DOWNWARD CLOSURE OF DEPTH IN COUNTABLE BOOLEAN ALGEBRAS

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Abstract. We study the set of depths of relative algebras of countable Boolean algebras, in particular the extent to which this set may not be downward closed within the countable ordinals for a fixed countable Boolean algebra. Doing so, we exhibit a structural difference between the class of arbitrary rank countable Boolean algebras and the class of rank one countable Boolean algebras.

1. Introduction

For most classes of common algebraic structures, there is a natural ordinal-valued or cardinal-valued measure that captures the *size* and/or *complexity* of a particular structure $\mathcal{S}$ in comparison to other members of the class. Examples include the dimension of a vector space, the Hausdorff rank of a scattered linear order, the rank of a torsion-free abelian group, the transcendence degree of a field, etc. These measurements of size and/or complexity share a common property: All are closed downwards in the sense that if $\mathcal{S}$ has size or complexity α and $\beta \leq \alpha$, then $\mathcal{S}$ has a substructure of size or complexity β.

For the class of countable Boolean algebras, there are several natural notions of size and complexity. After cardinality, which is not downward closed within the finite cardinalities, perhaps the simplest measure of size is the smallest ordinal α for which an α-atom (the interval algebra of ω^α) is not a relative algebra. As a α-atom bounds a β-atom exactly when $\beta \leq \alpha$, this is also downward closed.

If instead the desire is to measure complexity, perhaps the most appropriate measure is *depth*. Though perhaps surprising, there are straightforward examples that demonstrate depth is not downward closed within the relative algebras of an algebra. The purpose of this paper is to illustrate such examples and to study the extent to which depth fails to be downward closed. We do so both in the context of arbitrary rank Boolean algebras and rank one Boolean algebras. As these classes behave differently with respect to the extent to which depth fails to be downward closed, this exhibits a structural difference between arbitrary rank Boolean algebras and rank one Boolean algebras.

Date: June 26, 2012.

2010 *Mathematics Subject Classification.* Primary: 06E05; Secondary: 03G05.

Key words and phrases. Ketonen invariants, depth, downward closure.

Kach's research was partially supported via a Post-Doctoral Fellowship from the Marsden Fund of New Zealand. Lempp's research was supported by NSF Grant DMS-0555381 and Grant # 13407 by the John Templeton Foundation entitled *Exploring the Infinite by Finitary Means.* Lempp would like to thank Victoria University of Wellington for its hospitality during the beginning of the time this research was carried out. Both authors thank the anonymous referee for many helpful suggestions and corrections and Antonio Montalbán for simplifying the construction of Section 3.2.

For every countable ordinal α, we exhibit countable Boolean algebras whose depths of relative algebras are precisely the sets $\{0, \dots, \alpha, \alpha+2\}$ (Theorem 3.1), $\{0, 1, \omega^\alpha\}$ (Theorem 3.6), and $\{0, 2, \omega^\alpha\}$ (Theorem 3.7). We show that every Boolean algebra (of depth at least one) has, in addition to a depth zero relative algebra, a depth one relative algebra or a depth two relative algebra (Section 4). We also exhibit rank one Boolean algebras whose depths of relative algebras are precisely the set $\omega \cup \{\omega^\alpha\}$ (Theorem 5.4) and show the set of depths of relative algebras are even more heavily constrained for rank one Boolean algebras (Theorem 5.3).

Though we assume the reader has a working knowledge of measures, derivatives, and depth, we recall some background on measures and depth within Section 2. We defer the reader elsewhere (see [5], or see [3] or [7] for alternate expositions) for a full exposition of this material.

Another important measure of complexity is *Scott rank*. It, too, is not always downward closed. Alaev [1], for each countable ordinal α, constructs a Boolean algebra of Scott rank greater than α such that every relative algebra has Scott rank strictly less than $\omega + \omega$ or greater than or equal to α. Indeed, a careful analysis of his work (his constructions are very different from our constructions) might show that the set of depths of relative algebras is exactly $\{0, 1, \omega^\beta\}$ for some β. Similarly, a careful analysis of our work in Section 3.2 or Section 3.3 would probably yield the Scott ranks of relative algebras are exactly $\omega \cup \{\omega^\alpha\}$. We defer the reader elsewhere (see [6], for example) for background on Scott rank.

Throughout this paper, all Boolean algebras are countable. As is often done, we identify an element of Boolean algebras with the associated relative algebra. Thus, for $x \in \mathcal{B}$, we often write x for the subalgebra with universe $x \restriction \mathcal{B} := \{y \in B : y \leq x\}$ and say that x has property P if $x \restriction \mathcal{B}$ has property P.

2. Background and Notation

As part of the isomorphism invariants for countable (uniform) Boolean algebras, Ketonen [5] introduced a hierarchy of sets $\{K_\alpha\}_{\alpha \in \omega_1}$. The algebraic invariant assigned to a uniform Boolean algebra was a member of K_α for some $\alpha \in \omega_1$. The least possible α is a measure of the complexity of the uniform Boolean algebra.

Throughout this paper, we identify the countable atomless algebra $\mathcal{F}$ with the clopen algebra of 2^ω in the natural way. With this identification, we denote the element of the countable atomless algebra corresponding to a basic clopen set $[\sigma] := \{f \in 2^\omega : \sigma \preccurlyeq f\}$, where $\sigma \in 2^{<\omega}$, by σ. We therefore view measures as functions $\sigma \colon 2^{<\omega} \to \omega_1$, with $\sigma(x) = \max\{\sigma(\tau_1), \dots, \sigma(\tau_k)\}$ for an arbitrary element of the countable atomless algebra written as a sum $x = \tau_1 \oplus \cdots \oplus \tau_k$ of basic clopen sets.

Because measures are additive, these functions always satisfy

$$\sigma(\tau) = \max\{\sigma(\tau ^\frown 0), \sigma(\tau ^\frown 1)\}$$

for all $\tau \in 2^{<\omega}$, ensuring $\sigma(x)$ is well-defined.

Taking several steps backward, the algebraic invariants for arbitrary countable Boolean algebras begin with the better-known algebraic invariants for countable superatomic Boolean algebras. We recall that a Boolean algebra is *superatomic* if every subalgebra is atomic (bounds an atom). In the countable setting, the superatomic algebras are precisely the (left-closed, right-open) interval algebras of ordinals. We also recall that the *rank* of a nonzero countable superatomic Boolean algebra $\mathcal{B}$, denoted $\operatorname{rank}(\mathcal{B})$, is the ordinal $\alpha + 1$ for which there is an integer n

so that $\mathcal{B}$ is the interval algebra of $\omega^\alpha \cdot n$. By convention, the rank of the trivial algebra is zero.

The notion of rank is then generalized to arbitrary $\mathcal{B}$.

Definition 2.1. If $\mathcal{B}$ is any Boolean algebra and $x \in \mathcal{B}$, the *rank* of x, denoted $\mu_\mathcal{B}(x)$, is given by

$$\mu_\mathcal{B}(x) := \sup\{\mathrm{rank}(y) : y \leq x, y \in \mathrm{SA}(\mathcal{B})\},$$

where $\mathrm{SA}(\mathcal{B})$ is the set of superatomic elements of $\mathcal{B}$.

The element $x \in \mathcal{B}$ is *uniform* if $\mu_\mathcal{B}(x) = \mu_\mathcal{B}(x - z)$ for all $z \in \mathrm{SA}(x)$.

From the notion of rank for an arbitrary element, the notion of measures can be developed.

Definition 2.2. If $\mathcal{B}$ is any Boolean algebra, define its *measure (with domain $\mathcal{B}$)* to be the function $\hat{\sigma}_\mathcal{B} \colon \mathcal{B} \to \omega_1 \cup \{o\}$ given by

$$\hat{\sigma}_\mathcal{B}(x) := \begin{cases} o & \text{if } x \in \mathrm{SA}(\mathcal{B}), \\ \min\{\mu_\mathcal{B}(y) : y \leq x, x - y \in \mathrm{SA}(\mathcal{B})\} & \text{otherwise.} \end{cases}$$

Here o is a special symbol that (by definition) satisfies $o < \alpha$ for any ordinal $\alpha \in \omega_1$.

