in [Soa87], p. 315), the symbol $\mathcal{A}[s]$ denotes the evaluation of the expression $\mathcal{A}$ at stage s. If $\{W^s_i\}_{s\in\omega}$ is a computable approximation to W_i via finite sets (in the sense of [Soa87], p. 18), then we get a corresponding computable approximation $\{\Phi^s_i\}_{s\in\omega}$ to the e-operator

Φ_i. If Ψ is an e-operator with computable approximation $\{\Psi_s\}_{s\in\omega}$, and $\{X^s\}_{s\in\omega}$ is a Σ^0_2 approximation to a set X, then $\{\Psi^X[s]\}_{s\in\omega}$ denotes the Σ^0_2 approximation to the set Ψ^X described in Proposition 5 of [MC85]: first, for every finite set $D \subseteq X^s$, define $t(D,s)$ to be the least $t \le s$ for which $D \subseteq X^u$, for every u such that $t \le u \le s$; then, if $x \in {\Psi_s}^{X^s}$, let $D(x,s)$ be the least finite set D such that $\langle x, D\rangle \in \Psi_s$ and $D \subseteq X^s$, and $t(D,s) \le t(D',s)$ for every other finite set D' with the same property (let $D(x,s)$ be undefined if $x \notin {\Psi_s}^{X^s}$). Finally define by induction $\Psi^X[0] = \emptyset$ and

$$\Psi^X[s+1] = \{x \in {\Psi_{s+1}}^{X^{s+1}} : D(x,s+1) = D(x,s)\}.$$

(Notice that if there are axioms $\langle x, F^s\rangle \in \Psi$, such that, for cofinitely many s, $F^s \subseteq X^s$ but, for no s, $F^s \subseteq X$, one could have $x \in {\Psi_s}^{X^s}$ for cofinitely many s, but $x \notin \Psi^X$: The definition of $\{\Psi^X[s]\}_{s\in\omega}$ avoids this problem. Notice also that the sequence $\{\Psi^X[s]\}_{s\in\omega}$ is uniform in the given sequences $\{\Psi_s\}_{s\in\omega}$ and $\{X^s\}_{s\in\omega}$.)

1.1 The embeddability question

One of the most important open problems concerning the c.e. Turing degrees is the so-called *Embeddability Question*, i.e. the question of which finite lattices can be embedded into $\mathfrak{R}$. We briefly review the existing literature concerning this question. In the following, by an *embedding* of a lattice $\mathfrak{L}$ into an upper semilattice $\mathfrak{U}$ we shall mean a 1-1 mapping f from $\mathfrak{L}$ into $\mathfrak{U}$, preserving $\vee$ and $\wedge$ (hence for every $a, b \in \mathfrak{L}$, $h(a)\wedge h(b)$ must exist in $\mathfrak{U}$, and $h(a\wedge b) = h(a)\wedge h(b)$).

Thomason ([Tho71]) proved that every finite distributive lattice is embeddable into $\mathfrak{R}$. Lachlan (unpublished) and Lerman (unpublished) showed that the countable atomless Boolean algebra is embeddable into $\mathfrak{R}$ via a 0-preserving embedding (see for instance [Soa87] for a proof of this theorem). This implies that every countable distributive lattice can be embedded into $\mathfrak{R}$, preserving 0. On the other hand, Ambos-Spies ([AS80]) proved that the countable atomless Boolean algebra is embeddable into $\mathfrak{R}$ via a 1-preserving embedding. Thus every countable distributive lattice is so embeddable into $\mathfrak{R}$. Lachlan ([Lac72]) showed also that the two five-element nondistributive lattices M_5 and N_5 are embeddable into $\mathfrak{R}$ (hence $\mathfrak{R}$ is not a distributive upper semilattice).

As to negative results, Lachlan and Soare ([LS80]) proved that the lattice S_8 cannot be embedded into $\mathfrak{R}$. It was conjectured for some years (Downey's conjecture) that the only obstacle to embeddability should be the existence of a critical triple a, b, c in the lattice (i.e. a triple of pairwise incomparable elements a, b, c such that $a \cup c = a \cup b$ and $b \cap c \le a$), together with a pair p, q such that $b \le p \cap q \le b \cup c$. However, Lempp and Lerman (see [LL97]) showed that there exists a finite lattice (called L_{20} in [LL97]) without critical triples that is not embeddable into $\mathfrak{R}$.

As to $0, 1$-preserving embeddings, the question as to which finite distributive lattices can be embedded into $\mathfrak{R}$ has been settled. Ambos-Spies, Lempp and Lerman ([ASLL94]) showed that a finite distributive lattice can be embedded into $\mathfrak{R}$ preserving $0, 1$ if and only if the lattice contains a join-irreducible non-cappable element. This theorem supersedes several known nonembeddability results, including the famous Lachlan Nondiamond Theorem ([Lac66]), stating that the diamond (i.e. the four-element Boolean algebra) cannot be embedded into $\mathfrak{R}$ preserving $0, 1$.

What is the situation for $\mathfrak{S}$? Cooper (see [AL99]) asks for a characterization of the lattices which are embeddable in $\mathfrak{S}$. Useful information is provided by the following observation due to Cooper and McEvoy:

Theorem 1.2 ([MC85]) *If* $\mathbf{a}, \mathbf{b}$ *are* Π^0_1 *e-degrees and* $\mathbf{a}$ *is low then*

$$(\forall \mathbf{c})[\mathbf{c} \leq \mathbf{a}, \mathbf{b} \Rightarrow (\exists \mathbf{e} \in \Pi^0_1)[\mathbf{c} \leq \mathbf{e} \leq \mathbf{a}, \mathbf{b}]].$$

Thus, if $\mathfrak{L}$ is a lattice and $h : \mathfrak{L} \longrightarrow \mathfrak{R}$ is a lattice theoretic embedding such that range(h) contains only low Turing degrees then the composition $\iota \circ h : \mathfrak{L} \longrightarrow \mathfrak{S}$ is a lattice theoretic embedding as well.

Since every lattice which is known to be embeddable into $\mathfrak{R}$ is known to embed in fact into the low Turing degrees, one concludes that every lattice which is known to be embeddable into $\mathfrak{R}$ is also embeddable into $\mathfrak{S}$. For instance, it follows by the above mentioned results for $\mathfrak{R}$ that every countable distributive lattice is embeddable into $\mathfrak{S}$, and M_5 and N_5 are embeddable into $\mathfrak{S}$. On the other hand, Nies and Sorbi ([NS99]) showed that S_8 can be embedded into $\mathfrak{S}$, thus in fact the class of finite lattices that are embeddable into $\mathfrak{S}$ properly extends that of the finite lattices that are known to be embeddable into $\mathfrak{R}$.

If one is interested in $0, 1$-preserving embeddings, the starting point is the following result, proved by Ahmad, see [Ahm91]:

Theorem 1.3 *The diamond can be embedded into* $\mathfrak{S}$ *preserving* $0, 1$.

1.2 The theorem

We extend Ahmad's result to show that indeed every finite lattice is embeddable into $\mathfrak{S}$, preserving $0, 1$. As far as finite lattices are concerned, this answers the above mentioned question raised by Cooper.

Theorem 1.4 *Every finite lattice is embeddable into the* Σ^0_2 *enumeration degrees via an embedding which preserves* 0 *and* 1*. Moreover, the range of the embedding contains only low e-degrees, except for the image of* 1.

The following sections are devoted to the proof of this theorem.

2 The proof

Let $\mathfrak{L} = \langle L, \vee, \wedge, 0, 1, \leq \rangle$ be a finite lattice and let $\mathcal{J}$ be the set of join-irreducible elements of $\mathfrak{L}$. Define

$$\mathfrak{J} = \{J \subseteq \mathcal{J} : (\forall i, j)[j \in J \,\&\, i \leq j \Rightarrow i \in J]\}.$$

For every $a \in L$, let $J_a = \{j \in \mathcal{J} : j \leq a\}$: thus $J_a \in \mathfrak{J}$. Notice the following

Lemma 2.1 *For every $a \in L$, $a = \bigvee J_a$.*

Proof: Clearly $\bigvee J_a \leq a$. To show the converse, it is clear that in a finite lattice every element is the join of some set of join-irreducible elements. Thus, let $J \subseteq \mathcal{J}$ be such that $a = \bigvee J$; then $J \subseteq J_a$, and therefore $a \leq \bigvee J_a$. □

Lemma 2.2 *For every $a, b, c \in L$, if $a \wedge b = c$ then*

$$J_a \cap J_b = J_c.$$

Proof: If $a \wedge b = c$ then clearly $J_c \subseteq J_a$ and $J_c \subseteq J_b$. On the other hand, suppose that $j \in J_a \cap J_b$. Then $j \leq a, b$, so $j \leq c$ and thus $j \in J_c$. □

In the construction below, for every $j \in \mathcal{J}$ we define a Σ^0_2 set B_j (with $B_0 = \emptyset$ and $B_j \subseteq \omega^j (= \{x : (\exists y)[x = \langle j, y \rangle]\}$ if $j > 0$) through Σ^0_2 approximations $\{B^s_j : s \in \omega\}$ where B^s_j is defined at step s of the construction. For every $a \in L$, we let $A_a = \bigoplus_{j \in J_a} B_j$ (we think of $\mathcal{J}$ as a subset of ω; thus if $X \subseteq \mathcal{J}$ then it is convenient to identify $\bigoplus_{i \in X} B_i = \bigcup_{i \in X} B_i$; accordingly, if $\{F^i : i \in X\}$ where $X \subseteq \mathcal{J}$ is a family of sets and where $F^i \subseteq \omega^i$ for every $i \in X$, then we let $\bigoplus_{i \in X} F^i = \bigcup_{i \in X} F^i$). Given $X \subseteq \mathcal{J}$, we also let $A_X = \bigoplus_{i \in X} B_i$. If F is any set of numbers then we let $F^j = F \cap \omega^j$.

We observe that each A_a and each A_X are Σ^0_2 sets, with Σ^0_2 approximations determined by the Σ^0_2 approximations to the sets B_j with $j \in \mathcal{J}$. Let also $\overline{K}^s = \{x : x \notin K^s \,\&\, x \leq s\}$ where $\{K^s : s \in \omega\}$ is some nondecreasing computable sequence of finite sets whose union is K.

2.1 The requirements

For all $a, b, c \in L$, all $j \in \mathcal{J}$ and all $X \in \mathfrak{J}$, and for all pairs of e-operators Φ, Ψ, fix the requirements

$$\begin{array}{ll} \mathcal{J}_{j,X}: & j \leq \bigvee X \Rightarrow B_j = \Gamma^{A_X}_{j,X} \\ \mathcal{M}_{a,b,c,\Phi,\Psi}: & a \wedge b = c \Rightarrow [\Phi^{A_a} = \Psi^{A_b} \Rightarrow \Phi^{A_a} = \Delta^{A_c}_{a,b,c,\Phi,\Psi}] \\ \mathcal{Q}_{j,a,\Phi}: & j \not\leq a \Rightarrow B_j \neq \Phi^{A_a} \\ \mathcal{L}_{a,x,\Phi}: & a < 1 \Rightarrow \lim_s \Phi^{A^s_a}_s(x) \text{ exists} \end{array}$$

where $\Gamma_{j,X}$ and $\Delta_{a,b,c,\Phi}$ are e-operators to be constructed.

