INTERVAL DISMANTLABLE LATTICES

KIRA ADARICHEVA, JENNIFER HYNDMAN, STEFFEN LEMPP,
AND J. B. NATION

Abstract. A finite lattice is *interval dismantlable* if it can be partitioned into an ideal and a filter, each of which can be partitioned into an ideal and a filter, etc., until you reach 1-element lattices. In this note, we find a quasi-equational basis for the pseudoquasivariety of interval dismantlable lattices, and show that there are infinitely many minimal interval non-dismantlable lattices.

Define an *interval dismantling* of a lattice to be a partition of the lattice into two nonempty, complementary sublattices where one is an ideal and the other a filter. A finite lattice is said to be *interval dismantlable* if it can be reduced to 1-element lattices by successive interval dismantlings.

In order to work with these lattices, we note that the following are equivalent for a finite lattice $\mathbf{L}$:

(1) $L = I \,\dot\cup\, F$ for some disjoint proper ideal I and filter F.
(2) $\mathbf{L}$ contains a nonzero join prime element.
(3) $\mathbf{L}$ contains a non-one meet prime element.
(4) There is a surjective homomorphism $h : \mathbf{L} \to \mathbf{2}$.
(5) Some generating set X for $\mathbf{L}$ can be split into two disjoint nonempty subsets, $X = Y \,\dot\cup\, Z$, such that $\bigwedge Y \not\leq \bigvee Z$.
(6) Every generating set X for $\mathbf{L}$ can be split into two disjoint nonempty subsets, $X = Y \,\dot\cup\, Z$, such that $\bigwedge Y \not\leq \bigvee Z$.

So if a lattice $\mathbf{L}$ contains no join prime element, then it is interval non-dismantlable. If $\mathbf{L}$ contains no join prime element, but every proper sublattice does, then it is minimally interval non-dismantlable. If $\mathbf{L}$ contains an interval non-dismantlable sublattice, then $\mathbf{L}$ is interval non-dismantlable.

Note that it follows from (2) and (3) that every finite meet semidistributive or join semidistributive lattice is interval dismantlable. The

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atoms of a finite meet semidistributive lattice are join prime; dually, the coatoms of a finite join semidistributive lattice are meet prime.

In view of conditions (5) and (6) above, let us say that a subset X of a lattice $\mathbf{L}$ is *divisible* if it can be divided into two nonempty subsets Y and Z such that $\bigwedge Y \nleq \bigvee Z$; else X is *indivisible*.

It is straightforward to see that interval dismantlable lattices form a pseudoquasivariety, i.e., a class of finite algebraic structures closed under taking substructures and finite direct products. The basic theorem on pseudoquasivarieties is due to C. J. Ash [2]; see also Chapter 2 of V. A. Gorbunov [3].

Theorem 1. *Let $\mathcal{K}$ be a pseudoquasivariety of structures of finite type. Then $\mathcal{K}$ is the set of all finite structures in the quasivariety $\mathcal{Q} = \mathbb{SPU}(\mathcal{K})$, where $\mathbb{U}$ denotes the ultraproduct operator.*

Thus there is a set of quasi-equations that determines the set of finite interval dismantlable lattices. For each $n \geq 3$, let $X_n = \{x_1, \dots, x_n\}$ be a set of n variables. Consider the quasi-equations

$$(\delta_n) \qquad \&_{\varnothing \subset Y \subset X_n} \ \bigwedge Y \leq \bigvee (X_n \setminus Y) \to x_1 \approx x_2 \ .$$

Any indivisible subset A of a lattice $\mathbf{L}$ with $|A| \leq n$ satisfies the hypothesis of δ_n. On the other hand, by symmetry the conclusion could be replaced by $x_i \approx x_j$ for any $i \neq j$. Hence the quasi-equation δ_n expresses that $\mathbf{L}$ contains no indivisible subset of size k for $1 < k \leq n$. In particular, δ_n implies δ_{n-1}.