The observation that $\mathcal{F}$, the countable atomless Boolean algebra, is isomorphic to the quotient of any non-superatomic Boolean algebra $\mathcal{B}$ by the ideal $\mathrm{SA}(\mathcal{B})$ allows the domain of a measure (with domain $\mathcal{B}$) to be shifted from $\mathcal{B}$ to $\mathcal{F}$.

Definition 2.3. If $\mathcal{B}$ is any Boolean algebra and $\pi \colon \mathcal{B}/\mathrm{SA}(\mathcal{B}) \to \mathcal{F}$ is an isomorphism, define its *measure (with domain $\mathcal{F}$ with respect to π)* to be the function $\varsigma_\mathcal{B} \colon \mathcal{F} \to \omega_1 \cup \{o\}$ given by

$$\varsigma_\mathcal{B}(x) := \hat{\sigma}_\mathcal{B}(z),$$

where $z \in \mathcal{B}$ is such that $\pi([z]) = x$.

We note that $\varsigma_\mathcal{B}$ is well-defined for if $\pi([z]) = \pi([z'])$, then z and z' differ by at most a superatomic element, and consequently $\hat{\sigma}_\mathcal{B}(z) = \hat{\sigma}_\mathcal{B}(z')$. The choice of the isomorphism π is unimportant, so we will omit reference to it. Because of the natural embedding ι of $2^{<\omega}$ into $\mathcal{F}$, we can further simplify the domain of a measure.

Definition 2.4. If $\mathcal{B}$ is any Boolean algebra, define its *measure (with domain $2^{<\omega}$)* to be the function $\sigma_\mathcal{B} \colon 2^{<\omega} \to \omega_1$ given by

$$\sigma_\mathcal{B}(x) := \varsigma_\mathcal{B}(\iota(x)).$$

Conversely, if $\sigma \colon 2^{<\omega} \to \omega_1$ is any map satisfying the equality $\sigma(\tau) = \max\{\sigma(\tau ^\frown 0), \sigma(\tau ^\frown 1)\}$ for all $\tau \in 2^{<\omega}$, define a map $\varsigma_\sigma \colon \mathcal{F} \to \omega_1 \cup \{o\}$ by

$$\varsigma_\sigma(x) := \sup\{\sigma(\tau) : \tau \in 2^{<\omega}, \iota(\tau) \leq x\}$$

and $\varsigma_\sigma(0_\mathcal{F}) := o$. Let $\mathcal{B}_\sigma$ be the (unique) Boolean algebra having σ (equivalently ς_σ) as its measure.

We end this discussion of measures by introducting the notion of derivatives.

Definition 2.5. If $\sigma\colon \mathcal{F} \to \omega_1 \cup \{o\}$ is a measure, define by recursion on α its *αth derivative* to be the map $\Delta^\alpha\sigma$ with domain $\mathcal{F}$ given by $\Delta^0\sigma(x) := \sigma(x)$,

$$\Delta^{\alpha+1}\sigma(x) := \{(\Delta^\alpha\sigma(x_1), \dots, \Delta^\alpha\sigma(x_n)) : x = x_1 \oplus \cdots \oplus x_n\},$$

and $\Delta^\alpha\sigma(x) := \{(\cup_{\beta<\alpha}\Delta^\beta\sigma(x_1), \dots, \cup_{\beta<\alpha}\Delta^\beta\sigma(x_n)) : x = x_1 \oplus \cdots \oplus x_n\}$ for limit α.

Definition 2.6. The *depth* of a measure σ (with domain $\mathcal{F}$) is the least ordinal δ such that

$$\Delta^\delta\sigma(x) = \Delta^\delta\sigma(y) \implies \Delta^{\delta+1}\sigma(x) = \Delta^{\delta+1}\sigma(y)$$

for all $x, y \in \mathcal{F}$.

We note that this exposition largely follows Kach [4], which itself is largely based on the exposition in Heindorf [3] and Pierce [7].

Definition 2.7. If $\mathcal{B}$ is a Boolean algebra, we denote its depth by $\delta(\mathcal{B})$.

If $\mathcal{B}$ is a Boolean algebra, we denote the collection of depths $\{\delta(x) : x \in \mathcal{B}\}$ by $\Delta(\mathcal{B})$.

When studying the downward closure of depths of relative algebras, the depth zero measures (studied in [3] and [4]) and certain depth α measures (for $\alpha \in \omega_1$) will be ubiquitous. We introduce notation to describe them.

Definition 2.8. If $S \subset \omega_1$ is a nonempty set with maximal element, denote by $\mathcal{B}_{v(S)}$ the depth zero countable Boolean algebra having range$(\sigma) = S$ and having disjoint elements x and y with $\sigma(x) = \max S = \sigma(y)$; denote by $\mathcal{B}_{u(S)}$ the depth zero Boolean algebra having range$(\sigma) = S$ and not having disjoint elements x and y with $\sigma(x) = \max S = \sigma(y)$.

We will abuse notation slightly, sometimes writing $\mathcal{B}_{v(S)}$ for $\mathcal{B}_{u(S)}$ when $|S| = 1$.

Definition 2.9. For $\beta > 0$, denote by $\chi_{\beta,0}$ the rank β measure for which the preimage of β has order type 1 and the preimage of γ is empty for any $0 < \gamma < \beta$.

For $\beta > 0$ and $\alpha > 0$, denote by $\chi_{\beta,\alpha}$ the rank β measure for which the preimage of β has order type $\omega^\alpha + 1$ and the preimage of γ is empty for any $0 < \gamma < \beta$.

Heindorf showed that, for any countable ordinal α, the depth of $\chi_{1,\alpha}$ is α. We show more.

Lemma 2.10. *The depth of $\bigoplus_{i=1}^{i=N} \chi_{\beta,\alpha}$ is α if $N = 1$ and $\alpha + 1$ if $N > 1$.*

Proof. Heindorf showed $\delta(\chi_{1,\alpha}) = \alpha$ in Example 1.23.1 of Pierce [7]. As the depth of $\chi_{\beta,\alpha}$ does not depend on β, we have $\delta(\chi_{\beta,\alpha}) = \alpha$.

As $\bigoplus_{i=1}^{i=N} \chi_{\beta,\alpha}$ is a relative algebra of $\chi_{\beta,\alpha+1}$ for any N, we have that $\delta(\bigoplus_{i=1}^{i=N} \chi_{\beta,\alpha}) \leq \alpha + 1$. If $N > 1$, then $\delta(\bigoplus_{i=1}^{i=N} \chi_{\beta,\alpha}) > \alpha$ as $\Delta^\alpha\sigma(\chi_{\beta,\alpha}) = \Delta^\alpha\sigma(\chi_{\beta,\alpha} \oplus \chi_{\beta,\alpha})$ and $\Delta^{\alpha+1}\sigma(\chi_{\beta,\alpha}) \neq \Delta^{\alpha+1}\sigma(\chi_{\beta,\alpha} \oplus \chi_{\beta,\alpha})$. □

Proposition 2.11. *For any countable ordinal α, the $(\alpha+1)$st derivative of $\chi_{\beta,\alpha}$ is an isomorphism invariant. Indeed, the $(\alpha+1)$st derivative of any finite sum $\sum_{i=1}^{i=N} \chi_{\beta,\gamma_i}$ with $\gamma_i \leq \alpha$ is an isomorphism invariant.*

Proof. We start by noting that we need only consider the case when $\gamma_i = \alpha$ for all i. For if $\gamma_j < \gamma_k$ for some $1 \leq j, k \leq N$, then $\chi_{\beta,\gamma_j} \oplus \chi_{\beta,\gamma_k} \cong \chi_{\beta,\gamma_k}$. Consequently, we may assume $\gamma_j = \gamma_k$ for all $1 \leq j, k \leq N$. Moreover, if $\gamma_1 = \max\{\gamma_1, \dots, \gamma_N\} < \alpha$, then the $(\alpha+1)$st derivative is an isomorphism invariant if the $(\max\{\gamma_1, \dots, \gamma_N\} +$

1)st derivative is an isomorphism invariant. Consequently, we may assume $\gamma_i = \alpha$ for all $1 \leq i \leq N$.

If $\alpha = 0$, then the first derivative of $\sum_{i=1}^{i=N} \chi_{\beta,0}$ consists of all sequences with at most N many βs and any number of 0s. It is easy to see that this is an isomorphism invariant.

