Chains and antichains in the Weihrauch lattice

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Abstract

We study the existence and the distribution of "long" chains in the Weihrauch degrees, mostly focusing on chains with uncountable cofinality. We characterize when such chains have an upper bound and prove that there are no cofinal chains (of any order type) in the Weihrauch degrees. Furthermore, we show that the existence of coinitial sequences of non-zero degrees is equivalent to CH. Finally, we explore the extendibility of antichains, providing some necessary conditions for maximality.

1 Introduction

Weihrauch reducibility is a notion of reducibility between partial multi-valued functions that is useful to calibrate their uniform computational strength. The theory of Weihrauch reducibility has often been studied in connection with reverse mathematics, as it can be used to analyze the computability-theoretical content of $\forall\exists$-statements.

Many results in the literature on Weihrauch degrees focus on characterizing the degree of specific problems. In contrast, several natural questions on the structure of Weihrauch degrees are still open. In this paper, we study the existence of chains and antichains in the Weihrauch degrees.

In a given partial order $(P, \leq)$, a *chain* is a linearly ordered subset of P. Conversely, an *antichain* is a set of pairwise incomparable elements of P. We use $\{a_i\}_{i\in I}$ to denote a family indexed with elements in some set I. We stress that, for example, a chain $\{a_i\}_{i\in\omega}$ of elements of P could be ill-founded or even dense. On the other hand, if L is a linear order, we write $(a_x)_{x\in L}$ if the chain $\{a_x\}_{x\in L}$ is order-isomorphic to L via the map $x \mapsto a_x$.

A set $S \subseteq P$ is called *cofinal* (in P) if for every $p \in P$ there is $q \in S$ such that $p \leq q$. The *cofinality* of P, denoted $\operatorname{cof}(P)$, is the least cardinality of a cofinal chain. The dual notion of cofinality is *coinitiality*: a set $S \subseteq P$ is called *coinitial* if for every $p \in P$ there is $q \in S$ such that $q \leq p$. Equivalently, a coinitial set in P is a cofinal set in $P^* := (P, \geq)$. The *coinitiality* of P, denoted $\operatorname{coinit}(P)$, is the least cardinality of a coinitial chain. Both cofinality and coinitiality need not be well-defined for an arbitrary partial order (e.g. if the partial order is only made of two incomparable elements). A generalization of cofinality (resp. coinitiality) that is well-defined for every partial order P is *set-cofinality* (resp. *set-coinitiality*), namely the least cardinality of a cofinal (resp. coinitial) subset of P. The set-cofinality and the set-coinitiality of P are denoted by $\operatorname{setcof}(P)$ and $\operatorname{setcoinit}(P)$, respectively.

In this paper, we study the existence and extendibility of chains in the Weihrauch degrees. While we are mostly interested in the extendibility of chains of order type κ or κ^*, for some cardinal κ with

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$\mathrm{cof}(\kappa) > \omega$, many techniques apply to partially ordered families of degrees as well. We characterize when a set of Weihrauch degrees has an upper bound (Theorem 3.5) and, for every $\eta \leq \mathfrak{c} := 2^\omega$ with uncountable cofinality, we provide an explicit example of a chain of order type η with no upper bound (Theorem 3.7). This can then be used to show that there are no cofinal chains in the Weihrauch degrees of any order type (Theorem 3.17).

In contrast, the picture of the coinitiality of the Weihrauch degrees looks quite different: while there are descending chains of order type $\mathfrak{c}$ with no non-zero lower bound, the existence of a coinitial sequence in the Weihrauch degrees is independent of ZFC and equivalent to CH (Theorem 3.19). We also characterize when chains of order type $\kappa+\lambda^*$, for cardinals κ and λ with uncountable cofinality, admit an intermediate degree (Theorem 3.11), and show that each interval in the Weihrauch lattice is either finite or uncountable (Theorem 3.12).

Finally, we study the extendibility of antichains: we show that every antichain of size $< \mathfrak{c}$ can be extended (Corollary 4.4), and provide a necessary condition for an antichain to be maximal (Theorem 4.6).

1.1 Background

We now briefly recall the main notions and fix the notation that will be needed in the rest of the paper. With a small abuse of notation, we will often identify a Turing/Medvedev/Weihrauch degree with one of its representatives.

Given two partial multi-valued functions $f, g :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$, we say that f is *Weihrauch reducible* to g, and write $f \leq_\mathrm{W} g$, if there are two computable functionals $\Phi :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and $\Psi :\subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that, for every $p \in \mathrm{dom}(f)$,

$$\Phi(p) \in \mathrm{dom}(g) \wedge (\forall q \in g(\Phi(p)))(\Psi(p,q) \in f(p)).$$

We use $(\mathcal{W}, \leq_\mathrm{W})$ to denote the degree structure of the *Weihrauch degrees*, namely, the partial order induced by $\leq_\mathrm{W}$ on the equivalence classes. As a notational convenience, we also use $(\mathcal{W}_0, \leq_\mathrm{W})$ for the restriction of the Weihrauch degrees to non-empty problems. For a more comprehensive presentation of Weihrauch reducibility, we refer the reader to [2]. We mention that while Weihrauch reducibility is often defined in the more general context of partial multi-valued functions on represented spaces, every Weihrauch degree has a representative which is a partial multi-valued function on $\mathbb{N}^\mathbb{N}$ (see [2, Lem. 11.3.8]). In other words, in order to study the structure $(\mathcal{W}, \leq_\mathrm{W})$, there is no loss of generality in restricting our attention to problems on the Baire space. With a small abuse of notation, we can talk about multi-valued functions on spaces like $\mathbb{N}$, $\mathbb{N}^{<\mathbb{N}}$, or $(\mathbb{N}^\mathbb{N})^\mathbb{N}$, as their elements can be canonically represented with elements in $\mathbb{N}^\mathbb{N}$.

The Weihrauch degrees form a distributive lattice with a bottom element (the empty set) and no top element. The join and meet operators can be obtained, respectively, by lifting the following two operators on multi-valued functions to the degree structure:

- $(f_0 \sqcup f_1)(i,x) := f_i(x)$, where $\mathrm{dom}(f_0 \sqcup f_1) := \bigcup_{i<2} \{i\} \times \mathrm{dom}(f_i)$.
- $(f_0 \sqcap f_1)(x_0,x_1) := \bigcup_{i<2} \{i\} \times f_i(x_i)$, where $\mathrm{dom}(f_0 \sqcup f_1) := \mathrm{dom}(f_0) \times \mathrm{dom}(f_1)$.

We observe that neither of the above two operators can be generalized to obtain a countable join/meet. Indeed, while the operators $(f_i)_{i\in\mathbb{N}} \mapsto \bigsqcup_{i\in\mathbb{N}} f_i$ and $(f_i)_{i\in\mathbb{N}} \mapsto \bigsqcap_{i\in\mathbb{N}} f_i$ defined as $\bigsqcup_{i\in\mathbb{N}} f_i(i,x) := f_i(x)$ and $\bigsqcap_{i\in\mathbb{N}} f_i((x_j)_{j\in\mathbb{N}}) := \bigcup_{i\in\mathbb{N}} f_i(x_i)$, respectively, are well-defined and identify an upper and an lower bound for the family, they do not lift to the Weihrauch degrees and are not, in general, the supremum or the infimum of $\{f_i : i \in \mathbb{N}\}$. It is known that $(\mathcal{W}, \leq_\mathrm{W})$ is not an ω-complete join/meet semilattice ([6, Cor. 3.17]). In fact, it is also known that no non-trivial countable suprema exist.

Theorem 1.1 ([6, Prop. 3.15]). *A family $\{a_i : i \in \mathbb{N}\}$ of Weihrauch degrees has a supremum iff it is already the supremum of $\{a_i : i < n\}$ for some $n \in \mathbb{N}$. In particular, no (strictly ascending) chain $(a_i)_{i\in\omega}$ has a supremum.*

The dual result for infima does not hold: the infimum of a countable family does not always exist, but there are chains $(a_i)_{i\in\omega^*}$ with a greatest lower bound ([6, Example 3.19]).

Recently [8], the existence and distribution of (strong) minimal covers in the Weihrauch degrees has been fully characterized. In particular, it has been shown that the Weihrauch degrees do not have minimal degrees above $\emptyset$ but are dense (all the non-degenerate intervals are non-empty) only in the cone above the identity problem id. Empty intervals exist above every problem g with $\mathrm{id} \not\leq_{\mathrm{W}} g$, while strong minimal covers only exist in the cone below id.

2 Some results on Medvedev reducibility

There is a close connection between Weihrauch and Medvedev reducibility, and some structural results on the Weihrauch lattice can be obtained by analyzing the properties of the Medvedev degrees. In this section, we briefly recall the definition and the main properties of Medvedev reducibility. We also state and prove some results on the Medvedev degrees that, while being generalizations of already known facts, have not been explicitly observed in the literature before and are useful to obtain the main results on Weihrauch reducibility. For a more thorough introduction to Medvedev reducibility, the reader is referred to [7, 12].

Given $A, B \subseteq \mathbb{N}^{\mathbb{N}}$, sometimes referred to as *mass problems*, we say that A is *Medvedev reducible* to B, and write $A \leq_{\mathrm{M}} B$, if there is a computable functional $\Phi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $B \subseteq \mathrm{dom}(\Phi)$ and $\Phi(B) \subseteq A$. The *Medvedev degrees* are denoted by $(\mathcal{M}, \leq_{\mathrm{M}})$. They form a distributive lattice with a top element (the degree of $\emptyset$) and a bottom element (the degree of $\mathbb{N}^{\mathbb{N}}$ or, equivalently, the degree of any mass problem that contains a computable point). As for the Weihrauch degrees, we use $(\mathcal{M}_0, \leq_{\mathrm{M}})$ for the restriction of the Medvedev degrees to non-empty problems. Since $\emptyset$ is not the immediate successor of any mass problem, there are no maximal elements in $\mathcal{M}_0$. The join and the meet are induced from the following two operations, respectively:

- $A \vee B := \{\langle p, q\rangle : p \in A \text{ and } q \in B\}$;
- $A \wedge B := (0)^\frown A \cup (1)^\frown B$,

where $\langle \cdot, \cdot \rangle$ denotes the standard pairing function in the Baire space and $(0)^\frown A$ denotes the set obtained by concatenating the string (0) with all the strings in A.

