6. Control of Spaces. Our requirements will be of the form $(\varphi \to \psi)$ & $(\neg\varphi \to \chi)$. If φ seems to be true at a given stage of the construction, we take action to preserve the truth of φ, to make ψ true, and to preserve its truth. If φ seems to be false, we try to satisfy χ and to preserve its truth. We will define a recursive true path $\Lambda^0 \in [T^0]$ for the construction. Action taken for ψ and χ is determined by nodes $\xi \subset \Lambda^0$, which try to declare axioms for points in the space controlled by ξ, according to the apparent truth of φ. Thus we will assign spaces S (sets of points which have geometric dimension) to the node ξ, define a functional Δ_ξ, and try to arrange that the value m for the axiom $\Delta_\xi(A;\bar{x},x) = m$, where $OS(\xi) = A$, is determined by the truth or falsity of a sentence M_ξ associated with ξ for sufficiently many $\langle\bar{x},x\rangle$ such that $\langle\bar{x},s,x\rangle \in S$. (The coordinate s represents a stage of the construction rather than an argument for a functional, so we separate it.) In this case, ξ will *control* S. The exact definition of control will vary with the type of ξ, but we will try to present the definitions of control for the three types of requirements in as uniform a way as possible. Fix a requirement $R = R^{j,r}_{e,b,c}$ for the remainder of this section, and so consider the type j and the dimension r of this requirement to be fixed.

Definition 6.1: The spaces assigned to requirements of type j are specified as follows. Given $\xi \in T^k$, let $\zeta = up^n(\xi)$. Suppose that $R = R^{j,r}_{e,b,c}$ is assigned to ζ. The space S_ξ will be defined only if $k = r$, in which case we set $S_\xi = \mathbf{N}^r \times \{wt(\xi)\} \times \{\xi\}$, $wt(S_\xi) = wt(\xi)$, and $\dim(S_\xi) = r$ if $j = 0$; we set $S_\xi = \mathbf{N}^{r+1} \times \{e\} \times \{\xi\}$, $wt(S_\xi) = e$, and $\dim(S_\xi) = r+1$ if $j = 1$; and we set $S_\xi = \mathbf{N}^r \times \{e\} \times \{\xi\}$, $wt(S_\xi) = e$, and $\dim(S_\xi) = r$ if $j = 2$. Whenever we specify a *section* $S = \{\langle x_1,\dots,x_{r-k}\rangle\} \times \mathbf{N}^u \times \{\langle x,\xi\rangle\}$ of S_ξ, we define $\dim(S) = u$, and $wt(S) = x_{r-k}$ if $r > k$. For each $i \in [k,r]$, we let $up^i(S) = \{\langle x_1,\dots,x_{r\pm i}\rangle\} \times \mathbf{N}^i \times \{\langle x,\xi\rangle\}$ if $j \in \{0,2\}$, and $up^i(S) = \{\langle x_1,\dots,x_{r\pm i}\rangle\} \times \mathbf{N}^{i+1} \times \{\langle x,\xi\rangle\}$ if $j = 1$. Given u such that $S = up^u(S)$, we define $up(S) = up^{u+1}(S)$. We identify two spaces S_ξ and S_β whenever they agree in all but the last coordinate and $\xi \equiv \beta$, in which case we write $S_\xi \equiv S_\beta$. ∎

We will define the set of spaces *controlled by* ν^k *at* η^k *with initiator* δ^k *(and terminator* τ^k) below. Let S be a space assigned to a node of T^k. If $j \in \{1,2\}$, then there may be infinitely many nodes along a given path through T^r which are candidates for controlling S, so we may not be able to recursively identify the node which should control S. Thus we begin to define control on $T^{r\pm1}$ for requirements of types 1 or 2, spreading out the control of sections of S among many nodes. Implication chains will be used for such j to ensure that these nodes work together to produce the same output for the axioms they control on a subset of S which is large enough to ensure a particular iterated limit. We do define control on T^r when j = 0, as there is no ambiguity, in that case, as to which node should control the space.

