ON THE STRUCTURE OF THE ZIEGLER DEGREES

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Abstract. We first sketch the construction of a minimal degree under Ziegler reducibility (called $*$-reducibility by Ziegler), arising from his study of existentially closed groups and their finitely generated subgroups. We then extend this to show that every finite distributive lattice is an initial segment of the Ziegler degrees and deduce the undecidability of the $\forall\exists\forall$-theory of the Ziegler degrees (in the language of partial ordering).

1. Introduction

1.1. **The Theorem.** Ziegler [Zi80] defined what he called $*$-*reducibility*, and what we call *Ziegler reducibility*, in his comprehensive study of the finitely generated subgroups of existentially closed groups. This reducibility is a common refinement of both Turing and enumeration reducibility. In particular, he showed that if a finitely generated group H is a subgroup of an existentially closed group G and the word problem of a finitely generated group H_0 is Ziegler reducible to the word problem of H, then H_0 is (isomorphic to) a subgroup of G as well.

In this paper, we undertake what we believe to be the first systematic degree-theoretic study of the induced degree structure $\mathcal{D}^*$, the *Ziegler degrees*. We first show that there is a minimal Ziegler degree and then extend this to show that every finite distributive lattice is an initial segment of the Ziegler degrees. This allows us to conclude the undecidability of the $\forall\exists\forall$-theory of the Ziegler degrees (in the language of partial ordering).

Our formal results are as follows:

Theorem 1.1. (1) *There is a minimal Ziegler degree.*
(2) *There is an initial segment of the Ziegler degree isomorphic to the 3-element chain.*
(3) *Any finite distributive lattice is an initial segment of the Ziegler degrees.*
(4) *The $\forall\exists\forall$-theory of the Ziegler degrees (in the language of partial ordering) is undecidable.*

Our constructions for Theorem 1.1(1)-(3) follow the general outline of the corresponding embedding constructions for the Turing degrees, with some notable differences due to special features of Ziegler reducibility. We will present these in sections 2 and 4, with a brief sketch of how to embed the 3-element chain in section 3 as a warmup for the full embedding construction. We recommend readers to familiarize themselves with the Turing degree constructions before reading these. Finally, in section 5, we will present the proof of Theorem 1.1(4).

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1.2. **Preliminaries.** We first remind readers of some relevant definitions.

Definition 1.2. We use the notation A^c for the complement $\omega \setminus A$ of a set $A \subseteq \omega$.

- A set $A \subseteq \omega$ is *enumeration reducible* to a set $B \subseteq \omega$ (written $A \leq_e B$) if there is a c.e. set V such that
$$A = \{n \in \omega \mid \exists \langle n, i\rangle \in V\, [D_i \subseteq B]\}.$$
Here D_i is the finite set with *canonical index* i, namely,
$$i = \sum_{x \in D_i} 2^x.$$
- A set $C \subseteq \omega$ is *1-enumeration reducible* to a set $B \subseteq \omega$ (written $C \leq_e^1 B$, see Ziegler [Zi80, Definition II.3.11]) if there is a c.e. set W such that
$$C = \{n \in \omega \mid \exists \langle n, i, j\rangle \in W\, [D_i \subseteq B \wedge D_j \subseteq B^c \wedge |D_j| \leq 1]\}.$$
- A set $A \subseteq \omega$ is *Ziegler reducible* to a set $B \subseteq \omega$ (written $A \leq^* B$, see Ziegler [Zi80, Definition III.1.1]) if $A \leq_e B$ and $A^c \leq_e^1 B$.
- We say $A \leq^* B$ *via* $\langle e, i\rangle$ if $A \leq_e B$ via a c.e. set W_e, and $A^c \leq_e^1 B$ via a c.e. set W_i, and we write $A = (\Phi_e, \Psi_i)[B]$.

So, in words, $A \leq_e B$ holds if $n \in A$ can be enumerated from a finite collection of positive facts about B; and $A \leq^* B$ holds if $n \in A$ can be enumerated from a finite collection of positive facts about B, and $n \notin A$ can be enumerated from the conjunction of a finite collection of positive facts about B and at most one negative fact about B. Note that $A \leq_m B$ implies $A \leq^* B$; and that $A \leq^* B$ implies both $A \leq_T B$ and $A \leq_e B$. (In fact, all these implications are strict.)

2. A minimal Ziegler degree

We will make use of uniform f-trees and strongly uniform f-trees as defined in Lerman [Le83, Chapter VI], based on earlier work by Lachlan [La68] and Hugill [Hu69] as well as Lachlan [La71]. This will be critical in ensuring that we achieve $*$-minimality rather than just Turing minimality in the "splitting" stage. (In the next section, in Definition 3.2, we will have to weaken the notion of being strongly uniform slightly to being "very uniform".)

Definition 2.1. (1) An *f-tree* is a map $\mathcal{T} : 2^{<\omega} \to 2^{<\omega}$ such that for all $\pi, \rho \in 2^{<\omega}$, $\pi \prec \rho$ iff $\mathcal{T}(\pi) \prec \mathcal{T}(\rho)$ and such that $\pi <_{\text{lex}} \rho$ iff $\mathcal{T}(\pi) <_{\text{lex}} \mathcal{T}(\rho)$.

(2) A *sub-f-tree* of an f-tree $\mathcal{T}$ is an f-tree $\mathcal{S}$ with $\operatorname{ran}(\mathcal{S}) \subseteq \operatorname{ran}(\mathcal{T})$.

(3) A *uniform* f-tree is an f-tree $\mathcal{T} : 2^{<\omega} \to 2^{<\omega}$ satisfying:
- for all $\pi, \rho \in 2^{<\omega}$, $|\pi| = |\rho|$ implies $|\mathcal{T}(\pi)| = |\mathcal{T}(\rho)|$; and
- for all $n \in \omega$ and $k < 2$, there is τ_k^n such that for all $\rho \in 2^n$, $\mathcal{T}(\rho^\frown k) = \mathcal{T}(\rho)^\frown \tau_k^n$.

(4) A *strongly uniform* f-tree is a uniform f-tree $\mathcal{T} : 2^{<\omega} \to 2^{<\omega}$ such that for all $n \in \omega$ and $k < 2$, τ_0^n and τ_1^n (as defined above) differ on exactly one bit.

(5) A uniform f-tree $\mathcal{T}$ *has no explicit* $\langle e, i\rangle$*-contradictions* if, for every $\tau \in \operatorname{ran}(\mathcal{T})$ and $x \in \omega$, it is not the case that $\Phi_e[\tau](x) = 1$ and $\Psi_i[\tau](x) = 1$ (i.e., x is not enumerated into both the final set and its complement).

