THE PROOF-THEORETIC STRENGTH OF THE DUSHNIK-MILLER THEOREM FOR COUNTABLE LINEAR ORDERS

Rodney G. Downey
Steffen Lempp

Department of Mathematics
Victoria University of Wellington
Wellington
NEW ZEALAND

Department of Mathematics
University of Wisconsin
Madison, WI 53706-1388
USA

Abstract. We show that the Dushnik-Miller Theorem for countable linear orderings (stating that any countable linear ordering has a nontrivial self-embedding) is equivalent (over recursive comprehension (RCA_0)) to arithmetic comprehension (ACA_0).

This paper presents a result in reverse mathematics, a program initiated by H. Friedman and S. Simpson, trying to determine the weakest possible "set-theoretical" axiom (system) to prove a given theorem of "ordinary" mathematics by trying to prove the axiom from the theorem (over a weaker "base system").

The "set-theoretical" axiom systems we will be concerned with are weak subsystems of second-order arithmetic. (We refer to Simpson [Sita] for a detailed exposition of such systems.) In particular, we will use the axiom system RCA_0 of recursive comprehension (with Σ^0_1-induction) as a base system and exhibit a theorem which is equivalent (over the base system RCA_0) to the axiom system ACA_0 of arithmetic comprehension (with Σ^0_1-induction).

1991 *Mathematics Subject Classification.* 03F35, 06A05.

Key words and phrases. linear ordering, reverse mathematics, ACA_0, Dushnik-Miller Theorem, self-embedding.

The first author was partially supported by the New Zealand Marsden Fund for Basic Science, and both authors by a U. S./N. Z. Binational Grant. The second author's research was partially supported by NSF grant DMS-9504474.

Typeset by $\mathcal{AMS}$-TEX

The area of "ordinary" mathematics we study in the context of reverse mathematics is that of linear orderings. In particular, we will characterize the proof-theoretic strength of the

Dushnik-Miller Theorem on Countable Linear Orderings [DM40]**.** *Let $\mathcal{L}$ be a countably infinite linear ordering. Then there is a nontrivial* self-embedding *of $\mathcal{L}$, i.e., an order-preserving injection of $\mathcal{L}$ into itself which is not the identity.*

We will thus show the following

Theorem. *Over the base system RCA_0, the Dushnik-Miller Theorem on Countable Linear Orderings is equivalent to the axiom system ACA_0.*

Proof. We first prove the easy direction, namely, that ACA_0 is strong enough to prove the Dushnik-Miller Theorem on Countable Linear Orderings. Fix a countably infinite linear ordering $\mathcal{L}$. Call $C \subseteq L$ a *convex* subset of $\mathcal{L}$ if C contains any point z between any points $x, y \in C$.

First assume that L contains a convex subset C of order type ω. Then the map i which is the identity off C and moves every element of C to its immediate successor is a nontrivial self-embedding of $\mathcal{L}$ and can be defined by a first-order formula (in the language of arithmetic), thus can be shown to exist by ACA_0. The case where $\mathcal{L}$ contains a convex subset of order type ω^* (i.e., ω under the reverse ordering) is handled similarly.

So assume that $\mathcal{L}$ does not contain any convex subsets of order type ω or ω^*. Call a convex subset C of $\mathcal{L}$ discrete if every element of C (except the least, if any) has an immediate predecessor in C, and every element of C (except the greatest, if any) has an immediate successor in C. By our assumption, any discrete subset of $\mathcal{L}$ must be finite. By picking one element from each maximal discrete subset of $\mathcal{L}$ (except the first and last maximal discrete subset, if any), we see that there is an infinite subset of $\mathcal{L}$ which is densely ordered without endpoints. Since ACA_0 actually allows full first-order induction, we can now define the self-embedding as follows: List the points of $\mathcal{L}$ as $\{x_n\}_{n\in\omega}$. When picking an image $i(x_n)$ for x_n, simply ensure that $i(x_n)$ has infinitely many points to its left and right and is infinitely far apart from $i(x_0), \cdots i(x_{n-1})$. Since the choice of $i(x_n)$ can be made in an arithmetic way, i can be shown to exist by ACA_0. This concludes the proof of the easy direction.