Observe that we can restate the definition of Weihrauch reducibility as follows: $f \leq_{\mathrm{W}} g$ iff there are two computable functionals Φ, Ψ such that $\Phi(\mathrm{dom}(f)) \subseteq \mathrm{dom}(g)$ and, for every $p \in \mathrm{dom}(f)$, $\Psi(p, g(\Phi(p))) \subseteq f(p)$. In particular, this shows that $f \leq_{\mathrm{W}} g$ implies $\mathrm{dom}(g) \leq_{\mathrm{M}} \mathrm{dom}(f)$. This suggests that the relation between the domains of two multi-valued functions can be used to define an order-reversing embedding of the Medvedev degrees in the Weihrauch degrees[1]. Indeed, it is straightforward to show that $A \leq_{\mathrm{M}} B$ iff $\mathrm{id}_B \leq_{\mathrm{W}} \mathrm{id}_A$, where id_X is the restriction of the identity problem id to the set $X \subseteq \mathbb{N}^{\mathbb{N}}$.

Remark 2.1. We highlight that the mapping $d := A \mapsto \mathrm{id}_A$ induces an isomorphism between $\mathcal{M}^* = (\mathcal{M}, \geq_{\mathrm{M}})$ and the lower cone of id in the Weihrauch degrees. This is an important fact that will be frequently used in the rest of the paper.

The existence and distribution of minimal covers in the Medvedev degrees have been fully characterized. To describe them, let us define $\{p\}^+ := \{(e)^\frown q : \Phi_e(q) = p \text{ and } q \not\leq_{\mathrm{T}} p\}$.

[1] An order-preserving embedding of the Medvedev degrees in the Weihrauch degrees can be defined by exploiting the range of the computational problems. For more details, see [3, Section 5]. Infinitely many embeddings can be defined using the fact that the Weihrauch degrees admit a non-trivial injective endomorphism [1, Thm. 3.7].

Theorem 2.2 ([4, Cor. 2.5]). *For every $A <_{\mathrm{M}} B$, B is a minimal cover of A iff*

$$(\exists p \in A)(A \equiv_{\mathrm{M}} B \wedge \{p\} \text{ and } B \wedge \{p\}^+ \equiv_{\mathrm{M}} B).$$

Besides, for every $p \in \mathbb{N}^{\mathbb{N}}$, $\{p\}$ is a strong minimal cover of $\{p\}^+$ in $\mathcal{M}^*$. In other words, for every $A \subseteq \mathbb{N}^{\mathbb{N}}$, $\{p\} <_{\mathrm{M}} A$ implies $\{p\}^+ \leq_{\mathrm{M}} A$. In particular, since the bottom of $\mathcal{M}$ is equivalent to $\{p\}$ for every computable p, this implies that there is a first non-bottom degree in $\mathcal{M}$, namely, (the degree of) $\{0^{\mathbb{N}}\}^+$. Moreover, being a *degree of solvability* (i.e., being Medvedev-equivalent to a singleton) is equivalent to being the top of a strong minimal cover in $\mathcal{M}^*$, which proves the first-order definability of the Turing degrees in $(\mathcal{M}, \leq_{\mathrm{M}})$ ([4, Cor. 2.1]).

It is known that there are only three mass problems that are Medvedev-comparable with every other mass problem: these are $\{0^{\mathbb{N}}\}$, $\{0^{\mathbb{N}}\}^+$, and $\emptyset$. In other words, for every mass problem $A \notin \{\{0^{\mathbb{N}}\}, \{0^{\mathbb{N}}\}^+, \emptyset\}$, there is $B \subseteq \mathbb{N}^{\mathbb{N}}$ that is Medvedev-incomparable with A ([4, Thm. 1.1]). The antichains in $\mathcal{M}$ can have any cardinality up to $2^{\mathfrak{c}}$, the size of the whole lattice [12, Thm. 4.1]. However, maximal (non-trivial) antichains need not be infinite.

Proposition 2.3. *For every κ with $1 \leq \kappa \leq \mathfrak{c}$, there is a maximal antichain in $\mathcal{M}$ of size κ.*

In the following, we combine the cases of a finite and an infinite cardinal κ into one, so for convenience, we will use $\sup \kappa$ to denote $\kappa - 1$ for finite $\kappa > 0$, and κ itself for infinite κ.

Proof. The case $\kappa = 1$ was discussed just before the statement. For $\kappa > 1$, we generalize the technique used in [12, Example 4.2]: let $\{p_\alpha\}_{\alpha<\sup\kappa}$ be a non-maximal antichain in the Turing degrees. We define:

- for every $\alpha < \sup\kappa$, $A_\alpha := \{p_\alpha\}$;
- $A_{\sup\kappa} := \{q \in \mathbb{N}^{\mathbb{N}} : (\forall \alpha < \sup\kappa)(q \not\leq_{\mathrm{T}} p_\alpha)\}$.

It is easy to check that the sequence $(A_\alpha)_{\alpha\leq\sup\kappa}$ is an antichain in $\mathcal{M}$ of size κ. Indeed, for every $\alpha < \beta < \sup\kappa$, $A_\alpha \mid_{\mathrm{M}} A_\beta$ follows trivially from the fact that $\{p_\alpha\}_{\alpha<\sup\kappa}$ is a Turing-antichain. By definition, $A_{\sup\kappa} \not\leq_{\mathrm{M}} A_\alpha$, hence we only need to show that $A_\alpha \not\leq_{\mathrm{M}} A_{\sup\kappa}$. Since $\{p_\alpha\}_{\alpha<\sup\kappa}$ is not maximal, there is q such that, for every $\alpha < \sup\kappa$, $q \mid_{\mathrm{T}} p_\alpha$. Clearly any such q belongs to $A_{\sup\kappa}$, hence $A_\alpha \not\leq_{\mathrm{M}} A_{\sup\kappa}$.

To show that the antichain is maximal, fix $C \subseteq \mathbb{N}^{\mathbb{N}}$ and assume that $(\forall \alpha < \sup\kappa)(C \not\leq_{\mathrm{M}} A_\alpha)$. In particular, for every $q \in C$ and every $\alpha < \sup\kappa$, $q \not\leq_{\mathrm{T}} p_\alpha$. This implies that $C \subseteq A_{\sup\kappa}$, which in turn implies $A_{\sup\kappa} \leq_{\mathrm{M}} C$. □

2.1 Chains in the Medvedev degrees

We now give an overview of some results on the existence of "long" chains in the Medvedev degrees. In [13], it was observed that the existence of a chain of size κ in $(2^{\mathfrak{c}}, \subseteq)$ implies the existence of a chain of the same size in $\mathcal{M}$ (see the proof of [13, Thm. 4.2]). The converse direction was established in [11, Thm. 4.3.1]. Upon closer inspection, the proof of this equivalence can be adapted to obtain a slightly stronger theorem. We first highlight the following simple fact:

Proposition 2.4. *Let L be a linear order with $\mathrm{cof}(L) > \omega$. A chain $(A_x)_{x\in L}$ in $\mathcal{M}_0$ has an upper bound iff there is a cofinal subsequence $(A_{x_\alpha})_{\alpha<\mathrm{cof}(L)}$ such that $\bigcap_{\alpha<\mathrm{cof}(L)} A_{x_\alpha} \neq \emptyset$.*

Proof. Assume first that $B \neq \emptyset$ is an upper bound for $(A_x)_{x\in L}$. Then, there is $e \in \mathbb{N}$ and a cofinal subsequence $(A_{x_\alpha})_{\alpha<\mathrm{cof}(L)}$ such that for every $\alpha < \mathrm{cof}(L)$, $A_{x_\alpha} \leq_{\mathrm{M}} B$ via Φ_e. In particular, this implies that

$$\emptyset \neq \Phi_e(B) \subseteq \bigcap_{\alpha<\mathrm{cof}(L)} A_{x_\alpha}.$$

Conversely, assume that there is a cofinal subsequence $(A_{x_\alpha})_{\alpha<\mathrm{cof}(L)}$ such that $C := \bigcap_{\alpha<\mathrm{cof}(L)} A_{x_\alpha} \neq \emptyset$. Then, for every $\alpha < \mathrm{cof}(L)$, $A_{x_\alpha} \leq_{\mathrm{M}} C$, hence C is an upper bound for $(A_x)_{x\in L}$. □

Theorem 2.5 (essentially [11, Thm. 4.3.1])**.** *For any linear order L, the following are equivalent:*

(1) *there is a chain in $(2^{\mathfrak{c}}, \supseteq)$ of order type L;*

(2) *there is a chain in $\mathcal{M}$ of order type L;*

(3) *above every non-top Medvedev degree, there is a chain in $\mathcal{M}$ of order type L.*

Moreover, if $\operatorname{cof}(L) > \omega$, then the existence of a chain of order type L in $(2^{\mathfrak{c}}, \supseteq)$ implies the existence of an order-isomorphic chain in $\mathcal{M}$ with no (non-trivial) upper bound.

