DECIDABILITY AND UNDECIDABILITY IN THE ENUMERABLE TURING DEGREES

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Abstract. We survey some recent work on the (recursively) enumerable Turing degrees, with particular emphasis on work relating to decidability and undecidability.

Two fundamental notions of mathematics are those of a computable set and of an enumerable set. A set S is called *computable* (or *recursive*) if there is an effective algorithm which for any input x can compute whether x is an element of S. A set S is called *(recursively) enumerable* if there is an effective algorithm listing all elements of S.

Clearly, these notions only make sense for countable sets. However, all "basic" countable sets (such as the set of all k-tuples of natural numbers for some fixed k, the set of all finite strings of 0's and 1's, the set of words over some finite alphabet, or the set of first-order formulas over some finite language) are easily seen to be in effective 1–1 correspondence with the set of natural numbers. (The "code number" for an element of such a countable set is often called its *Gödel number*.) Thus, for investigating computability, one can restrict oneself to studying computability over the natural numbers.

The above intuitive notions of a computable set and an enumerable set were made precise in various equivalent ways in the mid-1930's and later. For example, Turing [Tu36] defined effective algorithm to mean what is now called a *Turing machine*, i.e., a very simple-minded computer with no limitations on run time or memory space. This (and its many equivalent definitions) is nowadays generally accepted as *the* way to define computability (see also Soare [Sota]).

There is a close connection between computability and definability in arithmetic. By Post's Theorem [Po48] (see [Kl52, p. 293]), a set is computable iff there are first-order formulas φ and ψ (in the language of arithmetic) in which all quantifiers are

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bounded such that for all x,

$$x \in S \quad \text{iff} \quad \exists y \varphi(x, y) \quad \text{iff} \quad \forall y \psi(x, y);$$

and a set is enumerable iff there is a first-order formula φ (in the language of arithmetic) in which all quantifiers are bounded such that for all x,

$$x \in S \quad \text{iff} \quad \exists y \varphi(x, y).$$

More obviously, and also historically, there is a close connection between computability/enumerability and decidability/axiomatizability. These connections often led to deep theorems, such as Matiyasevich's solution [Ma70, Ma93] to Hilbert's 10th Problem (based on previous work by Davis, J. Robinson, and others). He showed that a set S is enumerable iff there is a polynomial $p(\overline{x}, \overline{y}) \in \mathbf{Z}$ such that

$$S = \{\overline{x} \mid \exists \overline{y} p(\overline{x}, \overline{y}) = 0\}.$$

Given that most sets are noncomputable, the question arises as to how to compare them in terms of their information content, i.e., how to measure noncomputable information. The most general "effective" *reducibility* was first defined by Turing [Tu39]: A set S is *Turing reducible* to a set T (denoted by $S \leq_{\mathrm{T}} T$) if there is an *oracle Turing machine* computing S (with *oracle* T, i.e., such that the Turing machine can query membership information about the set T). This reducibility gives a prepartial ordering (i.e., a reflexive and transitive relation) on the power set of $\mathbf{N}$. We can now define two sets S and T to be *Turing equivalent* (denoted by $S \equiv_{\mathrm{T}} T$) if they are Turing reducible to each other. This gives an equivalence relation on the power set of $\mathbf{N}$. The equivalence class of a set S is called its *Turing degree* (denoted by $\deg_{\mathrm{T}} S$ or simply $\deg S$) and intuitively denotes the "information content" of the set S while stripping away all the facts about S inessential from a computational point of view, such as whether a particular number is an element of S.

These Turing degrees then form a quotient structure of the power set of $\mathbf{N}$, partially ordered by the relation induced on it by Turing reducibility. We denote this structure by $\mathbf{D}$, and the substructure of the Turing degrees of the enumerable sets by $\mathbf{E}$. Note that the latter structure can also be defined as the set of Turing degrees of solution sets of diophantine equations (by the above-mentioned result of Matiyasevich [Ma70, Ma93]) or as the set of Turing degrees of word problems of finitely presented groups (by Boone [Bo66], Clapham [Cl64], and Fridman [Fr67]).