As for the hard direction, we need to show that the second-order part of any model of "RCA_0 plus Dushnik-Miller Theorem on Countable Linear Orderings" is closed under Σ^0_1-comprehension. So fix any set $A \in \mathcal{S}$ (where $\mathcal{S}$ is the collection of subsets included in the given second-order model of "RCA_0 plus Dushnik-Miller"). We need to show that its "Turing jump" A' is also in $\mathcal{S}$. We do so by defining a countable linear ordering $\mathcal{L}$ computable in A such that any nontrivial self-embedding i of $\mathcal{L}$ can compute A'.

Fix an A-computable enumeration $\{A'_s\}_{s\in\omega}$ of A', and let $c(x) = \mu s \geq x(A' \upharpoonright (x+1) = A'_s \upharpoonright (x+1))$ be the associated A-computable *computation function* of A'. Since $A' \leq_T A \oplus c$, it suffices to ensure the following

Claim 1. *Any nontrivial self-embedding i of $\mathcal{L}$ can compute the computation function c.*

We define the linear ordering $\mathcal{L}$ of order type $(M,<)$ with universe M in stages and start by letting $\mathcal{L}_0$ be the ordering $0 \leq_{\mathcal{L}} 2 \leq_{\mathcal{L}} 4 \leq_{\mathcal{L}} \dots$ of all even integers in M. We establish Claim 1 by ensuring the existence of a function e satisfying the following

Claim 2. *There is a strictly increasing function $e : M \to L_0$ such that for all $x \in M$,*

$$\forall n_0, n_1 < c(x)\,(e(x) <_{\mathcal{L}} n_0 <_{\mathcal{L}} n_1 \to d(e(x), n_0) > d(n_0, n_1)),\ \textit{and} \tag{1}$$

$$e(x+1) = \mu y \in L_0 \forall n\,(n < c(x) \to n <_{\mathcal{L}} y), \tag{2}$$

where $d(n_0, n_1)$ is the ($\mathcal{M}$-finite) distance between n_0 and n_1 in $\mathcal{L}$.

We first establish Claim 1 from Claim 2: Fix any nontrivial self-embedding i of $\mathcal{L}$. By Σ^0_1-induction, e is monotonic, so the range of e is cofinal in $(M,<)$, and so there is $x_0 \in M$ such that for all $x \geq x_0$, we have $e(x) <_{\mathcal{L}} ie(x)$. Also, by Σ^0_1-induction and (1) of Claim 2, for all $x \geq x_0$, we have that one of $ie(x)$ and $i^2e(x)$ is $\geq c(x)$ (since $d(e(x), ie(x)) \leq d(ie(x), i^2e(x))$). So from $e(x)$ and i we can compute $c(x)$. Finally, by (2) of Claim 2, we can also compute $e(x+1)$. Thus i allows us to compute c as desired, establishing Claim 1 from Claim 2.

The proof of Claim 2 is a finite-injury priority argument (using Σ^0_1-induction). We have to maintain (1) and (2) of Claim 2 at any stage s for all $x \leq s$ (evaluating $c(x)$ for these x's at stage s). Note that the definition of the function e is fixed by (2) at any stage s (assuming $e(0) = 0$). The only problem arises if some number x enters A' at a stage $s > 0$, thus making (1) false. In that case, add all currently unused elements $y \leq s$ in $M - L_{s-1}$ into L_s just to the left of $e(x)$, and add sufficiently many unused elements $y > s$ in $M - L_{s-1}$ into L_s just to the right of $e(x)$ to make (1) true. Note that this action will not interfere with keeping (1) satisfied for any $x' < x$. It is now easy to verify that this construction will produce the desired linear ordering satisfying Claim 2.

References

[DM40] Dushnik, B. and Miller, E. W., *Concerning similarity transformations of linearly ordered sets*, Bull. Amer. Math. Soc. **46** (1940), 322–326.

[Ro82] Rosenstein, J. G., *Linear Orderings*, Academic Press, New York, 1982.

[Sita] Simpson, S. G., *Subsystems of second-order arithmetic*, available as preprint (to appear).