Proof. The implication (3) $\Rightarrow$ (2) is trivial, while (2) $\Rightarrow$ (1) can be readily obtained by inspecting the proof of [11, Thm. 4.3.1]. For (1) $\Rightarrow$ (3), let $(\mathcal{C}_x)_{x\in L}$ be a chain in $(2^{\mathfrak{c}}, \supseteq)$. Without loss of generality, we can assume that $\bigcap_{x\in L} \mathcal{C}_x = \emptyset$ (otherwise we can just replace $\mathcal{C}_x$ with $\mathcal{C}_x \setminus \bigcap_{y\in L} \mathcal{C}_y$). Fix a non-empty $A \subseteq \mathbb{N}^{\mathbb{N}}$ and let $p \in A$. Let also $\{p_\alpha\}_{\alpha<\mathfrak{c}}$ be the set of $\leq_{\mathrm{T}}$-minimal degrees above p. Clearly, for every α, $A <_{\mathrm{M}} \{p_\alpha\}$. For $x \in L$, let $A_x := \{p_\alpha : \alpha \in \mathcal{C}_x\}$. As in the proof of [11, Thm. 4.3.1], the family $\{A_x : x \in L\}$ is a chain in $\mathcal{M}$ of order type L.

Observe also that $\bigcap_{x\in L} A_x = \emptyset$. If $\operatorname{cof}(L) > \omega$ then $(A_x)_{x\in L}$ has no (non-trivial) upper bound. Indeed, for every cofinal subsequence $(A_{x_\alpha})_{\alpha<\operatorname{cof}(L)}$ of $(A_x)_{x\in L}$, $\bigcap_{\alpha<\operatorname{cof}(L)} A_{x_\alpha} = \bigcap_{x\in L} A_x = \emptyset$, hence the claim follows from Proposition 2.4. □

Corollary 2.6. *There is a chain in $\mathcal{M}_0$ of order type ω_1 with no upper bound.* □

An explicit example of such sequence can be built as follows: let $(d_\alpha)_{\alpha<\omega_1}$ be a $\leq_{\mathrm{T}}$-chain of order type ω_1. Clearly, this sequence does not have an upper bound, as lower Turing cones are countable. This also shows that the sequence of mass problems $(\{d_\alpha\})_{\alpha<\omega_1}$ has no upper bound in $\mathcal{M}_0$, as there are only countably many singletons in the lower Medvedev cone of every (non-empty) mass problem.

We highlight that we cannot fully characterize (in ZFC) the cardinals κ for which there is a chain of size κ in $\mathcal{M}$.

Theorem 2.7 ([11, Cor. 4.3.2])**.** *The existence of a chain of cardinality $2^{\mathfrak{c}}$ in the Medvedev degrees is independent of* ZFC. □

A natural problem is to characterize the cofinality of the Medvedev degrees. The question is only interesting when considering $\mathcal{M}_0$, as there is a top element in $\mathcal{M}$. We first recall the following well-known fact about the Turing degrees.

Theorem 2.8 ([9, Ex. V.2.4(c)])**.** *The following are equivalent:*

- CH*;*
- *there is a cofinal chain in $\mathcal{T}$;*
- *there is a cofinal chain in $\mathcal{T}$ of order type ω_1.* □

Recall also that ω_1-chains in the Turing degrees can be built in ZFC. However, under ZFC+$\neg$CH, no such chain can be cofinal. We now show that the analogue of Theorem 2.8 holds for $\mathcal{M}_0$. To this end, we first prove the following lemma.

Lemma 2.9. *If $(A_\alpha)_{\alpha<\kappa}$ is a cofinal chain in $\mathcal{M}_0$, then $\operatorname{cof}(\kappa) = \omega_1$.*

Proof. Recall first of all that for every mass problem A, there are ω many singletons that are Medvedev-reducible to A (as there are only countably many computable functionals). Assume that $(A_\alpha)_{\alpha<\kappa}$ is a cofinal sequence in $\mathcal{M}_0$ and that $\operatorname{cof}(\kappa) > \omega_1$. Let $(d_\beta)_{\beta<\omega_1}$ be a $\leq_{\mathrm{T}}$-chain. For every $\beta < \omega_1$, there is $\alpha_\beta < \kappa$ such that $\{d_\beta\} \leq_{\mathrm{M}} A_{\alpha_\beta}$. Since $\operatorname{cof}(\kappa) > \omega_1$, $\eta := \sup\{\alpha_\beta : \beta < \omega_1\} < \kappa$, and therefore, for every $\beta < \omega_1$, $\{d_\beta\} \le_{\mathrm{M}} A_{\eta+1}$, contradicting the fact that there can be only countably many singletons in the lower cone of $A_{\eta+1}$.

To conclude the proof, notice that no countable family (and, in particular, no countable chain) can be cofinal in $\mathcal{M}_0$ (the union of their lower cones does not contain all the singletons). If $(A_\alpha)_{\alpha<\kappa}$ is a cofinal chain and $\mathrm{cof}(\kappa) = \omega$, then there is a countable cofinal chain in $\mathcal{M}_0$, which is a contradiction. □

Theorem 2.10. *The following are equivalent:*

(1) CH*;*

(2) *there is a cofinal chain in* $\mathcal{M}_0$*;*

(3) *there is a cofinal chain in* $\mathcal{M}_0$ *of order type* ω_1*.*

Proof. Let us first show that (1) ⇒ (3). By Theorem 2.8, there is a cofinal chain $(d_\alpha)_{\alpha<\omega_1}$ in the Turing degrees (identifying a Turing degree with one of its representatives). Define the sequence $(A_\alpha)_{\alpha<\omega_1}$ as $A_\alpha := \{d_\alpha\}$. Clearly, for every $\alpha < \beta < \omega_1$, $A_\alpha <_{\mathrm{M}} A_\beta$. To prove that $(A_\alpha)_{\alpha<\omega_1}$ is cofinal in $\mathcal{M}_0$, let $C \subseteq \mathbb{N}^{\mathbb{N}}$ be non-empty and let $p \in C$. Since the sequence $(d_\alpha)_{\alpha<\omega_1}$ is cofinal in $\mathcal{T}$, there is $\alpha < \omega_1$ such that $p \le_{\mathrm{T}} d_\alpha$, which implies that $C \le_{\mathrm{M}} A_\alpha$.

The implication (3) ⇒ (2) is trivial, hence we only need to show that (2) ⇒ (1). Let $(B_\alpha)_{\alpha<\kappa}$ be a cofinal chain in $\mathcal{M}_0$. By Lemma 2.9, we can assume that $\kappa = \omega_1$. For each α, choose $p_\alpha \in B_\alpha$. Observe that the set $\{p_\alpha : \alpha < \omega_1\}$ is cofinal in $\mathcal{T}$. Indeed, for every $d \in 2^{\mathbb{N}}$, there is $\alpha < \omega_1$ such that $\{d\} \le_{\mathrm{M}} B_\alpha$, hence in particular $d \le_{\mathrm{T}} p_\alpha$. We now exploit the set $\{p_\alpha : \alpha < \omega_1\}$ to define a cofinal sequence $(q_\alpha)_{\alpha<\omega_1}$ in the Turing degrees. By Theorem 2.8, this suffices to conclude the proof.

For every α such that $0 < \alpha < \omega_1$, we fix a fundamental sequence for α, i.e., a sequence $(\alpha[n])_{n\in\mathbb{N}}$ such that

- $(\forall n \in \mathbb{N})(\alpha[n] \le \alpha[n+1] < \alpha)$; and
- $\alpha = \sup\{\alpha[n] + 1 : n \in \mathbb{N}\}$.

We then define $q_0 := p_0$ and, for every $\alpha > 0$, $q_\alpha := \left(\bigoplus_{n\in\mathbb{N}} q_{\alpha[n]}\right) \oplus p_\alpha$.

Observe that, if $\alpha < \beta$, then there are $n_0, \dots, n_k$ such that $\alpha = \beta[n_0][n_1]\dots[n_k]$, which implies that $q_\alpha \le_{\mathrm{T}} q_\beta$. Moreover, for every α, $p_\alpha \le_{\mathrm{T}} q_\alpha$, hence the sequence $(q_\alpha)_{\alpha<\omega_1}$ is cofinal in $\mathcal{T}$, and this concludes the proof. □

Theorem 2.11. $\mathrm{setcof}(\mathcal{M}_0) = \mathfrak{c}$.

Proof. Under CH, this is a corollary of Theorem 2.10 (as no countable family can be cofinal), so assume ¬CH. Notice first of all that the family of singletons is cofinal in $\mathcal{M}_0$, hence $\mathrm{setcof}(\mathcal{M}_0) \le \mathfrak{c}$. Let $\{A_\alpha\}_{\alpha<\kappa}$ be a cofinal family of non-empty mass problems with $\kappa \le \mathfrak{c}$. Let φ be the function that maps $p \in \mathbb{N}^{\mathbb{N}}$ to the least $\alpha < \kappa$ such that $\{p\} \le_{\mathrm{M}} A_\alpha$. The cofinality of $\{A_\alpha\}_{\alpha<\kappa}$ implies that

$$\mathbb{N}^{\mathbb{N}} = \bigcup_{\alpha<\kappa} \varphi^{-1}(\alpha).$$

Since every non-empty mass problem has countably many singletons in its lower cone, for every $\alpha < \kappa$ the set $\varphi^{-1}(\alpha)$ is countable. This implies $\kappa = \mathfrak{c}$ because

$$\mathfrak{c} = |\mathbb{N}^{\mathbb{N}}| = \left|\bigcup_{\alpha<\kappa} \varphi^{-1}(\alpha)\right| \le \kappa \cdot \omega = \kappa \le \mathfrak{c}.$$ □

3 Chains in the Weihrauch degrees

We now turn our attention to the study of chains in the Weihrauch degrees. As mentioned, some structural properties of the Weihrauch degrees are immediate consequences of the results obtained for the Medvedev degrees. For example, it is observed in [13, Prop. 4.2] that under $\mathsf{ZFC} + 2^{<\mathfrak{c}} = \mathfrak{c}$, there are chains of size $2^{\mathfrak{c}}$ in $\mathcal{M}$, and hence in $\mathcal{W}$. This follows from Theorem 2.5, as $2^{<\mathfrak{c}} = \mathfrak{c}$ implies that there are chains of size $2^{\mathfrak{c}}$ in $(2^{\mathfrak{c}}, \subseteq)$ ([13, Prop. 3.2]).