Controllers for S will be nodes which are derivatives of a node $\beta = \beta^r \in T^r$ such that S is a section of S_β. Control of a space S associated with a node of type 0 or 2 along a path $\Lambda^k \in [T^k]$ will be determined when we reach the first $\xi^k \subset \Lambda^k$ such that $wt(\xi^k) \geq wt(S)$ and $out^0(\xi^k)$ is pseudotrue. We impose the latter condition in order to prevent the specification of axioms while conflicts about the value of the axiom captured by the implication chain machinery remain to be resolved; so assume that $out^0(\eta^k)$ is pseudotrue. To determine control at $\eta^k \in T^k$, we see if there is such a $\xi^k \subseteq \eta^k$; if ξ^k exists, then the node controlling S at η^k is the same as the node controlling S at ξ^k. If $wt(\eta^k) < wt(S)$, then S is not controlled at η^k. Nevertheless, in the latter case, we define a (potential) controller ν^k for S at η^k; this node would be the controller were control to be defined. (Thus S may have a controller at η^k, but not be controlled at η^k.) The (potential) controller may be changed before we reach ξ^k, but will not change thereafter. (We choose this approach, rather than starting at ξ^k, because when we have to define control for requirements of type 1, we need to revise our determination of the controlling node beyond ξ^k.) As we want the controller ν^k for S at η^k to decide the value for axioms it controls, we require that $\nu^k \subset \eta^k$, so that η^k will have a guess at ν^k's outcome. Initiators determine when it becomes reasonable either to first define the control, or to define a new controller, because we see the value we want for the axioms being controlled.

Terminators for initiators will be defined if j = 2 and k = r-1. A terminator τ^k for the initiator δ^k will be the last node of a primary link $[\mu^k,\tau^k]$ such that $\mu^k \subset \delta^k \subseteq \tau^k$ and $wt(\tau^k) < wt(S)$, and will have the property that elements entering the target set for the terminator will enable us to correct axioms. (We specify that $\mu^k \subset \delta^k$ in order to be able to show that, under certain circumstances, the corresponding controller is also restrained by the same link.) Terminators will help us show that the notion of control defined allows the computation of iterated limits needed to satisfy requirements. When the initiator δ^k for the controller ν^k and the space S has a terminator τ^k, then ν^k forfeits its eligibility to control S. However, if there is no controller to replace ν^k, then we will still need to have derivatives of ν^k controlling sections of S. We say that ν^k *influences* S in this situation.

Control for requirements of type 1 will have a slightly different flavor. In this case, we have an extra dimension for the spaces controlled at each level, so in order to compute iterated limits, we can allow finitely many axioms to produce the incorrect value on each space of dimension 2 (one of the dimensions specifies stages for the construction, so we are really computing a single limit). This will be important, as we will not have the automatic correction feature which is available for requirements of type 2. To make use of this added flexibility, we allow terminators τ^k to be defined even if $wt(\tau^k) \geq wt(S)$, but do not allow new initiators to have large weight. We will thus eventually settle on a final initiator for S along any given path, or decide that no initiator exists along that path.

As mentioned above, we will have to keep track of primary links $[\mu^k,\pi^k]$ on T^k which restrain ν^k and are safe for ν^k, and those which are not safe. The links which may not be safe cause an element to be placed into some $A_a \in RS(\nu^k)$ by switching π^k, and are called ν^k-*injurious*. If such nodes also place elements into $A_c \in OS(\nu^k)$, they will allow axioms to be corrected. When this is the case, $[\mu^k,\pi^k]$ will be called ν^k-*correcting*. In order to remove $[\mu^k,\pi^k]$ while preserving the admissibility of strings, additional nodes may have to have their outcomes switched; these are the nodes in the set $\overline{PL}(\xi^k)$ defined below, where ξ^k is the immediate successor of π^k which determines that $[\mu^k,\pi^k]$ is a primary link along the given path. $\overline{PL}(\xi^k)$ is the set of nodes in $PL(\nu^k,\xi^k)$ which need to be switched to make ν^k free, and which come from a specified component of $PL(\nu^k,\xi^k)$, or from the end of a primary ξ^k-link restraining ν^k.