If 1 is join-reducible in $\mathcal{L}$ then we also have for each $z \in \omega$ the requirement

$$\mathcal{R}_z : \overline{K}(z) = \Gamma^{A_{\mathcal{J}}}(z)$$

where Γ is an e-operator to be constructed. (Of course, Γ must be constructed uniformly in z, but for technical reasons, we spread its construction over the entire tree of strategies.)

2.1.1 The embedding

Define $\kappa : \mathcal{L} \longrightarrow \mathfrak{S}$,

$$\kappa(a) = \begin{cases} \deg_e(A_a) & \text{if } a \neq 1 \\ \mathbf{0}'_e & \text{otherwise} \end{cases}$$

We now show that if all the requirements are satisfied then κ is a lattice-theoretic embedding, preserving 0 and 1.

(1) If $a \leq b$ then $J_a \subseteq J_b$, and thus $A_a \leq_e A_b$;

(2) assume that $a \wedge b = c$; then $A_c \leq_e A_a$ and $A_c \leq_e A_b$ by (1); on the other hand, if $Z \leq_e A_a$ and $Z \leq_e A_b$ then there exist e-operators Φ, Ψ such that $Z = \Phi^{A_a} = \Psi^{A_b}$; therefore by satisfaction of the requirement $\mathcal{M}_{a,b,c,\Phi,\Psi}$ we have that $Z \leq_e A_c$, thus getting

$$\deg_e(A_a) \cap \deg_e(A_b) = \deg_e(A_c)$$

as desired;

(3) assume that $a \vee b = c$; then $A_a \leq_e A_c$ and $A_b \leq_e A_c$ by (1); on the other hand, for every $j \in \mathcal{J}$,

$$\begin{aligned} j \in J_c \quad &\Rightarrow \quad j \leq a \vee b \\ &\Rightarrow \quad j \leq \bigvee J_a \vee \bigvee J_b = \bigvee X \end{aligned}$$

where $X = J_a \cup J_b$ and $X \in \mathfrak{J}$; hence $A_j \leq_e A_X$ by satisfaction of the requirement $\mathcal{J}_{j,X}$. In conclusion, $A_c \leq_e A_X \equiv_e A_a \oplus A_b$. Notice that this is true also if $c = 1$; indeed, if c is join-irreducible then $a = 1$ or $b = 1$.

Items (2) and (3) above show that κ is a lattice theoretic homomorphism. We now proceed with showing that κ is $1-1$ and preserves 0 and 1:

(4) assume that $a \not\leq b$; then by Lemma 2.1, there exists some $j \in J_a$ such that $j \not\leq b$; hence $B_j \not\leq_e A_b$ by satisfaction of the requirements $\mathcal{Q}_{j,b,\Phi}$; it follows that $A_a \not\leq_e A_b$;

(5) we have $\kappa(0) = \deg_e(A_0) = \deg_e(B_0) = \deg_e(\emptyset) = \mathbf{0}_e$, thus κ preserves 0;

(6) if 1 is join-irreducible then trivially $\kappa(1) = \mathbf{0}'_e$; otherwise, κ is 1-preserving by satisfaction of the requirements $\mathcal{R}_z$.

2.2 Strategies in isolation

We briefly explain our plan for satisfying the requirements.

Remark 2.3 In the following, we will make the following assumptions when considering a requirement:

1. when dealing with a requirement $\mathcal{J}_{j,X}$, we will always assume that $j \notin X$ (otherwise satisfaction of the requirement is automatic by our definition of A_X);

2. when dealing with a requirement $\mathcal{Q}_{j,a,\Phi}$, we will always assume that $j \not\leq a$;

3. when dealing with an $\mathcal{L}$-requirement $\mathcal{L}_{a,x,\Phi}$, we will always assume that $a < 1$;

4. finally, when dealing with an $\mathcal{M}$-requirement $\mathcal{M}_{a,b,c,\Phi,\Psi}$, we will assume that $c = a \wedge b$ and $a|b$ (if $a \leq b$ or $b \leq a$ then the requirement is automatically satisfied by our definition of A_a and A_b).

2.2.1 The $\mathcal{J}$-requirements

Consider a requirement $\mathcal{J}_{j,X}$. If we need to enumerate a number x into B_j then, as we also need $x \in \Gamma_{j,X}^{A_X}$, we select numbers $y_x^i \in \omega^i$ (with $y_x^j = x$), we add an axiom $\langle x, \bigoplus_{i\in X}\{y_x^i\}\rangle$ into $\Gamma_{j,X}$, and for every $i \in X$, we enumerate y_x^i into B_i. On the other hand, if $i \in X$ and $i \leq \bigvee Y$ with $i \notin Y$ and $Y \in \mathfrak{J}$, then we add also the axiom $\langle y_x^i, \bigoplus_{k\in Y}\{y_x^k\}\rangle$ into $\Gamma_{i,Y}$, and enumerate each y_x^k into B_k for every $k \in Y$, as we must get $y_x^i \in \Gamma_{i,Y}^{A_Y}$; we must proceed in this way, until all the relevant e-operators $\Gamma_{i,Y}$ are updated. We call this procedure the *functional updating procedure.*

We are therefore led to the following definition.

Definition 2.4 Given any $j \in \mathcal{J}$, let β^j be the least set β of pairs (i, Y) with $i \in \mathcal{J}, Y \in \mathfrak{J}$, $i \notin Y$ such that:

- $(j, Y) \in \beta$ for every Y such that $j \notin Y$ and $j \leq \bigvee Y$;

- if $(i, Y) \in \beta$ then $(i', Y') \in \beta$ for every pair (i', Y') such that $i' \in Y$, $i' \notin Y'$ and $i' \leq \bigvee Y'$.

Let also $\beta(j) = \{i : i = j \text{ or } (\exists k, Y)[(k, Y) \in \beta^j \,\&\, i \in Y]\}$.

The set β^j tells us which e-operators $\Gamma_{i,Y}$ must be updated, following the enumeration of a number in B_j. The set $\beta(j)$ tells us which sets B_i are involved in this updating procedure via enumeration of some number in B_i.

Since a number x may be enumerated into and extracted from some B_j finitely often, the operation of appointing new numbers y^i_x with $i \in \mathcal{J}$ may be repeated several times. For each i we thus end up appointing finitely many numbers y^i_x.

The construction will ensure that, whenever we need to extract x from B_j at some given stage, we are able to select some set $Z \subseteq \mathcal{J}$ such that extraction of sufficiently many numbers y^i_x from B_i for each $i \in Z$ not only gives $\Gamma_{j,X}$-rectification at x, i.e. $x \notin \Gamma^{A_X}_{j,X}$, but automatically provides all needed rectifications of all e-operators which are involved in this chain of extractions.

2.2.2 The $\mathcal{M}$-requirements

Consider the requirement $\mathcal{M}_{a,b,c,\Phi,\Psi}$. The strategy here basically consists in defining an e-operator $\Delta_{a,b,c,\Phi,\Psi}$ such that if $Z = \Phi^{A_a} = \Psi^{A_b}$ then $Z = \Delta^{A_c}_{a,b,c,\Phi,\Psi}$; stage by stage, if $x \in \Phi^{A_a} \cap \Psi^{A_b}$ via, say, axioms $\langle x, \bigoplus_{i \in J_a} G^i \rangle \in \Phi$ and $\langle x, \bigoplus_{i \in J_b} H^i \rangle \in \Psi$ where $G^i \subseteq B_i$ for every $i \in J_a$ and $H^i \subseteq B_i$ for every $i \in J_b$ then we define a suitable axiom $\langle x, \bigoplus_{i \in J_c} G^i \cup H^i \rangle \in \Delta_{a,b,c,\Phi,\Psi}$.

The extraction activity of both lower-priority $\mathcal{R}_z$- and lower-priority $\mathcal{Q}$-requirements may interfere with this strategy in that, at some given stage, it may entail some x being extracted from Z without entailing x being extracted from $\Delta^{A_c}_{a,b,c,\Phi,\Psi}$. If for instance x is extracted from Φ^{A_a} then, since $x \in \Delta^{A_c}_{a,b,c,\Phi,\Psi}$, there is no a priori obstacle to restraining $x \in \Psi^{A_b}$, following, if needed, reinsertion of $x \in \Psi^{A_b}$ via enumeration or re-enumeration of suitable sets F^i in B_i for every $i \in J_b$. We rely therefore on the possibility of restraining some such number x in either Φ^{A_a} or in Ψ^{A_b}, at the same time keeping such a number x out of the other set, thus getting $x \in \Phi^{A_a} - \Psi^{A_b}$ or $x \in \Psi^{A_b} - \Phi^{A_a}$.

More details on this action instigated by the extraction activity of lower-priority $\mathcal{R}_z$- or other lower-priority $\mathcal{Q}$-requirements will be given in 2.3.2.

On the other hand, the lowness of A_a and A_b will ensure that $\lim_s \Phi^{A_a}(x)[s]$ and $\lim_s \Psi^{A_b}(x)[s]$ exist. We therefore distinguish two possible outcomes of our strategy: Either we are able to force a permanent disagreement $x \in \Phi^{A_a} - \Psi^{A_b}$ or $x \in \Psi^{A_b} - \Phi^{A_a}$ for some x; or each such attempt fails, giving $Z = \Delta^{A_c}_{a,b,c,\Phi,\Psi}$ if $Z = \Phi^{A_a} = \Psi^{A_b}$.

2.2.3 The $\mathcal{Q}$-requirements

Consider $\mathcal{Q}_{j,a,\Phi}$. The strategy for satisfying the requirement is the following:

1. choose a witness c; enumerate $c \in B_j$ and apply the procedure of functional updating (thus selecting suitable numbers y^i_c with $y^j_c = c$);

2. await $c \in \Phi^{A_a}$;

3. restrain $c \in \Phi^{A_a}$ via restraining some finite sets $F^i \subseteq B_i$ for $i \in J_a$;

4. extract y^i_c from B_i for every i such that $i \notin J_a$.

We observe that the extractions in 4. not only provide $c \notin B_j$, since $j \not\leq a$, but also ensure automatic rectifications of all relevant e-operators, as can be seen easily from the following lemma:

Lemma 2.5 *For every $(i, Y) \in \beta^j$, if $i \not\leq a$ then there exists $y \in Y$ such that $y \not\leq a$.*

Proof: This is certainly true of all pairs $(j, Y) \in \beta^j$. Suppose now that this fails for a pair $(i, Y) \in \beta^j$ with $i \not\leq a$; then $y \leq a$ for every $y \in Y$, thus $\bigvee Y \leq a$, and so $i \leq a$, contradiction.

