Computably Enumerable Algebras, Their Expansions, and Isomorphisms

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Abstract. Computably enumerable algebras are the ones whose positive atomic diagrams are computably enumerable. Computable algebras are the ones whose atomic diagrams are computable. In this paper we investigate computably enumerable algebras and provide several algebraic and computable theoretic distinctions of these algebras from the class of computable algebras. We give a characterization of computably enumerable but not computable algebras in terms of congruences and effective conjunctions of Π^0_1-sentences. Our characterization, for example, shows that computable conjunctions of negative atomic formulas true in a given c.e. algebra can be preserved in infinitely many of its homomorphic images. We also study questions on how expansions of algebras by finitely many new functions affect computable isomorphism types. In particular, we construct a c.e. algebra with unique computable isomorphism type but which has no finitely generated c.e. expansion.

1 Preliminaries

Effectiveness issues in algebra and model theory have been investigated intensively in the last thirty years. One wishes to understand the effective content of model-theoretic and algebraic results, and the interplay between notions of computability, algebra, and model theory. A significant body of work has recently been done in the area, and this is attested by recent series of *Handbooks* and surveys in computable mathematics, computability, and algebra (see, e.g., [1], [7], [8]). In effective algebra and model theory an emphasis has been placed on the study of computable models and algebras. The study of computable isomorphism types of structures (especially those structures with unique computable isomorphism type), the relationship between computability and definability, and constructing models of theories have been central in the area. In this paper we continue this line of research and extend our study to a wider class of algebras called computably enumerable algebras. We place the emphasis on understanding the algebraic and computability-theoretic distinctions between computable algebras and computably enumerable algebras. Examples of computably enumerable algebras arise naturally in algebra and model theory. For example, finitely presented groups and rings or, in general, finitely presented algebras are computably enumerable, and so are Lindenbaum algebras of computably axiomatizable theories (e.g., Peano arithmetic). There has, of course, been some research on computably enumerable algebras, notably on computably enumerable Boolean algebras, starting with the work of Feiner [3], and on general properties of computably enumerable algebras studied by Kasymov [4], [5]. Computably enumerable algebras are also of interest in theoretical computer science in relation to issues on specifications of data structures and modeling programs as algebras. The underlying idea here is that many abstract data types can be identified with the isomorphism types of computably enumerable finitely generated algebras [2], [9].

Here is a brief outline of the paper. Further in this section, we give the basic definitions of computable and computably enumerable algebras, expansions, and provide some examples. In the next section we present a characterization theorem which distinguishes computably enumerable but not computable algebras from the class of all computable algebras. In particular, the theorem implies that any negative atomic sentence true in a c.e. but not computable algebra is preserved at infinitely many homomorphic images of the algebra. In the second section we define the notions of computable isomorphism and computable dimension. We explain expansions of c.e. algebras and give a sufficient condition for an algebra to have a finitely generated expansion. This provides a sufficient condition for an algebra to have an expansion with unique computable isomorphism type. We also provide a sufficient condition for a c.e. algebra not to have a finitely generated expansion (by computable functions). In the last section we study the interactions between finitely generated algebras, computable isomorphism types, and expansions of algebras. In particular, we construct an example of an infinite computably enumerable algebra which is locally finite in every expansion but which possesses exactly one computable isomorphism type.

An **algebra** is a structure of a finite purely functional language (signature). Thus, any algebra $\mathcal{A}$ is of the form $(A; f_0, \dots, f_n)$, where A is a nonempty set called the domain of the algebra, and each f_i is a function symbol which names a total operation on the domain A. Often the operation named by f_i is also denoted by the same symbol f_i. We refer to the symbols $f_0, \dots, f_n$ as the signature of the algebra. Often we call the operations $f_0, \dots, f_n$ **basic operations** or **functions** (of the algebra $\mathcal{A}$). Presburger arithmetic $(\omega; 0, S, +)$ is an algebra, so are groups, rings, lattices and Boolean algebras. Fundamental structures which arise in computer science such as lists, stacks, queues, trees, and vectors can all be viewed and studied as algebras. In some of the results of this paper we use **unary signatures**; these are signatures all function symbols of which are symbols for unary operations. In the case in which the signature consists of exactly n unary function symbols, we call algebras of this signature n**-unary algebras**.

Let $\mathcal{A} = (A, f_0, \dots, f_n)$ be an algebra. For each element $a \in A$ introduce a new constant symbol c_a that names the element a itself. Thus, we have an expansion of $\mathcal{A}$ by constants c. The **atomic diagram of** $\mathcal{A}$ is the set of all expressions of the type $f_i(c_{a_1}, \dots, c_{a_n}) = f_j(c_{b_1}, \dots, c_{b_k})$, $f_i(c_{a_1}, \dots, c_{a_n}) = c_b$, $c_a = c_b$ and their negations which are true in the algebra $\mathcal{A}$. The **positive atomic diagram of** $\mathcal{A}$ is the set of all expressions of the type $f_i(c_{a_1}, \dots, c_{a_n}) = f_j(c_{b_1}, \dots, c_{b_k})$, $f_i(c_{a_1}, \dots, c_{a_n}) = c_b$, and $c_a = c_b$ which are true in the algebra $\mathcal{A}$. The **negative atomic diagram of** $\mathcal{A}$ is the set of all expressions of the type $f_i(c_{a_1}, \dots, c_{a_n}) \neq f_j(c_{b_1}, \dots, c_{b_k})$, $f_i(c_{a_1}, \dots, c_{a_n}) \neq c_b$, and $c_a \neq c_b$ which are true in the algebra $\mathcal{A}$. The algebra $\mathcal{A} = (A; f_0, \ldots, f_n)$ is **computable** if its atomic diagram is a computable set. The algebra $\mathcal{A} = (A; f_0, \ldots, f_n)$ is **computably enumerable** if its positive atomic diagram is a computably enumerable set. The algebra $\mathcal{A} = (A; f_0, \ldots, f_n)$ is **co-computably enumerable** if its negative atomic diagram is a computably enumerable set. Here are some examples of computably enumerable algebras:

1. All computable algebras.
2. The Lindenbaum algebras of computably enumerable first-order theories, such as Peano arithmetic.
3. Finitely presented groups, and in fact all finitely presented algebras.

A typical example of a co-computably enumerable algebra is the group generated by a finite number of computable permutations $g_1, \ldots, g_k$ on the set of natural numbers. Indeed, if g and g' are elements of this group then their non-equality is confirmed by the existence of an $n \in \omega$ at which $g(n) \neq g'(n)$.

A computably enumerable algebra $\mathcal{A}$ can be explained as follows. As the positive atomic diagram of $\mathcal{A}$ can be computably enumerated, the set $E = \{(c_a, c_b) \mid a = b \text{ is true in the algebra } \mathcal{A}\}$, representing the equality relation in $\mathcal{A}$, is computably enumerable. Let f be a basic n-ary operation on $\mathcal{A}$. From the definition of a computably enumerable algebra, the operation f can be thought of as a function induced by a computable function, often also denoted by f, which **respects** the E-equivalence classes in the following sense: for all $x_1, \ldots, x_n, y_1, \ldots, y_n$ if $(x_i, y_i) \in E$, then $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in E$. Therefore, a natural way to think about $\mathcal{A}$ is that the elements of $\mathcal{A}$ are E-equivalence classes, and the operations of $\mathcal{A}$ are induced by computable operations. This reasoning suggests another equivalent approach to the definition of computably enumerable algebra explained in the next paragraph.