Definition 6.2: Fix $k < n$, $\nu^k \in T^k$, and $\mu^k \subset \pi^k = (\xi^k)^- \subset \xi^k \subseteq \eta^k \in T^k$ such that $[\mu^k,\pi^k]$ is a primary η^k-link. If π^k is the primary completion of some node σ^k, let $\overline{PL}(\xi^k) = PL((\sigma^k)^-,\xi^k)\cup\{(\sigma^k)^-\}$, and let $\overline{PL}(\xi^k) = \{\pi^k\}$ otherwise. We say that $[\mu^k,\pi^k]$ is ν^k-*injurious* if $RS(\nu^k)\cap TS(\beta^k) \neq \emptyset$ for some $\beta^k \in \overline{PL}(\xi^k)$, and is ν^k-*correcting* if $OS(\nu^k) \subseteq TS(\beta^k)$ for some $\beta^k \in \overline{PL}(\xi^k)$. ■

We note that if $[\mu^k,\pi^k]$ is a ν^k-injurious primary η^k-link, $\dim(\nu^k) = k$, and $tp(\nu^k) = 1$, then $[\mu^k,\pi^k]$ is ν^k-correcting. For as $\mu^k \neq \pi^k$ and $up(\mu^k) = up(\pi^k)$, it follows from (2.9) that $\dim(\mu^k) \geq k+1$. Hence by Lemma 2.2(iii) (Interaction), $[\mu^k,\pi^k]$ is ν^k-correcting.

We will determine the spaces *controlled by* ν^k *at* η^k below. This notion of control will have the following properties. If $\nu^k \in T^k$ is assigned the requirement R and controls S at η^k, then $\nu^k \subset \eta^k$, ν^k will be the unique node which controls S at η^k, and if $j \in \{0,2\}$, then ν^k will control S at all $\beta^k \supseteq \eta^k$ such that $out^0(\beta^k)$ is pseudotrue. The initiator for S at η^k will be the longest initiator appointed at any $\xi^k \subseteq \eta^k$ which has no terminator along η^k. Also, if X is a space of the proper dimension to have sections $X^{[i]}$ controlled on T^k, then either only finitely many sections of X will be controlled along any $\Lambda^k \in [T^k]$, or cofinitely many sections of X will be controlled along Λ^k by nodes which are derivatives of a fixed node $\nu^{k+1} \in T^{k+1}$; and if X is controlled along Λ^{k+1}, then ν^{k+1} will be the controller for X along Λ^{k+1}. The definition below is arranged to ensure these properties.

We proceed by induction on r-k if j = 0, and on r-k-1 if $j \in \{1,2\}$, and then by induction on $lh(\eta^k)$ for $\eta^k \subset \Lambda^k$. (Control will not be defined on T^r if $j \in \{1,2\}$; implication chains will ensure the existence of the iterated limit for r = k.) Let X be a section of the space for which R wants to define axioms, with dim(X) = k+1 if $j \in \{0,2\}$, and dim(X) = k+2 if j = 1. For each $i \in \mathbf{N}$ and $\eta^k \in T^k$, we determine the node $\nu^k \subset \eta^k$ which is the controller for $X^{[i]}$ at η^k, the node $\delta^k \subseteq \eta^k$ which is the initiator for $X^{[i]}$ at η^k, and those nodes $\subseteq \eta^k$ which are terminators for $X^{[i]}$ and some initiator for $X^{[i]}$ at η^k.

Definition 6.3 (Initiators, Controllers, and Terminators): Fix $k \leq r$ if j = 0, $k < r$ if $j \in \{1,2\}$, $\eta^k \in T^k$ such that $lh(\eta^k) > 0$, and a space S, and let $\bar{\delta}^k$ and $\bar{\nu}^k$ be, respectively, the initiator and controller for S at $(\eta^k)^-$, if these exist. We determine whether the the controller, initiator, and terminator for S at η^k exist, and if so, define those strings. We will assume by induction that:

(6.1) $\bar{\delta}^k$ exists iff $\bar{\nu}^k$ exists.