If $\alpha > 0$, then the $(\alpha+1)$st derivative of $\sum_{i=1}^{i=N} \chi_{\beta,\alpha}$ consists of finite sequences of αth derivatives. We note that no countable Boolean algebra having a perfect kernel of nodes x with $\sigma(x) = \beta$ can share this $(\alpha+1)$st derivative. The reason is that any such Boolean algebra has arbitrarily many disjoint elements with measure β, each of which can be split into arbitrarily many disjoint elements with measure β, each of which $\dots$, and so on α many times. The Boolean algebra $\sum_{i=1}^{i=N} \chi_{\beta,\alpha}$ has only N many elements with this property. Moreover, the $(\alpha+1)$st derivative reflects this difference.

Similarly, if $N' \neq N$, then the $(\alpha+1)$st derivatives of $\sum_{i=1}^{i=N} \chi_{\beta,\alpha}$ and $\sum_{i=1}^{i=N'} \chi_{\beta,\alpha}$ reflect this difference: If $N' < N$, then the latter does not have N many elements that can be sufficiently split; and if $N' > N$, then the latter has more than N elements that can be sufficiently split. □

Finally, we introduce some mechanisms to describe more complicated measures. The idea is to code information by having infinitely many Boolean algebras appear as relative algebras. These relative algebras can either appear "linearly" or "densely".

Definition 2.12. Denote the *empty string* by ε.

Definition 2.13. If $\{\mathcal{B}_i\}_{i\in\omega}$ is any infinite sequence of (not necessarily distinct) Boolean algebras (with σ_i a measure for $\mathcal{B}_i$), define their *spined sum* (denoted $\sum_{\text{sp}} \mathcal{B}_i$) to be the Boolean algebra with measure

$$\sigma(\tau) := \begin{cases} \sigma_i(\tau_1) & \text{if } \tau = 1^i \frown 0 \frown \tau_1, \\ \sup\{\sigma_j(\varepsilon) : j \geq i\} & \text{if } \tau = 1^i. \end{cases}$$

Define their *repeated spined sum* (denoted $\sum_{\text{rsp}} \mathcal{B}_i$) to be the spined sum of the sequence $\{\mathcal{B}_0, \mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \dots\}$.

If $\{\mathcal{B}_i\}_{i\leq K}$ is any finite sequence of Boolean algebras, define their *repeated spined sum* to be the spined sum of the sequence $\{\mathcal{B}_0, \dots, \mathcal{B}_K, \mathcal{B}_0, \dots, \mathcal{B}_K, \dots\}$ (denoted $\sum_{\text{rsp}} \mathcal{B}_i$).

We remark that the functions σ for spined sums and repeated spined sums are indeed measures as they satisfy $\sigma(\tau) = \max\{\sigma(\tau \frown 0), \sigma(\tau \frown 1)\}$ for all $\tau \in 2^{<\omega}$.

Definition 2.14. A string $\tau \in 2^{<\omega}$ is a *repeater string* if the length $|\tau|$ of τ is even and $\tau(2i) = \tau(2i+1)$ for all $i < |\tau|/2$. The string $\tau' = \tau(0) \frown \tau(2) \frown \dots \frown \tau(|\tau|/2-1)$ is a *witness* to τ being a repeater string.

A string $\tau \in 2^{<\omega}$ is an *almost repeater string* if τ is a repeater string or of the form $\tau = \tau' \frown 0$ or $\tau = \tau' \frown 1$ for some repeater string τ'.

A string $\tau \in 2^{<\omega}$ is a *xor string* if either $\tau = 01$ or $\tau = 10$.

We note that if τ is not an almost repeater string, there is a unique decomposition $\tau = \tau_1 \frown \tau_2 \frown \tau_3$ with τ_1 a repeater string, τ_2 a xor string, and τ_3 an arbitrary string. This uniqueness ensures later measures are well-defined.

The idea is for the relative measures rooted at a node $\tau_1 ^\frown \tau_2$ to code information, where τ_1 is a repeater string and τ_2 is a xor string. We therefore informally refer to these nodes as *coding locations*. It is possible to code this information at a coding location in different ways: either by utilizing depth (as in Definition 2.15) or by utilizing rank (as in Definition 2.16).

Definition 2.15. If $T : 2^{<\omega} \to \omega$ is any map, define the *rank one* $\mathcal{B}_T$ to be the Boolean algebra with measure

$$\sigma_T(\tau) := \begin{cases} 1 & \text{if } \tau \text{ is an almost repeater string,} \\ \\ \chi_{1,T(\tau')}(\tau_3) & \text{if } \tau = \tau_1 ^\frown \tau_2 ^\frown \tau_3 \text{ for some repeater string } \tau_1 \\ & \text{with witness } \tau' \text{ and xor string } \tau_2. \end{cases}$$

Definition 2.16. If $T : 2^{<\omega} \to \omega$ is any map, define the *arbitrary rank* $\mathcal{B}_T$ to be the Boolean algebra with measure

$$\sigma_T(\tau) := \begin{cases} \omega & \text{if } \tau \text{ is an almost repeater string,} \\ \\ \chi_{T(\tau'),0}(\tau_3) & \text{if } \tau = \tau_1 ^\frown \tau_2 ^\frown \tau_3 \text{ for some repeater string } \tau_1 \\ & \text{with witness } \tau' \text{ and xor string } \tau_2. \end{cases}$$

Of course, these maps can be generalized to maps T with ranges encompassing larger ordinals. For our purposes, it suffices to restrict attention to when the range is a subset of ω rather than a (bounded) subset of ω_1.

3. Arbitrary Rank Examples

In this section, we exhibit various countable Boolean algebras $\mathcal{B}$ for which $\Delta(\mathcal{B})$ is not a downward closed set within the countable ordinals. Before constructing more sophisticated examples, we start with perhaps the simplest example: The Boolean algebra $\mathcal{B} := \mathcal{B}_1 \oplus \mathcal{B}_2$ with $\mathcal{B}_1 := \mathcal{B}_{u(\{0,1\})}$ and $\mathcal{B}_2 := \mathcal{B}_{v(\{0,1\})}$ has $\Delta(\mathcal{B})$ not downward closed. This follows from the following readily verified facts:

- The depth of $\mathcal{B}$ is two.
- The depth of $\mathcal{B}_1$ is zero. The depth of $\mathcal{B}_2$ is zero.
- If $x \in B_1$ and $y \in B_2$, then $x \oplus y$ is isomorphic to a relative algebra of $\mathcal{B}_1$, a relative algebra of $\mathcal{B}_2$, or $\mathcal{B}$.

Consequently, we have $\Delta(\mathcal{B}) = \{0, 2\}$.

3.1. **The Set** $\{0, \dots, \alpha, \alpha + 2\}$**.** In order to realize $\Delta(\mathcal{B}) = \{0, \dots, \alpha, \alpha + 2\}$, it suffices to generalize the example just discussed. If α is a successor ordinal $\beta + 1$, rather than have the zero measure at the coding locations of $\mathcal{B}_1$ and $\mathcal{B}_2 \cong \chi_{1,0}$, we have $\chi_{1,\beta}$ at the coding locations of $\mathcal{B}_1$ and $\mathcal{B}_2 \cong \chi_{1,\alpha}$. If α is a limit ordinal, we instead have $\chi_{1,f_\alpha(n)}$ (where $\{f_\alpha(n)\}_{n\in\omega}$ is an increasing sequence cofinal in α) at the coding locations at height n of $\mathcal{B}_1$.

Theorem 3.1. *For every ordinal α, there is a countable Boolean algebra $\mathcal{B}$ with $\Delta(\mathcal{B}) = \{0, \dots, \alpha, \alpha + 2\}$.*

Proof. We start with the case when α is a successor ordinal $\beta + 1$. The algebra $\mathcal{B}$ is the sum of depth α algebras $\mathcal{B}_1$ and $\mathcal{B}_2$, where $\mathcal{B}_1$ is the rank one $\mathcal{B}_T$ with $T \equiv \beta$ and $\mathcal{B}_2$ is $\chi_{1,\alpha}$.

We establish $\Delta(\mathcal{B}) = \{0, \dots, \alpha, \alpha+2\}$ by showing $\delta(\mathcal{B}_1) = \alpha$, $\delta(\mathcal{B}) = \alpha+2$, and $x \oplus y$ is isomorphic to a relative algebra of $\mathcal{B}_1$, a relative algebra of $\mathcal{B}_2$, or $\mathcal{B}$ for all $x \in B_1$ and $y \in B_2$.