There is one more operation on the Turing degrees that will be used later, namely, the so-called *Turing jump*. Given a set S, its jump is defined as the set $S' = \{e \mid$ eth oracle Turing machine with oracle S halts on input $e\}$. This operation can be iterated: $S^{(0)} = S$, and $S^{(n+1)} = (S^{(n)})'$, and we have $S <_{\mathrm{T}} S'$, and that $S \leq_{\mathrm{T}} T$ implies $S' \leq_{\mathrm{T}} T'$. So this jump operation induces a well-defined operation on the Turing degrees: The jump of a degree $\mathbf{a}$ is A' for any set $A \in \mathbf{a}$. (A more intuitive definition of the nth jump of a set S is that it is the set of all Σ_n-sentences (in the language of arithmetic with a unary predicate for S) true in $\mathbf{N}$.)

1. Studying the enumerable Turing degrees. This partially ordered set of the enumerable Turing degrees has received a considerable amount of attention from researchers over the past fifty years, starting with the seminal paper of Post [Po44]. There are, loosely speaking, the following aspects under which one can consider this structure:

(1) *First-order aspects:* Typical questions here include the algebraic structure of $\mathbf{E}$ as a partially ordered set, especially the investigation of its finite substructures. Many of these were resolved in the 1960's and 1970's but some key problems remain open.
(2) *Logical aspects:* The main question here is the decidability of the first-order theory and of its fragments. This was the main topic of research in the 1980's; the most interesting remaining open question is the decidability of the Π_2-theory.
(3) *Model-theoretic aspects:* This aspect concerns mainly questions of definability and of the type structure. There are some exciting recent results but much remains to be done.
(4) *Second-order aspects:* The main topic here is that of automorphisms of the structure. Once Cooper's result [Cota] about a nontrivial automorphism is fully understood, this is likely to be the most active area in the near future.

2. First-order aspects. Very early work showed that the enumerable degrees form a countable *bounded upper semilattice* (i.e., a partially ordered set with least and greatest element in which any two elements have a least upper bound). The degree $\mathbf{0}$ of the computable sets and the degree $\mathbf{0}'$ of the halting problem are the least and the greatest element, respectively; the least upper bound of the degrees of S and T is $\deg(S \oplus T) = \deg(\{2x \mid x \in S\} \cup \{2x+1 \mid x \in T\})$.

The first nontrivial result was the theorem of Friedberg [Fr57] and Muchnik [Mu56] that the structure contains incomparable elements. This early period of algebraic investigations climaxed with the Sacks Density Theorem [Sa64] that the enumerable degrees are densely ordered. This theorem in particular led Shoenfield to his famous conjecture which can be phrased as follows:

Shoenfield Conjecture [Sh65]**.** *Given any two finite bounded upper semilattices $U \subset V$, any embedding of U into $\mathbf{E}$ (as an upper semilattice) can be extended to an embedding of V into $\mathbf{E}$.*

Since this conjecture would have allowed back-and-forth constructions, it would have implied a number of "nice" results about $\mathbf{E}$, such as the saturatedness of $\mathbf{E}$ and the $\aleph_0$-categoricity and decidability of its first-order theory. However, Shoenfield's Conjecture was almost immediately shown to fail quite dramatically. We mention here three

Refutations of Shoenfield's Conjecture.

(1) (Lachlan [La66] and Yates [Ya66]) *There is a minimal pair in $\mathbf{E}$, i.e., there are nonzero degrees with infimum 0. (This precludes that any embedding of the diamond lattice as an upper semilattice can be extended to an embedding of the five-element lattice obtained by adding a new element below the two atoms of the diamond lattice.)*
(2) (Yates (unpublished) and Cooper [Co74]) *There is a noncuppable degree in $\mathbf{E}$, i.e., there is a nonzero degree $\mathbf{a}$ such that no incomplete degree $\mathbf{b}$*

joins $\mathbf{a}$ *to* $\mathbf{0}'$. *(This precludes that any embedding of the three-element linear order can be extended to an embedding of the diamond lattice.)*