While the following observation is not an immediate consequence of what happens in the Medvedev degrees, its proof can be obtained by adapting the technique used in Corollary 2.6.

Proposition 3.1. *There is a chain in $\mathcal{W}$ of order type ω_1 with no upper bound.*

Proof. Let $(d_\alpha)_{\alpha<\omega_1}$ be a $\le_{\mathrm{T}}$-chain. For every α, define $f_\alpha \colon \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ as the constant map $p \mapsto d_\alpha$. Clearly, for every $\alpha < \beta$, $f_\alpha <_{\mathrm{W}} f_\beta$. Assume there is g such that for every $\alpha < \omega_1$, $f_\alpha \le_{\mathrm{W}} g$. Since there are only countably many computable functionals, there are computable functionals Φ, Ψ and distinct ordinals α, β such that the reductions $f_\alpha \le_{\mathrm{W}} g$ and $f_\beta \le_{\mathrm{W}} g$ are witnessed by Φ, Ψ. In particular, for every $p \in g(\Phi(0^\mathbb{N}))$ we have $d_\alpha = \Psi(0^\mathbb{N}, p) = d_\beta$, which is a contradiction. □

It is not hard to see that the previous proof can be adapted to show more than what is claimed. In the following, we will generalize the previous proposition and provide a characterization of when a chain of multi-valued functions admits an upper bound. We first observe the following fact:

Proposition 3.2. *Let $\mathcal{F}$ be an uncountable family of multi-valued functions. If*

$$(\forall f_0, f_1 \in \mathcal{F})(f_0 \neq f_1 \implies (\exists p \in \mathrm{dom}(f_0) \cap \mathrm{dom}(f_1))(f_0(p) \cap f_1(p) = \emptyset)),$$

then the family has no upper bound, i.e., for every g there is $f \in \mathcal{F}$ such that $f \not\le_{\mathrm{W}} g$.

Proof. Assume towards a contradiction that there is g such that for every $f \in \mathcal{F}$, $f \le_{\mathrm{W}} g$. Since $|\mathcal{F}| > \omega$, there are distinct $f_0, f_1 \in \mathcal{F}$ and computable functionals Φ, Ψ such that the reductions $f_0 \le_{\mathrm{W}} g$ and $f_1 \le_{\mathrm{W}} g$ are witnessed by Φ, Ψ. By hypothesis, we can fix $p \in \mathrm{dom}(f_0) \cap \mathrm{dom}(f_1)$ such that $f_0(p) \cap f_1(p) = \emptyset$. Clearly, for any solution $q \in g(\Phi(p))$, $\Psi(p, q)$ cannot belong to both $f_0(p)$ and $f_1(p)$, contradicting the definition of Weihrauch reduction. □

We show that a "long" chain (a chain whose order type has uncountable cofinality) admits an upper bound iff, roughly speaking, there is a cofinal subchain all of whose problems have common solutions for their inputs (Theorem 3.5). The proof is obtained by combining the following two lemmas.

Lemma 3.3. *Let P be a partial order with* $\mathrm{setcof}(P) > \omega$ *and let $\{f_x\}_{x\in P}$ be a family of partial multi-valued functions that is order-isomorphic to P. If $\{f_x\}_{x\in P}$ has an upper bound, then there is $E \subseteq P$ cofinal in P such that, letting $I_p^E := \{y \in E : p \in \mathrm{dom}(f_y)\}$,*

$$\left(\forall p \in \bigcup_{z\in E} \mathrm{dom}(f_z)\right)\left(\bigcap_{y\in I_p^E} f_y(p) \neq \emptyset\right).$$

Proof. Let g be an upper bound for $\{f_x\}_{x\in P}$. For every e, i, let $E_{e,i} := \{x \in P : f_x \le_{\mathrm{W}} g \text{ via } \Phi_e, \Phi_i\}$. Since $\bigcup_{e,i\in\mathbb{N}} E_{e,i} = P$ and $\mathrm{setcof}(P) > \omega$, there are $e, i \in \mathbb{N}$ such that $E_{e,i}$ is cofinal in P. For the sake of readability, let $E := E_{e,i}$, $\Phi := \Phi_e$, and $\Psi := \Phi_i$.

Notice that, by definition of Weihrauch reducibility, for every $p \in \bigcup_{z\in E} \mathrm{dom}(f_z)$, $\Phi(p) \in \mathrm{dom}(g)$. Moreover, for every $y \in I_p^E$ and every $q \in g(\Phi(p))$, $\Psi(p, q) \in f_y(p)$. In particular, $\bigcap_{y\in I_p^E} f_y(p) \neq \emptyset$, which concludes the proof. □

If $\mathrm{setcof}(P) = \omega$, then the previous result can be adapted obtaining that for every cardinal $\lambda < |P|$, there is $E \subseteq P$ with $|E| = \lambda$ such that for every $p \in \bigcup_{z \in E} \mathrm{dom}(f_z)$, $\bigcap_{x \in I_p^E} f_x(p) \neq \emptyset$.

We now provide a sufficient condition for a family $\{f_x\}_{x \in P}$ of problems to have an upper bound. We do not need to require that P has uncountable cofinality. However, the statement is trivial if $\mathrm{setcof}(P) = \omega$, as every countable family of problems has an upper bound.

Lemma 3.4. *Let P be a partial order and let $\{f_x\}_{x \in P}$ be a family of partial multi-valued functions that is order-isomorphic to P. For every $A \subseteq P$ and every $p \in \mathbb{N}^{\mathbb{N}}$, let $I_p^A := \{x \in A : p \in \mathrm{dom}(f_x)\}$.*

If there is a cofinal $E \subseteq P$ such that

$$\left(\forall p \in \bigcup_{z \in E} \mathrm{dom}(f_z)\right) \left(\bigcap_{y \in I_p^E} f_y(p) \neq \emptyset\right)$$

then $\{f_x\}_{x \in P}$ has an upper bound.

Proof. Notice first of all that, since E is cofinal in P, it is enough to show that $(f_z)_{z \in E}$ has an upper bound. Let $g :\subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be the problem with $\mathrm{dom}(g) := \bigcup_{z \in E} \mathrm{dom}(f_z)$ defined as

$$g(p) := \bigcap_{y \in I_p^E} f_y(p).$$

Observe that, for every $z \in E$ and every $p \in \mathrm{dom}(f_z)$, $\emptyset \neq g(p) \subseteq f_z(p)$. This shows that $(\forall z \in E)(f_z \leq_{\mathrm{W}} g)$. □

Combining the previous two lemmas, we obtain the following characterization:

Theorem 3.5. *Let P be an infinite partial order and let $\{f_x\}_{x \in P}$ be a family of partial multi-valued functions that is order-isomorphic to P. The following are equivalent:*

(1) *$\{f_x\}_{x \in P}$ has an upper bound in $\mathcal{W}$;*

(2) $\mathrm{setcof}(P) = \omega$ *or there is a cofinal $E \subseteq P$ such that for every $p \in \bigcup_{z \in E} \mathrm{dom}(f_z)$, $\bigcap_{y \in I_p^E} f_y(p) \neq \emptyset$, where $I_p^E := \{y \in E : p \in \mathrm{dom}(f_y)\}$.*

Proof. (1) ⇒ (2)**:** If $\mathrm{setcof}(P) = \omega$, then this is trivial; otherwise, it follows from Lemma 3.3.

(2) ⇒ (1)**:** Immediate by Lemma 3.4. □

For the sake of readability, we highlight the following particular case of Theorem 3.5.

Corollary 3.6. *Let $\kappa > \omega$ be a regular cardinal and let $(f_\alpha)_{\alpha < \kappa}$ be a chain of multi-valued functions. The following are equivalent:*

(1) *$(f_\alpha)_{\alpha < \kappa}$ has an upper bound in $\mathcal{W}$;*

(2) *there is a cofinal subsequence $(\overline{f}_\alpha)_{\alpha < \kappa}$ such that for every $p \in \bigcup_{\alpha < \kappa} \mathrm{dom}(\overline{f}_\alpha)$,*

$$\bigcap \{\overline{f}_\alpha(p) : p \in \mathrm{dom}(\overline{f}_\alpha)\} \neq \emptyset.$$ □

While Theorem 3.5 characterizes precisely when an infinite family of problems has an upper bound, we only have an example of a chain of order type ω_1 with no upper bound (Proposition 3.1). However, explicit examples can be easily obtained for every $\eta \leq \mathfrak{c}$ with uncountable cofinality. The same strategy can be used to obtain a chain (with no upper bound) isomorphic to any subchain of $(2^{\mathfrak{c}}, \supseteq)$ with uncountable cofinality.

Theorem 3.7. *For every ordinal $\eta \leq \mathfrak{c}$ with $\operatorname{cof}(\eta) > \omega$, there is a chain $(f_\alpha)_{\alpha<\eta}$ in $\mathcal{W}$ without upper bound.*

Proof. Let $\{p_\alpha\}_{\alpha<\eta}$ be a $\leq_{\mathrm{T}}$-antichain. For every $\alpha < \eta$, let f_α be the problem with $\operatorname{dom}(f_\alpha) := \{0^{\mathbb{N}}\}$ defined as $f_\alpha(0^{\mathbb{N}}) := \{p_\delta : \delta \geq \alpha\}$.