2.2.4 The $\mathcal{L}$-requirements

Consider the requirement $\mathcal{L}_{a,x,\Phi}$ with $a < 1$. The strategy for this simply consists in waiting for finite sets F^j with $j \in J_a$ such that $x \in \Phi^{F^a}$, and restraining such a finite set F^a in A_a.

2.2.5 The $\mathcal{R}_z$-requirements

If $z \in \overline{K}$ then we select j-traces y^j_z and we enumerate the axiom $\langle z, \bigoplus_{j \in \mathcal{J}} \{y^j_z\}\rangle$ into Γ together with enumerating y^j_z into B_j so as to get $z \in \Gamma^{A_{\mathcal{J}}}$. For every pair $i \in \mathcal{J}$ and $X \in \mathfrak{J}$ such that $i \notin X$ and $i \leq \bigvee X$, add also the axiom $\langle y^i_z, \bigoplus_{j \in X} \{y^j_z\}\rangle$ into $\Gamma_{i,X}$.

If and when z leaves $\overline{K}$, we extract all the numbers y^j_z from the respective sets B_j.

2.3 Combining the strategies

We examine here some of the main problems that arise from combining the strategies.

2.3.1 $\mathcal{R}_z$ injuring other strategies

Suppose that z leaves $\overline{K}$ and we still need to extract z from $\Gamma^{A_{\mathcal{J}}}$ at s (i.e. we *activate* the $\mathcal{R}_z$-strategy. We recall that the $\mathcal{R}_z$-strategy can be activated only at even stages). For any given j, it is, of course, possible that some y^j_z might have been extracted and inserted (for instance, due to the effects of the instigation actions explained later, or due to strategies for lowness) several times in and out of B_j, entailing corresponding procedures of functional updating. Let $Y^j(z)$ be the set of elements which are selected during this process and enumerated at some time in B_j. It is clear that the extraction of y^j_z might not be enough for rectifying all the relevant e-operators $\Gamma_{i,X}$. This rectification process would be, however, achieved if we could extract all of $Y^j(z)$ from B_j, for every j.

These extractions might, however, interfere with the restraining activity of some higher-priority requirements. More precisely, let $\mathcal{P}$ be a higher priority requirement on behalf of which we want to restrain a finite set, say $F^a \subseteq A_a$, and there are elements $j \in J_a$ such that $Y^j(z) \cap F^j \neq \emptyset$: We say in this case that $\mathcal{R}_z$ *injures* $\mathcal{P}$. Then we choose the highest-priority requirement which is injured in this way; for every $j \in \mathcal{J} - J_a$, we define $E^j = Y^j(z)$ (for the *current* value of $Y^j(z)$ at the current stage), we extract E^j from B_j, and we start the procedure of functional updating on behalf of those numbers which are restrained in A_a by $\mathcal{P}$ but might have lost some of their traces due to the above extraction activity. We notice that, following this updating procedure, some new traces might be appointed, and consequently some $Y^j(z)$ might become larger without, however, modifying our previous choice of E^j.

This action satisfies $\mathcal{R}_z$ (since we will see that we can always assume that $\mathcal{J} - J_a \neq \emptyset$). Moreover, it will be shown that at the same time it guarantees automatic rectification of all the operators involved in the required chain of extractions. The reason for this is roughly the following. Suppose that $j \leq \bigvee X$ and y is extracted from B_j, since $y \in Y^j(z)$ and $Y^j(z)(= E^j)$ is extracted from B_j. If we do not achieve $\Gamma_{j,X}-$rectification relatively to y (i.e. y is not extracted from $\Gamma^{A_X}_{j,X}$), then this is because the restraining activity required by the strategy for meeting $\mathcal{P}$ (i.e. the highest-priority requirement which is injured by $\mathcal{R}$), prevents us from extracting some $Y^i(z)$, with $i \in X$. But this can not be the case. Indeed, assume that $\mathcal{P}$ prevents $\Gamma_{j,X}-$correction. If $\mathcal{P} = \mathcal{L}_{a,x,\Phi}$, with $a < 1$, then we have that $X \subseteq J_a$ and $j \not\leq a$, but also $j \leq a$ since $j \leq \bigvee X$, thus getting a contradiction. On the other hand, if $\mathcal{P} = \mathcal{M}_{a,b,c,\Phi,\Psi}$ and $\mathcal{P}$ requires, say, restraining in A_b, then we have that $j \in J_a - J_c$ and $X \subseteq J_b$; but then $j \leq c$, again a contradiction. Finally, if $\mathcal{P} = \mathcal{Q}_{i,Y,\Phi}$, with $i \notin Y$, then $X \subseteq Y$, but then $i \notin X$, contrary to the assumption that $\mathcal{P}$ prevents us from rectifying $\Gamma^{A_X}_{j,X}(y)$.

Finally, we observe that each $\mathcal{P}$ may be injured only finitely many times by the requirements $\mathcal{R}_z$ since, as we shall see, the restraining activity of $\mathcal{P}$ only refers to some finite set, which therefore can contain only finitely many traces relative to finitely many numbers z.

2.3.2 Instigation

We now briefly look at another aspect of the interaction between an $\mathcal{M}$-requirement with $\mathcal{R}_z$, or between an $\mathcal{M}$-requirement and a $\mathcal{Q}$- requirement of lower priority. Let us consider $\mathcal{M} = \mathcal{M}_{a,b,c,\Phi,\Psi}$, and let $\mathcal{P}$ be either $\mathcal{R}_z$ or a $\mathcal{Q}$-requirement.

There are no a priori problems, of course, if the strategy for $\mathcal{M}$ has a finitary outcome corresponding to some x such that $\Phi^{A_a}(x) \neq \Psi^{A_b}(x)$. In this case $\mathcal{R}_z$ interacts with $\mathcal{M}$ just via the injury process described in the previous section.

What happens if $\mathcal{P}$ assumes that $\Phi^{A_a} = \Psi^{A_b}$? Suppose that $\mathcal{P}$ wants to extract E^j from B_j for some $j \in \mathcal{J}$. It might be that this extraction causes x to leave $\Phi^{A_a} \cap \Psi^{A_b}$ for some x but not x leaving $\Delta^{A_c}_{a,b,c,\Phi,\Psi}$. We say in this case that $\mathcal{P}$ *instigates* the $\mathcal{M}$-strategy. If, for instance, x leaves Φ^{A_a} then we have the opportunity, possibly at the expense of re-enumerating (with a consequent functional updating process) numbers in A_b, of restraining $x \in \Phi^{A_b}$, without interfering with the effects of the extractions from A_a which give $x \notin \Phi^{A_a}$; similarly, if x leaves Ψ^{A_b} then we have the opportunity of restraining $x \in \Phi^{A_a}$. (If x leaves $\Phi^{A_a} \cup \Psi^{A_b}$ then we simply choose which side we want to restrain.)

The effects of this instigation process remain valid until x re-enters $\Phi^{A_a} \cap \Psi^{A_b}$.

2.4 The tree of outcomes

Let $T = 2^{<\omega}$ be the *tree of outcomes*. We assume throughout a requirement assignment function R, effectively assigning to each node σ of the tree, a requirement $R(\sigma)$ such that along any path of T, each requirement is assigned exactly once, where $R(\sigma)$ is either an $\mathcal{R}_z$-, or a $\mathcal{Q}$-requirement, or an $\mathcal{M}$-requirement, or an $\mathcal{L}$-requirement. Finally we assume also that if $\sigma \subset \tau$ then $R(\sigma)$ has higher priority than $R(\tau)$.

Notation and terminology relative to strings are standard and can be found e.g. in [Soa87]. For clarity, we use $\preceq$ and $\prec$ for the nonstrict and strict lexicographical ordering on a tree $T \subseteq 2^{<\omega}$, respectively; and $\sigma \prec_L \tau$ to denote $\sigma \prec \tau$ but $\sigma \not\subseteq \tau$.

Let $\{\xi^j_\sigma\}_{\sigma \in T, j \in \omega}$ be a computable partition of the odd numbers into infinite computable sets, and let $\{\eta^j_z\}_{j,z \in \omega}$ be a computable partition of the even numbers such that $\xi^j_\sigma, \eta^j_z \subseteq \omega^j$ for every j, σ, z. Finally, if $\sigma \in T$ and $|\sigma| > 0$ then let $\sigma^- = \sigma \restriction (|\sigma| - 1)$.

2.5 Description of the construction along the true path

We explain in this section the meaning of the main auxiliary functions appearing in the construction, and we give a little more insight into the way in which the different strategies are combined along the true path f, with a brief description of some of their outcomes. The following refers to some stage s of the construction; consequently, the various auxiliary functions are understood to be evaluated at stage s; also we assume that by stage s, the higher-priority requirements have, so to speak, already settled down.

The strategies, directly or via instigation, may enumerate and restrain, or extract elements from, B_j with $j \in \mathcal{J}$. If o is the outcome at stage s of the strategy $R(\sigma)$ located at σ then, for every $j \in \mathcal{J}$, we use the symbol $F^j(\sigma\frown o)$ to denote the value of some finite set which $R(\sigma)$ wants to restrain in B_j, and we use the symbol $E^j(\sigma\frown o)$ to denote the value of some finite set which $R(\sigma)$ wants to extract from B_j.

The $\mathcal{R}_z$-nodes. For each z, there is an $\mathcal{R}_z$-node devoted to coding z in Γ^{A_X}.

If $z \in \overline{K}$ then $\sigma\frown 1 \subset f$ and $F^j(\sigma\frown 1) = \{y^j_z\}$. If $z \notin \overline{K}$ then we look (at even stages) for an opportunity to extract the set $Y^j(z)$ from B_j for $j \in \mathcal{J}$. There might be possible conflicts (through the injury mechanism, described in 2.3.1) with $F^j(\tau)$, with $\tau \subseteq \sigma$ (i.e. $Y^j(z) \cap F^j(\tau) \neq \emptyset$) for some j, as we also want to restrain at τ some finite sets $F^j(\tau)$ for, say, $j \in J_a$ for some a. We choose the least such τ; we define $E^i(\tau) = Y^i(z)$, and we extract $E^i(\tau)$ from B_i for each $i \in \mathcal{J} - J_a$; we apply the procedure of functional updating in relation to those numbers $y \in Y^j(z) \cap F^j(\tau)$ for which new traces are needed because of this extraction; we put z into $\lambda(\tau)$. (Thus, $\lambda(\tau)$ records the set of numbers $z \notin \overline{K}$ which τ takes the responsibility of keeping out of Γ^{A_X}. Since $F^j(\tau)$ is eventually finite, it follows that $\lambda(\tau)$ is eventually finite as well.) On the other hand, if no conflict arises with any $\tau \subseteq \sigma$ then we have $\sigma\frown 0 \subset f$ and we let $E^j(\sigma\frown 0) = Y^j(z)$. In this case, we let $\lambda(\sigma\frown 0) = \{z\}$. Notice that if $z \notin \overline{K}$ then there is exactly one node τ along f such that $z \in \lambda(\tau)$.