(3) (Cooper, Sui, Yi [CSYta]) *There is a superminimal pair in* $\mathbf{E}$, *i.e., there is a minimal pair* $\mathbf{a}_0$ *and* $\mathbf{a}_1$ *such that for any* $i \leq 1$, *every degree* $\mathbf{x} \in (\mathbf{0}, \mathbf{a}_i]$ *joins* $\mathbf{a}_{1-i}$ *to* $\mathbf{a}_0 \cup \mathbf{a}_1$. *(This precludes that any embedding of the diamond lattice can be extended to an embedding of the six-element lattice obtained by inserting two new elements, one below one old atom, and the other as the join of the first new element and the other old atom.)*

3. Logical aspects. Shoenfield's Conjecture, even though it failed, was crucial in that it stimulated the next generation of algebraic investigations which revealed the "not so nice" structure of $\mathbf{E}$ and culminated nearly two decades later in the proof of the undecidability of its theory by Harrington and Shelah [HS82]. Soon afterwards, Harrington and Slaman (unpublished, see [SWta]) succeeded in showing that in fact the first-order theory of $\mathbf{E}$ is as complicated as possible, namely, as complicated as first-order arithmetic.

Since the proof of this result is not readily available in the literature, we will briefly sketch it here, with some later simplifications due to Slaman and Woodin [SWta] as well as Nies, Shore, and Slaman [NSSta]. Clearly, the first-order theory of the enumerable degrees can be interpreted in first-order arithmetic in the usual way. The proof in the other direction proceeds in several steps:

Step 1: We code the natural numbers with addition and multiplication by a computable partial ordering $(P, \leq)$ such that the natural numbers n are coded by the minimal elements p_n of P and such that the operations are coded, e.g., by

$$\begin{aligned} m + n = k \text{ iff } \\ &\exists p, q \in P\ (p \text{ is minimal over } p_m \text{ and } p_n, \\ &\quad \text{and } q \text{ is maximal in } P \text{ and minimal over } p \text{ and over } p_k), \end{aligned}$$

and similarly for multiplication.

Step 2: Code this partial order $(P, \leq)$ into $\mathbf{E}$ with parameters (i.e., degrees) $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ such that the elements p of P are coded by the minimal degrees $\mathbf{x} = \mathbf{x}_p \leq \mathbf{a}$ with the property that $\mathbf{c} \leq \mathbf{x} \cup \mathbf{b}$, and such that $p \leq q$ in P iff $\mathbf{x}_p \cup \mathbf{d} \leq \mathbf{x}_q \cup \mathbf{d}$. We will call such a quadruple of parameters $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ a *coded standard model of arithmetic.*

The remaining steps are now needed to sort out the coded standard models of arithmetic from other coded models of (a finite fragment of) Peano arithmetic (PA).

Step 3: For any *promptly simple* degree $\mathbf{a} \in \mathbf{E}$, there are *low* parameters below $\mathbf{a}$ coding a standard model of arithmetic. (A degree $\mathbf{a}$ is promptly simple if $\mathbf{a} > \mathbf{0}$ and there is no degree $\mathbf{b} > \mathbf{0}$ with $\mathbf{a} \cap \mathbf{b} = \mathbf{0}$.) (A degree $\mathbf{a}$ is low if $\mathbf{a}' = \mathbf{0}'$.)

Step 4: Given two coded models M_0 and M_1 of (a finite fragment of) Peano arithmetic, we want to code an *embedding* (i.e., an order-preserving injection) f from M_0 into M_1 in a similar fashion. If M_0 is a standard model of arithmetic coded by low parameters, then, for any coded model M_1 of (a finite fragment of) Peano arithmetic, such a coded map f always exists.