Case 1: We define controllers when a new initiator is found. There are two subcases. Subcase 1.1 handles the base step, and Subcase 1.2 handles the inductive step.

Subcase 1.1: Either k = r, j = 0, and $S = S_{(\eta^k)^\pm}$; or k = r-1, $j \in \{1,2\}$, $wt(\eta^k) \leq wt(S)$, and $up(S) = S_{up((\eta^k)^\pm)}$; and in both cases, the principal derivative $(out^j(\eta^k))^-$ of $(\eta^k)^-$ along $out^j(\eta^k)$ is implication-free for all $j \leq k$, and $out^0(\eta^k)$ is pseudotrue. Then η^k is the *initiator* for S at η^k and $(\eta^k)^-$ is the *controller* for S at η^k.

Subcase 1.2: $k < r$ if j = 0, $k < r-1$ if $j \in \{1,2\}$, $wt(\eta^k) \leq wt(S)$, there is an initiator δ^{k+1} for up(S) at $\lambda(\eta^k)$, but δ^{k+1} is not the initiator for up(S) at $\lambda((\eta^k)^-)$. Let ν^{k+1} be the controller corresponding to δ^{k+1}. Then η^k is the *initiator* for S at η^k. The *controller* ν^k for S at η^k is the longest derivative of ν^{k+1} such that $\nu^k \subset \eta^k$. (By (6.2) below inductively, it will be the case that $\nu^{k+1} \subset \delta^{k+1}$, so such a derivative will exist.)

Case 2: (We switch controllers and initiators when a new derivative of $up(\bar{\nu}^k)$ is found.) Case 1 is not followed, either $k < r$ and j = 0 or $k < r-1$ and $j \in \{1,2\}$, $wt(\eta^k) \leq wt(S)$, $up(\bar{\nu}^k)$ controls up(S) at $\lambda(\eta^k)$, and $up((\eta^k)^-) = up(\bar{\nu}^k)$. Then η^k is the *initiator* for S at η^k and $(\eta^k)^-$ is the *controller* for S at η^k.

Case 3: Neither of the previous cases is followed, $j \in \{1,2\}$, $\bar{\nu}^k$ and $\bar{\delta}^k$ exist, and there is a primary $\bar{\nu}^k$-correcting η^k-link $[\mu^k,(\eta^k)^-]$ such that $\mu^k \subset \bar{\delta}^k \subseteq (\eta^k)^-$; and if $j = 2$, then $wt(\eta^k) \le wt(S)$ and $k = r-1$. (Again note, as in the earlier description of terminators, that we require that $\mu^k \neq \bar{\delta}^k$.) We call $(\eta^k)^-$ a *terminator* for S and $\bar{\delta}^k$ at η^k. (Note that if $j = 1$, then we allow $\bar{\nu}^k$-correcting primary links to cause a change of control, even if we discover them at a node whose weight exceeds wt(S). This is necessary, else we would not be able to correct axioms for a thick subset of up(S) when control is switched.)

Subcase 3.1: There is no controller for S at μ^k. If $j = 1$, then there is no controller or initiator for S at η^k. If $j = 2$, then $\bar{\nu}^k$ ($\bar{\delta}^k$, resp.) is the *controller* (*initiator,* resp.) for S at η^k.

Subcase 3.2: Otherwise. By (6.1) inductively, let $\tilde{\delta}^k$ and $\tilde{\nu}^k$ be, respectively, the initiator and controller for S at μ^k. Then $\tilde{\nu}^k$ is the *controller* for S at η^k; and the *initiator* for S at η^k is $\tilde{\delta}^k$ if $wt(\eta^k) > wt(S)$, and is η^k if $wt(\eta^k) \le wt(S)$.

Case 4: Otherwise. The *initiator* and *controller* for S at η^k are $\bar{\delta}^k$ and $\bar{\nu}^k$, respectively, if these exist, and fail to exist otherwise.