The instigation process. Let $\tau\frown 0 \subseteq \sigma$, and assume that $R(\tau) = \mathcal{M}_{a,b,c,\Phi,\Psi}$. Then, according to 2.3.2, σ instigates $R(\tau)$ if we have, for some x, X and $j \in X$,

$$\bigoplus_{j \in X} E^j_\sigma \text{ leaves } \bigoplus_{j \in X} B_j \Rightarrow x \text{ leaves } \Phi^{A_a} \cap \Psi^{A_b},$$

(where $E^j_\sigma = \bigcup_{\rho \preceq \sigma} E^j(\rho) - \bigcup_{\rho \subseteq \sigma} F^j(\rho)$) but x does not leave $\Delta^{A_c}_\tau$. If τ is the least node for which $R(\tau)$ is instigated by σ at the current stage then we accordingly define $x(\tau\frown 1)$, $E^j(\tau\frown 1)$, $F^j(\tau\frown 1)$ where the sets $E^j(\tau\frown 1)$ are designed to keep $x(\tau\frown 1)$ out of Φ^{A_a} (Ψ^{A_b}, respectively), whereas the sets $F^j(\tau\frown 1)$ are designed to keep $x(\tau\frown 1)$ in Ψ^{A_b} (Φ^{A_a}, respectively).

Following this instigation process we close step s and we move to next stage (thus, at each stage, we have at most one instigation).

We now consider the effects of the instigation for $R(\tau)$ at the next τ-true stage. If f is the true path then, since $\tau\frown 1 \not\subset f$, none of these instigations has a permanent effect. We must argue in this case that

$$\Phi^{A_a} = \Psi^{A_b} \Rightarrow \Phi^{A_a} = \Delta^{A_c}.$$

To this end, assume that $\Phi^{A_a} = \Psi^{A_b}$ and, for instance, $x \notin \Phi^{A_a}$ but $x \in \Delta^{A_c}$. To show that this is not possible, we use the fact that we eventually make A_a low. Indeed, if $\rho \subset f$ is such that $\tau\frown 0 \subseteq \rho$ and $R(\rho) = \mathcal{L}_{a,x,\Phi}$ then, when acting at ρ, we look for an opportunity of restraining $x \in \Phi^{A_a}$ via some finite sets $F^j(\rho^\frown 0)$ for $j \in J_a$ such that $F^j(\rho^\frown 0) \cap E^j_\rho = \emptyset$. If this is not possible then we can argue that some σ' with $\tau^\frown 0 \subseteq \sigma' \subseteq \rho$ can permanently instigate $R(\tau)$. But $\tau^\frown 1 \not\subset f$ and this implies that eventually $x \in \Phi^{A_a}$, contradiction.

The restraints are eventually permanent. We will need to show that $F^j(\sigma) \subseteq B_j$ for every $\sigma \subset f$ and $j \in \mathcal{J}$. After defining $F^j(\sigma)$ (for a given j), if we extract some $x \in F^j(\sigma)$ from B_j at some later stage then this can happen only because σ instigates some $R(\tau)$ with $\tau^\frown 0 \subseteq \sigma$. Indeed, σ might instigate $R(\tau)$ in order to create a diagonalization at some y, i.e. $y \in \Phi^{A_a} - \Psi^{A_b}$ or $y \in \Psi^{A_b} - \Phi^{A_a}$, and require reinsertion of elements of $E^i(\sigma)$ into some B_i, which are i-traces for some $z \notin \overline{K}$ so that, at some subsequent even stage, the requirement $\mathcal{R}_z$, facing the impossibility of extracting all of $E^i(\sigma)$ from B_i, might have no chance of achieving Γ-rectification for z other than extracting elements that are in $F^j(\sigma)$.

Notice that, because of these reinsertions and consequent functional updating procedures, we might have to provide new larger definitions of the sets $Y^i(z)$, and consequently, new definitions of $E^i(\sigma)$ and $F^j(\sigma)$, at subsequent σ^--true stages.

We shall show that this can happen only finitely many times. Indeed, there are only finitely many axioms $\langle y, G\rangle \in \Delta_\tau$ such that G contains elements from the sets $E^i(\sigma)$. First of all, notice that we define axioms $\langle y, G\rangle \in \Delta_\tau$ only at $\tau^\frown 0$-true stages. At these stages, we may assume that the elements of $E^i(\sigma)$, which have been reinserted in B_i in order to create a diagonalization and win $R(\tau)$, will be extracted again (on behalf of $R(\sigma)$, since the diagonalization has failed), so that no new axiom enumerated in Δ_τ will use these numbers. If this is so, then by our lowness strategy, we can argue as before that σ cannot instigate infinitely many times on behalf of the same number y and therefore σ can instigate only finitely many times. How do we achieve that Δ_τ-axioms defined when working below $\tau^\frown 0$ avoid elements from $E^i(\sigma)$? This task is taken care of by the auxiliary function $\mu^i(\tau)$: When we define and extract $E^i(\sigma)$ from B_i, we put the elements of $E^i(\sigma)$ into the sets $\mu^i(\rho)$, for every $\rho \subseteq \sigma$ (thus including $\mu^i(\tau)$) and we demand that possible future Δ_ρ-axioms avoid elements from $\mu^i(\rho)$. Of course, when the outcome at τ is 1, then we extract from $\mu^i(\rho)$, for every $\rho \subset \tau$, all the numbers that we fix in B_i: The reason for this is that if $\rho^\frown 0 \subseteq \tau$ and ρ is also an $\mathcal{M}$-node, then, at $\tau^\frown 1$-true stages s at which we reinsert elements from $E^i(\sigma)$ into B_i, Δ_ρ-axioms must be allowed to use these elements.

2.6 The construction

We assume, of course, that 1 is join-reducible in $\mathcal{L}$. (Otherwise, we can simply add an element x to $\mathcal{L}$ incomparable to all elements $y \in L - \{0,1\}$.) For every $\sigma \in T$ and $j \in \mathcal{J}$, at step s we use the following notation: We let

$$F^j_\sigma(s) = \bigcup_{\rho \subseteq \sigma} F^j(\rho, s)$$
$$E^j_\sigma(s) = \bigcup_{\rho \preceq \sigma} E^j(\rho, s) - F^j_\sigma(s)$$

($F^j_\sigma(s)$ is defined as the union over the strings $\tau \subseteq \sigma$, rather than over all the strings $\tau \preceq \sigma$, because the $\mathcal{R}_z$-strategies must be given the possibility to freely injure the fixing activity undertaken by actions to the left of the true path; on the other hand, notice that, apart from the $\mathcal{R}_z$-strategies, no strategy located to the right of the true path interferes, in the limit, with any fixing activity performed while acting on behalf of strategies located on the true path. Contrary to this, the extracting activity must be protected against the fixing activity performed by strategies, e.g. on behalf of lowness requirements, located at nodes to the right of the true path.)

We borrow the following definition from [Coo87]:

Definition 2.6 If Ψ is an enumeration operator (with finite approximations $\{\Psi^s : s \in \omega\}$) and $x \in \omega$ then let

$$\uparrow \epsilon(\Psi, x) = \{E : E \text{ finite and } x \notin \Psi^{\omega - E}\}$$
$$\uparrow \epsilon(\Psi, x, s) = \{E : E \text{ finite and } x \notin \Psi^{\omega - E}[s]\}$$

The next definition describes an action which will be repeatedly performed in the construction.

Definition 2.7 (Functional updating) Let $\mathfrak{F}$ be a family of finite sets. Assume that, for every $F \in \mathfrak{F}$, there exists $j \in \mathcal{J}$ such that $F \subseteq \omega^j$. Define *functional updating at* $\mathfrak{F}$ *at* s to be the following action. For every $F \in \mathfrak{F}$ with, say, $F \subseteq \omega^j$, and every $y \in F$ proceed as follows:

- if y is odd and $y = y^j(\tau, t)$ for some $\mathcal{Q}$-node τ and some least $t \leq s$ then for every $i \in \beta(j)$ define $F^i(y, s) = \{y^i(\tau, t)\}$;

- if y is even and $y \in Y^j(z, s-1)$ for some z, then for every $i \in \mathcal{J}$ choose a new $y^i \in \eta^i_z$ (called a j*-trace for* z). For every i, enumerate y^i into $Y^i(z, s)$ and, for every $i \in \beta(j)$, define $F^i(y, s) = \{y^i\}$. Add the axiom $\langle y^i, \bigoplus_{i' \in Y} \{y^{i'}\} \rangle$ into $\Gamma^s_{i,Y}$ for every pair $(i, Y) \in \beta^j$.

(When this action, which merely consists in defining some sets and possibly some axioms, is performed on behalf of some strategy $R(\sigma)$, at some stage s, and o is the outcome at σ at s, then we will let $F^i(y,s) \subseteq F^i(\sigma\widehat{\ }o, s)$, for every $i \in \beta(j)$, $F \in \mathfrak{F}$, $y \in F$.)

We now formally define the notion of instigation of an $\mathcal{M}$-strategy, already mentioned in the intuitive description of the construction.

Definition 2.8 (Instigation) Let σ and τ be nodes such that τ is an $\mathcal{M}$-node (with, say, $R(\tau) = \mathcal{M}_{a,b,c,\Phi,\Psi}$) and $\tau\widehat{\ }0 \subseteq \sigma$. We say that σ *instigates* $R(\tau)$ *at* x *at* s if $x \in \Delta^{\omega}_{\tau}[s]$ and one of the following holds:

1. $E^a_\sigma(s) \in\uparrow \epsilon(\Phi, x, s)$ and $E^c_\sigma(s) \notin\uparrow \epsilon(\Delta_\tau, x, s)$; or
2. $E^b_\sigma(s) \in\uparrow \epsilon(\Psi, x, s)$ and $E^c_\sigma(s) \notin\uparrow \epsilon(\Delta_\tau, x, s)$.

If (1) holds then we say that σ instigates $R(\tau)$ at x at s *via* a; otherwise we say that σ instigates $R(\tau)$ at x at s *via* b.

Definition 2.9 (Instigation action at x) If σ instigates $R(\tau) = \mathcal{M}_{a,b,c,\Phi,\Psi}$ at x at s via a (a similar definition will apply if σ instigates $R(\tau)$ via b, interchanging a with b) then (by Remark 2.10) choose the least (with respect to their canonical indices) finite sets F^i such that for every $i \in J_b$, $F^i \cap E^i_{\sigma^-}(s) = \emptyset$ and $x \in \Psi^{F^b}[s]$; define

$$x(\tau\widehat{\ }1, s) = x$$

for every $i \in J_b$, define

$$F^i(\tau\widehat{\ }1, s) = F^i$$

and for every $i \in J_a$ let

$$E^i(\tau\widehat{\ }1, s) = E^i_\sigma(s).$$

We say that $\tau\widehat{\ }1$ *needs functional updating* at any stage $t \geq s$ prior to the least stage $s' > s$ such that $\tau \subseteq \delta_{s'}$. We also say that $\tau\widehat{\ }1$ is *a-related* (respectively, *b-related* if σ instigates $R(\tau)$ via b) at any stage $t \geq s$ prior to the least stage $s' > s$ at which we declare that $\tau\widehat{\ }1$ is b-related (a-related, in the other case). Finally, for every $z \in \lambda(\sigma, s)$ we extract z from $\lambda(\rho, s+1)$ for every ρ. (The reason for this is that the instigation action may enumerate new elements into $Y^j(z,t)$, for some j, at a later stage through the process of functional updating so that z may require activation again of the $\mathcal{R}_z$-strategy if we take again action at σ at some later stage.)