Claim 3.1.1. *The depth of $\mathcal{B}_1$ is α.*

Proof. Since $\mathcal{B}_1$ contains $\chi_{1,\beta} \oplus \chi_{1,\beta}$ (which has depth $\beta+1$ by Lemma 2.10) as a relative algebra, the countable Boolean algebra $\mathcal{B}_1$ has depth at least $\alpha = \beta+1$. On the other hand, it has depth at most $\alpha = \beta + 1$ as the αth derivative of any element characterizes its isomorphism type (within those present in $\mathcal{B}_1$). This is a consequence of Proposition 2.11.

In more detail, we verify if $u, v \in B_1$ share common αth derivatives, then $u \cong v$. Proposition 2.11 implies u and v are isomorphic if neither u nor v have a perfect kernel of ones. If exactly one of u and v has a perfect kernel of ones (say u without loss of generality), then they do not share αth derivatives as u can be split $\alpha+1$ many times into elements with measure one whereas v cannot. If both u and v have a perfect kernel of ones, then u and v are isomorphic since T is a constant map. $\square$

Claim 3.1.2. *If $x \in B_1$ and $y \in B_2$, then $x \oplus y$ is isomorphic to a relative algebra of $\mathcal{B}_1$, a relative algebra of $\mathcal{B}_2$, or $\mathcal{B}$.*

Proof. Fix $x \in B_1$ and $y \in B_2$.

First assume that x contains a perfect kernel. Then $x \cong \mathcal{B}_1$ by Claim 3.1.1. If $y \cong \mathcal{B}_2$, then $x \oplus y \cong \mathcal{B}_1 \oplus \mathcal{B}_2$. If $y \not\cong \mathcal{B}_2$, then y is a finite sum of $\chi_{1,\gamma}$ for some $\gamma \leq \beta$. Then as $\mathcal{B}_1 \oplus y$ is isomorphic to $\mathcal{B}_1$, it must be that $x \oplus y$ is isomorphic to a relative algebra of $\mathcal{B}_1$.

Otherwise, i.e., if x does not contain a perfect kernel, then x is a finite sum of $\chi_{1,\gamma}$ for some $\gamma \leq \beta$. Then as $x \oplus \mathcal{B}_2$ is isomorphic to $\mathcal{B}_2$, it must be the case that $x \oplus y$ is isomorphic to a relative algebra of $\mathcal{B}_2$. $\square$

Claim 3.1.3. *The depth of $\mathcal{B}$ is $\alpha+2$.*

Proof. As the $(\alpha+1)$st derivative of $\mathcal{B}_2$ is an isomorphism invariant by Proposition 2.11, the $(\alpha+2)$nd derivative of an element $z \in B$ gives whether z has a relative algebra isomorphic to $\mathcal{B}_2$. The $(\alpha+2)$nd derivative of an element $z \in B$ also dictates whether z can be split $(\alpha+2)$-many times into measure one elements. This gives whether z has a relative algebra isomorphic to $\mathcal{B}_1$. Hence, by Claim 3.1.2 and Lemma 2.10, we have that the $(\alpha+2)$nd derivative of an element $z \in B$ dictates its isomorphism type within $\mathcal{B}$. $\square$

We finish by noting these claims imply $\Delta(\mathcal{B}) = \{0, \dots, \alpha, \alpha+2\}$. Since $\delta(\mathcal{B}) = \alpha+2$, we have $\alpha+2 \in \Delta(\mathcal{B})$. Since $\delta(\mathcal{B}_2) = \alpha$ and $\Delta(\mathcal{B}_2)$ is downward closed, we have $\{0, \dots, \alpha\} \subseteq \Delta(\mathcal{B})$. Finally, as a consequence of Claim 3.1.2, there are no elements with depth $\alpha+1$.

When α is a limit ordinal, it is no longer possible to take $\mathcal{B}_1$ to be $\mathcal{B}_T$ for some constant valued T. Instead, we take $\mathcal{B}_1$ to be the rank one $\mathcal{B}_T$, where $T(\tau) = f_\alpha(|\tau|)$ (where $\{f_\alpha(n)\}_{n\in\omega}$ is an increasing sequence cofinal in α), and take $\mathcal{B}_2$ to be $\chi_{1,\alpha}$. The analysis proceeds identically to the case when α is a successor ordinal. $\square$

3.2. **The Set $\{0, 1, \omega^\alpha\}$.** Fixing a countable ordinal α (throughout this subsection), the definition of a Boolean algebra $\mathcal{B}$ with $\Delta(\mathcal{B}) = \{0, 1, \omega^\alpha\}$ is rather complicated. Independent of α, the Boolean algebra will have a pure perfect kernel of rank ω.

At the coding locations, we will insert $\chi_{n,0}$ for $n \in \omega$. The positioning of these $\chi_{n,0}$ will depend on α and ensure every rank ω node has depth ω^α.

We describe the positioning of the $\chi_{n,0}$, developing the necessary terminology to do so via a sequence of definitions and lemmas.

Lemma 3.2. *If $\alpha > 0$, there is an increasing sequence of ordinals $\{\alpha_n\}_{n\in\omega}$ cofinal in ω^α satisfying $\alpha_{n+1} \geq \alpha_n \cdot 2 + 1$.*

Proof. If α is a successor ordinal $\alpha = \mu + 1$, then $\alpha_n = \omega^\mu \cdot (2^n - 1)$ suffices. If α is a limit ordinal, then $\alpha_n = \omega^{\mu_n}$ suffices, where $\{\mu_n\}_{n\in\omega}$ is any strictly increasing sequence cofinal in α. □

We fix such a sequence $\{\alpha_n\}_{n\in\omega}$.

Definition 3.3. Define a sequence $T_0 \subset T_1 \subset T_2 \subset \cdots \subset 2^{<\omega}$ of trees and a sequence of strings ϖ_0, ϖ_1, ϖ_2, ... by recursion as follows.

Define T_0 to be a subset of $2^{<\omega}$ homeomorphic to the ordinal $\omega^{\alpha_0} + 1$ and $\varpi_0 = \varepsilon$. If $T_0, \ldots, T_n$ and $\varpi_0, \ldots, \varpi_n$ have been defined, let ϖ_{n+1} be the length-lexicographically least string ϖ with $\varpi \notin T_n$. Let $f \in 2^\omega$ be the infinite path $\varpi_{n+1} ^\frown 0^\infty$. Define T_{n+1} to be a subset of $2^{<\omega}$ containing T_n homeomorphic to $\omega^{\alpha_{n+1}} + 1$ with the following properties:

- The subset T_{n+1} has major spine f, i.e., the (unique) path of rank $\alpha_{n+1}+1$ in T_{n+1} is f.
- If $\varrho \in T_n$ has a unique infinite path in T_n through it, then the subset $T_{n+1} \restriction \varrho$ is homeomorphic to the ordinal $\omega^{\alpha_n} + 1$.
- If $\varrho \in T_n$ has incomparable infinite paths in T_n through it, then $\varrho ^\frown 0$ ($\varrho ^\frown 1$, respectively) is in T_{n+1} if and only if $\varrho ^\frown 0$ ($\varrho ^\frown 1$, respectively) is in T_n.

We note several important properties about the sequence $\{T_n\}_{n\in\omega}$.

Lemma 3.4. *Every string $\varsigma \in 2^{<\omega}$ appears in cofinitely many T_n.*

For any $\varsigma \in 2^{<\omega}$ and for every $\ell \in \omega$, there is an integer $n \in \omega$ such that $T_n \restriction \varsigma$ is homeomorphic to an ordinal greater than $\omega^{\alpha_\ell} + 1$.

If $\varsigma \in T_n$ and $k \in \omega$ satisfies $k > n$, then the subset $T_k \restriction \varsigma$ is homeomorphic to an ordinal greater than or equal to $\omega^{\alpha_n} + 1$.

Proof. Fix a string ς. We prove each of the three statements in turn.

As the length-lexicographic ordering on $2^{<\omega}$ has order type ω, the choices of ϖ_n assures $\varsigma \in T_n$ for some n. Then $\varsigma \in T_m$ for all $m \geq n$.

Fix $\varsigma \in 2^{<\omega}$ and $\ell \in \omega$. For some $n > \ell$, the string ϖ_n will satisfy $\varsigma \prec \varpi_n$. This choice of n suffices as a consequence of the first condition.