Clearly, for every $\beta < \alpha$, $f_\beta \leq_{\mathrm{W}} f_\alpha$ (as $f_\alpha(0^{\mathbb{N}}) \subset f_\beta(0^{\mathbb{N}})$) and $f_\alpha \not\leq_{\mathrm{W}} f_\beta$ (as, by construction, p_β does not compute any element in $f_\alpha(0^{\mathbb{N}})$). Hence, $(f_\alpha)_{\alpha<\eta}$ is a strictly increasing chain in $\mathcal{W}$ of order type η. The fact that $(f_\alpha)_{\alpha<\eta}$ does not have an upper bound follows from Theorem 3.5. Indeed, for every $E \subseteq \eta$ cofinal in η, $\bigcap_{\alpha\in E} f_\alpha(0^{\mathbb{N}}) = \emptyset$. □

We now turn our attention to studying when a sequence of problems admits a lower bound. Since the Weihrauch degrees have a bottom element, the question is only interesting when working in $\mathcal{W}_0$, i.e., when restricting our attention to non-trivial lower bounds. Observe that, analogously to what happens for the upper bounds, every family $\{f_x\}_{x\in P}$ with $\operatorname{setcoinit}(P) = \omega$ has a (non-zero) lower bound.

As already mentioned, $\mathcal{M}^*$ is (isomorphic to) an initial segment of $\mathcal{W}$. As such, examples of chains in $\mathcal{W}_0$ with no lower bound can be readily obtained using Theorem 2.5. In particular, we highlight the following corollary:

Corollary 3.8. *For every cardinal $\kappa \leq \mathfrak{c}$ with $\operatorname{cof}(\kappa) > \omega$, there is a descending sequence $(f_\alpha)_{\alpha<\kappa}$ in $\mathcal{W}_0$ with no lower bound.* □

At the same time, it is not hard to show that there are descending sequences with no lower bound that do not intersect the lower cone of id. As an explicit example, consider the following: let $\{p_\alpha\}_{\alpha<\omega_1}$ and $q \in \mathbb{N}^{\mathbb{N}}$ be such that $(p_\alpha)_{\alpha<\omega_1}$ is a $\leq_{\mathrm{T}}$-chain and, for every α, $q \not\leq_{\mathrm{T}} p_\alpha$. Define $f_\alpha := p_\alpha \mapsto q$. Clearly, $\alpha < \beta$ implies $f_\beta <_{\mathrm{W}} f_\alpha$. The fact that $(f_\alpha)_{\alpha\in\omega_1^*}$ has no lower bound in $\mathcal{W}_0$ follows from the fact that, if $g \leq_{\mathrm{W}} f_\alpha$ then $\{p_\alpha\} \leq_{\mathrm{M}} \operatorname{dom}(g)$. However, every mass problem only has countably many singletons in its lower cone. Finally, for every α, $f_\alpha \not\leq_{\mathrm{W}} \mathrm{id}$ (as $q \not\leq_{\mathrm{T}} p_\alpha$).

This example suggests the following simple observation:

Proposition 3.9. *Let $\{f_\alpha\}_{\alpha<\kappa}$ be a chain in $\mathcal{W}_0$ and for each α, let $D_\alpha := \operatorname{dom}(f_\alpha)$. The chain $\{f_\alpha\}_{\alpha<\kappa}$ has a lower bound in $\mathcal{W}_0$ iff the chain $\{D_\alpha\}_{\alpha<\kappa}$ has an upper bound in $\mathcal{M}_0$.*

Proof. If g is a lower bound for $\{f_\alpha\}_{\alpha<\kappa}$, then $\operatorname{dom}(g)$ is a lower bound for $\{D_\alpha\}_{\alpha<\kappa}$. On the other hand, if B is an upper bound for $\{D_\alpha\}_{\alpha<\kappa}$, then $g := \mathrm{id}_B$ is a lower bound for $\{f_\alpha\}_{\alpha<\kappa}$. □

Despite the fact that a chain with no lower bound in $\mathcal{W}_0$ can live outside the lower cone of id, we now show that, when restricting our attention to chains of order type κ^* for some regular κ, there is a precise correspondence between the existence of chains in $\mathcal{W}_0$ with no lower bound and the existence of chains in $\mathcal{M}_0$ with no upper bound.

Theorem 3.10. *For every regular cardinal κ, the following are equivalent:*

(1) *there is a sequence in $\mathcal{W}_0$ of order type κ^* which is unbounded below;*

(2) *there is a sequence in $\mathcal{M}_0$ of order type κ which is unbounded above.*

Proof. The implication (2) ⇒ (1) is trivial as the lower cone of id in $\mathcal{W}_0$ is reverse isomorphic to $\mathcal{M}_0$. For the direction (1) ⇒ (2), let $(f_\alpha)_{\alpha\in\kappa^*}$ be a descending sequence in $\mathcal{W}_0$ with no lower bound. In particular, $\kappa > \omega$. Let $D_\alpha := \operatorname{dom}(f_\alpha)$. It follows from the definition of Weihrauch reducibility that the family $\{D_\alpha\}_{\alpha<\kappa}$ is a chain in $\mathcal{M}_0$. If $\{D_\alpha\}_{\alpha<\kappa}$ has order type κ, then we are done (by Theorem 2.5). Assume towards a contradiction that $\{D_\alpha\}_{\alpha<\kappa}$ is order-isomorphic to L with $|L| < \kappa$. In other words, let $(D_x)_{x\in L}$ be a subchain of $\{D_\alpha\}_{\alpha<\kappa}$ such that for every $\alpha < \kappa$, there is $x \in L$ such that $D_x \equiv_{\mathrm{M}} D_\alpha$. For every $x \in L$, let $M_x := \{\alpha < \kappa : D_x \equiv_{\mathrm{M}} D_\alpha\}$. Since κ is regular, there is $x \in L$ such that $|M_x| = \kappa$. In particular, since M_x is cofinal in κ and $\{D_\alpha\}_{\alpha<\kappa}$ is a chain, there is α_0 such that for every $\alpha > \alpha_0$, $D_x \equiv_{\mathrm{M}} D_\alpha$. This implies that x is the top element of L, and therefore the sequence $(D_x)_{x\in L}$ has an upper bound in $\mathcal{M}_0$. We have now reached a contradiction, as Proposition 3.9 implies that the sequence $(f_\alpha)_{\alpha\in\kappa^*}$ has a lower bound in $\mathcal{W}_0$. □

Having discussed the conditions under which a chain possesses an upper bound or a lower bound, we now briefly examine when two families of problems admit an intermediate degree.

Theorem 3.11. *Let P, Q be two partial orders with* $\mathrm{setcof}(P) > \omega$ *and* $\mathrm{setcoinit}(Q) > \omega$. *Let also $\{f_x\}_{x\in P}$ and $\{h_z\}_{z\in Q}$ be a two families in $\mathcal{W}$ that are order-isomorphic, respectively, to P and Q, and such that, for each $x \in P$ and $z \in Q$, $f_x \le_{\mathrm{W}} h_z$. Then the following are equivalent:*

1. *there is g such that for every $x \in P$ and $z \in Q$, $f_x \le_{\mathrm{W}} g \le_{\mathrm{W}} h_z$;*
2. *there are two computable functionals Φ, Ψ and two sets $X \subseteq P$ and $Z \subseteq Q$ cofinal in P and coinitial in Q, respectively, such that, for every $x \in X$ and $z \in Z$, $f_x \le_{\mathrm{W}} h_z$ via Φ, Ψ.*

Proof. For (1) ⇒ (2), the set X can be obtained as in the proof of Lemma 3.3 using the upper bound g for the family $\{f_x\}_{x\in P}$. In particular, there are $e, i \in \mathbb{N}$ such that for every $x \in X$, $f_x \le_{\mathrm{W}} g$ via Φ_e, Φ_i. The argument for obtaining Z is symmetrical: since g is a lower bound for $\{h_z\}_{z\in Q}$ and $\mathrm{setcoinit}(Q) > \omega$, there are $n, k \in \mathbb{N}$ and a coinitial set $Z \subseteq Q$ such that for every $z \in Z$, $g \le_{\mathrm{W}} h_z$ via Φ_n, Φ_k. The maps $\Phi := \Phi_n \circ \Phi_e$ and $\Psi := (p, q) \mapsto \Phi_i(p, \Phi_k(\Phi_e(p), q))$ (the compositions of the functionals witnessing the reductions $f_x \le_{\mathrm{W}} g$ and $g \le_{\mathrm{W}} h_z$) are the desired functionals.

To show that (2) ⇒ (1), fix Φ, Ψ, X, and Z as in the hypotheses. We can define g as follows: $\mathrm{dom}(g) := \bigcup_{x\in X} \mathrm{dom}(f_x)$ and $g(p) := \bigcup_{z\in Z} h(\Phi(p))$. Observe that for every $x \in X$, $f_x \le_{\mathrm{W}} g$ via the maps id and Ψ. Moreover, for every $z \in Z$, $g \le_{\mathrm{W}} h_z$ via Φ and $\pi_2 := (p, q) \mapsto q$. The claim follows from the fact that X and Z are cofinal in P and coinitial in Q, respectively. □

Observe that, unlike what happens when studying upper/lower bounds, the statement is not trivial if $\mathrm{setcof}(P) = \omega$ or $\mathrm{setcoinit}(Q) = \omega$. In particular, while there are chains of order type $\omega + 1 + \omega^*$ in $\mathcal{W}$, it is not clear whether every $\omega + \omega^*$ chain is extendible to an $\omega + 1 + \omega^*$ chain.

We mention, however, that no interval in the Weihrauch degrees can have cardinality ω.