Let E be an equivalence relation on ω. A computable n-ary function f **respects** E if for all natural numbers $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ so that $(x_i, y_i) \in E$, for $i = 1, \ldots, n$, we have $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in E$. Let $\omega(E)$ be the factor set obtained by factorizing ω by E, and let $f_0, \ldots, f_n$ be computable operations on ω which respect the equivalence relation E. An E**-algebra** is then the algebra $(\omega(E), F_0, \ldots, F_n)$, where each F_i is naturally induced by f_i. It is now not hard to show that an algebra $\mathcal{A}$ is computably enumerable if and only if $\mathcal{A}$ is an E-algebra for some computably enumerable equivalence relation E. In a similar way, one can show that an algebra $\mathcal{A}$ is co-computably enumerable if and only if $\mathcal{A}$ is an E-algebra for some co-computably enumerable equivalence relation E (that, is E is such that $\omega^2 \setminus E$ is computably enumerable).

The following can easily be checked. An algebra $\mathcal{A}$ is computable if and only if it is both computably enumerable and co-computably enumerable. Moreover, infinite computable algebras are the ones isomorphic to algebras of type $(\omega, f_0, \ldots, f_n)$, where each f_i is a computable function on ω. From now on we will concentrate on computably enumerable algebras.

The **isomorphism type** of an algebra $\mathcal{A}$ is the set of all algebras isomorphic to $\mathcal{A}$. We are interested in those algebras whose isomorphism types contain c.e. algebras. Informally, if the isomorphism type of $\mathcal{A}$ contains a c.e. algebra then this algebra has an effective realization. We formalize this in the following definitions. An algebra is **computably presentable** if it is isomorphic to a computable algebra. An algebra is **computably enumerably presentable**, which we often call **c.e. presentable**, if it is isomorphic to a computably enumerable algebra. Thus, computable or computably enumerable presentations of $\mathcal{A}$ can be thought as machine-theoretic (or program-theoretic) implementations of the algebra $\mathcal{A}$. Note that there is a distinction between c.e. algebras and c.e. presentable algebras. C.e. algebras are given explicitly by Turing machines representing the basic operations and the equality relation of the algebra, while c.e. presentability refers to the property of the isomorphism types of algebras.

Let $\mathcal{A}$ be a computably enumerable algebra, and let $f_1, \ldots, f_n$ be computable functions (which respect the equality relation on $\mathcal{A}$). Then the algebra $\mathcal{B} = (\mathcal{A}, f_1, \ldots, f_n)$ obtained by adding the operations $f_1, \ldots, f_n$ to $\mathcal{A}$ is called an **expansion** of $\mathcal{A}$. The signature $\sigma \cup \langle f_1, \ldots, f_n \rangle$ is an expansion of the original signature. The algebra $\mathcal{A}$ is called the **σ-reduct** (or simply a reduct) of $\mathcal{B}$. Note that if $\mathcal{A}$ is computably enumerable and f is a computable function then it is not always the case in which $(\mathcal{A}, f)$ is again an algebra because f may not respect the equality relation on $\mathcal{A}$. In this paper, expansions always refer to expansions of either computably enumerable or c.e. presentable algebras.

There are some notational conventions we need to make. Let $\mathcal{A}$ be a computably enumerable algebra. As the equality relation on $\mathcal{A}$ can be thought of as a computably enumerable equivalence relation on ω, we can refer to elements of $\mathcal{A}$ as natural numbers keeping in mind that each number n represents an equivalence class (that is, an element of $\mathcal{A}$). Thus, n can be regarded as either an element of $\mathcal{A}$, or the equivalence class containing n, or the natural number n. Which meaning we use will be clear from the content. Sometimes we denote elements of $\mathcal{A}$ by $[n]$, with $[n]$ representing the equivalence class containing the number n.

2 A Characterization Theorem

Let $\mathcal{A}$ and $\mathcal{B}$ be computably enumerable algebras. A homomorphism h from the algebra $\mathcal{A}$ into the algebra $\mathcal{B}$ is called a **computable homomorphism** if there exists a computable function $f : \omega \to \omega$ such that h is induced by f. In other words, for all $n \in \omega$, we have $h([n]) = [f(n)]$. We call f a representation of h. Clearly, if h is a computable homomorphism then its **kernel**, that is the set $\{(n, m) \mid h([n]) = h([m])\}$, is computably enumerable. We say that h is **proper** if there are distinct $[n]$ and $[m]$ in $\mathcal{A}$ whose images under h coincide. In this case the image $h(\mathcal{A})$ is called a proper homomorphic image of $\mathcal{A}$.

Our goal is to distinguish the class of computably enumerable but not computable algebras from the class of computable algebras. For this we need several notions. Let $\mathcal{A}$ be a computably enumerable algebra. Expand the language of $\mathcal{A}$ by adding constant symbols c_n for each n, so that c_n names the element $[n]$ in $\mathcal{A}$. A **fact** is a computably enumerable conjunction $\&_{i\in\omega}\phi_i(\bar{c})$ of sentences, where each $\phi_i(\bar{c})$ is of the form $\forall\bar{x}\psi_i(\bar{x}, \bar{c})$ with $\psi_i(\bar{x}, \bar{c})$ being a negative atomic formula. Call computably enumerable but not computable algebras **properly computably enumerable**. For example, any finitely generated computably enumerable algebra with undecidable equality problem is properly computably enumerable.

Definition 1. *An algebra $\mathcal{A}$* **preserves the fact** $\&_{i\in\omega}\phi_i(\bar{c})$ *if $\mathcal{A}$ satisfies the fact and there is a proper homomorphic image of $\mathcal{A}$ in which the fact is true.*

Here is now our characterization theorem. Informally, the theorem tells us that properly computably enumerable algebras possess many homomorphisms which are well behaved with respect to the facts true in $\mathcal{A}$.

Theorem 1. *A computably enumerable algebra $\mathcal{A}$ is properly computably enumerable if and only if $\mathcal{A}$ preserves all facts true in $\mathcal{A}$.*

Proof. Assume that $\mathcal{A}$ is a computable algebra. We can make the domain of $\mathcal{A}$ to be ω. Thus, in the algebra $\mathcal{A}$, the fact $\&_{i\neq j}(i \neq j)$ is clearly true. This fact cannot be preserved in any proper homomorphic image of $\mathcal{A}$.

For the other direction, we first note the following. Given elements m and n of the algebra, it is possible to effectively enumerate the minimal congruence relation, denoted by $\eta(m,n)$, of the algebra which contains the pair (m,n). Now note that if $[m] = [n]$ then $\eta(m,n)$ is the equality relation in $\mathcal{A}$. Denote $\mathcal{A}(m,n)$ the factor algebra obtained by factorizing $\mathcal{A}$ by $\eta(m,n)$. Clearly, $\mathcal{A}(m,n)$ is computably enumerable.

Now assume that $\mathcal{A}$ is properly computably enumerable and $\&_{i\in\omega}\phi_i(\bar{c})$ is a fact true in $\mathcal{A}$ which cannot be preserved. Hence, for any m and n in the algebra, if $[m] \neq [n]$ then in the factor algebra $\mathcal{A}(m,n)$, the fact $\&_{i\in\omega}\phi_i(\bar{c})$ cannot be satisfied. Therefore, for given m and n, there exists an i such that in the factor algebra $\mathcal{A}(m,n)$ the sentence $\neg\phi_i(\bar{c})$ is true. Now the sentence $\neg\phi_i(\bar{c})$ is equivalent to an existential sentence quantified over a positive atomic formula. Note that existential sentences quantified over positive atomic formulas true in $\mathcal{A}(m,n)$ can be computably enumerated. Hence, in the original algebra $\mathcal{A}$, for all m and n, either $[m] = [n]$ or there exists a an i such that $\neg\phi_i(\bar{c})$ is true in $\mathcal{A}(m,n)$. This shows that the equality relation in $\mathcal{A}$ is computable, contradicting the assumption that $\mathcal{A}$ is a properly computably enumerable algebra. $\square$

There are several interesting corollaries of the theorem above.