Step 5: We can interpret first-order arithmetic in the first-order theory of $\mathbf{E}$ as follows: Fix a sentence φ in the language of arithmetic. Then φ is true in $\mathbf{N}$ iff in **E**, the following sentence holds:

$$\begin{aligned}\forall \text{ promptly simple } \mathbf{a}\ \exists M_0 \vDash \text{ (finite fragment of) PA coded below } \mathbf{a}\\ ((\forall M_1 \vDash \text{ (finite fragment of) PA coded below } \mathbf{a}\\ \exists \text{ embedding } f: M_0 \to M_1 \text{ coded below } \mathbf{a}) \text{ and } M_0 \vDash \varphi).\end{aligned}$$

Note that the clause $(\forall M_1 \exists f \dots)$ here ensures that M_0 is a standard model of arithmetic since by Step 3, some standard model must be coded below **a** by low parameters, and by Step 4, M_0 is "more standard" than any other model of (a finite fragment of) Peano arithmetic coded below **a**.

Once undecidability of a theory has been established, the immediate next question is at what quantifier-level undecidability first occurs since mathematicians are usually only interested in statements with a small number of alternations of quantifiers.

By an old observation of Sacks [Sa63], the Π_1-theory of the enumerable degrees is decidable since any finite partial order can be embedded into **E**. By a recent result of Lempp, Nies, and Slaman [LNSta], the Π_3-theory is undecidable. This result is shown by coding finite bipartite graphs (in a language without equality) into **E** using only Σ_1-formulas with parameters, and then applying Nies's Transfer Lemma [Ni96] to transfer the hereditary undecidability of the Π_3-theory of bipartite graphs without equality to that of **E**. The gap remaining in this line of research thus is at the Π_2-theory, which we will discuss in more detail in the last section.

4. Model-theoretic aspects. The above research focusing on undecidability also led to some results concerning the type structure and questions of definability in **E**.

Lerman, Shore, and Soare [LSS84] exhibited infinitely many 3-types realized in **E** by embedding an infinite number of lattices into **E**, all generated under meet and join by three elements. This showed the non-$\aleph_0$-categoricity of the first-order theory of **E**, thus disproving another consequence of Shoenfield's Conjecture. Later, Ambos-Spies and Soare [AS89] found infinitely many 1-types realized in **E** (namely, degrees bounding n but not $n+1$ many degrees forming pairwise minimal pairs). And Ambos-Spies and Shore [AS93] showed that continuum many 1-types are consistent with the first-order theory of **E** (namely, that given any subset $S \subseteq \omega$ with at least three elements, there is a degree coding a partial ordering with maximal chain of length $k+1$ (incomparable to all other elements of the partial ordering) iff $k \in S$).

Clearly, the types of the least and the greatest element of **E** are isolated. But it is open whether there are any other isolated 1-types, and whether in fact all 1-types are isolated, i.e., whether **E** is a prime model of its theory.

This naturally leads to questions of definability. A fair number of results were shown in this respect over the years. The most exciting is probably the following recent

Theorem (Nies, Shore, Slaman [NSSta])**.** *If* $\mathbf{S} \subseteq \mathbf{E}$ *is definable in first-order arithmetic, and closed under double jump (i.e., for any degrees* $\mathbf{a}$ *and* $\mathbf{b}$*,* $\mathbf{a}'' = \mathbf{b}''$ *and* $\mathbf{a} \in \mathbf{S}$ *implies* $\mathbf{b} \in \mathbf{S}$*), then* $\mathbf{S}$ *is definable in* $\mathbf{E}$ *(in the language of partial ordering).*

This result has a number of interesting consequences; e.g., it shows the definability of the classes of the high_n and low_n enumerable degrees (for $n \geq 2$) in terms of the partial ordering only. By a small trick, they also obtained the definability of the class of the high (i.e., high_1) enumerable degrees. (They show that $\mathbf{a}$ is high iff for any $\mathbf{b}$ there is $\mathbf{c} \leq \mathbf{a}$ with $\mathbf{b}'' = \mathbf{c}''$. Here, an enumerable degree $\mathbf{a}$ is *low_n* if $\mathbf{a}^{(n)} = \mathbf{0}^{(n)}$, and *$high_n$* if $\mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}$.)