Theorem 3.12. *Every infinite interval in $\mathcal{W}$ is uncountable.*

Proof. Fix $h <_{\mathrm{W}} f$ and assume that the interval (h, f) is infinite. If the interval $(\mathrm{dom}(f), \mathrm{dom}(h))$ in $\mathcal{M}$ is infinite, then, by [13, Thm. 2.10], there is an antichain of size $2^{\mathfrak{c}}$ between $\mathrm{dom}(f)$ and $\mathrm{dom}(h)$. Fix any such antichain $\{A_\alpha\}_{\alpha<2^{\mathfrak{c}}}$. We can define $2^{\mathfrak{c}}$ many problems $\{g_\alpha\}_{\alpha<2^{\mathfrak{c}}}$ between h and f as follows: for every α, define $g_\alpha\colon \mathrm{dom}(f) \times A_\alpha \rightrightarrows \mathbb{N}^{\mathbb{N}}$ as $g_\alpha(p, q) := f(p)$. Observe that $\mathrm{dom}(g_\alpha) = \mathrm{dom}(f) \vee A_\alpha \equiv_{\mathrm{M}} A_\alpha$, hence $f \not\le_{\mathrm{W}} g_\alpha$ and $g_\alpha \not\le_{\mathrm{W}} h$. On the other hand, $g_\alpha \le_{\mathrm{W}} f$ (trivially) and $h \le_{\mathrm{W}} g_\alpha$ via the maps $t \mapsto (\Phi(t), \Gamma_\alpha(t))$ and Ψ, where Φ and Ψ are the functionals witnessing $h \le_{\mathrm{W}} f$ and Γ_α witnesses $A_\alpha \le_{\mathrm{M}} \mathrm{dom}(h)$. Moreover, the family $\{g_\alpha\}_{\alpha<2^{\mathfrak{c}}}$ is a $\le_{\mathrm{W}}$-antichain (because their domains are Medvedev-incomparable). This implies that the interval (h, f) has size $2^{\mathfrak{c}}$.

Assume now that the interval $(\mathrm{dom}(f), \mathrm{dom}(h))$ is finite. Assume towards a contradiction that $|(\mathrm{dom}(f), \mathrm{dom}(h))| = \omega$, and choose a representative g_n for each intermediate degree.

Claim: Without loss of generality, we can assume that f is not the minimal cover of any g_n.

Indeed, assume that this is not the case and let $I := \{i \in \mathbb{N} : f \text{ is a minimal cover of } g_i\}$. In particular, for every $i \neq j$, $f \equiv_{\mathrm{W}} g_i \sqcup g_j$ (negating this would lead to a contradiction with the fact that f is a minimal cover of g_i). As a consequence of [8, Thm. 1.4], if f is a minimal cover of g_i, then $\mathrm{dom}(f) <_{\mathrm{M}} \mathrm{dom}(g_i)$. If $\mathrm{dom}(g_i) \equiv_{\mathrm{M}} \mathrm{dom}(g_j)$ then $\mathrm{dom}(f) \equiv_{\mathrm{M}} \mathrm{dom}(g_i) \wedge \mathrm{dom}(g_j) \equiv_{\mathrm{M}} \mathrm{dom}(g_i)$, against the fact that $\mathrm{dom}(f) <_{\mathrm{M}} \mathrm{dom}(g_i)$. This implies that for every $i \neq j$, $\mathrm{dom}(g_i) \not\equiv_{\mathrm{M}} \mathrm{dom}(g_j)$, and therefore $|I| < \omega$ (otherwise, the Medvedev-interval $(\mathrm{dom}(f), \mathrm{dom}(h))$ would be infinite). By the pigeonhole principle, there must be $i \in I$ such that the Weihrauch-interval (h, g_i) is countable. If g_i is not the minimal cover of any problem in the interval (h, g_i), then we are done, as we can replace f with g_i and conclude the proof of the claim. Otherwise, we repeat the same argument with g_i in place of f. We observe that this procedure is bound to terminate. Indeed, if not, we are defining a sequence of minimal covers $f >_{\mathrm{W}} g_{i_0} >_{\mathrm{W}} g_{i_1} >_{\mathrm{W}} \dots$. In particular, since $\mathrm{dom}(f) <_{\mathrm{M}} \mathrm{dom}(g_{i_0}) <_{\mathrm{M}} \mathrm{dom}(g_{i_1}) <_{\mathrm{M}} \dots$, this would imply that the Medvedev-interval $(\mathrm{dom}(f), \mathrm{dom}(h))$ is infinite, which is a contradiction, and hence the claim is proved.

This implies that the family $\{g_n : n \in \mathbb{N}\}$ does not have a supremum (otherwise, by Theorem 1.1, there would be $m \in \mathbb{N}$ such that $g_m = \sup_n \{g_n : n \in \mathbb{N}\} <_{\mathrm{W}} f$ is a minimal cover). In particular, since f is not the supremum of $\{g_n : n \in \mathbb{N}\}$, there is $\overline{f} \not\geq_{\mathrm{W}} f$ such that, for every n, $g_n \leq_{\mathrm{W}} \overline{f}$, so $g_n <_{\mathrm{W}} \overline{f}$.

Consider now the problem $f \sqcap \overline{f}$: clearly, $f \sqcap \overline{f} <_{\mathrm{W}} f$ and for every n, $g_n <_{\mathrm{W}} f \sqcap \overline{f}$ (otherwise g_n would be the supremum of $\{g_m : m \in \mathbb{N}\}$). This contradicts the fact that every degree in the interval (h, f) was represented by some g_n, and therefore concludes the proof. □

Observe that examples of finite intervals can be easily obtained as a corollary of [8, Cor. 2.6]: indeed, for every $n \in \mathbb{N}$, there is a problem h with exactly n minimal covers. The Boolean algebra obtained by considering the joins of the finitely many minimal covers of h yields an interval of size 2^n.

Notice also that the previous proof uses Theorem 1.1 in a critical way to run a classical diagonal argument. However, the following result shows that Theorem 1.1 cannot be extended to $\mathfrak{c}$-sized chains, and hence the above strategy does not immediately yield that no $\mathfrak{c}$-sized interval exists in the Weihrauch degrees.

Proposition 3.13. *For every $\kappa \leq \mathfrak{c}$ with $\mathrm{cof}(\kappa) > \omega$, there is a chain of order type κ in $\mathcal{W}$ that admits a supremum.*

Proof. Let $\{p_\alpha\}_{\alpha<\kappa+1}$ be a κ-sized $\leq_{\mathrm{T}}$-antichain. For every $\alpha \leq \kappa$, define $A_\alpha := \{p_\gamma : \gamma < \alpha\}$. Observe that $\{A_\alpha\}_{0<\alpha<\kappa}$ is a chain in $\mathcal{M}_0$ of order type κ^* and $A_\kappa = \inf_{\leq_{\mathrm{M}}}\{A_\alpha : \alpha < \kappa\}$. Indeed, whenever $0 < \alpha < \beta < \kappa$, the reductions $A_\kappa \leq_{\mathrm{M}} A_\beta \leq_{\mathrm{M}} A_\alpha$ are trivial and the separation $A_\alpha \not\leq_{\mathrm{M}} A_\beta$ follows from the fact that $\{p_\alpha\}_{\alpha<\eta}$ is an antichain (in particular, $A_\alpha \not\leq_{\mathrm{M}} \{p_\beta\}$). Similarly, $A_\beta \not\leq_{\mathrm{M}} A_\kappa$. To show that A_κ is the greatest lower bound, assume that for every $\alpha < \kappa$, $B \leq_{\mathrm{M}} A_\alpha$. Then, since $\mathrm{cof}(\kappa) > \omega$, there is $e \in \mathbb{N}$ and a coinitial subsequence $(\overline{A_\alpha})_{\alpha<\mathrm{cof}(\kappa)}$ such that, for every α, $B \leq_{\mathrm{M}} \overline{A_\alpha}$ via Φ_e. Since $A_\kappa = \bigcup_{\alpha<\kappa} A_\alpha = \bigcup_{\alpha<\mathrm{cof}(\kappa)} \overline{A_\alpha}$, it follows that $B \leq_{\mathrm{M}} A_\kappa$.

For every $\alpha \leq \kappa$, we define $f_\alpha \colon A_\alpha \rightrightarrows \mathbb{N}^{\mathbb{N}}$ as $f_\alpha(p) := \{p_\delta : \delta \geq \alpha\}$. It is easy to see that $\{f_\alpha\}_{\alpha\leq\kappa}$ is a chain of order type $\kappa+1$. Indeed, if $\alpha < \beta \leq \kappa$, then $f_\alpha \leq_{\mathrm{W}} f_\beta$ is straightforward (for every $p \in A_\alpha$, $f_\alpha(p) \supset f_\beta(p) \neq \emptyset$), while $f_\beta \not\leq_{\mathrm{W}} f_\alpha$ follows from $A_\alpha \not\leq_{\mathrm{M}} A_\beta$.

To conclude the proof, let h be an upper bound for $(f_\alpha)_{\alpha<\kappa}$. As in the proof of Lemma 3.3, there are $e, i \in \mathbb{N}$ and a cofinal subsequence $(\overline{f}_\alpha)_{\alpha<\mathrm{cof}(\kappa)}$ such that for every $\alpha < \mathrm{cof}(\kappa)$, $\overline{f}_\alpha \leq_{\mathrm{W}} h$ via Φ_e, Φ_i.

We claim that $f_\kappa \leq_{\mathrm{W}} h$ via Φ_e, Φ_i. Indeed, for every $p \in \mathrm{dom}(f_\kappa) = \bigcup_{\alpha<\mathrm{cof}(\kappa)} \mathrm{dom}(\overline{f}_\alpha)$, $\Phi_e(p) \in \mathrm{dom}(h)$. Moreover, for every $\alpha < \mathrm{cof}(\kappa)$ and every $q \in h(\Phi_e(p))$, $\Phi_i(p, q) \in \overline{f}_\alpha(p)$, hence

$$\Phi_i(p, q) \in \bigcap_{\alpha<\mathrm{cof}(\kappa)} \overline{f}_\alpha(p) = \{p_\delta : \delta \geq \kappa\} = f_\kappa(p).$$ □

3.1 Cofinality and coinitiality

In this section, we study the cofinality and the coinitiality of the Weihrauch degrees. We first observe that, while for the Turing and the Medvedev degrees the existence of a cofinal chain is independent of ZFC and equivalent to CH, it is provable in ZFC that there are no cofinal chains (of any order type) in $\mathcal{W}$. To prove this, we first show that $\mathrm{setcof}(\mathcal{W}) > \mathfrak{c}$.