Corollary 1. *If $\mathcal{A}$ is properly computably enumerable then any two distinct elements m and n in $\mathcal{A}$ can be homomorphically mapped into distinct elements in a proper homomorphic image of $\mathcal{A}$.* $\square$

Indeed, take the fact $m \neq n$ true in $\mathcal{A}$, and apply the theorem.

Corollary 2. *If $\mathcal{A}$ is properly computably enumerable then any fact true in $\mathcal{A}$ is true in infinitely many distinct homomorphic images of $\mathcal{A}$. In particular, $\mathcal{A}$ cannot have finitely many congruences.*

Proof. Let ϕ be a fact. By theorem above, there is a homomorphism h_1 of $\mathcal{A}$ in which ϕ is true, and distinct elements m_1 and n_1 in $\mathcal{A}$ for which $h_1(m_1) = h_1(n_1)$. Now consider the fact $\phi\&(m_1 \neq n_1)$, and apply the theorem to this fact. There is a homomorphism h_2 of $\mathcal{A}$ in which $\phi\&(m_1 \neq n_1)$ is true, and distinct elements m_2 and n_2 in $\mathcal{A}$ for which $h_2(m_2) = h(n_2)$. Now consider the fact $\phi\&(m_1 \neq n_1)\&(m_2 \neq n_2)$, and apply the theorem to this fact. The corollary now follows by induction. $\square$

This theorem can now be applied to provide several algebraic conditions for c.e. algebras to be computable.

Corollary 3. *In each of the following cases an infinite computably enumerable algebra $\mathcal{A}$ is computable:*

1. *There exists a c.e. sequence (x_i, y_i) such that $[x_i] \neq [y_i]$) for all i and for any congruence relation η there is (x_j, y_j) for which $([x_j], [y_j]) \in \eta$.*
2. *$\mathcal{A}$ has finitely many congruences.*
3. *$\mathcal{A}$ is finitely generated and every congruence relation of $\mathcal{A}$ has a finite index.*

Proof. For Part 1), we see that the fact $\&_{i\in\omega}[x_i] \neq [y_i]$ is true in $\mathcal{A}$. The assumption states that this fact cannot be preserved in all proper homomorphic images of $\mathcal{A}$. Hence $\mathcal{A}$ must be a computable algebra by the theorem above. For part 2), let $\eta_0, \ldots, \eta_k$ be all non-trivial congruences of $\mathcal{A}$; for each η_i take (x_i, y_i) such that $[x_i] \neq [y_i]$ and $([x_i], [y_i])$. Then the fact $\&_{i\leq k}([x_i] \neq [y_i])$ is true in $\mathcal{A}$ but cannot be preserved in all proper homomorphic images of $\mathcal{A}$. Thus $\mathcal{A}$ is a computable algebra. For Part 3), consider any two elements $[m]$ and $[n]$ in $\mathcal{A}$ and consider the congruence relation $\eta([m], [n])$ defined in the proof of the theorem. By assumption, $[m] \neq [n]$ iff the algebra $\mathcal{A}(m, n)$ is finite. The set $X = \{(m, n) \mid \mathcal{A}(m, n)$ is finite $\}$ is computably enumerable. Hence, the fact $\&_{(m,n)\in X}([m] \neq [n])$ is true in $\mathcal{A}$ but cannot be preserved in any homomorphic image of $\mathcal{A}$. □

3 Computable Isomorphisms and Finitely Generated Expansions

Let h be a computable homomorphism from a c.e. algebra $\mathcal{A}$ into a c.e. algebra $\mathcal{B}$. Naturally, a computable homomorphism h is called a **computable isomorphism** if h is a bijection. Clearly, the composition of computable isomorphisms is a computable isomorphism. It is not hard to see that the inverse of a computable isomorphism is also a computable isomorphism. Indeed, let $h : \mathcal{A} \to \mathcal{B}$ be a computable isomorphism whose representation is f. A representation g of h^{-1} is defined as follows. For a given n find the first m such that $f(m) = n$ in the algebra $\mathcal{B}$. Then $g(n) = m$. Note that the computable functions which are representatives of isomorphisms need not be bijections on the natural numbers. We say that $\mathcal{A}$ and $\mathcal{B}$ have the same **computable isomorphism type** if there is a computable isomorphism from $\mathcal{A}$ into $\mathcal{B}$. Here is one of the definitions which will be used in this paper:

Definition 2. *The* **computable dimension** *of $\mathcal{A}$ is the number of its computable isomorphism types. A computably enumerable algebra $\mathcal{A}$ is* **computably categorical** *if its computable dimension is* 1.

Thus, the definition informally tells us that all possible implementations of a computably categorical algebra are equivalent. In other words, if $\mathcal{A}$ and $\mathcal{B}$ are computably enumerable presentations of a computably categorical algebra then there must exist a computable isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

The notion of computable isomorphism and topics related to it have been studied intensively in computable algebra. We note that most of the results related to computable isomorphisms of computable algebras can be transferred to the class of computably enumerable algebras. This can, for example, be done as follows. Let $\mathcal{A} = (A; f_0, \ldots, f_n)$ be a computable algebra. Consider its extension $\mathcal{B} = (\mathcal{A} \cup \{a, b\}; g_0, \ldots, g_n, E, S)$, where a, b are new symbols not in A, defined as follows. For each original basic operations f, define $g(\bar{x}) = f(\bar{x})$ if $\bar{x}$ is in A; otherwise $g(\bar{x}) = a$. Also, set $E(x, y) = a$ if $x = y$, and $E(x, y) = b$ otherwise. Finally, $S(x) = x$ in case $x \in A$; in other cases set $S(a) = b$, $S(b) = a$. Now $\mathcal{B}$ preserves many computability-theoretic and algebraic properties of $\mathcal{A}$. Note that any computably enumerable presentation of the extended algebra is a computable presentation. Therefore there is a one-to-one correspondence between computable presentations of $\mathcal{A}$ and computably enumerable presentations of the expansion $\mathcal{B}$. Hence, most of the results about computable presentations of $\mathcal{A}$ and its computable isomorphism types hold true for computably enumerable presentations of the extension. For example if $\mathcal{A}$ has computable dimension n in the class of all computable presentations then the computable dimension of $\mathcal{B}$ is n (in the class of all c.e. presentations). We also note that $\mathcal{B}$ preserves the automorphism group of $\mathcal{A}$. In addition, all congruences of $\mathcal{A}$ are also congruences of $\mathcal{B}$, and there is only one congruence relation of $\mathcal{B}$ which is not a congruence of $\mathcal{A}$.

A typical example of a computably categorical computably enumerable algebra is provided in the next proposition.

Proposition 1. *Any finitely generated computably enumerable algebra is computably categorical.*

Proof. Let $a_1, \ldots, a_n$ be the generators of the algebra. Let $k_1, \ldots, k_n$ and $m_1, \ldots, m_n$ be natural numbers representing $a_1, \ldots, a_n$ in two computably enumerable presentations $\mathcal{A}$ and $\mathcal{B}$, respectively. The mapping $k_i \to m_i$ can be effectively be extended to an isomorphism due to the fact that $a_1, \ldots, a_n$ are generators. □

Clearly, every computable algebra $\mathcal{A}$ can be made computably categorical in an expansion by adding the successor function to the signature of the algebra. The following proposition provides examples of computably enumerable algebras which can be made computably categorical by using expansions of the original language.