Remark 2.10 Since at each step, the construction instigates at most once, then, in the situation illustrated by Definition 2.9, we have that no ρ with $\tau \subset \rho \subset \sigma$ instigates $R(\tau)$, thus $E^b_{\sigma^-}(s) \notin\uparrow \epsilon(\Psi, x, s)$ or $E^c_{\sigma^-}(s) \in\uparrow \epsilon(\Delta_\tau, x, s)$, and so we can find finite sets F^i as in the definition.

We now proceed with the construction. Together with the auxiliary functions described in 2.5, we will define at stage s a string δ_s, called the *true path at stage s*. Given any string σ, we say that s is a σ-*true stage* if $\sigma \subseteq \delta_s$.

Stage 0) Let $\delta_0 = \emptyset$. For every $\sigma \in T$, $j \in \mathcal{J}$, and $x \in \omega$, let

$$F^j(\sigma,0) = E^j(\sigma,0) = G^j(\sigma,x,0) = H^j(\sigma,x,0) = \mu^j(\sigma,0) = \lambda(\sigma,0) = \emptyset.$$

Let also $x(\sigma,0) = c(\sigma,0) = y^j(\sigma,0) =\uparrow$ for every σ and j.

Stage $s+1$) Unless explicitly redefined by the construction, each auxiliary function retains the same value as at step s. We distinguish two cases, according to whether $s+1$ is odd or even.

Case 1: $s+1$ odd: Suppose that we have already defined $\sigma = \delta_s \upharpoonright n$. If $n = s+1$ then close stage $s+1$ (see 2.6.7) and go to stage $s+2$. Otherwise, in order to define $\sigma^+ = \delta_{s+1} \upharpoonright n+1$, we distinguish the following cases.

2.6.1 σ is an $\mathcal{M}$-node

Assume $R(\sigma) = \mathcal{M}_{a,b,c,\Phi,\Psi}$. We distinguish the following two cases.

1. Suppose first that for every j,

   $$F^j(\sigma\hat{\ }1,s) \cap E^j_\sigma(s+1) = E^j(\sigma\hat{\ }1,s+1) \cap F^j_\sigma(s) = \emptyset,$$

   $x(\sigma\hat{\ }1,s)$ is defined, and

   (i) $\sigma\hat{\ }1$ is a-related and

   $$E^a_{\sigma^+}(s+1) \in\uparrow \epsilon(\Phi,x,s+1)$$

   or

   (ii) $\sigma\hat{\ }1$ is b-related and

   $$E^b_{\sigma^+}(s+1) \in\uparrow \epsilon(\Psi,x,s+1).$$

   Then define $\sigma^+ = \sigma\hat{\ }1$.

   Functional updating. If σ^+ needs functional updating at $s+1$ then (assume for definiteness that σ^+ is a-related) let $\mathfrak{F} = \{F^j(\sigma^+,s) : j \in J_b\}$ and apply the procedure of functional updating at $\mathfrak{F}$ at $s+1$.

   Define for every $i \in \mathcal{J}$,

   $$F^i(\sigma^+,s+1) = F^i(\sigma^+,s) \cup \bigcup\{F^i(y,s+1) : y \in F, F \in \mathfrak{F}\}.$$

   (Thus, at $s+1$, σ^+ ceases to need functional updating.) Finally, for every $\tau \subset \sigma$ and $i \in \mathcal{J}$, extract all elements of $F^i(\sigma^+,s+1)$ from $\mu^i(\tau,s+1)$.

2. Otherwise, let $\sigma^+ = \sigma\widehat{\ }0$, and choose the least $y \in \Phi^{A_a}[s] \cap \Psi^{A_b}[s]$ such that $y \notin \Delta_\sigma^{A_c}[s]$. For every $j \in J_a \cup J_b$, choose the least (according to their canonical indices) finite sets $G^j = G^j(\sigma, y, s+1)$ (for $j \in J_a$) and $H^j = H^j(\sigma, y, s+1)$ (for $j \in J_b$) such that

$$\langle y, \bigoplus_{j \in J_a} G^j \rangle \in \Phi^s,$$

$$\langle y, \bigoplus_{j \in J_b} H^j \rangle \in \Psi_s,$$

$$G^j \cup H^j \subseteq (B_j^s \cup F_\sigma^j(s+1)) - E_\sigma^j(s+1),$$

and

$$(G^j \cup H^j) \cap \mu^j(\sigma, s+1) = \emptyset.$$

Then enumerate the axiom $\langle y, \bigoplus_{j \in J_c}(G^j \cup H^j)\rangle$ into Δ_σ^{s+1}.

Whatever case has happened to hold (i.e. whether yielding outcome 1 or 0), now close the σ^+-action (see 2.6.6. Notice that this does not mean that we move directly to stage $s+2$! Here and in the following cases, if no action has been taken at σ, i.e. item 2. of 2.6.6 does not hold, then we move on to define $\delta_{s+1} \restriction n+2$.)

2.6.2 σ is a $\mathcal{Q}$-node

Assume $R(\sigma) = \mathcal{Q}_{j,a,\Phi}$. We distinguish the following two cases.

1. If $c = c(\sigma, s)$ is defined and therefore the corresponding $y^i(\sigma, s)$ are defined then:

   (a) If $c \notin \Phi^{A_a}[s]$ then define $\sigma^+ = \sigma\widehat{\ }1$ and for every $i \in \beta(j)$, let $F^i(\sigma^+, s+1) = \{y^i(\sigma, s)\}$.

   (b) If $c \in \Phi^{A_a}[s]$ then let $\sigma^+ = \sigma\widehat{\ }0$; choose the least finite sets F^i (for $i \in J_a$) such that $F^i \cap E_\sigma^i(s+1) = \emptyset$, $F^a \subseteq A_a[s]$, $c \in \Phi^{F^a}[s]$; and let $F^i(\sigma^+, s+1) = F^i$ for each such i; finally, for every i such that $i \not\leq a$, let $E^i(\sigma^+, s+1) = \{y^i(\sigma, s)\}$, and put $E^i(\sigma^+, s+1)$ into $\mu^i(\rho, s+1)$ for every $\rho \subseteq \sigma$.

2. Otherwise, for every $i \in \mathcal{J}$, choose new numbers $y^i = y^i(\sigma, s+1) \in \xi_\sigma^i$ and define $c(\sigma, s+1) = y^j$.

   Functional updating via $\mathcal{Q}$ at σ. Add the axiom $\langle y^i, \bigoplus_{i' \in Y}\{y^{i'}\}\rangle$ into $\Gamma_{i,Y}^{s+1}$ for every pair $(i, Y) \in \beta^j$; finally, let $F^i(\sigma^+, s+1) = \{y^i(\sigma, s)\}$ for every $i \in \beta(j)$.

If σ^+ instigates some $R(\tau)$ at some x at $s+1$ then consider the least such τ and the least such x, and perform the corresponding instigation action relative to τ and x, as in Definition 2.9.

Close the σ^+-action (see 2.6.6).

2.6.3 σ is an $\mathcal{L}$-node

Assume $R(\sigma) = \mathcal{L}_{a,x,\Phi}$.

1. If there exist finite sets $F^j \subseteq \omega^j$ such that for every $j \in \mathcal{J}$,

$$F^j \cap E^j_\sigma(s+1) = \emptyset,$$

and $x \in \Phi_s^{Fa}$, then choose the least such finite sets and let $\sigma^+ = \sigma\widehat{\ }0$.

- If $F^j \subseteq B^s_j$ for all $j \in J_a$ then define $F^j(\sigma\widehat{\ }0, s+1) = F^j$.
- Otherwise, define $\mathfrak{F} = \{F^i : i \in J_a\}$ and apply the procedure of functional updating at $\mathfrak{F}$ at $s+1$.
  Define for every $j \in \mathcal{J}$,

$$F^j(\sigma\widehat{\ }0, s+1) = F^j \cup \bigcup \{F^j(y, s+1) : y \in F, F \in \mathfrak{F}\}.$$

2. Otherwise, let $\sigma^+ = \sigma\widehat{\ }1$.

Close the σ^+-action (see 2.6.6).

2.6.4 σ is an $\mathcal{R}_z$-node

1. If $z \in \overline{K}^s$ then let $\sigma^+ = \sigma\widehat{\ }1$.

   Functional updating via $\mathcal{R}_z$ at σ. If no axiom $\langle z, G\rangle \in \Gamma$ has been defined so far then for every $j \in \mathcal{J}$, choose $y^j_z \in \eta^j_z$ to be some new number; let $Y^j(z, s+1) = \{y^j_z\}$ for every $j \in \mathcal{J}$.

   Add the axiom $\langle z, \bigoplus_{j\in\mathcal{J}}\{y^j_z\}\rangle$ into Γ^{s+1}, and for every pair (i, X) such that $i \notin X$ and $i \leq \bigvee X$, add the axiom $\langle y^i_z, \bigoplus_{i'\in X}\{y^{i'}_z\}\rangle$ into $\Gamma^{s+1}_{i,X}$. Finally, let $F^j(\sigma\widehat{\ }1, s+1) = \{y^j_z\}$.

2. Otherwise, let $\sigma^+ = \sigma\widehat{\ }0$.

Close the σ^+-action (see 2.6.6).

We remind the reader that functional correction for $\mathcal{R}_z$, following extraction of z from $\overline{K}$, only takes place at even stages.

Case 2: $s+1$ even: At even stages we rectify the various e-operators $\Gamma_{i,X}$ which need be rectified following extraction of numbers from $\overline{K}$. If $\sigma \subseteq \delta_s$ is a z-th node then we write $\sigma = \sigma_z$.

2.6.5 Activating the $\mathcal{R}_z$ strategy

Consider the least $z \leq s$ with $z \notin \overline{K}^s$, if any, such that $\sigma_z \subset \delta_s$, and $z \notin \lambda(\tau, s)$ for all $\tau \subseteq \sigma_z{}^\frown 0$ or $\sigma_z{}^\frown 1 \subseteq \delta_s$.

If no such z exists then go to stage $s + 2$.

Otherwise, proceed as follows. We first give a definition.

Definition 2.11 (Injury) Given a node σ such that $\sigma \subseteq \sigma_z$, we say that $\mathcal{R}_z$ *injures* $R(\sigma^-)$ *through* z *at* $s + 1$ if, for some $j \in \mathcal{J}$,

$$Y^j(z, s) \cap F^j_\sigma(s) \neq \emptyset.$$

Find the least σ, if any, such that $R(\sigma^-)$ is injured this way.