If $\varsigma \in T_n$ has a unique infinite path in T_n through it, then the fact that $T_k \restriction \varsigma$ is homeomorphic to an ordinal greater than $\omega^{\alpha_n}+1$ for $k > n$ follows from the second condition on T_{n+1} in the construction for $k = n+1$. More generally, for $k > n+1$, this follows from the containment $T_{n+1} \subseteq T_k$. And if $\varsigma \in T_n$ does not have a unique infinite path in T_n through it, it follows by restricting attention to a subtree above such a node. □

Definition 3.5. For each $\varsigma \in 2^{<\omega}$, define n_ς to be the least integer k such that n_ς appears in T_k.

Having done the necessary preparation, we are ready to construct the desired countable Boolean algebra.

Theorem 3.6. *There is a countable Boolean algebra $\mathcal{B}$ with $\Delta(\mathcal{B}) = \{0, 1, \omega^\alpha\}$.*

Proof. The algebra $\mathcal{B}$ is

$$\sigma(\tau) = \begin{cases} \omega & \text{if } \tau \text{ is an almost repeater string,} \\ \\ \chi_{n_{\tau'},0}(\tau_3) & \text{if } \tau = \tau_1 \frown \tau_2 \frown \tau_3 \text{ for some repeater string } \tau_1 \\ & \text{with witness } \tau' \text{ and xor string } \tau_2. \end{cases}$$

Since any nontrivial finite sum of countable Boolean algebras $\chi_{n,0}$ has depth one (whether or not the n are distinct), it suffices to show that if τ is an almost repeater string, then the depth of τ is ω^α. We demonstrate this by showing $\delta(\tau) \leq \omega^\alpha$ and $\delta(\tau) \geq \omega^\alpha$ separately.

Claim 3.6.1. *If τ is an almost repeater string, then $\delta(\tau) \leq \omega^\alpha$.*

Proof. Fix elements $x, y \in B$ with $\Delta^{\omega^\alpha}\sigma(x) = \Delta^{\omega^\alpha}\sigma(y)$. Then $x \cong y$ as the order type of the relative algebras $\chi_{n,0}$ above x and above y must be equal for every $n \in \omega$. For if there were an integer n for which these order types were distinct, the ω^αth derivative would distinguish this as the order types would be strictly less than ω^α (being not greater than $\omega^{\alpha_n} + 1$). □

Claim 3.6.2. *If τ is an almost repeater string, then $\delta(\tau) \geq \omega^\alpha$.*

Proof. For a contradiction to $\delta(\tau) \geq \omega^\alpha$, suppose $\delta(\tau) = \beta < \omega^\alpha$. Fix the witness τ' to τ being an almost repeater string. Fix an integer n for which $\tau' \preccurlyeq \varpi_n$ and $\alpha_n > \beta$ (this is possible by Lemma 3.4). We reach the desired contradiction by showing $\delta(\hat{\tau}) > \beta$, where $\hat{\tau}$ is the almost repeater string (of even length) having witness ϖ_n.

Let $\tilde{\tau}$ satisfy $\hat{\tau} \prec \tilde{\tau}$ and $T_n \restriction \tilde{\tau} \cong \omega^\beta + 1$. Then $\Delta^\beta\sigma(\hat{\tau}) = \Delta^\beta\sigma(\tilde{\tau})$ as the order type of T_m (for each $m \in \omega$) above both $\hat{\tau}$ and $\tilde{\tau}$ is either empty for both or greater than or equal to $\omega^\beta + 1$ for both. On the other hand, $\Delta^{\beta+1}\sigma(\hat{\tau}) \neq \Delta^{\beta+1}\sigma(\tilde{\tau})$ as the order type of T_n above $\hat{\tau}$ is strictly greater than β, whereas this is not the case for $\tilde{\tau}$. □

Thus, we have $\Delta(\mathcal{B}) = \{0, 1, \omega^\alpha\}$ as desired. □

3.3. **The Set $\{0, 2, \omega^\alpha\}$.** Fixing an ordinal α (throughout this subsection), the idea for constructing a measure σ with $\Delta(\sigma) = \{0, 2, \omega^\alpha\}$ is similar to the idea for Section 3.2. The key difference is the usage of depth zero measures $\mathcal{B}_{u(n)}$ at the coding locations rather than $\chi_{n,0}$.

Theorem 3.7. *For each countable ordinal α, there is a countable Boolean algebra $\mathcal{B}$ with $\Delta(\mathcal{B}) = \{0, 2, \omega^\alpha\}$.*

Proof. As with Theorem 3.6, we rely heavily on the definition of n_ζ for $\zeta \in 2^{<\omega}$ (see Definition 3.5). The algebra $\mathcal{B}$ is

$$\sigma(\tau) = \begin{cases} \omega & \text{if } \tau \text{ is an almost repeater string,} \\ \\ \sigma_{u(n_{\tau'}+1)}(\tau_3) & \text{if } \tau = \tau_1 \frown \tau_2 \frown \tau_3 \text{ for some repeater string } \tau_1 \\ & \text{with witness } \tau' \text{ and xor string } \tau_2, \end{cases}$$

where $\sigma_{u(n_{\tau'}+1)}$ is the measure for the depth zero Boolean algebra $\mathcal{B}_{u(n_{\tau'}+1)}$.

Since any relative algebra of any finite sum of countable Boolean algebras of the form $\mathcal{B}_{u(n)}$ has depth zero or two (whether or not the n are distinct), it suffices to show that if τ is an almost repeater string, then the depth of τ is ω^α. As the justification for this is identical to Theorem 3.6, we omit it. □

3.4. **More Complicated Sets.** By utilizing the constructions of the previous subsections, more complicated sets can be realized as $\Delta(\mathcal{B})$.

Theorem 3.8. *For all countable ordinals α and β with $\omega^\alpha < \beta$, there is a countable Boolean algebra $\mathcal{B}$ with $\Delta(\mathcal{B}) = \{0,1\} \cup \{\omega^\alpha, \dots, \beta\}$.*

Proof. We argue that a β-spined sum of the Boolean algebra from Theorem 3.6 suffices. More precisely, let $T \subset 2^\omega$ be homeomorphic to $\omega^\beta + 1$. The Boolean algebra $\mathcal{B}$ is

$$\sigma(\tau) = \begin{cases} \omega & \text{if } \tau \in T, \\ & \\ \sigma_\alpha(\tau_3) & \text{if } \tau = \tau_1 \frown \tau_2 \frown \tau_3 \\ & \text{with } \tau_1 \in T,\ \tau_1 \frown \tau_2 \notin T, \text{ and } |\tau_2| = 1. \end{cases}$$

where σ_α is the measure of the Boolean algebra $\mathcal{B}_\alpha$ constructed in Theorem 3.6 with $\Delta(\mathcal{B}_\alpha) = \{0, 1, \omega^\alpha\}$.

We note that $\gamma \notin \Delta(\mathcal{B})$ for any ordinal γ with $1 < \gamma < \omega^\alpha$. The reason is that every relative algebra of $\mathcal{B}$ is either a (possibly trivial) finite sum of depth zero relative algebras of $\mathcal{B}_\alpha$ (thus depth zero or one) or bounds a depth ω^α element of $\mathcal{B}_\alpha$ (thus depth at least ω^α).

We note that a γ-spined sum of $\mathcal{B}$ has depth γ for $\gamma \geq \omega^\alpha$. The reason is essentially the same reason as the reason why $\delta(\chi_{\beta,\alpha}) = \alpha$, namely, that the γth derivative cannot distinguish a γ-spined sum from a $(\gamma+1)$-spined sum. We emphasize that the hypothesis $\gamma \geq \omega^\alpha$ is necessary, else the order type of $\chi_{n_{\tau'},0}$ (where τ' is as in Theorem 3.6) would be less than ω^γ, allowing the γth derivative to distinguish a γ-spined sum and a $(\gamma + 1)$-spined sum.