Theorem 2. *A finite lattice is interval dismantlable if and only if it satisfies δ_n for all $n \geq 3$, that is, the lattice contains no indivisible subset of more than two elements.*

Proof. First, assume that $\mathbf{L}$ is interval dismantlable. For every $n \geq 3$ and $\mathbf{a} \in L^n$, we want to show that δ_n holds under the substitution $x_i \mapsto a_i$. If $a_1 = a_2$, then the conclusion of δ_n holds. If $a_1 \neq a_2$, then the sublattice $\mathbf{S} = \mathrm{Sg}(a_1, \dots, a_n)$ is nontrivial and interval dismantlable, and hence $\mathbf{S}$ has a decomposition $S = I \,\dot{\cup}\, F$ into a proper ideal and filter. Let $Y = \{a_i : a_i \in F\}$ and $Z = \{a_j : a_j \in I\}$. Then $\bigwedge Y \in F$ and $\bigvee Z \in I$, whence $\bigwedge Y \nleq \bigvee Z$, so that the corresponding inclusion in the hypothesis of δ_n fails. Thus δ_n holds for every substitution.

Conversely, let us show that every finite lattice that satisfies all δ_n is interval dismantlable. We do so by induction on $|L|$. To begin, the 1-element lattice satisfies every δ_n and is trivially interval dismantlable. So consider a finite lattice $\mathbf{L}$ with $|L| > 1$. Choose a generating set $X = \{a_1, a_2, \dots, a_k\}$ for $\mathbf{L}$ with $a_1 \neq a_2$. Since $\mathbf{L}$ satisfies δ_k and the conclusion fails, there is a nontrivial splitting $X = Y \dot\cup Z$ with $\bigwedge Y \not\leq \bigvee Z$. This splits $\mathbf{L}$ into a proper ideal and filter, $L = I \dot\cup F$, and each of these is a smaller lattice that satisfies δ_n for all n. By induction, both I and F are interval dismantlable, and so $\mathbf{L}$ is as well. □

Any class of finite lattices closed under sublattices can be characterized by the exclusion of its minimal non-members. Examples of minimal interval non-dismantlable lattices include $\mathbf{M}_3$ and the lattices in Figure 1, which fail δ_4. We would like to show that the pseudoquasivariety of finite interval dismantlable lattices is not finitely based, for which we need an infinite sequence of minimal interval non-dismantlable lattices, such that any finite collection of the quasi-equations δ_j is satisfied by at least one of them. The next theorem provides this by generalizing the top right example of Figure 1.

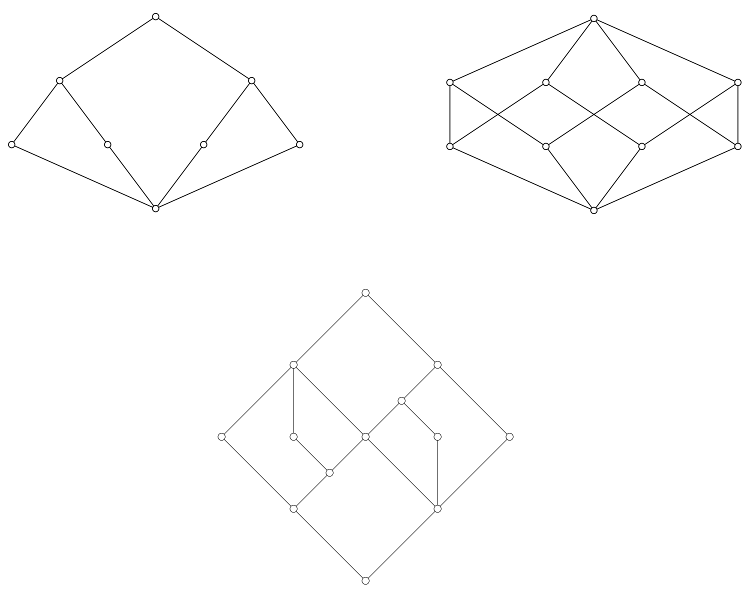

FIGURE 1. Three minimal interval non-dismantlable lattices.