(i) If there is such a σ, then there are four cases:

- If σ is an $\mathcal{M}$-node, $R(\sigma) = \mathcal{M}_{a,b,c,\Phi,\Psi}$, $\sigma = \sigma^{-\frown}1$, and σ is a-related, then let $X = J_b$.
- If σ is an $\mathcal{M}$-node, $R(\sigma) = \mathcal{M}_{a,b,c,\Phi,\Psi}$, $\sigma = \sigma^{-\frown}1$, and σ is b-related, then let $X = J_a$.
- If σ is a $\mathcal{Q}$-node and $R(\sigma) = \mathcal{Q}_{j,a,\Phi}$ then let $X = J_a$ (notice that this entails $\sigma = \sigma^{-\frown}0$).
- If σ is an $\mathcal{L}$-node and $R(\sigma) = \mathcal{L}_{a,x,\Phi}$ then let $X = J_a$ (again, this entails $\sigma = \sigma^{-\frown}0$).

Let $\mathfrak{F} = \{F^j(\sigma, s) : j \in X\}$ and apply the procedure of functional updating at $\mathfrak{F}$ at $s + 1$.

Define for every $i \in \mathcal{J}$,

$$F^i(\sigma, s + 1) = F(\sigma, s) \cup \bigcup \{F^i(y, s + 1) : y \in F, F \in \mathfrak{F}\}.$$

Finally, for every $j \in \mathcal{J} - X$, let

$$E^j(\sigma, s + 1) = Y^j(z, s);$$

put z into $\lambda(\sigma, s + 1)$; and for every $\rho \neq \sigma$ such that ρ is not an $\mathcal{R}_z$-node or $\sigma \subset \rho$, extract z from $\lambda(\rho, s + 1)$.

(ii) If there is no such σ, and $z \notin \lambda(\sigma_z{}^\frown 0, s)$ or $\sigma_z{}^\frown 1 \subseteq \delta_s$, then (write $\sigma = \sigma_z{}^\frown 0$): for every $j \in \mathcal{J}$, let

$$E^j(\sigma, s + 1) = Y^j(z, s);$$

put z into $\lambda(\sigma, s + 1)$; and for every $\rho \neq \sigma$ such that ρ is not an $\mathcal{R}_z$-node or $\sigma \subset \rho$, extract z from $\lambda(\rho, s + 1)$.

Put $E^j(\sigma, s + 1)$ into $\mu^j(\rho, s + 1)$ for every $\rho \subseteq \sigma$.

λ**-closure.** Finally, put z into $\lambda(\rho^\frown 0, s+1)$ for every ρ such that $\sigma \prec_L \rho$ and ρ is an $\mathcal{R}_z$-node, and define $E^j(\rho^\frown 0, s+1) = E^j(\sigma, s+1)$. We say in this case that $z \in \lambda(\rho^\frown 0, t)$ *because of* σ at any stage $t > s$ prior to the least stage $s' > s$, if any, such that $\lambda(\rho^\frown 0, s') = \emptyset$. (We will show that extraction of the sets $E^j(\sigma, s+1)$ from B_j is enough for the rectification of Γ and of all relevant e-operators $\Gamma_{j,X}$, if these sets are restrained out of the B_j's. Thus, we place z into $\lambda(\rho^\frown 0, s+1)$ so that when we move to the right of σ in the tree of outcomes at some later stage t, z does not require activating the $\mathcal{R}_z$-strategy as long as $z \in \lambda(\rho^\frown 0, t)$. Thus, eventually, z requires only finitely many activations of the $\mathcal{R}_z$-strategy. Notice that z may require again activation of the $\mathcal{R}_z$-strategy only if some $R(\tau)$ with $\tau^\frown 0 \subseteq \sigma$ is instigated at some later stage t so that this implies $\lambda(\rho^\frown 0, t) = \emptyset$ as required by Definition 2.9.)

If σ instigates some $R(\tau)$ at some x at $s+1$ then consider the least such τ and the least such x, and perform the corresponding instigation action relative to τ and x as in Definition 2.9.

Finally, define $\delta_{s+1} = \sigma$, close the σ-action and close stage $s+1$.

2.6.6 Closing a σ-action

We distinguish the following two possibilities:

1. if, for some i, $F^i(\sigma, s+1) \neq F^i(\sigma, s)$, or $E^i(\sigma, s+1) \neq E^i(\sigma, s)$, or $\mu^i(\sigma, s+1) \neq \mu^i(\sigma, s)$, or $\lambda(\sigma, s+1) \neq \lambda(\sigma, s)$, or σ instigates some $R(\tau)$ at some x at $s+1$, or we apply the procedure of functional updating; then *reset* all nodes τ with $\sigma \prec \tau$, by making, for each such τ, $x(\tau, s+1)$, $y^i(\tau, s+1)$, $c(\tau, s+1)$ undefined, and setting all of $F^i(\tau, s+1)$, $E^i(\tau, s+1)$, $\mu^j(\tau, s+1)$, and Δ_τ^{s+1} equal to $\emptyset$. Let also $\lambda(\tau, s+1) = \emptyset$ for every τ such that $\sigma \prec \tau$ and τ is not an $\mathcal{R}_z$-node, for any z. Close stage $s+1$ (see 2.6.7) and go to stage $s+2$;

2. otherwise, move on to define $\delta_{s+1} \upharpoonright (n+2)$ if $n+1 < s+1$.

2.6.7 Closing stage $s+1$

For every $j \in \mathcal{J}$, let

$$B_j^{s+1} = (B_j^s \cup \bigcup_{\sigma \subseteq \delta_{s+1}} F^j(\sigma, s+1)) - \bigcup_{\sigma \subseteq \delta_{s+1}} E^j(\sigma, s+1).$$

Let also Γ^{s+1} consist of Γ^s plus all the axioms added to Γ at stage $s+1$, and for every i, X with $i \in \mathcal{J}$, $X \in \mathfrak{J}$, $i \notin X$ and $i \leq \bigvee X$, let $\Gamma_{i,X}^{s+1}$ consist of $\Gamma_{i,X}^s$ plus all the axioms added to $\Gamma_{i,X}$ at stage $s+1$.

2.7 The verification

We begin with the following definition. For every z, let

$$Y^i(z) = \bigcup_s Y^i(z,s);$$

recall that, for every s, $Y^i(z,s) \subseteq Y^i(z,s+1)$.

Definition 2.12 For every n, let $f_n = \liminf_s \delta_s \upharpoonright n$. (Here the lim inf is taken with respect to the ordering $\preceq$ of the strings. We show below that in fact this lim inf exists.)

Lemma 2.13 *For every n:*

1. *f_n is defined, and for almost all f_n-true stages s, we have $|\delta_s| > |f_n|$;*
2. *f_n instigates only finitely many times;*
3. *for every $j \in \mathcal{J}$, $F^j(f_n) = \lim_s F^j(f_n, s)$ exists and $F^j(f_n)$ is finite, and $F^j(f_n) \subseteq B_j$;*
4. *for every $j \in \mathcal{J}$, $E^j(f_n) = \lim_s E^j(f_n, s)$ exists and is finite;*
5. *$c(f_n) = \lim_s c(f_n, s)$ and $x(f_n) = \lim_s x(f_n, s)$ exist;*
6. *$\lambda(f_n) = \lim_s \lambda(f_n, s)$ exists and is finite;*
7. *for every $z \in \lambda(f_n)$, and for every i, $Y^i(z)$ is finite.*

Proof: The proof is by induction on n.

Step 0*:* Clearly, $f_0 = \emptyset$; for every $j \in \mathcal{J}$ and s, $F^j(\emptyset, s) = E^j(\emptyset, s) = \lambda(\emptyset, s) = \emptyset$; for every $s > 0$, $|\delta_s| > 0$; $c(\emptyset, s)$ is undefined, since $\emptyset$ is not a $\mathcal{Q}$-node; $x(\emptyset, s)$ is always undefined; finally, $\emptyset$ never instigates.

Step $n+1$: Assume that f_n exists, and f_n satisfies 1. through 7., and assume by induction that for every $\tau \preceq f_n$ and for every $j \in \mathcal{J}$, $F^j(\tau) = \lim_s F^j(\tau, s)$, $E^j(\tau) = \lim_s E^j(\tau, s)$, $\lambda(\tau) = \lim_s \lambda(\tau, s)$ exist and are finite, and $F^j(\tau) \subseteq B_j$ for every $\tau \subseteq f_n$.

For every $\sigma \preceq f_n$ and every j, also set

$$F^j_\sigma = \bigcup_{\tau \subseteq \sigma} F^j(\tau)$$

and

$$E^j_\sigma = \bigcup_{\tau \preceq \sigma} E^j(\tau) - F^j_\sigma.$$

Thus, let t_n be the least stage such that

(i) for every $\tau \subseteq f_n$ and every $s \geq t_n$, if $\tau \subseteq \delta_s$ then $|\delta_s| > |\tau|$;

(ii) for every $s \geq t_n$ and every $\tau \prec_L f_n$, $\tau \not\subseteq \delta_s$;

(iii) no $\tau \subseteq f_n$ instigates after t_n;

(iv) for every $s \geq t_n$, every $\tau \preceq f_n$, and every $j \in \mathcal{J}$, $F^j(\tau) = F^j(\tau, s)$, $E^j(\tau) = E^j(\tau, s)$, and $\lambda(\tau) = \lambda(\tau, s)$.

Using these assumptions, we show below that f_{n+1} exists and that we can find a stage t_{n+1} satisfying the above conditions (i) through (iv) for f_{n+1}.

If, for some $j \in \mathcal{J}$, the values of $\lim_s F^j(f_{n+1}, s)$ and $\lim_s E^j(f_{n+1}, s)$ are not explicitly indicated in the proof below then it will immediately follow from the construction that $\lim_s F^j(f_{n+1}, s) = \emptyset$ and $\lim_s E^j(f_{n+1}, s) = \emptyset$.

By assumption (i), relative to t_n, it is clear that f_{n+1} exists. Let $u_0 \geq t_n$ be the least f_{n+1}-true stage such that for every $u \geq u_0$, if $\tau \prec_L f_{n+1}$ then $\tau \not\subseteq \delta_u$.

As to the rest of the verification, we first need the following sublemmas.

Sublemma 1 *If $\lambda(f_{n+1})$ exists and is finite then there exists an f_{n+1}-true stage $\hat{u}$ such that for all $s \geq \hat{u}$, f_{n+1} does not instigate at s.*

Proof: Let $u_1 \geq u_0$ be a stage such that, for every $u \geq u_1$, $\lambda(f_{n+1}) = \lambda(f_{n+1}, u)$. It follows by construction that $E^i(f_{n+1}, s) \subseteq E^i(f_{n+1}, s+1)$, for every $s \geq u_1$ and i. We might indeed have $\subset$ because of some instigation action initiated by f_{n+1}, which may make $Y^i(z, t)$ larger at some later t for some $z \in \lambda(f_{n+1})$, and thus require $E^i(f_{n+1}, s) \subset E^i(f_{n+1}, s+1)$ at some f_{n+1}-true stage $s > t$.

Let $\rho_0 \supset \rho_1 \supset \cdots \supset \rho_k$ be the list of all $\mathcal{M}$-nodes ρ such that $\rho^\frown 0 \subseteq f_{n+1}$.