(6) τ_0 and τ_1 *witness that* $\mathcal{T}$ $\langle e, i\rangle$*-splits at* σ if τ_0, τ_1, $\sigma \in \operatorname{ran}(\mathcal{T})$; τ_0, $\tau_1 \succ \sigma$; and
$$x \notin (\Phi_e, \Psi_i)[\tau_0] \text{ and } x \in (\Phi_e, \Psi_i)[\tau_1].$$

(7) A uniform f-tree $\mathcal{T}$ is *$\langle e, i\rangle$-splitting* if, for every $\rho \in \mathrm{dom}(\mathcal{T})$, $\mathcal{T}(\rho^\frown 0)$ and $\mathcal{T}(\rho^\frown 1)$ witness that $\mathcal{T}(\rho)$ $\langle e, i\rangle$-splits.

Observe that if τ_0, τ_1 witness that $\mathcal{T}(\rho)$ $\langle e, i\rangle$-splits (and has no explicit $\langle e, i\rangle$-contradictions), then we can find τ_0', τ_1' that differ only on one bit and witness that $\mathcal{T}(\rho)$ $\langle e, i\rangle$-splits. This is because the negative part Ψ_i of a Ziegler operator only depends on one bit of negative information. So, with one exception, we can move all the segments of τ_0 and τ_1 to the right to obtain τ_0', τ_1'. By our definition of strongly uniform f-tree, if τ_0', τ_1' differ only on one segment then they differ only in one bit. Furthermore, observe that we could not have extended the definition of explicit $\langle e, i\rangle$-contradictions to both computations giving 0 since that cannot be determined by finite strings.

We will make use of the following lemma, whose proof is modeled on [Le83, Theorem V.2.7], which guarantees that if $\mathcal{T}$ is a computable f-tree which has no cone without $\langle e, i\rangle$-splittings or explicit $\langle e, i\rangle$-contradictions, then we can find a computable sub-f-tree which is $\langle e, i\rangle$-splitting.

Lemma 2.2. *Let $\mathcal{T}$ be a computable strongly uniform f-tree, and let $e, i \in \omega$. If $\mathcal{T}$ has no explicit $\langle e, i\rangle$-contradictions and for every $\sigma \in \mathrm{ran}(\mathcal{T})$, there are τ_0, $\tau_1 \in \mathrm{ran}(\mathcal{T})$ which witness that σ $\langle e, i\rangle$-splits, then there is a computable strongly uniform $\langle e, i\rangle$-splitting sub-f-tree $\mathcal{S}$ of $\mathcal{T}$.*

Proof. We proceed by induction on n. Set $\mathcal{S}(\langle\rangle) = \mathcal{T}(\langle\rangle)$. For $n > 0$, let τ_0 and τ_1 witness that $\mathcal{T}$ $\langle e, i\rangle$-splits at $\mathcal{S}(0^{n-1})$. As observed above, we may assume that τ_0 and τ_1 differ only on one bit. Suppose that for $k < 2$, we have $\tau_k = \mathcal{S}(0^{n-1})^\frown\rho_k$. Then, for each $\pi \in 2^{n-1}$ and $k < 2$, we define $\mathcal{S}(\pi^\frown k) = \mathcal{S}(\pi)^\frown\rho_k$.

We now verify that $\mathcal{S}$ is indeed a computable $\langle e, i\rangle$-splitting strongly uniform f-tree. Clearly, $\mathcal{S}$ is a computable strongly uniform f-tree. Suppose that n is minimal such that for some $\rho \in 2^n$, $\mathcal{S}(\rho^\frown 0)$ and $\mathcal{S}(\rho^\frown 1)$ do not witness that $\mathcal{S}(\rho)$ $\langle e, i\rangle$-splits. Recall, however, by construction, that $\mathcal{S}(0^{n\frown}0)$ and $\mathcal{S}(0^{n\frown}1)$ do witness that $\mathcal{S}(0^n)$ $\langle e, i\rangle$-splits, say, with witness argument x, so $x \notin (\Phi_e, \Psi_i)[\mathcal{S}(0^{n\frown}0)]$ and $x \in (\Phi_e, \Psi_i)[\mathcal{S}(0^{n\frown}1)]$.

Suppose first, for a contradiction, that the computation $x \notin (\Phi_e, \Psi_i)[\mathcal{S}(0^{n\frown}0)]$ uses a negative bit in $\mathcal{S}(0^{n\frown}0)$ of length $< |\mathcal{S}(0^n)|$: Then $x \in \Psi_i[\mathcal{S}(0^{n\frown}1)]$, but also $x \in \Phi_e[\mathcal{S}(0^{n\frown}1)]$, contradicting there not being any explicit $\langle e, i\rangle$-contradictions.

So the computation $x \notin (\Phi_e, \Psi_i)[\mathcal{S}(0^{n\frown}0)]$ uses at most a negative bit in $\mathcal{S}(0^{n\frown}0)$, which is of length $\geq |\mathcal{S}(0^n)|$. But then, by the monotonicity of both Φ_e and Ψ_i and the strong uniformity of $\mathcal{T}$, we have $x \notin (\Phi_e, \Psi_i)[\rho^\frown 0]$ and $x \in (\Phi_e, \Psi_i)[\rho^\frown 1]$ as desired. □

We can now prove the first part of our main theorem, the existence of a minimal Ziegler degree:

Proof of Theorem 1.1(1). The proof closely follows the classical proof of the existence of a minimal degree in the Turing degrees. At stage 0, let $\mathcal{T}_0$ be the identity f-tree.

At odd stages $s + 1 = 2e + 1$, the requirements for $A >^* \emptyset$ are handled as usual by going to a full sub-f-tree in order to ensure that $A \neq \varphi_e$ for all e. (Recall that $A \equiv^* \emptyset$ is equivalent to $A \equiv_T \emptyset$.) More precisely, we set $\mathcal{T}_{s+1}(\rho) = \mathcal{T}_s(k^\frown\rho)$ for some $k < 2$ and all $\rho \in 2^{<\omega}$, where φ_e and $\mathcal{T}_s(k)$ disagree. (Notice that this preserves strong uniformity.)

At even stages $s = 2\langle e, i\rangle + 2$, we satisfy the requirement that if $(\Phi_e, \Psi_i)[A]$ is a valid computation, then either it $*$-computes A or is computable.

Case 1: There is $\sigma \in \operatorname{ran}(\mathcal{T}_s)$ such that there is $x \in \omega$ with $x \in \Phi_e[\sigma]$ and $x \in \Psi_i[\sigma]$. In this case, go to the full sub-f-tree above σ. More precisely, let $\pi \in 2^{<\omega}$ with $\mathcal{T}_s(\pi) = \sigma$ and set $\mathcal{T}_{s+1}(\rho) = \mathcal{T}_s(\pi^\frown\rho)$ for all $\rho \in 2^{<\omega}$. (Notice that this preserves strong uniformity.) This ensures that $(\Phi_e, \Psi_i)[A]$ is not a valid computation for any branch A on $\mathcal{T}_{s+1}$.

Case 2: Case 1 does not hold and there is $\sigma \in \operatorname{ran}(\mathcal{T}_s)$ with no τ_0 and τ_1 extending $\sigma \in \operatorname{ran}(\mathcal{T})$ which witness $\langle e, i\rangle$-splitting. Let $\mathcal{T}_{s+1}$ be the full sub-f-tree above σ. This will ensure that if $(\Phi_i, \Psi_j)[A]$ is a valid computation, then the result is computable.