Obviously, the least and the greatest element of $\mathbf{E}$ are definable. It is open whether any other enumerable degrees are definable in $\mathbf{E}$ (in the language of partial ordering without parameters). The following question addresses an interesting partial result in this direction which seems more accessible to current methods:

Question (Li Angsheng (see [Sl])). *Are there degrees $\mathbf{a} < \mathbf{b} < \mathbf{c}$ in $\mathbf{E}$ and a formula $\varphi(x)$ in the language of partial ordering without parameters such that for all enumerable degrees $\mathbf{x} \in (\mathbf{a}, \mathbf{c})$,*

$$(\mathbf{E}, \leq) \models \varphi(\mathbf{x}) \quad \textit{iff} \quad \mathbf{x} = \mathbf{b}?$$

The strongest possible definability result would be the following

Biinterpretability Conjecture (Slaman, Woodin [Sl91]). *There is a map f from $\mathbf{E}$ into a standard model coded in $\mathbf{E}$ such that for all $\mathbf{a} \in \mathbf{E}$, $\deg W_{f(\mathbf{a})} = \mathbf{a}$.*

This conjecture would have implied in particular the *rigidity* of $\mathbf{E}$ (i.e., that the only automorphism of $\mathbf{E}$ is the identity). This consequence, and thus the Biinterpretability Conjecture, was refuted by Cooper as described in the next section.

5. Second-order aspects. The earliest results on automorphisms of $\mathbf{E}$ concerned *automorphism bases* (i.e., sets $\mathbf{S} \subseteq \mathbf{E}$ such that any automorphism which is the identity on $\mathbf{S}$ must be the identity on $\mathbf{E}$). A number of nontrivial automorphism bases were found, mainly in the 1980's. An interesting recent result here is due to Ambos-Spies [Amta] that any nontrivial initial segment of $\mathbf{E}$ forms an automorphism base.

However, the question of whether there are any automorphisms of $\mathbf{E}$ other than the identity remained open until Cooper's recent results [Cota] about the existence of such automorphisms. He also showed that there is an automorphism mapping a low to a nonlow enumerable degree. Thus the low enumerable degrees are not definable in $\mathbf{E}$ from the partial ordering alone, in contrast to all other jump classes as mentioned above.

Once a full proof of Cooper's results is available, a whole number of questions will arise: How many automorphisms are there (e.g., are there continuum many)? Are all automorphisms arithmetical? Is there a finite automorphism base?

6. The Π_2-theory. The main open question about the enumerable Turing degrees at this time, accessible to currently available methods, is in our opinion the decidability of the Π_2-theory of the enumerable degrees. It is not hard to see that the Π_2-theory can be rephrased in purely algebraic terms as follows:

Equivalent formulation of the decidability of the Π_2-theory. *Decide if given any finite partial orders $P \subseteq Q_0, \ldots, Q_n$ (for some $n \geq 0$), any embedding of P into $\mathbf{E}$ can be extended to an embedding of Q_i into $\mathbf{E}$ for some $i \leq n$. (Note here that i may depend on the embedding of P into $\mathbf{E}$.)*

A natural subproblem of the above is obtained by setting $i = 0$, i.e., deciding whether any embedding of a finite partial order P into $\mathbf{E}$ can be extended to an embedding of a finite partial order $Q \supset P$ into $\mathbf{E}$. (This is usually called the *extension of embeddings problem.*) A solution to this problem was given by Slaman and Soare, which we state in a modified version due to Lempp and Lerman:

Extension of Embeddings Theorem (Slaman, Soare [SS95], rephrased). *Fix a finite lattice P and a finite upper semilattice Q extending P as an upper semilattice and respecting the lattice structure of P. Then any embedding of P into $\mathbf{E}$ (as a lattice) can be extended to an embedding of Q into $\mathbf{E}$ (as an upper semilattice) iff*

$$\forall a, b \in P\, \forall x \in Q - P (a = \min\{c \in P \mid c > x\} \;\&\; x \not\leq b \rightarrow x \vee b = a \vee b).$$

Lerman calls the above the "Saturation Axiom". It is a generalization of the phenomenon encountered in the superminimal pair mentioned earlier. Lerman [Le96] then suggests the following general approach to deciding the Π_2-theory: Expand the language of partial ordering to include