We show by induction on $r \leq k$ that f_{n+1} instigates $R(\rho_r)$ only finitely many times.

Thus, let $R(\rho_r) = \mathcal{M}_{a_r, b_r, c_r, \Theta_r, \Omega_r}$, and assume by induction on r that $v_r \geq u_1$ be a stage such that for no $s \geq v_r$ and no $\rho \in \{\rho_m : m < n\}$ does f_{n+1} instigate $R(\rho)$. If $v' \geq v_r$ is the least stage at which $f_{n+1} \subseteq \delta_{v'}$, then for every $s \geq v'$ and $i \in \mathcal{J}$, we have that

$$E^i(f_{n+1}, s) \subseteq \mu^i(\rho_r, s);$$

therefore the elements of $E^i(f_{n+1}, s)$ will not be used for defining Δ_{ρ_r} axioms at s. Moreover, we always reset when we move past $\rho_r{}^\frown 1$. Thus there are only finitely many numbers x such that, at some stage s, we add an axiom $\langle x, \oplus_{i \in J_{c_r}}(G^i \cup H^i)\rangle$ into Δ_{ρ_r} with $(G^i \cup H^i) \cap E^i(f_{n+1}, s) \neq \emptyset$ for some $s \geq v_r$. Let x be any such number. If f_{n+1} instigates $R(\rho_r)$ at x at any $v \geq v'$ then since $\rho_r{}^\frown 0 \subseteq f_{n+1}$ there must be a stage $v'' \geq v'$ such that $f_{n+1} \subseteq \delta_{v''}$ and this instigation is no longer valid at v'', and thus at any $u \geq v''$, since for every $u \geq v''$, $E^{a_r}(f_{n+1}, u) \notin\uparrow \epsilon(\Theta_r, x, u)$ (if the instigation was via a_r) or $E^{b_r}(f_{n+1}, u) \notin\uparrow \epsilon(\Omega_r, x, u)$ (if the instigation was via b_r).

We can therefore conclude that a stage $\hat{u}$ with the desired features exists.

Sublemma 2 *If $\lambda(f_{n+1})$ exists and is finite, and $z \in \lambda(f_{n+1})$, then for every j, $Y^j(z)$ is finite.*

Proof: If s is such that $s \geq \hat{u}$ and $z \in \lambda(f_{n+1}, t)$ for every $z \in \lambda(f_{n+1})$ and $t \geq s$, then $z \in \lambda(\rho^\frown 0, t)$ for all such z and t, and every $\mathcal{R}_z$-node ρ such that $f_{n+1} \prec_L \rho$, and thus no $z \in \lambda(f_{n+1})$ does activate the $\mathcal{R}_z$-strategy any more after stage s.

It follows that after stage s, we do not enumerate any element in any of the sets $Y^j(z)$. Indeed, such an enumeration could only be the consequence of applications of the procedure of functional updating, following some instigating action initiated by f_{n+1}.

Sublemma 3 *If $\lambda(f_{n+1})$ exists and is finite, and $\hat{u}$ is as in Sublemma 1, then for every $s \geq \hat{u}$, $t \geq s$, and every $j \in \mathcal{J}$, $F^j(f_{n+1}, s) \subseteq B_j^t$.*

Proof: Let $j \in \mathcal{J}$ be given. We distinguish the following cases.

f_n is an $\mathcal{M}$-node. If $f_{n+1} = f_n{}^\frown 0$ then the claim is trivially true by construction. Thus assume that $f_{n+1} = f_n{}^\frown 1$. Since $\hat{u}$ is a stage after which f_{n+1} does not instigate any more (indeed, such an instigation would be the only possibility which would demand to extract from B_j some element of $F^j(f_{n+1}, s)$, for $s \geq \hat{u}$), we conclude that $F^j(f_{n+1}, s) \subseteq B_j^t$, for every $t \geq s \geq \hat{u}$.

f_n is a $\mathcal{Q}$-node. The proof is similar to that for $\mathcal{M}$-nodes.

f_n is an $\mathcal{L}$-node. The proof is similar to that for $\mathcal{M}$-nodes.

f_n is an $\mathcal{R}_z$-node. Trivial since for every $j \in \mathcal{J}$, $F^j(f_{n+1}) = \emptyset$ if $f_{n+1} = f_n{}^\frown 0$; or for every $j \in \mathcal{J}$, $F^j(f_{n+1}) = Y^j(x) = \{y_x^j\} \subseteq B_j$ if $f_{n+1} = f_n{}^\frown 1$.

We now continue the proof of Lemma 2.13, showing that f_{n+1} satisfies 1. through 7. of the lemma. We analyze several cases according to the requirement $R(f_{n+1})$ assigned to f_{n+1}.

Case 1: f_n is an $\mathcal{M}$-node: Assume $R(f_n) = \mathcal{M}_{a,b,c,\Phi,\Psi}$.

If $f_{n+1} = f_n{}^\frown 0$ then for every s and i, $F^i(f_{n+1}, s) = E^i(f_{n+1}, s) = \emptyset$ and $x(f_{n+1}, s)$ is undefined. In this case $\lambda(f_{n+1}, s) = \emptyset$.

Assume now that $f_{n+1} = f_n{}^\frown 1$, and suppose for definiteness that $R(f_{n+1})$ is a-related. Then there exists a least stage $v \geq u_0$ such that at v we instigate $R(f_n)$ at some x, and we define $x = x(f_{n+1}, v)$, $F^i = F(f_{n+1}, v)$ and $E^i = E(f_{n+1}, v)$ for some finite sets F^i and E^i; moreover $x = x(f_{n+1}, s)$ for every $s \geq v$.

Therefore,

$$\lambda(f_{n+1}) = \{z \notin \overline{K} : z \notin \bigcup_{\tau \subseteq f_n} \lambda(\tau) \,\&\, (\exists i \in \mathcal{J})[Y^i(z) \cap F^i \neq \emptyset]\}$$

and this set is finite since we always choose new elements for the various sets $Y^i(z)$. Thus, elements that are put into $Y^i(z,s)$ at stages $s > t$ are not in F^i.

It follows by Sublemma 2 that $Y^i(z)$ is finite for every $z \in \lambda(f_{n+1})$; therefore $F^i(f_{n+1}, s)$ and $E^i(f_{n+1}, s)$ reach a limit, and these limits are finite. By Sublemma 3, $F^i(f_{n+1}) \subseteq B_i$.

Let $t \geq \hat{u}$ be the least stage at which all the limits relative to the auxiliary functions pertaining to f_{n+1} are reached. Consequently, at no f_{n+1}-true stage $s > t$ do we close stage s. Then t is the required stage t_{n+1}.

Case 2: f_n is an $\mathcal{Q}$-node: Assume $R(f_n) = \mathcal{Q}_{j,a,\Phi}$. If $s \geq t_n$ is the least odd stage s such that $f_n \subseteq \delta_s$ then for every $t \geq s$ and for every $i \in \beta(j)$,

$$y^i(f_n, t) = y^i(f_n, s) = y^i(f_n)$$

and, consequently

$$c(f_n, t) = c(f_n, s) = c(f_n)$$

since $c(f_n, t) = y^j(f_n, t)$.

If $f_{n+1} = f_n{}^\frown 1$ then for every $i \in \beta(j)$, for every $u \geq s$,

$$\begin{aligned} F^i(f_{n+1}) &= F^i(f_{n+1}, u) = \{y^i(f_n)\} \\ E^i(f_{n+1}) &= E^i(f_{n+1}, u) = \emptyset. \end{aligned}$$

If $f_{n+1} = f_n{}^\frown 0$ then there exists a least stage $u \geq t_n$ such that for every $i \in \mathcal{J}$ we select finite sets F^i such that $c \in \Phi^{F^a}$. We can argue as we have done for the outcome $f_n{}^\frown 1$ for $\mathcal{M}$-nodes to conclude that $\lambda(f_{n+1})$ exists and is finite, and that $F^i(f_{n+1}, s)$ and $E^i(f_{n+1}, s)$ reach a limit and these limits are finite. Again, $F^i(f_{n+1}) \subseteq B_i$.

Notice that $y^i(f_n) \in E^i(f_{n+1})$ for every $i \not\leq a$.

From this we can infer the existence of the desired stage t_{n+1}, and at no f_{n+1}-true stage $s \geq t_{n+1}$ do we close stage s.

Case 3: f_n is an $\mathcal{L}$-node: Assume $R(f_n) = \mathcal{L}_{a,x,\Phi}$, and $a < 1$.

If $f_{n+1} = f_n{}^\frown 1$ then for every $j \in \mathcal{J}$,

$$F^j(f_{n+1}) = E^j(f_{n+1}) = \emptyset,$$

and the claim is trivial.

If $f_{n+1} = f_n{}^\frown 0$ then there exists a least stage $u \geq t_n$ such that for every $i \in \mathcal{J}$, we select finite sets F^i such that $x \in \Phi^{F^a}$. Again, we can argue as we have done for $\mathcal{M}$-nodes to conclude that $\lambda(f_{n+1})$ exists and is finite. This, together with Sublemma 3, which tells us that we eventually stop functional updating, implies that $F^i(f_{n+1}, s)$ and $E^i(f_{n+1}, s)$ reach a limit, and these limits are finite. Again, $F^i(f_{n+1}) \subseteq B_i$.

From this we can infer the existence of the desired stage t_{n+1}.

Case 4: f_n is an $\mathcal{R}_z$-node.

If $z \in \overline{K}^s$ then $f_{n+1} = \sigma\frown 1$ and for every $j \in \mathcal{J}$, $F^j(f_{n+1}) = \{y_z^j\}$ and $E^j(f_{n+1}) = \emptyset$. Thus the claim is trivial.

Otherwise, $f_{n+1} = \sigma\frown 0$. Either $\lambda(f_{n+1}) = \emptyset$, and again the claim is trivial; or $\lambda(f_{n+1}) = \{z\}$.

In the latter case, if at some last f_{n+1}-true even stage $s+1$ we have $z \in \lambda(f_{n+1}, s+1) - \bigcup_{\tau \subseteq f_n} \lambda(\tau, s)$ (i.e. we activate the $\mathcal{R}_z$-strategy through (ii)) and for every $t \geq s+1$, $z \in \lambda(f_{n+1}, t)$ then for every $j \in \mathcal{J}$,

$$E^j(f_{n+1}) = Y^j(z)$$

and this set is finite by Sublemma 2.

There is a third possibility, i.e. by λ-closure, z is eventually in $\lambda(f_{n+1})$ because of some $\tau \prec_L f_{n+1}$. If the corresponding action of λ-closure has taken place at some last stage t, then we have defined $E^j(f_{n+1}, t) = E^j(\tau, t)$, and, for every $s \geq t$, we have $E^j(f_{n+1}, s) = E^j(f_{n+1}, t)$. From this, the claim easily follows.