Proposition 2. *Let $\mathcal{A}$ be a computably enumerable algebra which satisfies the following property. There is a sequence $\{X_i\}_{i \in \omega}$ of disjoint nonempty subsets of ω so that:*

1. *The sets $\{(x, i) \mid x \in X_i\}$ and $X = \cup_{\in \omega} X_i$ are computable.*
2. *For all $x, y, i \in \omega$ if $x \in X_i$ and $[x] = [y]$ in $\mathcal{A}$ then $y \in X_i$.*

Then $\mathcal{A}$ has a finitely generated, hence computably categorical, expansion.

Proof. We define two new unary functions f and g in the following manner. The functions f and g coincide with the identity function outside of X. On X the functions f and g are defined as follows. Let x_i be the minimal element in X_i, where the order is the order on natural numbers. For all elements n of X_i, set $f(n) = x_{i+1}$ and $g(n) = i$. Now it is not hard to see that f, g respect the equality relation of $\mathcal{A}$. Indeed, if $[n]$ equals $[m]$ in the algebra $\mathcal{A}$, then we have the following. If $n \notin X$ then, by the definition of f and g,

$f([n]) = [n] = [m] = f([m])$ and $g([n]) = [n] = [m] = g([m])$. If $n \in X$ then $n \in X_i$ for some i. Therefore, by the definition of f and g, we have $f([n]) = [x_{i+1}] = f([m])$, and $g([n]) = g([m]) = i$. Also, note that for every $i \in \omega$ we have $gf^i([x_0]) = [i]$, where $f^0(a) = a$ for all a. Thus, $(\mathcal{A}, f, g)$ is finitely generated. The generator is the minimal element of X_0. The proposition is proved.

Corollary 4. *If a computably enumerable algebra $\mathcal{A}$ possesses a homomorphism h whose kernel is computable then $\mathcal{A}$ has a finitely generated expansion.*

Indeed, since the kernel of h is a computable set, one can extract a sequence $\{X_i\}_{i\in\omega}$, where each X_i is the equivalence class determined by the kernel of h, which satisfies the conditions of the proposition. □

We are interested in finding computably categorical expansions of c.e. presentable algebras. One way to do this would be to find finitely generated c.e. presentable expansions. It turns out, which we show below, that this is not always possible. One then asks whether or not it is possible to find computably categorical c.e. algebras which do not have finitely generated expansions. We provide examples of such algebras. We now need some definitions and a tool for showing that a given c.e. algebra is not finitely generated.

Definition 3. *An infinite algebra $\mathcal{A}$ is* **locally finite** *if every finitely generated subalgebra of $\mathcal{A}$ is finite. We say that a computably enumerable algebra is* **absolutely locally finite** *if all its expansions are locally finite.*

Next we define a set of natural numbers related to the complexity of the equality relation of a given computably enumerable algebra.

Definition 4. *The* **transversal** *of a computably enumerable algebra $\mathcal{A}$, denoted by $tr(\mathcal{A})$, is the set $\{n \mid \forall y(y < n \rightarrow [y] \neq [n])\}$.*

It is not hard to see that the equality relation $E = \{(m, n) \mid [m] = [n])\}$ in the computably enumerable algebra $\mathcal{A}$ is Turing equivalent to $tr(\mathcal{A})$. We say that a set X of natural numbers is **hyperimmune** if there does not exist a computable function m such that $m(i) \geq x_i$ for all i, where $x_0 < x_1 < x_2 < \ldots$ is a listing of the elements of X in strictly increasing order. In this case, the function m is said to **majorize** the set X. We also say that a set M is **hypersimple** if M is computably enumerable and its complement is hyperimmune. Note that hypersimple sets exist (e.g. see Soare [10]).

Proposition 3. *[6] If the transversal of a c.e. algebra is hyperimmune then the algebra is absolutely locally finite.*

Proof. Let $\mathcal{A}$ be any c.e. algebra whose transversal is hyperimmune. Consider any finitely generated subalgebra of $\mathcal{A}$, and assume that the subalgebra is infinite. Let $n_0, \ldots, n_k$ be the generators of the subalgebra. Define the following sequence: $X_0 = \{n_0, \ldots, n_k\}$, $X_{i+1} = X_i \cup \{f(\bar{x}) \mid \bar{x} \in X_i, f \in \sigma\}$, where $\bar{x}$ is an n-tuple of X_i and f is an n-ary operation of the language σ of the algebra. Clearly, each X_i is a finite subset of natural numbers. Now let m_i be the maximal element of X_i. Note that for each i there exists an x_{i+1} in X_{i+1} such that

$[x_{i+1}] \neq [y]$ for all $y \in X_i$ because the subalgebra is infinite. Hence, the function $m(i) = m_i$ is computable and gives a counterexample for $tr(\mathcal{B})$ being hyperimmune. Thus, $\mathcal{A}$ and all its expansions are locally finite. Hence $\mathcal{A}$ is absolutely locally finite. □

4 An Absolutely Locally Finite and Computably Categorical Algebra

The goal of this section is to construct a computably enumerable and computably categorical algebra which does not have finitely generated c.e. presentable expansions. We need the following definitions and notations.

Let $\mathcal{A}$ be a finite algebra. The algebra $\mathcal{A}$ **collapses** into an algebra $\mathcal{B}$ if there exists a finite algebra $\mathcal{A}'$ containing $\mathcal{A}$ as a subalgebra such that $\mathcal{B}$ is a homomorphic image of $\mathcal{A}'$. In the case in which $\mathcal{A}$ can be collapsed into $\mathcal{B}$ but $\mathcal{B}$ is not a homomorphic image of $\mathcal{A}$, we say that $\mathcal{A}$ **properly collapses** into $\mathcal{B}$. We note that all our homomorphisms are surjective.

Let $\mathcal{A}$ be a finite 3-unary algebra of signature f, g_1, g_2. The f-**orbit** of an element $a \in A$ is the sequence $a, f(a), f^2(a), \ldots$. This orbit is called an f-**cycle of length** n if all elements $a, f(a), \ldots, f^{n-1}(a)$ are pairwise distinct and $f^n(a) = a$. An example of an algebra $\mathcal{A}$ which properly collapses into $\mathcal{B}$ is the following. We present this example as it is typical of algebras we use in our main theorem of this section. The algebra is a finite unary algebra of signature f, g_1, g_2. Assume that $\mathcal{A}$ has $m \cdot n$ elements and the algebra forms an f-cycle of length $m \cdot n$ such that $g_i(x) = x$, $i = 1, 2$, for each element x of the cycle. Consider a 3-unary algebra $\mathcal{B}$ satisfying the following conditions:

1. B consists of the disjoint union of two f-cycles of lengths n and 1.
2. For each $x \in B$ in the f-cycle of length n we have $g_1(x) = g_2(x) = x$.
3. Let b be the element which forms an f-cycle of length 1, that is $f(b) = b$. Then the elements $g_1(b)$ and $g_2(b)$ belong to the f-cycle of length n, and $g_1(b) \neq g_2(b)$.

It is easy to see that $\mathcal{A}$ properly collapses into $\mathcal{B}$.

The idea of collapsing can be used in constructing computably enumerable algebras. Here is a module of a such construction. Assume a finite algebra $\mathcal{A}$ can be collapsed into $\mathcal{B}$, and the languages of both algebras have unary operations only. Let $\mathcal{A}'$ be a finite algebra containing $\mathcal{A}$ as a subalgebra and h be a homomorphism from $\mathcal{A}'$ into $\mathcal{B}$. Assume that the domains of $\mathcal{A}'$ and $\mathcal{B}$ are disjoint. Define the new algebra $\mathcal{C}$ obtained by taking the unions of the domains and operations of $\mathcal{A}'$ and $\mathcal{B}$, respectively. Then the transitive closure of the relation $\{(x, y) \mid h(x) = h(y) \vee y = h(x) \vee x = y\}$ is a congruence relation on $\mathcal{C}$. The factor algebra obtained is isomorphic to $\mathcal{B}$.