Lemma 3.14. *For every family $\mathcal{F}$ of multi-valued functions with $|\mathcal{F}| \le \mathfrak{c}$ and every infinite $A \subseteq \mathbb{N}^\mathbb{N}$ with $|A| \ge |\mathcal{F}|$, there is $g\colon A \rightrightarrows \mathbb{N}^\mathbb{N}$ such that for every $f \in \mathcal{F}$, $g \not\le_{\mathrm{W}} f$.*

Proof. Let $X \subseteq \mathbb{N}^\mathbb{N}$ be such that $|X| = |\mathcal{F}|$ and let $\mathcal{F} = \{f_x : x \in X\}$. Let also $\varphi\colon \mathbb{N} \times \mathbb{N} \times X \to A$ be an injective function.

Intuitively, we define $g\colon A \rightrightarrows \mathbb{N}^\mathbb{N}$ so that $\varphi(e, i, x)$ is the input for g witnessing the fact that $g \not\le_{\mathrm{W}} f_x$ via Φ_e, Φ_i. More precisely, for every $e, i \in \mathbb{N}$ and every $x \in X$, we define g on $p := \varphi(e, i, x)$ as follows: if $\Phi_e(p) \notin \mathrm{dom}(f_x)$ or there is $q \in f_x(\Phi_e(p))$ such that $(p, q) \notin \mathrm{dom}(\Phi_i)$ then $g(p) := \mathbb{N}^\mathbb{N}$. Otherwise, fix $q \in f_x(\Phi_e(p))$ and define $g(p) := \mathbb{N}^\mathbb{N} \setminus \{\Phi_i(p, q)\}$. Finally, define $g(p) := \mathbb{N}^\mathbb{N}$ for every $p \notin \mathrm{ran}(\varphi)$.

By construction, it is immediate to check that for every $x \in X$ and every $e, i \in \mathbb{N}$, $g \not\le_{\mathrm{W}} f_p$ via Φ_e, Φ_i as witnessed by the input $\varphi(e, i, x)$ for g. □

As an immediate consequences of Lemma 3.14, we obtain:

Corollary 3.15. $\mathrm{setcof}(\mathcal{W}) > \mathfrak{c}$. □

Corollary 3.16. *No embedding of $\mathcal{M}_0$ in the Weihrauch degrees can be cofinal in the Weihrauch degrees.*

Proof. This follows from the previous result and the fact that $\mathrm{setcof}(\mathcal{M}_0) = \mathfrak{c}$ (Theorem 2.11). □

Theorem 3.17. *There are no cofinal chains in $\mathcal{W}$.*

Proof. Observe first of all that every cofinal chain contains a well-ordered cofinal chain, hence, without loss of generality, we can restrict our attention to well-ordered chains.

Let $(f_\beta)_{\beta<\kappa}$ be a cofinal chain in $\mathcal{W}$. Notice that if $\mathrm{cof}(\kappa) < \kappa$, then the chain $(f_\beta)_{\beta<\kappa}$ contains a cofinal chain of size $\mathrm{cof}(\kappa)$. Assume therefore that κ is a regular cardinal. Since $\mathrm{setcof}(\mathcal{W}) > \mathfrak{c}$ (Corollary 3.15), $\kappa > \mathfrak{c}$. Fix a chain $(g_\alpha)_{\alpha<\omega_1}$ in $\mathcal{W}$ with no upper bound (as in Proposition 3.1 or Theorem 3.7). Since the chain $(f_\beta)_{\beta<\kappa}$ is cofinal in $\mathcal{W}$, for every $\alpha < \omega_1$ there is $\beta_\alpha < \kappa$ such that $g_\alpha \le_{\mathrm{W}} f_{\beta_\alpha}$. Since $(g_\alpha)_{\alpha<\omega_1}$ has no upper bound, the sequence $(\beta_\alpha)_{\alpha<\omega_1}$ must be cofinal in κ, contradicting the fact that $\kappa = \mathrm{cof}(\kappa) > \mathfrak{c}$. □

Finally, we observe that the coinitiality of the Weihrauch degrees is closely connected with the cofinality of the Medvedev degrees. We first highlight the following fact:

Proposition 3.18. *There is a continuum-sized family $\{f_p\}_{p\in\mathbb{N}^\mathbb{N}}$ such that for every p, $f_p \le_{\mathrm{W}} \mathrm{id}$ and for every non-empty g, there is $p \in \mathbb{N}^\mathbb{N}$ such that $f_p \le_{\mathrm{W}} g$.*

Proof. For every $p \in \mathbb{N}^\mathbb{N}$, we define $f_p\colon \{p\} \rightrightarrows \mathbb{N}^\mathbb{N}$ as $f_p(p) := \mathbb{N}^\mathbb{N}$. It is trivial to see that $f_p \le_{\mathrm{W}} \mathrm{id}$ and, for every non-empty g and for every $p \in \mathrm{dom}(g)$, $f_p \le_{\mathrm{W}} g$. □

Theorem 3.19. *The set-coinitiality of $\mathcal{W}_0$ is $\mathfrak{c}$. Moreover, the following are equivalent:*

1. CH*;*
2. *there is a coinitial chain in $\mathcal{W}_0$;*
3. *there is a coinitial chain in $\mathcal{W}_0$ of order type ω_1.*

Proof. As a corollary of Proposition 3.18, the lower cone of id is a coinitial subset of $\mathcal{W}_0$. Since the lower cone of id is isomorphic to $\mathcal{M}^*$, and every coinitial set in $\mathcal{W}_0$ must, a fortiori, be coinitial in the lower cone of id, we immediately have $\mathrm{setcoinit}(\mathcal{W}_0) = \mathrm{setcof}(\mathcal{M}_0) = \mathfrak{c}$.

The second part of the statement is a simple consequence of Theorem 2.10, as the existence of a coinitial chain in $\mathcal{W}_0$ is equivalent to the existence of a cofinal chain in $\mathcal{M}_0$. □

4 Antichains

In this section, we prove some results on the size and extendibility of antichains in the Weihrauch degrees. Clearly, given that the Medvedev degrees embed as a lattice in the Weihrauch degrees, every antichain in the Medvedev degrees immediately gives an antichain in the Weihrauch degrees. In particular, since there are antichains of size $2^{\mathfrak{c}}$ in $\mathcal{M}$ ([10]), we immediately obtain that the same holds for $\mathcal{W}$ as well. In fact, such antichains can be found "everywhere" in the Weihrauch lattice.

Proposition 4.1. *For every problem $f \neq \emptyset$, there is an antichain $\mathcal{A}$ in $\mathcal{W}$ of size $2^{\mathfrak{c}}$ with $f \in \mathcal{A}$.*

Proof. Fix a problem f and let $p \in \mathbb{N}^{\mathbb{N}}$ be such that $\operatorname{dom}(f) <_{\mathrm{M}} \{p\}$. Let also $\{p_\alpha : \alpha < \mathfrak{c}\}$ be the set of minimal degrees above p. As in [10] (see also [12, Thm. 4.1]), we can define a $\leq_{\mathrm{M}}$-antichain $\{A_\beta\}_{\beta<2^{\mathfrak{c}}}$ where $A_\beta \subset \{p_\alpha : \alpha < \mathfrak{c}\}$. For every $\beta < 2^{\mathfrak{c}}$, define g_β as the problem obtained applying Lemma 3.14 to $\mathcal{F} = \{f\}$ and $A = \{(e,i)^\frown p : e, i \in \mathbb{N} \text{ and } p \in A_\beta\}$. In particular, this guarantees that $g_\beta \not\leq_{\mathrm{W}} f$.

To conclude the proof, observe that, for every $\beta \neq \gamma$, $g_\beta \mid_{\mathrm{W}} g_\gamma$ because $\operatorname{dom}(g_\beta) \equiv_{\mathrm{M}} A_\beta \mid_{\mathrm{M}} A_\gamma \equiv_{\mathrm{M}} \operatorname{dom}(g_\gamma)$. Similarly, $f \not\leq_{\mathrm{W}} g_\beta$ because $A_\beta \not\leq_{\mathrm{M}} \operatorname{dom}(f)$. □

At the same time, it is trivial to observe that the image of a maximal antichain in $\mathcal{M}$ need not be maximal in $\mathcal{W}$. In fact, Dzhafarov, Lerman, Patey, and Solomon proved:

Proposition 4.2 ([5])**.** *For every countable family $\{f_n\}_{n\in\mathbb{N}}$ of non-trivial problems, there is g such that for every n, $g \mid_{\mathrm{W}} f_n$.* □

In particular, this implies that every countable antichain in $\mathcal{W}$ is not maximal, hence the analog of Proposition 2.3 does not hold for the Weihrauch degrees.

However, observe that the previous proposition cannot be extended to $\mathfrak{c}$-sized families. This is an immediate consequence of Proposition 3.18. At the same time, the set of problems defined in the proof of Proposition 3.18 is far from being an antichain. This is because no antichain can be coinitial in $\mathcal{W}_0$, as the bottom $\emptyset$ is meet-irreducible (in other words, the meet of any two elements of the antichain is always a non-empty problem that is not above any element of the antichain).

We now provide some sufficient conditions that guarantee the extendibility of an antichain in $\mathcal{W}$. For the sake of the presentation, we first state and prove the following Theorem 4.3, while weaker sufficient conditions will be provided in Theorem 4.6.