In all cases, we say that τ^k is a *terminator* for S and δ^k *along* η^k ($\Lambda^k \in [T^k]$, resp.) if τ^k is a terminator for S and δ^k at some $\xi^k \subseteq \eta^k$ ($\xi^k \subset \Lambda^k$, resp.). ∎

The following properties are easily verified by induction on $lh(\eta^k)$, as is (6.1). (6.5)(ii) follows from Lemma 4.1 (Nesting), (6.2), and Case 3 of Definition 6.3, where terminators are defined to restrain the previous initiator.

(6.2) If ν^k controls S at η^k with initiator δ^k, then $\nu^k \subset \delta^k \subseteq \eta^k$.

(6.3) If δ^k is the initiator for S at both η^k and $\tilde{\eta}^k$ and ν^k and $\tilde{\nu}^k$ are the controllers for S at η^k and $\tilde{\eta}^k$, respectively, then $\nu^k = \tilde{\nu}^k$.

(6.4) If δ^k is the initiator for S at η^k, then $wt(\delta^k) \le wt(S)$.

(6.5) Suppose that $\eta^k \subset \tilde{\eta}^k$, and δ^k and $\tilde{\delta}^k$ are the initiators for S at η^k and $\tilde{\eta}^k$, respectively. Then:

(i) If $wt(\tilde{\eta}^k) \le wt(S)$, then $\delta^k \subseteq \tilde{\delta}^k$.

(ii) If $wt(S) \leq wt(\eta^k)$, then $\tilde{\delta}^k \subseteq \delta^k$; and if $j \in \{0,2\}$, then $\tilde{\delta}^k = \delta^k$.

We are now ready to define control. Recall that control is supported only on pseudotrue nodes, as defined in Definition 5.9. There is a corresponding notion at non-pseudotrue nodes which we call weak control. Control is replaced by influence for requirements of type 2, when the initiator has a terminator.

Definition 6.4 (Control): We say that ν^k *weakly controls* S at η^k if ν^k is the controller for S at η^k corresponding to the initiator δ^k, there is no terminator for δ^k and S along η^k, and

(6.6) $wt(S) \leq wt(\eta^k)$.

If ν^k is the controller for S at η^k with initiator δ^k, there is a terminator for δ^k and S along η^k, and (6.6) holds, then we say that ν^k *weakly influences* S at η^k. ν^k *controls* (*influences,* resp.) S at η^k if ν^k weakly controls (influences, resp.) S at η^k and $out^0(\eta^k)$ is pseudotrue. Given $\Lambda^k \in T^k$, we say that ν^k *weakly controls (weakly influences, resp.)* S (δ^k is the *initiator* for S, resp.) *along* Λ^k if ν^k weakly controls S (δ^k is the initiator for S, resp.) at all sufficiently long $\eta^k \subset \Lambda^k$; and that ν^k *controls (influences,* resp.) S *along* Λ^k if there are infinitely many $\eta^k \subset \Lambda^k$ such that $out^0(\eta^k)$ is pseudotrue, and ν^k controls (influences, resp.) S at all sufficiently long $\eta^k \subset \Lambda^k$ such that $out^0(\eta^k)$ is pseudotrue. ∎

We note that control along Λ^k and weak control along Λ^k coincide if there are infinitely many pseudotrue $\eta^k \subset \Lambda^k$.

Suppose that $\Lambda^k \in [T^k]$. The following fact now follows easily from (2.1), Lemma 4.1 (Nesting), (6.5), and (6.6), as there must be a longest initiator along any path if there is any initiator along that path:

(6.7) Suppose that $\xi^k \subset \Lambda^k$ and ξ^k extends all initiators and properly extends all terminators for S at any $\eta^k \subset \Lambda^k$. (If $j \in \{0,2\}$, this will be the case if $wt(\xi^k) \geq wt(S)$.) Then ν^k weakly controls (weakly influences, resp.) S (δ^k is the initiator for S, resp.) along Λ^k iff ν^k weakly controls (weakly influences, resp.) S (δ^k is the initiator for S, resp.) at ξ^k iff ν^k weakly controls (weakly influences, resp.) S (δ^k is the initiator for S, resp.) at every η^k such that $\xi^k \subseteq \eta^k \subset \Lambda^k$. Furthermore, if ν^k weakly controls S along Λ^k, $\xi^k \subseteq \eta^k \subset \Lambda^k$, and δ^k is the initiator for S at η^k, then

δ^k is the longest node which is an initiator for S at some $\gamma^k \subseteq \eta^k$ and which has no terminator along η^k.