Case 3: Neither Case 1 nor Case 2 holds. Then for every $\sigma \in \operatorname{ran}(\mathcal{T}_s)$, there are $\tau_0, \tau_1 \succ \sigma$ in $\operatorname{ran}(\mathcal{T}_s)$ such that τ_0 and τ_1 are incomparable and witness an $\langle e, i\rangle$-splitting. Define $\mathcal{T}_{s+1}$ to be the computable, strongly uniform $\langle e, i\rangle$-splitting sub-f-tree of $\mathcal{T}$ as given by Lemma 2.2.

Note that by the odd steps, $\lim_s |\mathcal{T}_s(\langle\rangle)| = \infty$, and clearly the $\mathcal{T}_s$ form a descending sequence of f-trees, so we can define $A = \bigcup_s \mathcal{T}_s(\langle\rangle)$. We will show that A has minimal Ziegler degree. The same argument as for the Turing degrees shows that $A \not\equiv^* \emptyset$. We now show that, for every e and i, $(\Phi_e, \Psi_i)[A]$ satisfies one of the following, by our action at stage $s = 2\langle e, i\rangle + 2$ of the construction:

- $(\Phi_i, \Psi_j)[A]$ is not a valid computation, or
- $(\Phi_i, \Psi_j)[A]$ is computable, or
- $(\Phi_i, \Psi_j)[A] \geq^* A$.

In Case 1 of Stage s, the computation $(\Phi_i, \Psi_j)[A]$ does not $*$-compute any set since there is some x computed to be both in it and not in it.

In Case 2, if $(\Phi_e, \Psi_i)[A]$ computes a set B, then $B \equiv_T \emptyset$ and thus $B \equiv^* \emptyset$ (unless there is x which is neither computed to be in the set nor its complement; however, in that case, $(\Phi_e, \Psi_i)[A]$ is not a valid computation). To see this, note that to check if $x \in B$, find $\sigma \in \operatorname{ran}(\mathcal{T}_{s+1})$ with $(\Phi_e, \Psi_i)[\sigma](x)$ defined (which must exist since $(\Phi_e, \Psi_i)[A](x)$ is defined). If there were $\tau \in \operatorname{ran}(\mathcal{T}_{s+1})$ with $(\Phi_e, \Psi_i)[\sigma](x) \neq (\Phi_e, \Psi_i)[\tau](x)$, this would contradict the hypothesis of Case 2; thus $(\Phi_e, \Psi_i)[\sigma](x) = (\Phi_e, \Psi_i)[A](x)$.

Finally, in Case 3, if the pair (Φ_e, Ψ_i) does $*$-compute a set B, say, then we need to show that $A \leq^* B$. Given $x \in \omega$, we first fix $k \in \omega$ with $|\mathcal{T}_{s+1}(0^k)| \leq x < |\mathcal{T}_{s+1}(0^{k+1})|$. (If $x < |\mathcal{T}_{s+1}(\langle\rangle)|$, then $A(x) = \mathcal{T}_{s+1}(\langle\rangle)(x)$ is outright computable.)

We first verify that $A \leq_e B$. Namely, $x \in A$ iff

$$\begin{aligned}
&\Big(\mathcal{T}_{s+1}(0^k 1)(x) = \mathcal{T}_{s+1}(0^{k+1})(x) = 1\Big) \vee \\
&\qquad \exists y \in B\, \exists u\, \exists v\, \exists w \Big(\langle y, u\rangle \in W_e \wedge D_u \subseteq \mathcal{T}(0^k 1) \wedge \\
&\qquad\qquad \langle y, v, w\rangle \in W_i \wedge D_v \subseteq \mathcal{T}(0^{k+1}) \wedge |D_w| \leq 1 \wedge \\
&\qquad\qquad\quad [D_w \neq \emptyset \to \max(D_w) < |\mathcal{T}(0^{k+1})|] \wedge D_w \cap \mathcal{T}(0^{k+1}) = \emptyset\Big).
\end{aligned}$$

In words, $x \in A$ if either $\mathcal{T}_{s+1}(\sigma)(x) = 1$ for all σ for which $\mathcal{T}_{s+1}(\sigma)(x)$ is defined; or if x is at the "splitting level" where $\mathcal{T}_{s+1}(0^{k+1})$ and $\mathcal{T}_{s+1}(0^k 1)$ witness the $\langle e, i\rangle$-splitting with witness y, enumerating y into the complement of the computation and into the computation, respectively, and $y \in B$.

Furthermore, we verify that $A^c \leq_e^1 B$: Namely, $x \in A^c$ iff

$$\Big(\mathcal{T}_{s+1}(0^k1)(n) = \mathcal{T}_{s+1}(0^{k+1})(n) = 0\Big) \vee$$
$$\exists y \notin B\, \exists u\, \exists v\, \exists w\, \Big(\langle y, u\rangle \in W_e \wedge D_u \subseteq \mathcal{T}(0^k1) \wedge$$
$$\langle y, v, w\rangle \in W_i \wedge D_v \subseteq \mathcal{T}(0^{k+1}) \wedge |D_w| \leq 1 \wedge$$
$$[D_w \neq \emptyset \rightarrow \max(D_w) < |\mathcal{T}(0^{k+1})|] \wedge D_w \cap \mathcal{T}(0^{k+1}) = \emptyset\Big).$$

In words, $x \in A^c$ if either $\mathcal{T}_{s+1}(\sigma)(x) = 0$ for all σ for which $\mathcal{T}_{s+1}(\sigma)(x)$ is defined; or if x is at the "splitting level" where $\mathcal{T}_{s+1}(0^{k+1})$ and $\mathcal{T}_{s+1}(0^k1)$ witness the $\langle e, i\rangle$-splitting with witness y, enumerating y into the complement of the computation and into the computation, respectively, and $y \notin B$. □

Remark 2.3. As Downey has pointed out, the set obtained here is also of minimal m-degree.

3. The 3-element chain

As a warmup for the next section, we now modify the above construction to build a 3-element chain as an initial segment in the Ziegler degrees. The idea of the construction, which for the Turing degrees can be found in Odifreddi [Od89, Chapter V.6], is to build a set A such that the *even part* of A has minimal degree, and such that, for every valid computation $(\Phi_e, \Psi_i)[A]$, the resulting set is either computable, $*$-equivalent to the even part of A, or $*$-computes A. (Odifreddi uses the odd part instead of the even part in his presentation; but to be compatible with the next section, we switch the even and odd parts.) So the *even part* of A will represent the middle element in the 3-element chain.

Definition 3.1. For $A \subseteq \omega$,

$$\mathrm{Even}(A) = \{x \mid 2x \in A\}.$$

As in the previous section, the embedding result for the 3-element chain will be proved by building a decreasing sequence of computable uniform f-trees. We first prove the necessary lemmas which establish that it is possible to build such a sequence. However, we will have to give up strong uniformity, so showing that $(\Phi_e, \Psi_i)[A]$ is $*$-computable from another set will be more complicated as for the negative information, we are allowed only one negative query; we will get around this difficult by imposing the restriction that any disagreements on a segment have to be the same, as stated in the following definition.