Theorem 3. *There is a sequence of minimal interval non-dismantlable lattices* $\mathbf{K}_n$ $(n \geq 4)$ *such that each* $\mathbf{K}_n$ *satisfies* δ_j *for* $3 \leq j < n$*, but fails* δ_n*.*

Proof. For $n \geq 4$, we construct a lattice $\mathbf{K}_n$ as follows. The carrier set is $n \times (n-2) = \{(i,j) : 0 \leq i < n \text{ and } 0 \leq j < n-2\}$, with the order given by $(i,j) \leq (k,\ell)$ if $j \leq \ell$ and either $0 \leq k-i \leq \ell-j$ or

$n+k-i \leq \ell-j$, plus a top element T and bottom element B. Thus we are thinking of the first coordinates modulo n, as if wrapped around a cylinder. The covers in the middle portion of the lattice are given by $(i,j) < (i,j+1)$ and $(i,j) < (i+1 \bmod n, j+1)$ where $0 \leq i < n$ and $0 \leq j < n-3$. The middle portion of the lattice $\mathbf{K}_5$ is illustrated in Figure 2.

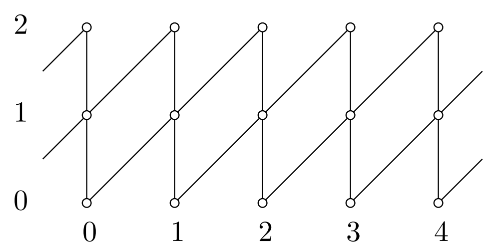

FIGURE 2. Middle portion of the lattice $\mathbf{K}_5$; add top and bottom elements for the whole lattice.

For a generating set, we can take $X = \{(i,0) : i < n\}$. This has the property that any pair of distinct elements of X meets to B, while the join of any $n-1$ is T. Thus $\mathbf{K}_n$ fails δ_n and is interval non-dismantlable.

In view of the circular symmetry, we may consider the maximal sublattices not containing the generator $(0,0)$. These are easily seen to be $\mathbf{S}_0 = K_n \setminus \{(0,j) : j < n-2\}$ and $\mathbf{T}_0 = K_n \setminus \{(j,j) : j < n-2\}$. Both these are interval dismantlable. For $S_0 = \uparrow(1,0) \,\dot\cup\, \downarrow(n,n-3)$, with the filter being dually isomorphic to the lattice $\mathrm{Co}(\mathbf{n-2})$ of convex subsets of an $n-2$ element chain, and hence meet semidistributive, and the ideal being isomorphic to $\mathrm{Co}(\mathbf{n-2})$ and hence join semidistributive. Likewise $T_0 = \uparrow(n-1,0) \,\dot\cup\, \downarrow(n-2,n-3)$, with the filter being meet semidistributive and the ideal being join semidistributive.

To see that $\mathbf{K}_n$ satisfies δ_j for $3 \leq j < n$, consider an arbitrary generating set X for $\mathbf{K}_n$. For each k with $0 \leq k < n$, the set $\mathbf{S}_k = K_n \setminus \{(k,\ell) : \ell < n-2\}$ is a proper sublattice of $\mathbf{K}_n$. Hence $X \not\subseteq S_k$, i.e., X contains an element of the form (k,ℓ) for each $k < n$. Thus $|X| \geq n$. So every subset of K_n with fewer than n elements generates a proper sublattice, which is interval dismantlable. Therefore $\mathbf{K}_n$ satisfies δ_j for $j < n$. □

Discussion. The original notion of dismantlability is that a finite lattice is *dismantlable* if it can be reduced to a 1-element lattice by successively removing doubly irreducible elements. These lattices were characterized independently by Ajtai [1] and Kelly and Rival [4], as those lattices not containing an n-crown for $n \geq 3$. Dismantlable lattices do not form a pseudoquasivariety, as they are not closed under finite direct products.

More generally, we can define a *sublattice dismantling* of a lattice to be a partition of the lattice into two nonempty, complementary sublattices. A finite lattice is said to be *sublattice dismantlable* if it can be reduced to 1-element lattices by successive sublattice dismantlings. Clearly both the original dismantlable lattices and interval dismantlable lattices are sublattice dismantlable, and this class does form a pseudoquasivariety. It would be interesting to characterize sublattice dismantlable lattices.

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Department of Mathematics, Hofstra University, Hempstead, NY 11549, USA

E-mail address: `kira.adaricheva@hofstra.edu`

Department of Mathematics and Statistics, University of Northern British Columbia, Prince George, BC, Canada

E-mail address: `jennifer.hyndman@unbc.ca`

Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

E-mail address: `lempp@math.wisc.edu`

Department of Mathematics, University of Hawai'i, Honolulu, HI 96822, USA

E-mail address: `jb@math.hawaii.edu`