It follows $\delta(\mathcal{B})$ is as desired. □

Lemma 3.9. *If $\mathcal{B}_1$ and $\mathcal{B}_2$ satisfy $\max\{\delta(\mathcal{B}_1), \delta(\mathcal{B}_2)\} \geq 2$ and have measures with disjoint range, then $\delta(\mathcal{B}_1 \oplus \mathcal{B}_2) = \max\{\delta(\mathcal{B}_1), \delta(\mathcal{B}_2)\}$.*

Proof. Let $\delta = \max\{\delta(\mathcal{B}_1), \delta(\mathcal{B}_2)\}$. As $\delta(\mathcal{B}_1 \oplus \mathcal{B}_2) \geq \delta$ is immediate, we show that $\delta(\mathcal{B}_1 \oplus \mathcal{B}_2) \leq \delta$. Fixing $x_1, y_1 \in \mathcal{B}_1$ and $x_2, y_2 \in \mathcal{B}_2$, we show $\Delta^\delta\sigma(x_1 \oplus x_2) = \Delta^\delta\sigma(y_1 \oplus y_2)$ implies $x_1 \cong y_1$ and $x_2 \cong y_2$. For a contradiction, we suppose (without loss of generality) that $x_1 \not\cong y_1$.

Since $x_1 \not\cong y_1$ and $\delta(\mathcal{B}_1) \leq \delta$, there must be (without loss of generality) a decomposition $x_1 = x_{10} \oplus \cdots \oplus x_{1k}$ of x and an ordinal $\beta \geq 1$ such that

$$(\Delta^\beta\sigma(x_{10}), \dots, \Delta^\beta\sigma(x_{1k})) \notin \Delta^{\beta+1}\sigma(y_1).$$

As $x_{10} \oplus \cdots \oplus x_{1k} \oplus x_2$ is a decomposition of $x_1 \oplus x_2$, we conclude from $\Delta^\delta\sigma(x_1 \oplus x_2) = \Delta^\delta\sigma(y_1 \oplus y_2)$ that there is a decomposition $y_{10} \oplus \cdots \oplus y_{1k} \oplus y_2'$ of $y_1 \oplus y_2$ with $\Delta^\beta\sigma(x_{1i}) = \Delta^\beta\sigma(y_{1i})$ and $\Delta^\beta\sigma(x_2) = \Delta^\beta(y_2')$. As $\beta \geq 1$ and the measures have disjoint range, this implies $y_{1i} \leq y_1$ for $0 \leq i \leq k$ and $y_2' \leq y_2$. Consequently, it must be the case that $y_1 = y_{10} \oplus \cdots \oplus y_{1k}$ and $y_2 = y_2'$. Thus $(\Delta^\beta\sigma(x_{10}), \dots, \Delta^\beta\sigma(x_{1k})) = (\Delta^\beta\sigma(y_{10}), \dots, \Delta^\beta\sigma(y_{1k})) \in \Delta^{\beta+1}\sigma(y_1)$, a contradiction. □

Theorem 3.10. *For all countable ordinals $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_m$ with $\omega^{\alpha_1} < \beta_1 < \cdots < \omega^{\alpha_m} < \beta_m$, there is a countable Boolean algebra $\mathcal{B}$ with $\Delta(\mathcal{B}) = \{0,1\} \cup \{\omega^{\alpha_1}, \ldots, \beta_1\} \cup \cdots \cup \{\omega^{\alpha_m}, \ldots, \beta_m\}$.*

Proof. We construct such a countable Boolean algebra by induction on m. When $m = 2$, the idea is to replicate Theorem 3.8 (using α_2 and β_2 for α and β), replacing the instances of σ_α with instances of Theorem 3.8 (using α_1 and β_1 for α and β). For $m > 2$, the idea is to replicate Theorem 3.8 (using α_m and β_m for α and β), replacing the instances of σ_α with the inductively constructed instances with depths $\{0,1\} \cup \{\omega^{\alpha_1}, \ldots, \beta_1\} \cup \cdots \cup \{\omega^{\alpha_{m-1}}, \ldots, \beta_{m-1}\}$.

We start by explicitly constructing a countable Boolean algebra $\mathcal{B}$ with $\delta(\mathcal{B}) = \{0,1\} \cup \{\omega^{\alpha_1}, \ldots, \beta_1\} \cup \{\omega^{\alpha_2}\}$. As preparation, partition the integers ω into infinite disjoint sets $\{Q_n\}_{n\in\omega}$. Let $\mathcal{B}_n$ be the Boolean algebra from Theorem 3.8 with $\delta(\mathcal{B}_n) = \{0,1\} \cup \{\omega^{\alpha_1}, \ldots, \beta_1\}$, where $\chi_{i,0}$ exists as a relative algebra only for $i \in Q_n$ rather than all $i \in \omega$.

The countable Boolean algebra $\mathcal{B}$ is

$$\sigma(\tau) = \begin{cases} \omega & \text{if } \tau \text{ is an almost repeater string,} \\ \\ \sigma_{\mathcal{B}_{n_{\tau'}}}(\tau_3) & \text{if } \tau = \tau_1 {}^\frown \tau_2 {}^\frown \tau_3 \text{ for some repeater string } \tau_1 \\ & \text{with witness } \tau' \text{ and xor string } \tau_2, \end{cases}$$

where $\sigma_{n_{\tau'}}$ is the measure associated with $\mathcal{B}_{n_{\tau'}}$.

We verify $\Delta(\mathcal{B}) = \{0,1\} \cup \{\omega^{\alpha_1}, \ldots, \beta_1\} \cup \{\omega^{\alpha_2}\}$. By Lemma 3.9 and Theorem 3.8, any finite sum of relative algebras of elements not bounding an almost repeater string has depth in $\{0,1\} \cup \{w^{\alpha_1}, \ldots, \beta_1\}$. Of course, as Lemma 3.9 only applies if $\max\{\delta(\mathcal{B}_1), \delta(\mathcal{B}_2)\} \geq 2$, the case $\delta(\mathcal{B}_1), \delta(\mathcal{B}_2) < 2$ needs to be treated separately: Both $\mathcal{B}_1$ and $\mathcal{B}_2$ are finite sums of Boolean algebras $\chi_{i,0}$, so the sum $\mathcal{B}_1 \oplus \mathcal{B}_2$ has depth one. If τ is an almost repeater string, then $\delta(\tau) = \omega^{\alpha_2}$ as in Claim 3.6.1 and Claim 3.6.2. It follows that $\Delta(\mathcal{B}) = \{0,1\} \cup \{\omega^{\alpha_1}, \ldots, \beta_1\} \cup \{\omega^{\alpha_2}\}$.

In order to realize the set of depths $\{0,1\} \cup \{\omega^{\alpha_1}, \ldots, \beta_1\} \cup \{\omega^{\alpha_2}, \ldots, \beta_2\}$, we take a β_2-spined sum (using a tree homeomorphic to $\omega^{\beta_2} + 1$) of the Boolean algebra $\mathcal{B}$ just constructed. As in Theorem 3.8, it is not difficult to verify that this has the desired depths.

For $m > 2$, we repeat this process using the Boolean algebras constructed inductively for the set $\{0,1\} \cup \{\omega^{\alpha_1}, \ldots, \beta_1\} \cup \cdots \cup \{\omega^{\alpha_{m-1}}, \ldots, \beta_{m-1}\}$ to construct a Boolean algebra $\mathcal{B}$ with $\Delta(\mathcal{B}) = \{0,1\} \cup \{\omega^{\alpha_1}, \ldots, \beta_1\} \cup \cdots \cup \{\omega^{\alpha_{m-1}}, \ldots, \beta_{m-1}\} \cup \{\omega^{\alpha_m}\}$. We take a β_m-spined sum of this to obtain $\{0,1\} \cup \{\omega^{\alpha_1}, \ldots, \beta_1\} \cup \cdots \cup \{\omega^{\alpha_{m-1}}, \ldots, \beta_{m-1}\} \cup \{\omega^{\alpha_m}, \ldots, \beta_m\}$. □

If the constructions instead use Theorem 3.7 rather than Theorem 3.6, the following sets are realized.

Theorem 3.11. *For all countable ordinals $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_m$ with $\omega^{\alpha_1} < \beta_1 < \cdots < \omega^{\alpha_m} < \beta_m$, there is a countable Boolean algebra $\mathcal{B}$ with $\Delta(\mathcal{B}) = \{0,2\} \cup \{\omega^{\alpha_1}, \ldots, \beta_1\} \cup \cdots \cup \{\omega^{\alpha_m}, \ldots, \beta_m\}$.*

4. Arbitrary Rank Constraints

Though the examples of Section 3 demonstrate that there is some flexibility within the set of depths of relative algebras of a countable Boolean algebra, there are constraints. For example, as a consequence of the fact that the depth of every constant measure is zero, every countable Boolean algebra has a depth zero relative algebra. More subtle is that every countable Boolean algebra of depth greater than zero has either a depth one relative algebra or a depth two relative algebra. Thus, in terms of trying to make $\Delta(\mathcal{B})$ minimal, Theorem 3.6 and Theorem 3.7 are optimal.