In our result below we construct computably enumerable algebras which will be disjoint unions of infinitely many finite 3-unary algebras. The following definitions describe these finite 3-unary algebras.

Let $\mathbf{r} = r_0, r_1, \ldots, r_k$ be a finite sequence of prime numbers. We define algebras $\mathcal{A}(\mathbf{r}, i)$, with $i < n_k$ and $n_k = r_0 \cdot \ldots \cdot r_k$, of signature f, g_1, g_2 as follows.

1. The domain $A(\mathbf{r}, i)$ of the algebra $\mathcal{A}(\mathbf{r}, i)$ is $\{x_0, \ldots, x_{n_k-1}\} \cup \{y_0, \ldots, y_i\}$.
2. $A(\mathbf{r}, i)$ contains an f-cycle of length n_k so that $f(x_i) = x_{i+1}$ and $f(x_{n_k-1}) = x_0$ for all $i < n_k - 1$. The functions g_1 and g_2 are each the identity function on the set $\{x_0, \ldots, x_{n_k-1}\}$.
3. For every $t \leq i-1$, we have $g_1(y_t) = x_t$ and $g_2(y_t) = x_{t+1}$. Call each y_i a **top** element. The function f is the identity function on all the top elements.

The number of top elements of the algebra $\mathcal{A}(\mathbf{r}, i)$ is $i+1$ and does not exceed the length of the f-cycle. In the case when the number of top elements equals n_k, we call the algebra an **open algebra** and denote it by $\mathcal{A}(\mathbf{r})$. Otherwise, we call the algebra a **partially open algebra**.

We say that a partially open (or open) algebra **omits** a prime number p if p is not a divisor of the length of the f-cycle of the algebra. Otherwise, the algebra **realizes** p. Here are some algebraic properties of open algebras.

Lemma 1. *Consider two sequences $\mathbf{r} = r_0, r_1, \ldots, r_k$ and $\mathbf{r}' = r'_0, r'_1, \ldots, r'_l$ of prime numbers. Then*

1. *If $\mathcal{A}(\mathbf{r})$ realizes p but not p' and $\mathcal{A}(\mathbf{r}')$ realizes p' but not p, then neither of these algebras collapses into the other.*
2. *Assume that $r_0 = r'_0$, …, $r_i = r'_i$ for some $i \leq k, l$. Consider $\mathbf{q} = r_0, \ldots, r_i$. Then the algebra $\mathcal{A}(\mathbf{q})$ is a homomorphic image of $\mathcal{A}(\mathbf{r})$ and $\mathcal{A}(\mathbf{r}')$.*
3. *If every p realized in $\mathcal{A}(\mathbf{r})$ is also realized in $\mathcal{A}(\mathbf{r}')$ then $\mathcal{A}(\mathbf{r}')$ can be homomorphically mapped onto $\mathcal{A}(\mathbf{r})$.*
4. *No open algebra $\mathcal{A}(\mathbf{r})$ can be homomorphically mapped onto a partially open algebra.*

Proof. For part 1 note the following. On the one hand, the length of the f-cycle of every homomorphic image of $\mathcal{A}(\mathbf{r})$ is not a multiple of p'. On the other hand, the length of the f-cycle of $\mathcal{A}(\mathbf{r}')$ is a multiple of p'. Part 2 of the lemma is immediate. Part 3 follows from Part 2. Finally, the last part follows from the fact that in the open algebra the length of the f-cycle coincides with the number of top elements which clearly does not hold for partially open algebras. □

For the sequence $\mathbf{r} = r_0, r_1, \ldots, r_k$ set $\mathbf{r}[j]$ be $r_0, r_1, \ldots, r_j$, where $j \leq k$. The following lemma further describes some of the properties of open and partially open algebras and the relationship between them.

Lemma 2. *The partially open algebras $\{\mathcal{A}(\mathbf{r}, i)\}_{i<n_k}$ have the following properties:*

1. *Each $\mathcal{A}(\mathbf{r}, i)$ is not a homomorphic image of $\mathcal{A}(\mathbf{r}, j)$ for distinct i and j.*
2. *Each $\mathcal{A}(\mathbf{r}, i)$ can be extended to the open algebra $\mathcal{A}(\mathbf{r})$.*
3. *Each $\mathcal{A}(\mathbf{r}, i)$ can be collapsed into the open algebra $\mathcal{A}(\mathbf{r}[j])$.*
4. *Neither of the partial algebras $\mathcal{A}(\mathbf{r}[j_1], i_1)$ and $\mathcal{A}(\mathbf{r}[j_2], i_2)$ for $(j_1, i_1) \neq (j_2, i_2)$ is a homomorphic image of the other.*

Proof. Part 1) follows from the following observation. Say $i < j$. It is not hard to see that $\mathcal{A}(\mathbf{r}, j)$ and $\mathcal{A}(\mathbf{r}, i)$ cannot be mapped homomorphically onto each other because the number of top elements in $\mathcal{A}(\mathbf{r}, i)$ and in $\mathcal{A}(\mathbf{r}, j)$ is distinct. Part 2) is easy to see by introducing a sufficient number of new top elements for $\mathcal{A}(\mathbf{r}, i)$. For Part 3) note that $\mathcal{A}(\mathbf{r}[j])$ is an open algebra. Now the rest follows from Part 2), and the second part of the previous lemma. Finally, for the last part note the following. If $j_1 = j_2$ then we can apply Part 1) of the lemma. If $j_1 < j_2$, then it is clear that $\mathcal{A}(\mathbf{r}[j_1], i_1)$ cannot be homomorphically mapped onto $\mathcal{A}(\mathbf{r}[j_2], i_2)$. Now there is no homomorphism from $\mathcal{A}(\mathbf{r}[j_2], i_2)$ into $\mathcal{A}(\mathbf{r}[j_1], i_1)$ because these two algebras have a distinct number of top elements. □

To give an initial intuition to the reader, we would like to say a few words about the algebra $\mathcal{A}$ constructed in the next theorem. The algebra will be a disjoint union $\mathcal{A}_0 \cup \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots$ of algebras such that each $\mathcal{A}_i$ is either an open algebra or a partially open algebra. Thus, the $\mathcal{A}$ constructed will be a 3-unary algebra. In addition, no $\mathcal{A}_i$ will be a homomorphic image of any other $\mathcal{A}_j$. The algebra $\mathcal{A}$ will be absolutely locally finite. In order to guarantee this property of $\mathcal{A}$, the construction ensures that the transversal $tr(\mathcal{A})$ is hyperimmune. Hyperimmunity is satisfied on those $\mathcal{A}_i$'s which are open algebras. Now we formulate the theorem.

Theorem 2. *There exists an infinite computably categorical, absolutely locally finite, and computably enumerable algebra.*

Proof. Our goal is to construct a computably enumerable algebra $\mathcal{A}$ such that $tr(\mathcal{A})$ is hyperimmune. In addition, we want to make sure that any computably enumerable algebra $\mathcal{G}$ isomorphic to $\mathcal{A}$ is computably isomorphic to $\mathcal{A}$. Due to Proposition 3, these two properties guarantee that the algebra $\mathcal{A}$ will be absolutely locally finite, and thus as desired. We fix effective enumerations of all partial computable unary functions $\phi_0, \phi_1, \ldots$, and all partial computably enumerable 3-unary algebras $\mathcal{G}_0$, $\mathcal{G}_1$, ... In some of these algebras their unary operations may not be total functions. We need to construct a computably enumerable algebra $\mathcal{A}$ which satisfies the following requirements:

D_e: ϕ_e does not majorize the transversal $tr(\mathcal{A})$ of $\mathcal{A}$, and

R_j: If $\mathcal{G}_j$ is isomorphic to $\mathcal{A}$ then G_j is computably isomorphic to $\mathcal{A}$,

where $e, j \in \omega$. We list these requirements as $D_0, R_0, D_1, R_1, \ldots$ and call this sequence the priority list. Requirements listed earlier **have higher priority** than those listed later in this sequence.