Definition 2.14 Let $f = \bigcup_n f_n$. The infinite path f is called the *true path.*

Lemma 2.15 *For every a, x, Φ, the requirement $\mathcal{L}_{a,x,\Phi}$ is satisfied.*

Proof: Let $f_n \subset f$ be such that $R(f_n) = \mathcal{L}_{a,x,\Phi}$, and assume that there exist infinitely many stages s such that $x \in \Phi^{A_a}[s]$. By Lemma 2.13, if at some stage $s \geq t_n$, we have that $x \in \Phi^{A_a}[s]$ then there exists a finite set F such that for every $j \in \mathcal{J}$, $F^j \cap E^j(f_n) = \emptyset$ and $x \in \Phi^{F^a}$. Therefore, we are eventually able to select at some f_n-true stage such a finite set F so that $F^j(f_n\frown 0) \supseteq F^j$ and $F^j(f_n\frown 0) \subseteq B_j$ by Lemma 2.13. Thus $x \in \Phi^{A_a}$.

Lemma 2.16 *For every a, b, c, Φ, Ψ, the requirement $\mathcal{M}_{a,b,c,\Phi,\Psi}$ is satisfied.*

Proof: Let f_n be such that $R(f_n) = \mathcal{M}_{a,b,c,\Phi,\Psi}$.

If $f_{n+1} = f_n\frown 1$ then it is immediate to see that $x(f_{n+1}) = \lim_s x(f_{n+1}, s)$ exists. If, for instance, f_{n+1} is eventually a-related and thus $F^j(f_{n+1}) \subseteq B_j$ for all $j \in \mathcal{J}_b$ by Lemma 2.13, then $x(f_{n+1}) \in \Psi^{A_b} - \Phi^{A_a}$ since

$$\bigoplus_{i \in J_a} E^a_{f_{n+1}} \in\uparrow \epsilon(\Phi_a, x(f_{n+1})).$$

Otherwise, if f_{n+1} is eventually b-related, a similar argument shows that $x(f_{n+1}) \in \Phi^{A_a} - \Psi^{A_b}$.

It follows that $\mathcal{M}_{a,b,c,\Phi,\Psi}$ is satisfied.

Assume now that $f_{n+1} = f_n{}^\frown 0$. Let

$$\Delta = \bigcup_{s \geq t_n} \Delta_{f_n,s}$$

where t_n is as in the proof of Lemma 2.13. Let x_0 be such that for every $x \geq x_0$, $\mathcal{L}_{a,x,\Phi}$ and $\mathcal{L}_{b,x,\Psi}$ have lower priority than $\mathcal{M}_{a,b,c,\Phi,\Psi}$. We claim that

$$\Phi^{A_a} = \Psi^{A_b} \Rightarrow (\forall x \geq x_0)[\Phi^{A_a}(x) = \Delta^{A_c}(x)].$$

Thus, assume that $Z = \Phi^{A_a} = \Psi^{A_b}$. We first show that if $x \in Z$ then $x \in \Delta^{A_c}$. To this end, it is enough to observe that if $x \in Z$ then there exists a stage t such that for every $s \geq t$, $x \in \Phi^{A_a}[s] \cap \Psi^{A_b}[s]$. Then for every $j \in \mathcal{J}_a \cup \mathcal{J}_b$,

$$\lim_s G^j(f_n, x, s) = G(f_n, x) \quad \text{and} \quad \lim_s H^j(f_n, x, s) = H(f_n, x)$$

exist (the fact that, by Lemma 2.15, A_a and A_b are of low e-degrees guarantees that our choice of $G^j(f_n, x, s)$ and $H^j(f_n, x, s)$ as the least such finite subsets stabilizes in the limit), and $G^j(f_n, x, s) \cup H^j(f_n, x, s) \subseteq B_j$, giving

$$\bigoplus_{j \in \mathcal{J}_c} (G^j(f_n, x) \cup H^j(f_n, x)) \subseteq A_c,$$

hence $x \in \Delta^{A_c}$.

Assume now that $x \geq x_0$ and $x \in \Delta^{A_c} - Z$, e.g. $x \notin \Phi^{A_a}$. Let $\sigma \subset f$ be a node such that $R(\sigma) = \mathcal{L}_{a,\Phi,x}$. Since $x \notin \Phi^{A_a}$, by construction the extraction activity performed on behalf of the requirements $R(\tau)$ with $f_n \subset \tau \subset \sigma$ (directly, or by activating $\mathcal{R}_z$, via injury of $R(\tau)$), interferes with the strategy for fixing $x \in \Phi^{A_a}$ on behalf of $R(\sigma)$. Then there is a least τ that instigates $R(f_n)$ at x. Hence τ will eventually instigate $R(f_n)$ at some such x, and the construction would make $f_{n+1} = f_n{}^\frown 1$, contradiction.

Lemma 2.17 *For every j, a, Φ, the requirement $\mathcal{Q}_{j,a,\Phi}$ is satisfied.*

Proof: We need to show that if $j \not\leq a$ then $B_j \neq \Phi^{A_a}$. Let f_n be such that $R(f_n) = \mathcal{Q}_{j,a,\Phi}$. By Lemma 2.13, $c = \lim_s c(f_n, s)$ and $\lim y^i(f_n, s)$ exist.

If $c \notin \Phi^{A_a}$ then $f_{n+1} = f_n{}^\frown 1$ and $c \in F^j(f_{n+1}) \subseteq B_j$; hence $c \in B_j - \Phi^{A_a}$.

If $c \in \Phi^{A_a}$ then $f_{n+1} = f_n{}^\frown 0$ and $E^j(f_{n+1}) = \{c\}$, hence $c \in \Phi^{A_a} - B_j$.

Lemma 2.18 *For each z, the requirement $\mathcal{R}_z$ is satisfied.*

Proof: We need to show that $\overline{K} = \Gamma^{A_{\mathcal{J}}}$.

($\subseteq$) Let $z \in \overline{K}$; then there exists a stage t such that for every $s \geq t$, $z \in \overline{K}^s$; then at some even stage s, we enumerate an axiom $\langle z, \bigoplus_{j \in \mathcal{J}} \{y^j_z\}\rangle$ into Γ^{s+1}, and the construction ensures that for every j, $y^j_z \in B^s_j$. Hence $z \in \Gamma^{A_{\mathcal{J}}}$.

($\supseteq$) Assume that $z \notin \overline{K}$, and let f_n be such that $R(f_n) = \mathcal{R}_z$; hence $f_{n+1} = f_n{}^\frown 0$. Let $\sigma \subseteq f_{n+1}$ be the unique string such that $z \in \lambda(\sigma)$ (recall that $f_n{}^\frown 0 \subset f$ and if $z \notin \lambda(\tau)$ for any $\tau \subseteq f_n$, then the construction places $z \in \lambda(f_n{}^\frown 0)$). Then there exists some $J \subseteq \mathfrak{J}$, $J \neq \emptyset$, such that $y^j_x \in E^j(\sigma)$, for all $j \in J$ where y^j_x is the original j-trace for x. Thus $x \notin \Gamma^{A_{\mathcal{J}}}$.

Lemma 2.19 *For every j and X, the requirement $\mathcal{J}_{j,X}$ is satisfied.*

Proof: For every j and X such that $j \notin X$ and $j \leq \bigvee X$, we need to show that $B_j = \Gamma^{A_X}_{j,X}$. It is clear that whenever we enumerate an element y in some B_j then the procedure of functional updating will (at least temporarily) enumerate y in $\Gamma^{A_X}_{j,X}$.

Notice that if $y \in B_j$ is a $\mathcal{Q}$-witness then, in fact, at some stage s we add an axiom $\langle y, \bigoplus_{i \in X}\{y^i\}\rangle$ into $\Gamma_{j,X}$, and $y \in B_j$ if and only if for every $i \in X$, $y^i \in B_i$ (see the first bullet of Definition 2.7), thus $y \in \Gamma^{A_X}_{j,X}$. On the other hand, if $y \notin B_j$ then by $1(b)$ of 2.6.2 and by Lemma 2.5, $y \notin \Gamma^{A_X}_{j,X}$.

If y has been enumerated into B_j on behalf of $\mathcal{R}_z$ at some even stage then there is some z such that $y \in Y^j(z)$. It is immediate to check that if $z \in \overline{K}$ then $y \in B_j \cap \Gamma^{A_X}_{j,X}$.

Thus, assume that $z \notin \overline{K}$. Let $f_n \subset f$ be the $\mathcal{R}_z$-node along the true path, and consider the unique $\tau \subseteq f_n{}^\frown 0$ such that $z \in \lambda(\tau)$. Suppose first that $y \in B_j - \Gamma^{A_X}_{j,X}$. If our extraction activity, activated at some last stage s by $\mathcal{R}_z$, does not guarantee that $y \notin B_j$, then y is involved in a procedure of functional updating which yields $y \in \Gamma^{A_X}_{j,X}$, contradiction.

Finally, assume that $y \in \Gamma^{A_X}_{j,X} - B_j$. Let us consider the following possibilities:

1. $\tau = f_n{}^\frown 0$; at some last τ-true even stage $s+1$, $z \in \lambda(\tau, s+1) - \lambda(\tau, s)$; and for every $t \geq s$, $z \in \lambda(\tau, t)$. In this case, $Y^i(z) \subseteq E^i(\sigma)$ for every i, hence $y \notin \Gamma^{A_X}_{j,X}$;

2. $\tau \subset f_n{}^\frown 0$: We must distinguish the following three cases:

   - If $R(\tau^-) = \mathcal{L}_{a,x,\Phi}$ with $a < 1$, then $X \subseteq J_a$ and $j \not\leq a$. On the other hand, we have that $j \leq a$ since $j \leq \bigvee X$, contradiction.
   - If $R(\tau^-) = \mathcal{M}_{a,b,c,\Phi,\Psi}$ and, say, τ is permanently b-related then $j \in J_a - J_c$ and $X \subseteq J_b$ but then $j \leq c$, contradiction.
   - If $R(\tau^-) = \mathcal{Q}_{i,Y,\Phi}$, with $i \notin Y$, then $X \subseteq Y$, but then $i \notin X$.

In any of these cases, we see that there exists $i \in X$ such that $E^i(\tau) \neq \emptyset$. By Lemma 2.13, let t be the last stage such that $E^j(\tau, t+1) \neq E^j(\tau, t)$. Then $y \in Y^j(z,t)$. If $\langle y, \bigoplus_{k \in X} G^k \rangle \in \Gamma_{j,X}$ then $G^k \subseteq Y^k(z,t) \subseteq E^k(\tau)$. Hence $y \notin \Gamma_{j,X}^{A_X}$, contradiction.

3. There is a last stage s at which we define $z \in \lambda(f_n{}^\frown 0, s)$ by λ-closure, because of some $\rho \prec_L f_n$, and for every $t \geq s$, $z \in \lambda(f_n{}^\frown 0, t)$. Then for every $t \geq s$ and for every i, $E^i(f_{n+1}, t) = E^i(\rho, s)$, and $E^i(\rho, s) \cap B_i = \emptyset$, and we can argue as in the preceding case to conclude that $y \notin \Gamma_{j,X}^{A_X}$, contradiction.

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