Theorem 4.1. *If $\mathcal{B}$ has depth greater than zero, then either $1 \in \Delta(\mathcal{B})$ or $2 \in \Delta(\mathcal{B})$, i.e., there is a depth one or depth two relative algebra of $\mathcal{B}$.*

This theorem follows quickly from the following technical lemma.

Lemma 4.2. *Fix an ordinal α and a rank downward closed countable Boolean algebra $\mathcal{B}$ with $\rho := \rho(\mathcal{B})$. If:*

- *$\rho \geq \alpha$,*
- *the preimage of each $\beta < \alpha$ is a union of pure perfect kernels, and*
- *$\sigma(x) = \beta$ implies $x \cong \mathcal{B}_{v(\{0,\dots,\beta\})}$ for $\beta < \alpha$,*

then either

- *$\mathcal{B}$ contains a depth one relative algebra,*
- *$\mathcal{B}$ contains a depth two relative algebra, or*
- *$\rho \geq \alpha + 1$, the preimage of each $\beta \leq \alpha$ is a union of pure perfect kernels, and $\sigma(x) = \beta$ implies $x \cong \mathcal{B}_{v(\{0,\dots,\beta\})}$ for $\beta \leq \alpha$.*

Proof. Assuming $\mathcal{B}$ contains no depth one relative algebra and no depth two relative algebra, we show $\rho \geq \alpha + 1$, the preimage of each $\beta \leq \alpha$ is a union of pure perfect kernels, and $\sigma(x) = \beta$ implies $x \cong \mathcal{B}_{v(\{0,\dots,\beta\})}$ for $\beta \leq \alpha$.

Before performing some case analysis, we show every ordinal $\beta < \alpha$ must appear above any element with rank α. If not, there would be an element $x \in B$ of rank α and some $\beta < \alpha$ with $\sigma(y) \neq \beta$ for all $y \leq x$ (choose β minimal with this property). Passing to a relative algebra of x as necessary, we may assume that the preimage of α in x is either a pure perfect kernel or an isolated path. Passing to a relative algebra of x again as necessary, we may assume that $x \cong \mathcal{B}_{u(S)}$ or $x \cong \mathcal{B}_{v(S)}$, where S is the set of ordinals strictly less than β together with α. Then $x \oplus \mathcal{B}_{v(\{0,\dots,\beta\})}$ is a depth one relative algebra, contrary to the hypothesis.

We continue by treating separately the cases when the preimage of α is exactly one isolated path, contains at least two isolated paths, is exactly one isolated path and a union of pure perfect kernels, and is a union of pure perfect kernels.

If the preimage of α were to be exactly one isolated path, it would be the case that $\rho > \alpha$ as $\delta(\mathcal{B}) > 0$. As $\mathcal{B}$ is rank downward closed, there would then be an element $x \in B$ with $\rho(x) = \alpha + 1$. Passing to a relative algebra of x as necessary, we may assume that the preimage of $\alpha + 1$ in x is either a pure perfect kernel or an isolated path. Passing to a relative algebra of x again as necessary, we may assume that $x \cong \mathcal{B}_{u(S)}$ or $x \cong \mathcal{B}_{v(S)}$ for some set $S \subseteq \{0, \dots, \alpha, \alpha + 1\}$ containing $\alpha + 1$ but not containing α. Then $x \oplus \mathcal{B}_{u(\{0,\dots,\alpha\})}$ would be a depth one or depth two relative algebra (depending on S), contrary to the hypothesis.

If the preimage of α were to contain at least two isolated paths, then there would be ordinals μ_1 and μ_2 (possibly equal) such that $\mathcal{B}$ contains the relative algebra $\mathcal{B}_{u(0,\dots,\mu_1,\alpha)} \oplus \mathcal{B}_{u(0,\dots,\mu_2,\alpha)}$. This is depth one, contrary to the hypothesis.

If the preimage of α were to contain exactly one isolated path and a union of pure perfect kernels, then $\mathcal{B}$ would contain the depth two relative algebra $\mathcal{B}_{u(0,\dots,\alpha)} \oplus \mathcal{B}_{v(0,\dots,\alpha)}$, contrary to the hypothesis.

Consequently, the preimage of α is a union of pure perfect kernels. Combined with the discussion above, this implies $x \cong \mathcal{B}_{v(\{0,\dots,\alpha\})}$ if $\rho(x) = \alpha$. Finally, it must be the case that $\rho \geq \alpha + 1$ as otherwise $\mathcal{B}$ would be depth zero. □

Proof of Theorem 4.1. We start by observing that it suffices to consider rank downward closed countable Boolean algebras. The reason is the depth of a relative algebra does not depend on the ranks of its relative algebras but rather the relative sizes of the ranks of its relative algebras (i.e., we can take the Mostowski collapse of $\{\sigma(x) : x \in B\}$ and preserve depths).

Since $\mathcal{B}$ has depth greater than zero, it has rank greater than zero. Consequently, the hypotheses of Lemma 4.2 are satisfied for $\alpha = 1$. Thus $\mathcal{B}$ either contains a depth one relative algebra or a depth two relative algebra (in which case nothing more needs to be shown) or $\rho \geq 2$ and the hypotheses of Lemma 4.2 are satisfied for $\alpha = 2$. Continuing by induction, as $\rho = \rho(\mathcal{B})$ is a fixed countable ordinal, at some point the third possibility in Lemma 4.2 cannot happen. We conclude $\mathcal{B}$ has either a depth one relative algebra or a depth two relative algebra. □

5. Rank One Examples and Constraints

As there are natural maps from the class of arbitrary rank countable Boolean algebras to the class of rank one countable Boolean algebras, these classes share many properties. For example, Camerlo and Gao [2] showed the isomorphism problem for the class of arbitrary rank countable Boolean algebras is no more difficult than the isomorphism problem restricted to the class of rank one countable Boolean algebras.

However, there are significant differences between which sets of countable ordinals are realized as $\Delta(\mathcal{B})$ for some arbitrary rank countable Boolean algebra and for some rank one countable Boolean algebra. For example, there is no rank one countable Boolean algebra $\mathcal{B}$ with $\Delta(\mathcal{B}) = \{0, 1, \omega^\alpha\}$ or $\Delta(\mathcal{B}) = \{0, 2, \omega^\alpha\}$. This is particularly interesting as this is the first structural difference, to the authors' knowledge, between the arbitrary rank countable Boolean algebras and the rank one countable Boolean algebras.

Lemma 5.1. *Let $\mathcal{B} := \sum_{rsp} \mathcal{B}_i$ for an arbitrary sequence of countable Boolean algebras $\{\mathcal{B}_i\}_{i\in\omega}$ (not necessarily rank one). If x and y are* spined *elements, then x and y are isomorphic.*

Proof. We show the αth derivatives of x and y are equal for all α. Equality of the 0th derivatives follows from

$$\sigma(x) = \sup\{\rho(\mathcal{B}_i)\}_{i\in\omega} = \sigma(y)$$

as both x and y bound a copy of $\mathcal{B}_i$ for all i.

Assuming equality of the βth derivative for all $\beta < \alpha$, we show equality of the αth derivative. Fix a partition $(x_0, x_1, \dots, x_n)$ of x, where without loss of generality x_0 is spined and x_i is not spined for $1 \leq i \leq n$. Since y is spined and the x_i for $1 \leq i \leq n$ are not spined, there are disjoint non-spined y_i for $1 \leq i \leq n$ with $y_i \leq y$ and $y_i \cong x_i$. Then $y_0 := y - \bigoplus_{1\leq i\leq n} y_i$ is spined. The inductive hypothesis ensures $\Delta^\beta\sigma(x_0) = \Delta^\beta\sigma(y_0)$ for any $\beta < \alpha$, so $(\Delta^\beta\sigma(x_0), \Delta^\beta\sigma(x_1), \dots, \Delta^\beta\sigma(x_n)) = (\Delta^\beta\sigma(y_0), \Delta^\beta\sigma(y_1), \dots, \Delta^\beta\sigma(y_n)) \in \Delta^\alpha\sigma(y)$. The reverse containment $\Delta^\alpha\sigma(y) \subseteq \Delta^\alpha\sigma(x)$ is identical. □

It is important to note that the isomorphism type of $\mathcal{B} = \sum_{\mathrm{rsp}} \mathcal{B}_i$ is not dependent on the choice of the sequence $\mathcal{B}_0$, $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_2$, ... Indeed, as long as every $\mathcal{B}_i$ appears infinitely often, the proof of Lemma 5.1 shows that it has the same isomorphism type as $\mathcal{B}$.