The next lemma specifies some properties of the control process.

Lemma 6.1 (Finite Control Lemma): Fix $k \leq n$, an admissible $\Lambda^k \in [T^k]$, and a space S assigned to a node working for requirement R, where $k \leq \dim(R)$ if $j = tp(R) = 0$, and $k < \dim(R)$ if $tp(R) \in \{1,2\}$. Then:

(i) $\{\nu^k \in T^k: \exists\eta^k(\nu^k$ weakly controls or weakly influences S at $\eta^k)\}$ is finite.

(ii) If $j \in \{0,2\}$ then:

(a) $|\{\nu^k \subset \Lambda^k: \exists\eta^k(\eta^k \subset \Lambda^k$ & ν^k weakly controls or weakly influences S at $\eta^k)\}| \leq 1$; and

(b) $|\{\delta^k \subset \Lambda^k: \exists\eta^k \subset \Lambda^k(\delta^k$ is an initiator for S at η^k & S is weakly controlled or weakly influenced at $\eta^k)\}| \leq 1$.

(iii) Suppose that $k < \dim(R)$. Let F be the set of initiators for S on T^k. Then F is finite and for all $\Lambda \in [T^k]$, S is weakly controlled along Λ iff there is a $\delta^k \in F$ such that $\delta^k \subset \Lambda$ and there is no terminator for δ^k and S at any $\eta^k \subset \Lambda$.

(iv) If $\nu^k \subset \delta^k \subset \Lambda^k$, $(\delta^k)^- = \nu^k$, $k = \dim(R)-1$, and ν^k is a controller at some $\eta^k \subset \Lambda^k$, then δ^k is an initiator at δ^k.

Proof: (i): If $k = \dim(R)$, then $tp(R) = 0$, and there is a unique node on T^k which controls S. Suppose that $k < \dim(R)$, and that $\nu^k \in T^k$ and ν^k weakly controls or weakly influences S at η^k. By (2.1), (6.2), and (6.4), $wt(\nu^k) \leq wt(S)$. But as the weight function is one-to-one, there are only finitely many $\nu^k \in T^k$ such that $wt(\nu^k) \leq wt(S)$.

(ii): If $k = \dim(R)$, then $tp(R) = 0$, and there is a unique controller ν^k for S on T^k. Furthermore, for any $\eta^k \in T^k$, if S is weakly controlled at η^k with initiator δ^k, then $\nu^k \subset \eta^k$ and δ^k is the immediate successor of ν^k along η^k.

Suppose that $k < \dim(R)$. By (6.6) and Definition 6.4, if S is weakly controlled or weakly influenced at η^k, then $wt(S) \leq wt(\eta^k)$. (ii)(b) now follows from (6.5)(ii). (ii)(a) follows from (6.3).

(iii): Suppose that $k < \dim(R)$. If $\delta^k \in F$ then by (6.4), $wt(\delta^k) \leq wt(S)$. As the weight function is one-to-one, F is finite. By Definitions 6.3 and 6.4, if ν^k weakly controls S along $\Lambda \in [T^k]$ then Λ must extend some element δ^k of F such that there is no terminator for δ^k and S along Λ. Conversely, suppose that Λ extends an element δ^k of F such that there is no terminator for δ^k and S along Λ. By (6.7) and Definitions 6.3 and 6.4, S is weakly controlled along Λ.