- the language of bounded upper semilattices (i.e., $\leq$, $\vee$, 0, and 1);
- $(n+2)$-ary meet predicates $M(a, b_0, \dots, b_n)$ (for all $n \geq 1$): This denotes that all $x \leq b_0, \dots, b_n$ are also $\leq a$ and takes into account that the meet of two degrees need not always exist;
- saturation predicates generalizing the phenomenon of the Saturation Axiom mentioned above; and
- a unary predicate for the so-called promptly simple degrees. (The class of the promptly simple degrees was shown by Ambos-Spies, Jockusch, Shore, and Soare [AJSS84] to coincide with a number of other interesting classes (such as the degrees which do not form one half of a minimal pair). Prompt simplicity interacts nontrivially with saturation, e.g., Cooper, Slaman, and Yi (unpublished) observed that saturation cannot occur between a promptly simple degree $\mathbf{a}$ and a non-promptly simple degree $\mathbf{b}$, namely, given any such degrees $\mathbf{a}$ and $\mathbf{b}$, there is an enumearable degree $\mathbf{x} < \mathbf{a}$ with $\mathbf{x} \not\leq \mathbf{b}$ and $\mathbf{x} \cup \mathbf{b} < \mathbf{a} \cup \mathbf{b}$.)

Lerman's approach then consists in deciding the Π_1-theory in this expanded language in order to give a decision procedure for the Π_2-theory in the language of partial ordering.

Lerman's approach thus highlights another natural subproblem, the so-called *Lattice Embeddings Problem*, namely, to characterize the finite lattices which are embeddable into $\mathbf{E}$. This is actually a very old open problem going back to the early investigations in the 1960's of the algebraic structure of $\mathbf{E}$ and, in particular, of its finite substructures.

Of course, the minimal pair theorem of Lachlan and Yates mentioned earlier yields an embedding of the diamond lattice into $\mathbf{E}$. This was soon generalized to showing that all finite distributive lattices are embeddable into $\mathbf{E}$ by Lerman (unpublished) and Thomason [Th71]. Lachlan [La72] proved that the two nondistributive five-element lattices M_5 and N_5 are embeddable into $\mathbf{E}$. However, Lachlan and Soare [LS80] exhibited a finite lattice, S_8, which is not embeddable into $\mathbf{E}$. The known embedding and non-embedding techniques up to the late 1980's were then distilled into two conditions (the so-called *Embeddability Condition (EC)* and *Nonembeddability Condition (NEC)*) by Ambos-Spies and Lerman [AL86, AL89].

EC turned out to be a rather unwieldy condition and received little attention. NEC, on the other hand, was a nice, algebraic condition and was conjectured by many to be the correct condition characterizing exactly the non-embeddable finite lattices. In particular, NEC requires the existence of a so-called critical triple in the lattice.

Definition. A triple $\langle a, b, c\rangle$ of elements of a finite lattice L is called a *critical triple* if a, b, and c are pairwise-incomparable, $a \vee b = a \vee c$, and $b \wedge c \leq a$.

Ambos-Spies and Lerman [AL86] observed that in a finite lattice, the absence of critical triples is equivalent to another property which is easier to verify.

Proposition. *A finite lattice L with least element 0 fails to have critical triples if for all $a < d$ in L such that the interval (a, d) is empty, the difference of the intervals $[0, d] - [0, a]$ has a (unique) least element.*

The above proposition allows one to organize the enumeration of elements for an embeddings proof for a lattice without critical triple much more easily, so the absence of critical triples was conjectured by many to ensure the embeddability of a finite lattice. This was recently refuted, however, by Lempp and Lerman [LLta], who exhibited a finite lattice, L_{20}, which is not embeddable into the enumerable degrees but does also not contain a critical triple.

The search for a characterization of the finite lattices embeddable into the enumerable degrees continues and is likely to be hard; it involves analyzing the obstructions to embeddability in a typical pinball machine proof and using them to produce a nonembeddability proof in an effective fashion if possible. Only then, a decision procedure for the Π_2-theory of the enumerable degrees can reasonably be attempted.

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