Lemma 5.2. *Let $\mathcal{B} := \sum_{sp} \mathcal{B}_i$, where $\rho(\mathcal{B}_i) = 1$ and $\delta(\mathcal{B}_i) \leq N$ for some fixed $N \in \omega$. Then $\delta(\mathcal{B}) \leq N + 10$.*

Proof. Let $\{\mathcal{B}^0, \dots, \mathcal{B}^K\}$ enumerate the rank one countable Boolean algebras of depth at most N, noting the important fact that this set is finite by Heindorf [3]. Fix an integer m with the property that

$$(\forall k \leq K) \left[(\exists i \geq m) \left[\mathcal{B}_i \cong \mathcal{B}^k \right] \implies (\exists^\infty i) \left[\mathcal{B}_i \cong \mathcal{B}^k \right] \right],$$

noting that such an integer m must exist. It follows from the preceding remarks that $\sum_{\mathrm{sp} \geq m} \mathcal{B}_i$ is a repeated spined sum of some finite subset of $\{\mathcal{B}^0, \dots, \mathcal{B}^K\}$.

We show the depth of $\mathcal{B}$ is at most $N + 10$ by showing the depth of $\sum_{\mathrm{sp} \geq m} \mathcal{B}_i$ is at most $N + 6$. If $\sum_{\mathrm{sp} \geq m} \mathcal{B}_i$ is isomorphic to a relative algebra of a finite sum of the $\mathcal{B}_i$, then in fact it has depth at most $N + 4$. Otherwise, the $(N + 6)$th derivative of any spined element is distinct from the $(N + 6)$th derivative of any nonspined element (the latter all having depth at most $N + 4$, so the $(N + 6)$th derivative is an isomorphism invariant). As all spined elements of $\sum_{\mathrm{sp} \geq m} \mathcal{B}_i$ are isomorphic by Lemma 5.1 and the discussion following, it follows that $\sum_{\mathrm{sp} \geq m} \mathcal{B}_i$ has depth at most $N + 6$. Since $\mathcal{B}$ is the sum of $\sum_{\mathrm{sp} \geq m} \mathcal{B}_i$ and $\sum_{< m} \mathcal{B}_i$, we have $\delta(\mathcal{B}) \leq \max\{N + 4, N + 6\} + 4 = N + 10$. □

Theorem 5.3. *If $\mathcal{B}$ is rank one and has depth at least ω, then $n \in \Delta(\mathcal{B})$ for integers n cofinal in ω.*

Proof. Fix a rank one countable Boolean algebra $\mathcal{B}$ with $\delta(\mathcal{B}) \geq \omega$. Towards a contradiction, we suppose that there is an integer $N \in \omega$ with $n \notin \Delta(\mathcal{B})$ for any integer $n > N$.

Let $S \subset 2^\omega$ be the set of strings τ with $\delta(\tau) \geq \omega$. This set S, being downward closed, forms a tree. The set S has no dead ends as a consequence of the inequality $\delta(\mathcal{B}_1 \oplus \mathcal{B}_2) \leq \max\{\delta(\mathcal{B}_1), \delta(\mathcal{B}_2)\} + 4$ for all $\mathcal{B}_1$ and $\mathcal{B}_2$ in Heindorf [3]. The set S also has no isolated paths as a consequence of Lemma 5.2. Thus, the set S is a pure perfect kernel.

We therefore view $\mathcal{B}$ as having rank one countable Boolean algebras of depth at most N at coding locations and depth at least ω at every almost repeater string. From this representation of $\mathcal{B}$, we define trees T^i for $0 \leq i \leq K$, where $\tau \in T^i$ if $\mathcal{B}^i$ appears at some coding location above τ.

Then there is a string τ such that if τ' extends τ, then $\tau' \in T^i$ whenever $\tau \in T^i$ (for all $0 \leq i \leq K$). This has depth at most $N + 10$ via the same argument as in Lemma 5.2, a contradiction. □

The key ingredient in the proof of Theorem 5.3 is the existence of at most finitely many rank one algebras of depth at most N (see the proof of Lemma 5.2). Hence, the hypothesis of Theorem 5.3 can be weakened to $\mathcal{B}$ having finite rank. Thus, the countable Boolean algebras of Theorem 3.6 and Theorem 3.7 are simplest possible (in terms of rank). The conclusion of Theorem 5.3 cannot be strengthened with the proof provided as there are infinitely many rank one algebras of finite depth. Indeed, it cannot be strengthened at all.

Theorem 5.4. *For each countable ordinal α, there is a rank one countable Boolean algebra $\mathcal{B}$ with $\Delta(\mathcal{B}) = \omega \cup \{\omega^\alpha\}$.*

Proof. As with Theorem 3.6 and Theorem 3.7, we rely heavily on the definition of n_ζ for $\zeta \in 2^{<\omega}$ (see Definition 3.5). The measure for $\mathcal{B}$ is

$$\sigma(\tau) = \begin{cases} 1 & \text{if } \tau \text{ is an almost repeater string,} \\ \\ \chi_{1,n_{\tau'}}(\tau_3) & \text{if } \tau = \tau_1 \frown \tau_2 \frown \tau_3 \text{ for some repeater string } \tau_1 \\ & \text{with witness } \tau' \text{ and xor string } \tau_2. \end{cases}$$

Since $\delta(\chi_{1,n}) = n$, it follows $n \in \Delta(\mathcal{B})$ for all $n \in \omega$. Since any finite sum of $\chi_{1,n}$ has finite depth, it suffices to show that if τ is an almost repeater string, then the depth of τ is ω^α. As the justification for this is similar to Theorem 3.6, we omit it. □

6. Questions

Unfortunately, we do not completely understand which sets of countable ordinals are within the set $\{\Delta(\mathcal{B}) : \mathcal{B}$ is a countable Boolean algebra$\}$. Though a complete understanding of this set may or may not be interesting, certain questions seem fundamental to understanding the structure of the set of relative algebras of an arbitrary countable Boolean algebra.

With the exception of Theorem 3.1, the examples within this paper make strong usage of the combinatorial properties of an ordinal of the form ω^α. It is natural to ask if similar examples exist for ordinals not sharing these combinatorial properties (e.g., a successor ordinal like $\omega + 1$ or a limit ordinal like $\omega + \omega$).

Question 6.1. Is there a countable Boolean algebra $\mathcal{B}$ with $\Delta(\mathcal{B}) = \{0, 1, \omega + 1\}$ or $\Delta(\mathcal{B}) = \{0, 2, \omega + 1\}$? With $\Delta(\mathcal{B}) = \{0, 1, \omega + \omega\}$ or $\Delta(\mathcal{B}) = \{0, 2, \omega + \omega\}$?

We also do not know whether it is possible to have arbitrarily large but finite gaps in $\Delta(\mathcal{B})$.

Question 6.2. Is there, for each integer $N > 2$, a countable Boolean algebra $\mathcal{B}$ and an ordinal λ with $\lambda, \lambda + N \in \Delta(\mathcal{B})$ and $\lambda + k \notin \Delta(\mathcal{B})$ for any $0 < k < N$?

In a similar vein, we wonder if Theorem 5.3 can be strengthened.

Question 6.3. If $\mathcal{B}$ is rank one and has depth at least ω, is $n \in \Delta(\mathcal{B})$ for (almost) all integers $n \in \omega$?

Finally, we wonder if Theorem 5.3 is in essence the only structural difference between $\Delta(\mathcal{B})$ for arbitrary rank $\mathcal{B}$ and rank one $\mathcal{B}$.

Question 6.4. Is there, for any set S with $\omega \subset S$, a rank one countable Boolean algebra $\mathcal{B}$ with $\Delta(\mathcal{B}) = S$ whenever there is an arbitrary rank countable Boolean algebra $\mathcal{B}$ with $\Delta(\mathcal{B}) = S$?

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