An R_j-strategy devoted to satisfying just one requirement R_j is the following. Start constructing the algebra $\mathcal{A}$ so that the algebra consists of the disjoint union of f-cycles of prime length 2, 3, 5, ... Roughly speaking, as long as $\mathcal{G}_j$ provides f-cycles of length p_i for each prime p_i, where the f-cycle of length p_i has already been built in $\mathcal{A}$, then effectively establish an isomorphism from f-cycles in $\mathcal{G}_j$ into the f-cycles of $\mathcal{A}$ of the same length. Thus, if $\mathcal{G}_j$ is indeed isomorphic to the $\mathcal{A}$ which is being built then $\mathcal{G}_j$ is computably isomorphic to $\mathcal{A}$.

A possible strategy, call it a D_e-strategy, which satisfies just one requirement D_e is easy to describe. Start constructing the algebra $\mathcal{A}$ so that $\mathcal{A}$ consists of disjoint f-cycles of length 2, $2 \cdot 3$, $2 \cdot 5$, ..., where each f-cycle of size $2 \cdot p$ appears just once. Wait until $\phi_e(e)$ is defined.

If $\phi_e(e)$ is defined and is greater than the eth element of $\mathcal{A}$ then collapse all f-cycles which contain 0, 1, …, $\phi_e(e)$ into an f-cycle of length 2. Thus, in this case, the resulting algebra will have one f-cycle of length 2 and infinitely many f-cycles of pairwise distinct lengths for which 2 is a factor. Otherwise, the algebra will have f-cycles of lengths 2, $2 \cdot 3$, $2 \cdot 5$, …. In either case the requirement D_e is clearly satisfied.

Problems arise when one tries to satisfy both requirements D_e and R_j simultaneously. One possible reason is the following. While the R_j-strategy builds an isomorphism, the D_e-strategy, as described above, may collapse several f-cycles in $\mathcal{A}$ into an f-cycle X, say, of length 2. It may well be the case that R_j had already built a partial isomorphism into some f-cycles, and those f-cycles have now collapsed into X. Thus, $\mathcal{G}_j$ can now easily defeat the R_j-strategy as follows: $\mathcal{G}_j$ provides an f-cycle C of length $2 \cdot p$ isomorphic to an f-cycle in $\mathcal{A}$. Once an isomorphism from C into the f-orbit in $\mathcal{A}$ has been established, $\mathcal{G}_j$ now collapses C into X' which is the image of the f-orbit X in $\mathcal{A}$. This behavior of $\mathcal{G}_j$ forces the image of C to be collapsed into X in order to save the established partial isomorphism. Hence, the algebra $\mathcal{A}$ may become a finite one which is clearly not desirable. Now the reader can imagine the difficulties involved in trying to satisfy *all* requirements D_e and R_j for $e, j \in \omega$.

In order to overcome the difficulties described above we use open and partially open algebras and their algebraic properties described in Lemma 1 and Lemma 2. During the construction presented below, we put certain sentences in double brackets [[like this]]. These are designed to explain the construction and ideas and are not part of the algorithm which constructs the algebra $\mathcal{A}$. The algebra constructed will be a 3-unary algebra of signature f, g_1, g_2. At stage t, we will have a finite algebra $\mathcal{A}_t$ which consists of the union of open or partially open algebras. In addition, our construction will be involved with satisfying requirements D_e only. In other words, the construction does not deal with satisfying requirements R_j directly. We argue later that all the requirements R_j are satisfied due to the algebraic nature of the example constructed. Now we proceed to our construction.

Stage 0. Let $\mathcal{A}_0$ be isomorphic to the disjoint union of the partially open algebras $\mathcal{A}(\mathbf{r}_0, 0)$ and $\mathcal{A}(\mathbf{r}_1, 1)$, where $\mathbf{r}_0$ is the sequence 2, and $\mathbf{r}_1$ is the sequence $2, 3$. Thus, $\mathcal{A}(\mathbf{r}_0, 0)$ contains an f-cycle of length 2 associated with one top element, and $\mathcal{A}(\mathbf{r}_1, 1)$ contains an f-cycle of length 6 associated with two top elements. Associate the function ϕ_0 with $\mathcal{A}(\mathbf{r}_1, 1)$ with the goal of meeting requirement D_0 on this subalgebra. The equality relation E_0 in $\mathcal{A}_0$ is the identity relation.

Stage $t+1$. At the end of the previous stage we have the algebra $\mathcal{A}_t$ which is a disjoint union $\mathcal{A}_{0,t} \cup \mathcal{A}_{1,t} \cup \ldots \cup \mathcal{A}_{t,t}$ of subalgebras. The inductive assumptions put on the algebra $\mathcal{A}_t$ are the following. Each $\mathcal{A}_{j,t}$ is isomorphic to either an open algebra or a partially open algebra. The partial function ϕ_j, $j \leq t-1$, is associated with the algebra $\mathcal{A}_{j+1,t}$ with the goal of meeting requirement D_j if D_j has not already been met. In addition, there exist an increasing sequence $r(0,t), r(1,t), \ldots, r(t,t)$ and its subsequence $r(i_0,t), \ldots, r(i_{k(t)},t)$ of prime numbers which satisfy the following conditions. For $j \leq t$, let $\mathbf{r}(j)$ be obtained by dropping all $r(i_0,t), \ldots, r(i_{k(t)},t)$ from the sequence $r(0,t), r(1,t), \ldots, r(j-1,t)$, and putting $r(j,t)$ at the end.

1. If $j \in \{i_0, \ldots, i_{k(t)}\}$ then $A_{j,t}$ is isomorphic to the open algebra $\mathcal{A}(\mathbf{r}(j))$.
2. If $j \notin \{i_0, \ldots, i_{k(t)}\}$ then $A_{j,t}$ is isomorphic to the partially open algebra $\mathcal{A}(\mathbf{r}(j), j)$.

Thus, the algebra $A_{j,t}$ is open if and only if $j \in \{i_0, \ldots, i_{k(t)}\}$. [[The idea is that requirements D_j have currently been met (but later could be injured) for $j \in \{i_0, \ldots, i_{k(t)}\}$.]] We also note that if $\mathcal{A}_{j,t}$ is a partially open algebra then it has exactly $j+1$ many top elements. In addition, the algebra $\mathcal{A}_{j,t}$ omits all prime numbers $r(i_s, t)$ with $i_s < j$, and realizes all prime numbers in the sequence $\mathbf{r}(j)$. Finally, $\mathcal{A}_{j,t}$ can be collapsed into any of the open algebras obtained by extending partially open algebras $\mathcal{A}_{s,t}$ with $s \leq j$, and cannot be homomorphically mapped into the open algebras $\mathcal{A}_{s',t}$ with $s' < j$ [[This is because the open algebras $\mathcal{A}_{s',t}$ realize some prime numbers omitted by $\mathcal{A}_{j,t}$.]] Finally, note that all the algebras $\mathcal{A}_{0,t} \cup \mathcal{A}_{1,t} \cup \ldots \cup \mathcal{A}_{t,t}$ cannot be mapped onto each other homomorphically due to Lemma 1 and Lemma 2.