(iv): We note that if tp(R) = 0, then $\lambda(\delta^k) \supset \mathrm{up}(\nu^k)$, so $\lambda(\delta^k)$ extends an immediate successor of $\mathrm{up}(\nu^k)$, and so S is weakly controlled along $\lambda(\delta^k)$. Thus (iv) can fail for $\mathrm{tp}(R) \le 2$ only if ν^k is defined as the controller for some space through Case 3 of Definition 6.3. Suppose that ν^k is defined by that case. Then ν^k must be a controller at some $\xi^k \subset \eta^k$. Hence if we fix the shortest $\xi^k \subset \Lambda^k$ at which ν^k is a controller, then Subcase 1.1, Subcase 1.2, or Case 2 of Definition 6.3 must be followed at ξ^k. But then $\xi^k = \delta^k$, and δ^k is the initiator corresponding to ν^k at δ^k. ∎

Our next lemma spells out some important relationships between initiators, terminators, and weak control for requirements of type 1.

Lemma 6.2 (Terminator Lemma): Fix $k < n-1$ and $\Lambda^k \in [T^k]$, and let $\Lambda^{k+1} = \lambda(\Lambda^k)$. Fix a space X which is assigned to a requirement of type 1 and is weakly controlled by some node of T^{k+1}, and fix $i \in \mathbf{N}$. Then:

(i) If $\delta^k \subset \Lambda^k$ is an initiator for $X^{[i]}$ at δ^k, and $u \ge i$, then δ^k is an initiator for $X^{[u]}$ at δ^k.

(ii) Suppose that $\delta^{k+1} \subseteq \eta^{k+1} \subset \Lambda^{k+1}$ are given such that δ^{k+1} is the initiator for X at all γ^{k+1} such that $\eta^{k+1} \subseteq \gamma^{k+1} \subset \Lambda^{k+1}$, and there is no initiator $\tilde{\delta}^{k+1} \supset \eta^{k+1}$ for X (the latter condition includes those $\tilde{\delta}^{k+1}$ which may not lie along Λ^{k+1}). Let $\eta^k = \mathrm{out}(\eta^{k+1})$. Suppose that $\eta^k \subseteq \delta^k \subset \Lambda^k$, δ^k is an initiator for $X^{[i]}$, and δ^{k+1} is not the initiator for X at $\lambda(\delta^k)$. Then there is a terminator for $X^{[i]}$ and δ^k along Λ^k.

Proof: (i): By (6.4), $\mathrm{wt}(\delta^k) \le \mathrm{wt}(X^{[i]}) = i$; so as $u \ge i$, $\mathrm{wt}(\delta^k) \le \mathrm{wt}(X^{[u]}) = u$. By induction on $\mathrm{lh}(\delta^k)$ and (2.1), if an initiator for one of $X^{[i]}$ or $X^{[u]}$ exists at $(\delta^k)^-$, then that node is the initiator for both $X^{[i]}$ and $X^{[u]}$ at $(\delta^k)^-$. (i) now follows from Definition 6.3.

(ii): Let $\tilde{\delta}^{k+1}$ be the initiator for X at $\lambda(\delta^k)$. As $\delta^k \supseteq \eta^k = \mathrm{out}(\eta^{k+1})$ and $\eta^{k+1} \subset \Lambda^{k+1}$, $\lambda(\delta^k) \supseteq \eta^{k+1}$ by (2.4) and (2.6). Hence by choice of η^{k+1}, $\tilde{\delta}^{k+1} \subseteq \eta^{k+1}$. Now by (6.7), $\bar{\delta}^{k+1}$ is the initiator for X at γ^{k+1} iff $\bar{\delta}^{k+1}$ is the longest node which is an initiator at some $\xi^{k+1} \subseteq \gamma^{k+1}$ and which does not have a terminator along γ^{k+1}. Thus $\tilde{\delta}^{k+1} \not\supset \delta^{k+1}$, else $\tilde{\delta}^{k+1}$ would have a terminator along η^{k+1}. Hence as $\lambda(\delta^k) \supseteq \eta^{k+1}$ and δ^{k+1} is not the initiator for X at $\lambda(\delta^k)$, $\tilde{\delta}^{k+1} \subset \delta^{k+1}$. By (6.7), this is only possible if there is a $\gamma^{k+1} \subseteq \lambda(\delta^k)$ such that $(\gamma^{k+1})^-$ is a terminator for X and δ^{k+1} along γ^{k+1}. Let ν^k be the initial derivative of $(\gamma^{k+1})^-$ along δ^k, and note, by Lemma 3.1(i) (Limit Path), that $\nu^k \subset \delta^k$. As