Definition 3.2. Recalling the notation from Definition 2.1, we define a *very uniform* f-tree to be a uniform f-tree $\mathcal{T} : 2^{<\omega} \rightarrow 2^{<\omega}$ satisfying that for all $n \in \omega$, $k < 2$, and $x < |\tau_k^n|$, if $\tau_0^n(x) \neq \tau_1^n(x)$ then $\tau_0^n(x) < \tau_1^n(x)$. (Thus any disagreements on any fixed τ_k^n "agree" with each other.)

Lemma 3.3. *Let $e \in \omega$ and let $\mathcal{T}$ be a computable very uniform f-tree such that for some σ,* $\mathrm{Even}(\mathcal{T}(\sigma^\frown 0)) \neq \mathrm{Even}(\mathcal{T}(\sigma^\frown 1))$. *Then there is a computable very uniform sub-f-tree $\mathcal{S}$ of $\mathcal{T}$ such that for every A on $\mathcal{T}$,*

$$\mathrm{Even}(A) \neq \varphi_e.$$

Proof. The proof is the same as for Proposition V.6.11 from [Od89]. □

Lemma 3.4. *Let $e, i \in \omega$ and let $\mathcal{T}$ be a computable very uniform f-tree such that for some $\sigma \in 2^{<\omega}$, $\mathrm{Even}(\mathcal{T}(\sigma^\frown 0)) = \mathrm{Even}(\mathcal{T}(\sigma^\frown 1))$. Then there is a computable very uniform sub-f-tree $\mathcal{S} \subseteq \mathcal{T}$ such that for every A on $\mathcal{S}$, $(\Phi_e, \Psi_i)[\mathrm{Even}(A)] \neq A$ (or $(\Phi_e, \Psi_i)[\mathrm{Even}(A)]$ is not even a valid computation).*

Proof. The proof is essentially the same as for Proposition V.6.12 from [Od89]. □

Put together, these lemmas suggest that we will be interested in f-trees which have cofinally many levels which disagree on their even parts and cofinally many levels which agree on their even parts. The simplest way to ensure this is to require that the levels of $\mathcal{T}$ alternate between agreeing and disagreeing on their even parts.

Definition 3.5. A very uniform f-tree $\mathcal{T}$ is *alternating* if, for every $\rho \in 2^{<\omega}$,

- if $|\rho|$ is even then $\mathrm{Even}(\mathcal{T}(\rho^\frown 0)) \neq \mathrm{Even}(\mathcal{T}(\rho^\frown 1))$; and
- if $|\rho|$ is odd then $\mathrm{Even}(\mathcal{T}(\rho^\frown 0)) = \mathrm{Even}(\mathcal{T}(\rho^\frown 1))$.

Lemma 3.6. *Let $e, i \in \omega$ and let $\mathcal{T}$ be a computable very uniform f-tree such that every $\langle e, i\rangle$-splitting is witnessed only by sequences which have different even parts. Then for every A on $\mathcal{T}$ such that $(\Phi_e, \Psi_i)[A]$ is a valid computation,*

$$(\Phi_e, \Psi_i)[A] \leq^* \mathrm{Even}(A).$$

Proof. Let $B = (\Phi_e, \Psi_i)[A]$. We first show that $B \leq_e \mathrm{Even}(A)$. This follows since by our assumption on $\langle e, i\rangle$-splittings being witnessed only by sequences which have different even parts,

$$x \in B \iff \exists u \left(\langle x, u\rangle \in W_e \wedge \mathrm{Even}(D_u) \subseteq \mathrm{Even}(A)\right).$$

Similarly, $B^c \leq_e^1 \mathrm{Even}(A)$ because, again by our assumption on $\langle e, i\rangle$-splittings being witnessed only by sequences which have different even parts,

$$x \notin B \iff \exists v, w\big(\langle x, v, w\rangle \in W_i \wedge \mathrm{Even}(D_v) \subseteq \mathrm{Even}(A) \wedge |D_w| \leq 1 \wedge \mathrm{Even}(D_w) \cap \mathrm{Even}(A) = \emptyset\big).$$

□

Lemma 3.7. *Let $e, i \in \omega$ and let $\mathcal{T}$ be a computable very uniform f-tree such that for every $\rho \in 2^{<\omega}$, if $\mathrm{Even}(\mathcal{T}(\rho^\frown 0)) \neq \mathrm{Even}(\mathcal{T}(\rho^\frown 1))$, then $\mathcal{T}(\rho^\frown 0)$ and $\mathcal{T}(\rho^\frown 1)$ $\langle e, i\rangle$-split. Then for every A on $\mathcal{T}$ for which $(\Phi_e, \Psi_i)[A]$ is a valid computation,*

$$\mathrm{Even}(A) \leq^* (\Phi_e, \Psi_i)[A].$$

Proof. Let $B = (\Phi_e, \Psi_i)[A]$. We first check that $\mathrm{Even}(A) \leq_e B$. Fix $x \in \omega$. If there is $k < 2$ such that every $\sigma \in \mathrm{ran}(\mathcal{T})$ of minimal length $> 2x$ has $\sigma(2x) = k$, then $\mathrm{Even}(A)(x) = k$. Otherwise, there are $\sigma, \tau \in \mathrm{ran}(\mathcal{T})$ (of minimal length) such that $\sigma(2x) \neq \tau(2x)$. Since $\mathcal{T}$ is very uniform and σ, τ were chosen of minimal length, we may in addition assume that there is $\rho \in 2^{<\omega}$ such that $\mathcal{T}(\rho^\frown 0) = \sigma$ and $\mathcal{T}(\rho^\frown 1) = \tau$. Thus, by assumption on $\mathcal{T}$, σ and τ $\langle e, i\rangle$-split. By the very uniformity of $\mathcal{T}$, we have $\sigma(2x) = 0$. Search for y witnessing this splitting. Since $\sigma(2x) = 0$, it must be that $(\Phi_e, \Psi_i)[\sigma](y) = 0$. Then $x \in \mathrm{Even}(A)$ iff $B(y) = 1$.

Similarly, if there are $\sigma, \tau \in \mathrm{ran}(\mathcal{T})$ (of minimal length) such that $\sigma(2x) \neq \tau(2x)$, then under the same hypotheses as in the previous paragraph, we have $x \notin \mathrm{Even}(A)$ iff $B(y) = 0$; otherwise $A(2x)$ is computable. So $(\mathrm{Even}(A))^c \leq_e^1 B$. □

Thus, by the previous two lemmas, if A is on a computable very uniform f-tree $\mathcal{T}$ such that for all $\sigma \in 2^{<\omega}$, $\mathcal{T}(\sigma^\frown 0)$ and $\mathcal{T}(\sigma^\frown 1)$ $\langle e, i\rangle$-split iff they disagree on their even parts, and if $(\Phi_e, \Psi_i)[A]$ is a valid computation, then $\mathrm{Even}(A) \equiv^* (\Phi_e, \Psi_i)[A]$.