In addition, we have the equality relation E_t in the algebra $\mathcal{A}_t$ telling us which numbers represent the same elements of $\mathcal{A}_t$. Let $m(e,t)$ be the minimal number in $\mathcal{A}_{e+1,t}$, that is $m(e,t)$ is the minimal number among all numbers representing elements of $\mathcal{A}_{e+1,t}$. Let $p(e,t)$ be the position at which $m(e,t)$ appears in the listing of the transversal $tr(\mathcal{A}_t)$ in the strictly increasing order. The idea is to meet D_e on number $m(e,t)$ in case $\phi_e(p(e,t)) \geq m(e,t)$.

Here is now a description of the stage. Compute $\phi_j^{t+1}(p(j,t))$ for all $j \leq t$, where $\phi_j^{t+1}(x)$ denotes the $(t+1)$th approximation of ϕ in computing the value $\phi_j(x)$. Let e be the minimal j with $j \leq t-1$ for which $\phi_j^{t+1}(p(j,t)) \geq m(j,t)$ (if such j exists). Act as follows with the goal of meeting D_e.

Consider the algebra $\mathcal{A}_{e,t}$. It is a partially open algebra [[this can be assumed by induction]]. Extend the algebra $\mathcal{A}_{e,t}$ to an open algebra [[to do this, introduce sufficiently many top elements]]. Collapse all algebras $\mathcal{A}_{s,t}$ with $s \geq e$ into the open algebra thus obtained. If need be, by enumerating elements of $\mathcal{A}_{e,t}$ by new numbers, make sure that all numbers in ω less than or equal to $\phi_e^{t+1}(p(e,t))$ have been used. [[Thus, all the subsequent elements enumerated into the algebra will be greater than $\phi_e^{t+1}(p(e,t))$, and hence D_e is permanently satisfied as long as no requirement of higher priority injures D_e.]]

Define the sequence $r(0,t+1), \ldots, r(e,t+1), r(e+1,t+1), \ldots, r(t+1,t+1)$ as follows:

1. Set $r(0,t+1) = r(0,t), r(1,t+1) = r(1,t), \ldots, r(e,t+1) = r(e,t)$.
2. All $r(j,t+1)$, where $e+1 \leq j \leq t+1$, are unused prime numbers greater than all numbers and the lengths of all f-cycles which have appeared in the construction so far. [[In particular, these numbers are omitted by all open and partially open algebras which have appeared in the construction so far. We note that this is an important part of the construction, guaranteeing that all prime numbers between $r(e,t+1)$ and $r(e+1,t+1)$ will be omitted in all partially open and open subalgebras of $\mathcal{A}$. This property is then used to show that $\mathcal{A}$ is computably categorical.]]

Among $i_0, \ldots, i_{k(t)}$, keep all i_s with $i_s < e$, cancel all the others, and set $k(t+1) = e$.

Construct $\mathcal{A}_{t+1}$ as follows. Let $\mathbf{r}'(j)$ be obtained by dropping $r(i_0, t+1), \ldots, r(i_{k(t+1)}, t+1)$ from the sequence $r(0,t+1), r(1,t+1), \ldots, r(j-1,t+1)$, and putting $r(j,t+1)$ at the end.

1. If $j \in \{i_0, \dots, i_{k(t+1)}\}$ then $A_{j,t+1}$ is isomorphic to the open algebra $\mathcal{A}(\mathbf{r}'(j))$.
2. If $j \notin \{i_0, \dots, i_{k(t+1)}\}$ then $A_{j,t+1}$ is isomorphic to the partially open algebra $\mathcal{A}(\mathbf{r}'(j), j)$.

Make sure that $\mathcal{A}_t$ is embedded into $\mathcal{A}_{t+1}$ so that the identity mapping represents the embedding. [[Note that all the elements of the algebras $\mathcal{A}_{e+1,t}, \dots, \mathcal{A}_{t,t}$ have been collapsed into $\mathcal{A}_{e,t}$ which has become an open algebra. So all the numbers appearing in $\mathcal{A}_{e+1,t+1}$, ..., $\mathcal{A}_{t+1,t+1}$ are now new.]]

If there is no j with $j \leq t$ for which $\phi_j^{t+1}(p(j,t)) \geq m(j,t)$ then proceed as follows. Keep all the parameters and associated objects (e.g. $r(j,t)$, $\mathcal{A}_{j,t}$, $i_k(t)$) but increment the parameter t to $t+1$. Consider a new prime number $r(t+1,t+1)$ not used and greater than all numbers and the lengths of all f-cycles which have appeared so far. Construct $\mathcal{A}_{t+1,t+1}$ to be isomorphic to the partially open algebra $\mathcal{A}(\mathbf{r}'(t+1), t)$, where $\mathbf{r}'(t+1)$ is defined above. Embed $\mathcal{A}_t$ into $\mathcal{A}_{t+1}$ so that the identity mapping represents the embedding.

Thus, $\mathcal{A}_{t+1}$ is such that $\mathcal{A}_t$ embeds into $\mathcal{A}_{t+1}$, and the identity mapping is the embedding. Therefore the equality relation E_t is a subset of the equality relation E_{t+1} in $\mathcal{A}_{t+1}$. The algebra $\mathcal{A}_{t+1}$ has now been constructed. The algebra constructed preserves all the inductive assumptions. This completes the stage.

Now the algebra $\mathcal{A}$ is defined to be the limit of all the algebras $\mathcal{A}_t$. More formally, for all $n, m \in \omega$, the numbers n and m represent the same element in $\mathcal{A}$ if and only if $(n,m) \in E_t$ for some t. Moreover, for all $n, m, m_1, m_2 \in \omega$, we have $f(n) = m$, $g_1(n) = m_1$ and $g_2(n) = m_2$ in the algebra $\mathcal{A}$ if and only if the equalities $f(n) = m$, $g_1(n) = m_1$ and $g_2(n) = m_2$ hold true in some algebra $\mathcal{A}_t$. Hence, the algebra $\mathcal{A}$ is computably enumerable.

Now our goal is to show that $\mathcal{A}$ constructed is indeed a desired algebra.

Lemma 3. *The transversal $tr(\mathcal{A})$ is a hyperimmune set.*

We need to show that each requirement D_e is satisfied. This is shown by induction on e. Assume that $\phi_0(p(0,t)) \geq m(0,t)$ at some stage $t_0 + 1$ (if no such stage exists then D_0 is clearly satisfied). Then at stage t_0+1, requirement D_0 is satisfied. Moreover, the construction guarantees the following properties:

1. $\mathcal{A}_{0,t'}$ is open and no new number is being made equal to any of the elements of $\mathcal{A}_{0,t'}$ for any $t' \geq t_0 + 1$. In other words, the construction does not enumerate any new number into $\mathcal{A}_{0,t'}$ after stage $t_0 + 1$.
2. $m(0,t') = m(0,t'+1)$ and $p(0,t') = p(0,t'+1)$ and $\phi_0(p(0,t')) < m(0,t')$ for all $t' \geq t_0+1$.

Indeed, after stage $t_0 + 1$, the algebra $\mathcal{A}_{0,t'}$ is never collapsed further, and thus $m(0,t') = \lim_{t \to \infty} m(0,t)$. This implies that the second property holds. Hence D_0 is permanently satisfied after stage $t_0 + 1$. Now, by induction, assume that D_0, ..., D_{e-1} are all permanently satisfied. Also assume that there exists a stage t_{e-1} such that no numbers are enumerated into the algebras $\mathcal{A}_{0,t}, \dots, \mathcal{A}_{e-1,t}$, and hence these algebras never change after stage t_{e-1}. Assume that $\phi_e(p(e,t_e)) \geq m(0,t_e)$ at some stage $t_e + 1 > t_{e-1}$ (if no such stage exists then D_e is clearly satisfied). Then at stage $t_e + 1$, requirement D_e is satisfied. In addition, the construction guarantees the following properties:

1. $\mathcal{A}_{e,t'}$ is open and no number is enumerated into $\mathcal{A}_{e,t'}$ for any $t' \geq t_e + 1$.
2. $m(e,t') = m(e,t'+1)$ and $p(e,t') = p(e,t'+1)$ and $\phi_e(p(e,t')) < m(e,t')$ for all $t' \geq t_e+1$.