Theorem 4.3. *If $\{f_\alpha\}_{\alpha<\mathfrak{c}}$ is an antichain in $\mathcal{W}$ such that $\{\operatorname{dom}(f_\alpha)\}_{\alpha<\mathfrak{c}}$ is not cofinal in $\mathcal{M}_0$, then there is g such that for every $\alpha < \mathfrak{c}$, $g \mid_{\mathrm{W}} f_\alpha$.*

Proof. Since the family $\{\operatorname{dom}(f_\alpha)\}_{\alpha<\mathfrak{c}}$ is not cofinal in $\mathcal{M}_0$, there is $D \subset \mathbb{N}^{\mathbb{N}}$ such that for every $\alpha < \mathfrak{c}$, $D \not\leq_{\mathrm{M}} \operatorname{dom}(f_\alpha)$. Without loss of generality, we can assume that $|D| = \mathfrak{c}$ (as $D \equiv_{\mathrm{M}} D \times \mathbb{N}^{\mathbb{N}}$).

Thus, if g is a problem with $\operatorname{dom}(g) = D$, then for every $\alpha < \mathfrak{c}$, $f_\alpha \not\leq_{\mathrm{W}} g$. To conclude the proof, it is enough to define g as the problem obtained applying Lemma 3.14 to $\mathcal{F} = \{f_\alpha\}_{\alpha<\mathfrak{c}}$ and $A = D$. □

Corollary 4.4. *No antichain $\{f_\alpha\}_{\alpha<\kappa}$ in $\mathcal{W}$ with $\kappa < \mathfrak{c}$ is maximal.*

Proof. This follows from the fact that no family of size $< \mathfrak{c}$ is cofinal in $\mathcal{M}_0$ (Theorem 2.11). □

Observe also that no antichain $\mathcal{A}$ can be cofinal in $\mathcal{M}_0$. Indeed, since there is no maximal element in $\mathcal{M}_0$, for every $A \in \mathcal{A}$ there is a non-empty mass problem B with $A <_{\mathrm{M}} B$. In particular, since $\mathcal{A}$ is an antichain, for every $C \in \mathcal{A}$ we have $B \not\leq_{\mathrm{M}} C$. This simple observation, together with Theorem 4.3, immediately yields:

Corollary 4.5. *For every antichain $\mathcal{F}$ in $\mathcal{W}$ with $|\mathcal{F}| \leq \mathfrak{c}$, if $\{D \subseteq \mathbb{N}^{\mathbb{N}} : (\exists f \in \mathcal{F})(D = \operatorname{dom}(f))\}$ is an antichain in $\mathcal{M}$, then $\mathcal{F}$ is not maximal.* □

We conclude this section by stating and proving the following generalization of Theorem 4.3.

Theorem 4.6. *Let $\{f_\alpha\}_{\alpha<\mathfrak{c}}$ be an antichain in $\mathcal{W}$ and let $D_\alpha := \operatorname{dom}(f_\alpha)$. If the set $\mathcal{B} := \{D_\beta : (\forall x \in D_\beta)(f_\beta(x)$ has a computable element$)\}$ is not cofinal in $\mathcal{M}_0$, then $\{f_\alpha\}_{\alpha<\mathfrak{c}}$ is not maximal.*

Proof. Assume first of all that $\{D_\alpha\}_{\alpha<\mathfrak{c}}$ is a cofinal set in $\mathcal{M}_0$ (otherwise, the claim follows by Theorem 4.3). Let $\mathcal{C} := \{D_\delta : (\forall\alpha)(D_\alpha \not\equiv_{\mathrm{M}} D_\delta \to D_\alpha \not\leq_{\mathrm{M}} D_\delta)\}$ be the set of $\leq_{\mathrm{M}}$-minimal elements in $\{D_\alpha\}_{\alpha<\mathfrak{c}}$.

Fix γ so that $D_\gamma \notin \mathcal{C}$ and for every $D_\beta \in \mathcal{B}$, $D_\gamma \not\leq_{\mathrm{M}} D_\beta$. Observe that such γ exists: since $\{D_\alpha\}_{\alpha<\mathfrak{c}}$ is a cofinal set in $\mathcal{M}_0$, $\mathcal{C}$ cannot be cofinal in $\{D_\alpha\}_{\alpha<\mathfrak{c}}$, as the $\leq_{\mathrm{M}}$-minimal elements in $\{D_\alpha\}_{\alpha<\mathfrak{c}}$ form a $\leq_{\mathrm{M}}$-antichain, and no antichain can be cofinal in $\mathcal{M}_0$. This implies that $\mathcal{B}\cup\mathcal{C}$ is not cofinal in $\mathcal{M}_0$ (in an upper semilattice, the union of two non-cofinal sets is not cofinal), and hence we can find γ as above.

Let $D \equiv_{\mathrm{M}} D_\gamma$ be such that $|D| = \mathfrak{c}$, and fix a bijection $\varphi\colon \mathbb{N} \times D_\gamma \times \mathfrak{c} \times \mathbb{N} \times \mathbb{N} \to D$. We define the problem $g\colon D \to \mathbb{N}$ as follows: fix $p \in D_\gamma$, $\alpha < \mathfrak{c}$, and $e, i \in \mathbb{N}$. If, for every $n \in \mathbb{N}$, $\Phi_e(\varphi(n, p, \alpha, e, i)) \downarrow\in D_\alpha$ and there is $b = b(n)$ such that

$$(\forall y \in f_\alpha \circ \Phi_e \circ \varphi(n, p, \alpha, e, i))(\Phi_i(\varphi(n, p, \alpha, e, i), y) = b(n)),$$

then define $g(\varphi(n, p, \alpha, e, i)) := 1 - b(n)$. Otherwise, for every n, we define $g(\varphi(n, p, \alpha, e, i)) := 0$.

We now show that, for every $\alpha < \mathfrak{c}$, $g \mid_{\mathrm{W}} f_\alpha$. We first show that for every α, $f_\alpha \not\leq_{\mathrm{W}} g$. Observe that if $D_\alpha \notin \mathcal{B}$, then f_α has an input x with no computable element (while g only has computable solutions), hence $f_\alpha \not\leq_{\mathrm{W}} g$. If $D_\alpha \in \mathcal{B}$, then $f_\alpha \not\leq_{\mathrm{W}} g$ follows immediately from $\operatorname{dom}(g) \equiv_{\mathrm{M}} D_\gamma \not\leq_{\mathrm{M}} D_\alpha$.

It remains to prove that for every α, $g \not\leq_{\mathrm{W}} f_\alpha$. Fix α and assume that $g \leq_{\mathrm{W}} f_\alpha$ via Φ_e, Φ_i. Observe that, if for every $n \in \mathbb{N}$ there is $b = b(n)$ such that for every $y \in f_\alpha \circ \Phi_e \circ \varphi(n, p, \alpha, e, i)$, $\Phi_i(\varphi(n, p, \alpha, e, i), y) = b(n)$, then by construction, $g(\varphi(n, p, \alpha, e, i)) \neq b(n)$. On the other hand, let $n \in \mathbb{N}$ and $y_1, y_2 \in f_\alpha \circ \Phi_e \circ \varphi(n, p, \alpha, e, i)$ be such that $\Phi_i(\varphi(n, p, \alpha, e, i), y_1) \neq \Phi_i(\varphi(n, p, \alpha, e, i), y_2)$. Since g is single-valued, at least one of the two produced solutions is incorrect, and this concludes the proof. □

5 Open questions

In this paper, we explored some questions on the existence of chains and antichains in the Weihrauch degrees, but many more remain open and require further investigation. In Section 3, we provided a characterization of when "long" chains are extendible (Corollary 3.6). However, Theorem 3.7 and Proposition 3.13 only provide explicit examples of chains of size $\mathfrak{c}$. While it follows from the results on the Medvedev degrees that the existence of a chain of size $2^{\mathfrak{c}}$ is consistent with ZFC, a natural question is the following:

Open Question 5.1. Under ZFC, is there a chain of size $2^{\mathfrak{c}}$ in $\mathcal{W}$? More generally, is every chain of size $\kappa < 2^{\mathfrak{c}}$ extendible?

Likewise, while we established a connection between chains with no lower bound in $\mathcal{W}_0$ and chains with no upper bound in $\mathcal{M}_0$ (Theorem 3.10), this only applies to well-founded sequences. It is known that there are no well-founded sequences in $\mathcal{M}$ of size $2^{\mathfrak{c}}$ (as a consequence of Theorem 2.5). This suggests the following question:

Open Question 5.2. What are the possible order types of chains in $\mathcal{W}_0$ with no lower bound? Is there an order type L such that the existence of a chain in $\mathcal{W}_0$ of order type L with no lower bound is independent of ZFC?

Another interesting question concerns the chains of order type $\omega+\omega^*$. Theorem 3.11 characterizes when two families of problems $\{f_x\}_{x\in P}$ and $\{h_z\}_{z\in Q}$ have an intermediate degree (i.e., some g such that, for every x and z, $f_x \leq_{\mathrm{W}} g \leq_{\mathrm{W}} h_z$). However, this only applies to families with uncountable cofinality/coinitiality.

Open Question 5.3. Is there an $\omega + \omega^*$ chain in $\mathcal{W}$ with no intermediate degree?

In Section 3.1, we showed that there are no cofinal chains in $\mathcal{W}$ (Theorem 3.17) and that the existence of coinitial chains is equivalent to CH (Theorem 3.19). While the set-coinitiality of $\mathcal{W}_0$ is $\mathfrak{c}$, we only obtained a lower bound for its set-cofinality.

Open Question 5.4. Is $\operatorname{setcof}(\mathcal{W}) = 2^{\mathfrak{c}}$?

In Section 4, we studied the extendibility of antichains in $\mathcal{W}$ and provided some sufficient conditions under which a $\mathfrak{c}$-sized antichain is extendible (Theorem 4.6). A natural question that has proven challenging to fully resolve is the following:

Open Question 5.5. Characterize the maximal antichains in the Weihrauch degrees. Is there a maximal antichain of size $\mathfrak{c}$?

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