Recall that $\mathcal{T}$ being an alternating very uniform f-tree implies that the levels of $\mathcal{T}$ alternate between agreement and disagreement on their even parts. The next two lemmas ensure that we can build computable alternating $\langle e, i\rangle$-splitting very uniform f-trees.

Lemma 3.8. *Let $e, i \in \omega$ and let $\mathcal{T}$ be a computable very uniform alternating f-tree such that for some $\rho \in 2^{<\omega}$, for every $\sigma \succeq \mathcal{T}(\rho)$ in the range of $\mathcal{T}$, there is an $\langle e, i\rangle$-splitting above σ, and for each such $\langle e, i\rangle$-splitting above σ, the even parts disagree. If there are no explicit $\langle e, i\rangle$-contradictions on $\mathcal{T}$, then there is a computable very uniform alternating sub-f-tree $\mathcal{S}$ of $\mathcal{T}$ such that for all $\pi \in 2^{<\omega}$ with $|\pi|$ even, $\mathcal{S}(\pi^\frown 0)$ and $\mathcal{S}(\pi^\frown 1)$ $\langle e, i\rangle$-split.*

Proof. The proof is similar to the construction in the proof of Lemma 2.2: Assume, without loss of generality, that $|\rho|$ is even, and start with $\mathcal{S}(\langle\rangle) = \mathcal{T}(\rho)$. Then for $\pi \in 2^{<\omega}$ of even length, let $\mathcal{S}(\pi^\frown 0)$ and $\mathcal{S}(\pi^\frown 1)$ be an $\langle e, i\rangle$-splitting of $\mathcal{S}(\pi)$ on $\mathcal{T}$ such that both have odd length, proceeding as in the proof of Lemma 2.2 to ensure that $\mathcal{S}$ remains very uniform; for $\pi \in 2^{<\omega}$ of odd length, suppose that $\mathcal{S}(\pi) = \mathcal{T}(\pi')$, and let $\mathcal{S}(\pi^\frown k) = \mathcal{T}(\pi'^\frown k)$ for $k < 2$. So for $\pi \in 2^{<\omega}$ of odd length, we are guaranteed disagreement on the even parts by hypothesis, and for $\pi \in 2^{<\omega}$ of even positive length, the fact that $\mathcal{T}$ is computable very uniform alternating helps ensure that so is $\mathcal{S}$. □

Finally, we need to prove the following lemma which takes care of the remaining case:

Lemma 3.9. *Let $e, i \in \omega$ and let $\mathcal{T}$ be a computable very uniform alternating f-tree such that for every $\sigma \in \mathrm{ran}(\mathcal{T})$, there is an $\langle e, i\rangle$-splitting above σ on which the even parts agree. If there are no explicit $\langle e, i\rangle$-contradictions on $\mathcal{T}$, then there is a computable very uniform alternating sub-f-tree $\mathcal{S}$ of $\mathcal{T}$ such that for all $\pi \in 2^{<\omega}$, $\mathcal{S}(\pi^\frown 0)$ and $\mathcal{S}(\pi^\frown 1)$ $\langle e, i\rangle$-split.*

Proof. The construction of $\mathcal{S}$ is again similar to that in the proof of Lemma 2.2, except that for $\pi \in 2^{<\omega}$ of even length, we may need to extend $\mathcal{S}(\pi^\frown 0)$ and $\mathcal{S}(\pi^\frown 1)$ more to create even disagreements. □

We are now ready to put the pieces together:

Proof of Theorem 1.1(2). We build a decreasing sequence of computable very uniform alternating f-trees $\mathcal{T}_s$, and A will be the unique branch on all of them.

We need to ensure the following:

1. $\mathrm{Even}(A)$ is not computable;
2. $A \not\leq^* \mathrm{Even}(A)$; and
3. for $e, i \in \omega$, one of the following holds:
    - (a) $(\Phi_e, \Psi_i)[A]$ is not a valid computation;
    - (b) $(\Phi_e, \Psi_i)[A]$ is computable;
    - (c) $(\Phi_e, \Psi_i)[A] \equiv^* \mathrm{Even}(A)$; or
    - (d) $(\Phi_e, \Psi_i)[A] \geq^* A$.

Construction:

Let $\mathcal{T}_0 : 2^{<\omega} \to 2^{<\omega}$ be the identity f-tree, noting that this f-tree is computable, very uniform and alternating.

Let $s = 3e + 1$. Set $\mathcal{T}_{s+1}$ to be the computable sub-f-tree of $\mathcal{T}_s$ provided by Lemma 3.3. Notice that this sub-f-tree is found as a full sub-f-tree of $\mathcal{T}_s$ above some ρ and hence can be made computable, very uniform and alternating.

Let $s = 3\langle e, i\rangle + 2$. Set $\mathcal{T}_{s+1}$ to be the computable sub-f-tree of $\mathcal{T}_s$ provided by Lemma 3.4. As above, $\mathcal{T}_{s+1}$ can be made computable, very uniform and alternating.

Let $s = 3\langle e, i\rangle + 3$. If there are $\sigma \in \mathrm{ran}(\mathcal{T}_s)$ and $x \in \omega$ with $\Phi_e[\sigma](x) = 1 = \Psi_i[\sigma](x)$, then set $\mathcal{T}_{s+1}$ to be the full sub-f-tree of $\mathcal{T}_s$ above ρ (chosen of even length). Again, $\mathcal{T}_{s+1}$ is computable, very uniform and alternating.

Else, if there is $\sigma \in \mathrm{ran}(\mathcal{T}_s)$ with no $\langle e, i\rangle$-splitting above it, set $\mathcal{T}_{s+1}$ to be the full sub-f-tree of $\mathcal{T}_s$ above σ. As above, $\mathcal{T}_{s+1}$ can be made computable, very uniform and alternating.

If neither of the above cases hold, then, for every $\sigma \in \mathrm{ran}(\mathcal{T}_s)$, there is an $\langle e, i\rangle$-splitting on $\mathcal{T}$. If there is $\rho \in 2^{<\omega}$ such that for every $\sigma \succeq \mathcal{T}(\rho)$ in the range of $\mathcal{T}$ and for each $\langle e, i\rangle$-splitting above σ, the even parts disagree, then define $\mathcal{T}_{s+1}$ as given by Lemma 3.8.

Otherwise, for every $\sigma \in \mathrm{ran}(\mathcal{T}_s)$, there is an $\langle e, i\rangle$-splitting above σ which agrees on the evens. Then define $\mathcal{T}_{s+1}$ as given by Lema 3.9.

Now let A be the unique branch in all $\mathcal{T}_s$.

Verification:

By Lemma 3.3, the stages $s + 1 \equiv 1 \bmod 3$ ensure that $\mathrm{Even}(A)$ is not computable.

By Lemma 3.4, the stages $s + 1 \equiv 2 \bmod 3$ ensure that $A \not\leq^* \mathrm{Even}(A)$.