Hence D_e is permanently satisfied after stage $t_e + 1$. Continuing this reasoning we see that each D_j is satisfied. □

Lemma 4. *The algebra $\mathcal{A}$ is computably categorical.*

Let $\mathcal{G}$ be a computably enumerable algebra isomorphic to $\mathcal{A}$. Fix an approximation $\mathcal{G}^0, \mathcal{G}^1, \ldots, \mathcal{G}^t, \ldots$ of $\mathcal{G}$. We need to construct a computable isomorphism h from $\mathcal{G}$ into $\mathcal{A}$. We reason as follows. Take a natural number n. Since $\mathcal{G}$ and $\mathcal{A}$ are isomorphic there must exist a stage $t+1$, and a subalgebra $\mathcal{A}_{i,t+1}$ in $\mathcal{A}_{t+1}$ which is currently isomorphic to a subalgebra $\mathcal{C}_{t+1}$ in $\mathcal{G}^{t+1}$ containing n. The subalgebra $\mathcal{A}_{i,t+1}$ is unique at this stage and cannot be homomorphically mapped onto any of the other open or partially open subalgebras of $\mathcal{A}_{t+1}$. Note that $\mathcal{C}_{t+1}$ may further collapse at a later stage while $\mathcal{G}$ is being enumerated. The image of n under the current isomorphism from $\mathcal{C}_{t+1}$ onto $\mathcal{A}_{i,t+1}$ represents an element $[n']$ in $\mathcal{A}$, and $[n']$ belongs either to an open or a partially open subalgebra of $\mathcal{A}$. Let us denote this subalgebra by $\mathcal{B}$ (so $\mathcal{A}_{i,t+1}$ collapses into $\mathcal{B}$). Consider the subalgebra $\mathcal{C}$ of $\mathcal{G}$ which contains all $[m]$ so that m is enumerated into $\mathcal{C}_{t+1}$ (hence $[n] \in C$) and all top elements y such that $g_1(y)$ belongs to the f-cycle determined by $[n]$. The algebra $\mathcal{C}$ is also either open or partially open as $\mathcal{G}$ is isomorphic to $\mathcal{A}$. We claim that $\mathcal{B}$ and $\mathcal{C}$ are isomorphic.

Indeed, if $\mathcal{A}_{i,t+1}$ never collapses into any of the algebras $\mathcal{A}_{j,t'}$ for any $t' \geq t+1$ and $j < i$, then clearly $\mathcal{B}$ is isomorphic to $\mathcal{A}_{i,t+1}$. In this case, by construction and Lemma 1 and Lemma 2 for all $t' \geq t+1$, we have the following:

1. None of the subalgebras $\mathcal{A}_{0,t'}, \mathcal{A}_{1,t'}, \ldots, \mathcal{A}_{i,t'}$ collapse into each other.
2. None of the subalgebras $\mathcal{A}_{0,t'}, \mathcal{A}_{1,t'}, \ldots, \mathcal{A}_{i,t'}$ is a homomorphic image of any other algebra in this list.
3. None of $\mathcal{A}_{j,t'}$ with $j > i$ collapses onto any of the algebras in $\mathcal{A}_{0,t'}, \mathcal{A}_{1,t'}, \ldots, \mathcal{A}_{i-1,t'}$.

Hence, the algebra $\mathcal{C}_{t+1}$ cannot be collapsed into any of the algebras $\mathcal{A}_{j,t'}$ with $t' \geq t+1$ and $j \leq i-1$. Therefore, $\mathcal{C}_{t+1}$ is in fact isomorphic to $\mathcal{C}$, and hence to $\mathcal{B}$.

Now assume that s is the last stage at which numbers in $\mathcal{A}_{i,t+1}$ are collapsed into the subalgebra $\mathcal{A}_{j,s}$ for some j. Thus, $\mathcal{B}$ is $\mathcal{A}_{j,s}$. Note that $\mathcal{A}_{j,t'}$ is open for $t' \geq s$. Let n_1 be the length of the f-cycle of $\mathcal{A}_{j,s}$. The construction guarantees that there will be no f-cycle whose length is strictly between the n_1 and the length of the f-cycle of $\mathcal{C}_{t+1}$. Hence, the only way in which $\mathcal{C}$ can be isomorphic to any of the subalgebras of $\mathcal{A}_s$ is that $\mathcal{C}$ is isomorphic to one of the algebras $\mathcal{A}_{0,s}$, …, $\mathcal{A}_{j,s}$. By the last parts of Lemma 1 and Lemma 2, the subalgebra $\mathcal{C}$ cannot be isomorphic to partially open algebras. Hence $\mathcal{C}$ is an open algebra. In addition, $\mathcal{C}$ cannot be isomorphic to open algebras among $\mathcal{A}_{0,s}$, …, $\mathcal{A}_{s-1,j}$ because, by the first part of Lemma 1 and the construction, the subalgebra $\mathcal{C}$ omits some numbers realized in these open algebras. Therefore, $\mathcal{C}$ is forced to be isomorphic to $\mathcal{A}_{s,j}$. Hence $\mathcal{C}$ and $\mathcal{B}$ are isomorphic subalgebras. In fact, we have shown that any isomorphism from $\mathcal{C}_t$ into $\mathcal{A}_{i,t+1}$ induces, in a natural way, an isomorphism from $\mathcal{C}$ onto $\mathcal{B}$.

Now we construct an isomorphism h from $\mathcal{G}$ into $\mathcal{A}$. Assume that for all $0, \ldots, n-1$, the values $h(0), \ldots, h(n-1)$ have been defined. Since $\mathcal{G}$ and $\mathcal{A}$ are isomorphic there must exist a stage $t+1$, and a subalgebra $\mathcal{A}_{i,t+1}$ in $\mathcal{A}_{t+1}$ which is currently isomorphic to the subalgebra $\mathcal{C}_t$ in $\mathcal{G}$ containing n. We now want to define the value $h(n)$. If h already maps some numbers less than n into $\mathcal{A}_{i,t+1}$ then compute a $j < n$ such that $j = n$ in $\mathcal{G}$ and set $h(n) = h(j)$. Note that such j must exist. Otherwise, effectively set up an isomorphism from $\mathcal{C}_{t+1}$ onto $\mathcal{A}_{i,t+1}$. As we have already noted above, this isomorphism induces an isomorphism from $\mathcal{C}$ into $\mathcal{B}$. Now it is clear that this mapping can be effectively extended to an isomorphism. □

Finally, we note that no c.e. presentable expansion of the algebra $\mathcal{A}$ is locally finite. Otherwise, since $\mathcal{A}$ is computably categorical, the algebra $\mathcal{A}$ would have an expansion which is not locally finite. This would contradict the fact that $tr(\mathcal{A})$ is hyperimmune. This concludes the proof of our theorem. □

At the end of this section we state the following open question: Does there exist a c.e. algebra no c.e. presentable expansion of which is computably categorical? The existence of such an algebra would show that the method of expansions is not powerful enough to control computable isomorphism types of c.e. algebras.

5 Acknowledgments

All authors acknowledge the support of the University of Heidelberg, Germany. Bakhadyr Khoussainov thanks the Marsden Fund of New Zealand and the Humboldt Foundation for their support. Steffen Lempp acknowledges support by NSF grant DMS-0140120 and a Mercator Guest Professorship of the Deutsche Forschungsgemeinschaft. Theodore A. Slaman acknowledges the support of NSF grant DMS-9988644 and the Humboldt Foundation.

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