Finally, the stages $s + 1 \equiv 0 \bmod 3$ ensure that one of the disjuncts in item (3) above holds, using Lemmas 3.6-3.9.

This concludes the proof sketch for the 3-element chain. □

4. Finite distributive lattices

This section introduces the modifications to the previous section to show Theorem 1.1(3), namely, that any finite distributive lattice embeds into $\mathcal{D}^*$ as an initial segment. This will allow us to prove in the next section as a corollary the undecidability of the theory $\mathrm{Th}(\mathcal{D}^*)$, and indeed of its $\forall\exists\forall$-fragment.

A key ingredient of our argument is Birkhoff's representation theorem for finite distributive lattices.

Theorem 4.1 (Birkhoff's Representation Theorem [Bi37])**.** *Every finite distributive lattice L is isomorphic to the lattice of downward-closed subsets of the partial order of nonzero join-irreducible elements of L (under set inclusion).*

For the remainder of this section, we fix a finite distributive lattice L and use the following

Notation 4.2. Identify the (nonzero) join-irreducible elements of L with the set $\{0, \dots n-1\}$. Write $D(L)$ for the subsets of $\{0, \dots, n-1\}$ corresponding to the Birkhoff representation of L, i.e., $D(L)$ is the set of downwards closed subsets of $\{0, \dots n-1\}$. Denote the lattice order on $D(L)$ by $<_L$, and define the set

$$S_m = \{m' < n \mid m' \not\geq_L m\}.$$

Intuitively, the set $S_m \in D(L)$ corresponds to the join of all elements $m' \not\geq_L m$ in L (under the Birkhoff representation of L).

Definition 4.3. For $m < n$, let

$$\mathrm{Mod}_n^m(A) = \{x \in A \mid x \equiv m \bmod n\}.$$

Extend this to $S \subseteq \{0, \dots, n-1\}$ by

$$\mathrm{Mod}_n^S(A) = \bigcup_{m \in S} \mathrm{Mod}_n^m(A).$$

To link this definition to the previous section, note that for the 3-element chain, we can use $n = 2$ and $D(L) = \{\emptyset, \{0\}, \{0,1\}\}$, where these sets represent the bottom, middle and top element of the 3-element chain, respectively. Now "agreeing on the even parts" corresponds to agreeing on the numbers 0 mod 2. Furthermore, in this case, $S_1 = \{0\}$, and $S_0 = \emptyset$.

Theorem 1.1(3) will be proved by building a decreasing sequence of computable very uniform alternating f-trees. The notion of "very uniform" will remain the same as in Definition 3.2, but the notion of "alternating" has to be modified.

Definition 4.4. A very uniform f-tree $\mathcal{T}$ is *alternating (for L)* if, for every $m < n$ and for every $\rho \in 2^{<\omega}$, we have

$$|\rho| \equiv m \bmod n \implies$$
$$\mathrm{Mod}_n^{S_m}(\mathcal{T}(\rho^\frown 0)) = \mathrm{Mod}_n^{S_m}(\mathcal{T}(\rho^\frown 1)) \wedge \mathrm{Mod}_n^m(\mathcal{T}(\rho^\frown 0)) \neq \mathrm{Mod}_n^m(\mathcal{T}(\rho^\frown 1)),$$

i.e., $\rho^\frown 0$ and $\rho^\frown 1$ agree on all numbers $x \equiv m' \bmod n$ for all $m' \in S_m$ but not on all numbers $x \equiv m \bmod n$.

We first prove the necessary lemmas which establish that it is possible to build such a sequence of f-trees. We start with a lemma that establishes that we can diagonalize, generalizing the techniques of Lemmas 3.3 and 3.4.

Lemma 4.5. *Let $e, i \in \omega$, let $m < n$ and let $\mathcal{T}$ be a computable very uniform alternating f-tree. Then there is a computable very uniform alternating sub-f-tree $\mathcal{S}$ of $\mathcal{T}$ such that if for every A on $\mathcal{S}$, $(\Phi_e, \Psi_i)[\mathrm{Mod}_n^{S_m}(A)]$ is a valid computation, then*

$$\mathrm{Mod}_n^m(A) \neq (\Phi_e, \Psi_i)[\mathrm{Mod}_n^{S_m}(A)].$$

Proof. Fix $k < 2$ and $\rho \in 2^{<\omega}$ such that $m + 1 + |\rho|$ is a multiple of n, and

$$\mathrm{Mod}_n^m(\mathcal{T}(0^{m\frown} k)) \restriction (|\mathcal{T}(0^{m+1})| + 1) \neq (\Phi_e, \Psi_i)[\mathrm{Mod}_n^{S_m}(\mathcal{T}(0^{m\frown} k^\frown \rho))] \restriction (|\mathcal{T}(0^{m+1})| + 1) \downarrow .$$

Note that such ρ and k must exist since $(\Phi_e, \Psi_i)[\mathrm{Mod}_n^{S_m}(A)]$ is a valid computation for all A on $\mathcal{T}$ (and so is total), and since

$$(\Phi_e, \Psi_i)[\mathrm{Mod}_n^{S_m}(0^{m+1\frown} \rho))] \restriction (|\mathcal{T}(0^{m+1})| + 1) = (\Phi_e, \Psi_i)[\mathrm{Mod}_n^{S_m}(0^{m\frown} 1^\frown \rho))] \restriction (|\mathcal{T}(0^{m+1})| + 1),$$

using $m \notin S_m$ and by the definition of alternating f-tree, whereas

$$\mathrm{Mod}_n^m(\mathcal{T}(0^{m+1})) \restriction (|\mathcal{T}(0^{m+1})| + 1) \neq \mathrm{Mod}_n^m(\mathcal{T}(0^{m\frown} 1)) \restriction (|\mathcal{T}(0^{m+1})| + 1)$$

by the definition of alternating f-tree.

Now define $\mathcal{S}$ to be the full sub-f-tree of $\mathcal{T}$ above $\mathcal{T}(0^m{}^\frown k^\frown\rho)$. By our assumptions, this $\mathcal{S}$ is a computable very uniform alternating f-tree ensuring the lemma's claim. □

The next two lemmas specify conditions under which $(\Phi_e, \Psi_i)[A]$ and $\mathrm{Mod}_n^S(A)$ have the same Ziegler degree.

Lemma 4.6. *Let $e, i \in \omega$, let $S \in D(L)$, and let $\mathcal{T}$ be a computable very uniform alternating f-tree such that every $\langle e, i\rangle$-splitting is only witnessed by sequences which have different Mod_n^S-parts. Then, for every A on $\mathcal{T}$ such that $(\Phi_e, \Psi_i)[A]$ is a valid computation,*

$$(\Phi_e, \Psi_i)[A] \leq^* \mathrm{Mod}_n^S(A).$$

Proof. We first show that $(\Phi_e, \Psi_i)[A] \leq_e \mathrm{Mod}_n^S(A)$. This follows since

$$x \in (\Phi_e, \Psi_i)[A] \iff \exists u\big(\, \langle x, u\rangle \in W_e \wedge \mathrm{Mod}_n^S(D_u) \subseteq \mathrm{Mod}_n^S(A)\big).$$

Similarly, $((\Phi_e, \Psi_i)[A])^c \leq_e^1 \mathrm{Mod}_n^S(A)$ because

$$x \notin (\Phi_e, \Psi_i)[A] \iff \exists v, w\big(\, \langle x, v, w\rangle \in W_i \wedge \\ \mathrm{Mod}_n^S(D_v) \subseteq \mathrm{Mod}_n^S(A) \wedge |D_w| \leq 1 \wedge \mathrm{Mod}_n^S(D_w) \cap \mathrm{Mod}_n^S(A) = \emptyset\big).$$

□

Lemma 4.7. *Let $e, i \in \omega$, let $S \in D(L)$, and let $\mathcal{T}$ be a computable very uniform alternating f-tree with no explicit $\langle e, i\rangle$-contradictions such that for every $\rho \in 2^{<\omega}$, if $\mathrm{Mod}_n^S(\mathcal{T}(\rho^\frown 0)) \neq \mathrm{Mod}_n^S(\mathcal{T}(\rho^\frown 1))$, then they $\langle e, i\rangle$-split. Then, for every A on $\mathcal{T}$ for which $(\Phi_e, \Psi_i)[A]$ is a valid computation,*

$$\mathrm{Mod}_n^S(A) \leq^* (\Phi_e, \Psi_i)[A].$$

Proof. We first check that $\mathrm{Mod}_n^S(A) \leq_e (\Phi_e, \Psi_i)[A]$. Fix $m \in S$ and consider $nx + m$. If there is $k < 2$ and some ℓ such that every $\sigma \in \mathrm{ran}(\mathcal{T})$ of length ℓ has $\sigma(nx + m) = k$, then $\mathrm{Mod}_n^S(A)(nx + m) = k$. Otherwise, there are $\sigma, \tau \in \mathrm{ran}(\mathcal{T})$ such that $\sigma(nx + m) = 0 < 1 = \tau(nx + m)$. Since $\mathcal{T}$ is very uniform, we may in addition assume that there is $\rho \in 2^{<\omega}$ such that $\mathcal{T}(\rho^\frown 0) = \sigma$ and $\mathcal{T}(\rho^\frown 1) = \tau$. Thus, by our assumption on $\mathcal{T}$, σ and τ $\langle e, i\rangle$-split at $\mathcal{T}(\rho)$. Search for y witnessing this splitting. Then $(\Phi_e, \Psi_i)[\sigma](y) = 0$ and $(\Phi_e, \Psi_i)[\tau](y) = 1$. Thus $nx+m \in \mathrm{Mod}_n^S(A)$ iff $(\Phi_e, \Psi_i)[A](y) = 1$.

Under the same hypotheses as in the previous paragraph, we have $nx + m \notin \mathrm{Mod}_n^S(A)$ iff $(\Phi_e, \Psi_i)[A](y) = 0$, so $(\mathrm{Mod}_n^S(A))^c \leq_e^1 (\Phi_e, \Psi_i)[A]$. □

Thus, by the previous two lemmas, if A is on a computable very uniform alternating f-tree $\mathcal{T}$ such that $\mathcal{T}(\rho^\frown 0)$ and $\mathcal{T}(\rho^\frown 1)$ $\langle e, i\rangle$-split iff they disagree on their Mod_n^S-parts and such that $(\Phi_e, \Psi_i)[A]$ is a valid computation, then $\mathrm{Mod}_n^S(A) \equiv^* (\Phi_e, \Psi_i)[A]$.

The next lemma ensures that we can build computable very uniform alternating f-trees witnessing such splittings.

Lemma 4.8. *Let $e, i \in \omega$ and let $\mathcal{T}$ be a computable very uniform alternating f-tree such that for some $\rho \in 2^{<\omega}$, for every $\sigma \succeq \mathcal{T}(\rho)$ in the range of $\mathcal{T}$, there is an $\langle e, i\rangle$-splitting above σ. Let*

$$S_0 = \{m < n \mid \exists^\infty m' \equiv_n m\, [\mathcal{T}\ \langle e, i\rangle\text{-splits at level } m']\},$$

and let S be the downward closure of S_0 in L. (In other words, S is the smallest set in $D(L)$ such that above any $\sigma \succeq \mathcal{T}(\rho)$ there is an $\langle e,i\rangle$-splitting disagreeing on their Mod_n^S-parts.) If there are no explicit $\langle e,i\rangle$-contradictions on $\mathcal{T}$, then there is a computable very uniform alternating sub-f-tree $\mathcal{S}$ of $\mathcal{T}$ such that for all $\pi \in 2^{<\omega}$, $\mathcal{S}(\pi^\frown 0)$ and $\mathcal{S}(\pi^\frown 1)$ $\langle e,i\rangle$-split iff $\mathrm{Mod}_n^S \mathcal{S}(\pi^\frown 0) \neq \mathrm{Mod}_n^S \mathcal{S}(\pi^\frown 1)$.

Notice that any $\langle e,i\rangle$-splitting above $\mathcal{T}(\rho)$ is witnessed by sequences of the form $\mathcal{T}(\pi^\frown 0)$, $\mathcal{T}(\pi^\frown 1)$ by the discussion preceding Lemma 2.2.

Proof. The proof is similar to the construction in the proof of Lemma 3.8. We build $\mathcal{S}$ by induction on ℓ. Set $\mathcal{S}(\langle\rangle) = \mathcal{T}(\rho)$.

Suppose we have already defined $\mathcal{S}(\pi)$ for all π of length $m' = nx + m$; and so we need to define $\mathcal{S}(\pi^\frown k)$ for all $\pi \in 2^{m'}$ and $k < 2$ such that

$$\mathrm{Mod}_n^{S_m}(\mathcal{S}(\pi^\frown 0)) = \mathrm{Mod}_n^{S_m}(\mathcal{S}(\pi^\frown 1)) \wedge \tag{4.1}$$

$$\mathrm{Mod}_n^{m}(\mathcal{S}(\pi^\frown 0)) \neq \mathrm{Mod}_n^{m}(\mathcal{S}(\pi^\frown 1)) \wedge \tag{4.2}$$

$$\big(\mathcal{S}(\pi^\frown 0) \text{ and } \mathcal{S}(\pi^\frown 1)\ \langle e,i\rangle\text{-split} \iff \mathrm{Mod}_n^S \mathcal{S}(\pi^\frown 0) \neq \mathrm{Mod}_n^S \mathcal{S}(\pi^\frown 1)\big) \tag{4.3}$$

to preserve $\mathcal{S}$ being an alternating f-tree and to satisfy the conditions of Lemmas 4.6 and 4.7.

We distinguish two cases, depending on whether $S \subseteq S_m$ or not: If $S \subseteq S_m$, we can find π' with $|\pi'| \equiv_n m$ such that $\mathcal{T}(\pi'^\frown 0)$ and $\mathcal{T}(\pi'^\frown 1)$ extend $\mathcal{S}(\pi)$ and don't $\langle e,i\rangle$-split. Setting $\mathcal{S}(\pi^\frown 0)$ and $\mathcal{S}(\pi^\frown 1)$ to be these sequences, we satisfy (4.1)-(4.3). Since $S \subseteq S_m$, $\mathrm{Mod}_n^S \mathcal{S}(\pi^\frown 0) = \mathrm{Mod}_n^S \mathcal{S}(\pi^\frown 1)$.

On the other hand, suppose $S \not\subseteq S_m$. Let $\ell \in S \backslash S_m$. Then $\ell \geq_L m$, and hence, since S is downward closed, $m \in S$. Let $\ell' \geq_L m$ be in S_0. To build $\mathcal{S}(\pi^\frown 0)$ and $\mathcal{S}(\pi^\frown 1)$, find an $\langle e,i\rangle$-splitting at some level equivalent to $\ell' \bmod n$. We then further extend to the next level equivalent to $m \bmod n$ to ensure disagreement on the Mod_n^m-part. Thus, we can satisfy (4.1)-(4.3). □

We now have all the pieces we need in order to establish the proof promised for this section:

Proof of Theorem 1.1(3). We build a decreasing sequence of computable very uniform alternating f-trees $\mathcal{T}_s$ and let A be the unique branch on all of them.

We need to ensure the following:

1. $\mathrm{Mod}_n^m(A) \not\leq^* \mathrm{Mod}_n^{S_m}(A)$ for every $m < n$; and
2. For $e, i \in \omega$, one of the following holds:
   (a) $(\Phi_e, \Psi_i)[A]$ is not a valid computation, or
   (b) $(\Phi_e, \Psi_i)[A] \equiv^* \mathrm{Mod}_n^S(A)$ for some $S \in D(L)$.

Note here that (1) suffices to show that $\mathrm{Mod}^S(A) \not\leq^* \mathrm{Mod}_n^{S'}(A)$ for all $S, S' \in D(L)$ with $S \not\subseteq S'$ since we can choose some $m \in S - S'$ and so have $\mathrm{Mod}_n^m(A) \not\leq^* \mathrm{Mod}_n^{S_m}(A)$ while also $\mathrm{Mod}_n^m(A) \leq^* \mathrm{Mod}_n^S(A)$ and $\mathrm{Mod}_n^{S'}(A) \leq^* \mathrm{Mod}_n^{S_m}(A)$.

We start with $\mathcal{T}_0 : 2^{<\omega} \to 2^{<\omega}$ as the full identity tree, which is certainly computable, very uniform and alternating.

At stage $s = 2(n \cdot \langle e,i\rangle + m) + 1$, we meet (1) for the reduction (Φ_e, Ψ_i) using Lemma 4.5: The $\mathcal{S}$ obtained from this lemma will be our next $\mathcal{T}_{s+1}$ and clearly meets the requirement.

At stage $s = 2\langle e, i\rangle + 2$, we meet (2) for the reduction (Φ_e, Ψ_i) using Lemma 4.8: Pick $S \in D(L)$ maximal in $D(L)$ with the condition stated in the lemma's hypothesis. Then the $\mathcal{S}$ obtained from this lemma with this S will be our next $\mathcal{T}_{s+1}$ and again clearly meets the requirement unless there are explicit $\langle e, i\rangle$-contradictions on $\mathcal{T}_s$, in which case we can go to a full sub-f-tree.

In either case, the resulting f-tree $\mathcal{T}_{s+1}$ can be made computable, very uniform and alternating so that we can proceed with our construction. Since $\lim_s |\mathcal{T}_s(\langle\rangle)| = \infty$, there will be a unique branch A through all the $\mathcal{T}_s$ as desired.

This concludes the proof of Theorem 1.1(3). □

5. Undecidability

This section will sketch the proof of Theorem 1.1(4), that the ∀∃∀-theory of the Ziegler degrees (in the language of partial ordering) is undecidable.

This follows by the same argument as for the many-one degrees and the Turing degrees. (The original undecidability argument of the full theory of the Turing degrees is due to Lachlan [La68] and uses that every finite distributive lattice is an initial segment of the Turing degrees; the undecidability of the ∀∃∀-theory of the Turing degrees was only established by Schmerl (published in Lerman [Le83, Corollary Vii.4.6]) and uses that every finite lattice is an initial segment of the Turing degrees. It was only later that Nies [Ni96] established the hereditary undecidability of the ∀∃∀-theory of the finite distributive lattices [Ni96, Theorem 4.8] and used it and his Transfer Lemma [Ni96, Lemma 3.1] to establish the undecidability of the ∀∃∀-theory of the many-one degrees that it became known that embedding only the finite distributive lattices as initial segments would have sufficed to establish the undecidability of the ∀∃∀-theory of the Turing degrees as well.)

The proof of the undecidability of the ∀∃∀-theory of the Ziegler degrees follows exactly as Nies's proof for the many-one degrees [Ni96, Theorem 4.8]: The finite distributive lattices have a hereditarily undecidable ∀∃∀-theory and are by our Theorem 1.1 Σ_1-elementarily definable with parameters in the Ziegler degrees.

We note here that the c.e. Ziegler degrees are closed downward in the Ziegler degrees (since $A \leq^* B$ implies $A \leq_e B$, and so B being c.e. implies A being c.e.). Thus the undecidability result for the first-order theory of the c.e. Q-degrees (which coincide with the c.e. Ziegler degrees) proved by Downey, LaForte and Nies [DLN98] also yields the undecidability of the first-order theory of the Ziegler degrees. However, this proof does not yield our sharper result on the undecidability of the ∀∃∀-theory of the Ziegler degrees since their coding seems to be considerably more complicated than ours.

6. Open questions

We conclude here with a couple of open questions related to the results above. The first concerns whether our result on the undecidability of the Ziegler degrees is tight.

Question 6.1. *Is the ∀∃-quantifier theory of $\mathcal{D}^*$ decidable?*

While it is known that the ∀∃-theory of $\mathcal{D}_T$ is decidable, the corresponding question in the enumeration degrees is still open. Moreover, $\mathcal{D}_e$ embeds into $\mathcal{D}^*$ via the map that sends a set A to the uniform join of the set of $B \leq_e A$, but it is not known whether the image of this embedding is definable.

Finally, we finish by stating what we currently consider probably the most interesting open question related to the results above:

Question 6.2. *Is there a finite closed ideal in the Ziegler degrees?*

Recall here that the notion of a *closed ideal* in the Ziegler degrees was defined by Ziegler [Zi80, Definition III.3.10] and, loosely speaking, corresponds to the notion of a Scott ideal in the Turing degrees. Ziegler [Zi80, Theorem III.3.12(2)] showed that the closed ideals correspond precisely to the collections of Ziegler degrees encoding all finitely generated subgroups of a fixed existentially closed group. Thus a positive answer to our question would yield an existentially closed group such that its finitely generated subgroups have their word problems reside in only finitely many Ziegler degrees.

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