$(\gamma^{k+1})^-$ is a terminator for X and δ^{k+1} along γ^{k+1}, $(\gamma^{k+1})^-$ must have infinite outcome along γ^{k+1}, so ν^k must have finite outcome along δ^k. Now $\gamma^{k+1} \not\subset \Lambda^{k+1}$ as there is no terminator for X and δ^{k+1} along Λ^{k+1}, else δ^{k+1} would not be the initiator for X along Λ^{k+1}. Furthermore, as $\delta^{k+1} \subset \Lambda^{k+1}$, by (2.10), some extension of δ^k along Λ^k must switch $(\gamma^{k+1})^-$, so there must be a derivative $\xi^k \supseteq \delta^k$ of $(\gamma^{k+1})^-$ along Λ^k which has infinite outcome along Λ^k. It now follows that ξ^k is a terminator for δ^k along Λ^k via the primary Λ^k-link $[\nu^k, \xi^k]$. ∎

The next definition is notational in nature. Given $\Lambda^k \in [T^k]$, a node ν^{k+1} of T^{k+1}, and a space X whose sections $X^{[i]}$ may be weakly controlled by nodes of T^k, we define $\mathrm{CON}(\nu^{k+1},\Lambda^k,X)$ to be the set of sections of X which are weakly controlled by derivatives ν^k of ν^{k+1} such that $\nu^k \subset \Lambda^k$. This set is partitioned into two sets, $\mathrm{ACT}(\nu^{k+1},\Lambda^k,X)$ corresponding to the derivatives of ν^{k+1} which are activated along Λ^k, and $\mathrm{VAL}(\nu^{k+1},\Lambda^k,X)$ corresponding to the derivatives of ν^{k+1} which are validated along Λ^k.

Definition 6.5: Let $k < n$, $\nu^{k+1} \in T^{k+1}$, $\Lambda^k \in [T^k]$, and a space X be given. We define

$$\mathrm{CON}(\nu^{k+1},\Lambda^k,X) = \cup\{S \subseteq X: \exists \nu^k \subset \Lambda^k(\mathrm{up}(\nu^k) = \nu^{k+1} \ \&\ \nu^k \text{ weakly controls } S \text{ along } \Lambda^k)\},$$

$$\mathrm{VAL}(\nu^{k+1},\Lambda^k,X) = \cup\{S \subseteq X: \exists \nu^k \subset \Lambda^k(\mathrm{up}(\nu^k) = \nu^{k+1} \ \&\ \nu^k \text{ weakly controls } S \text{ along } \Lambda^k \ \&\ \nu^k \text{ is validated along } \Lambda^k)\}, \text{ and}$$

$$\mathrm{ACT}(\nu^{k+1},\Lambda^k,X) = \cup\{S \subseteq X: \exists \nu^k \subset \Lambda^k(\mathrm{up}(\nu^k) = \nu^{k+1} \ \&\ \nu^k \text{ weakly controls } S \text{ along } \Lambda^k \ \&\ \nu^k \text{ is activated along } \Lambda^k)\}.$$ ∎

In the next definition, we introduce *thick* and *thin subsets*. Thick subsets of a space S of dimension k+1 are the union of cofinitely many sections $S^{[i]}$ of S. Thin subsets are the complements of thick subsets.

Definition 6.6: Fix a space S of dimension k. We say that $\widetilde{S}$ is a *thick subset* of S if $\widetilde{S} = \cup\{S^{[i]}: i \in I\}$ where I is a cofinite set of natural numbers. We say that $\widetilde{S}$ is a *thin subset* of S if $\widetilde{S} \subseteq S$ and $S \backslash \widetilde{S}$ is a thick subset of S. ∎

We now show that a node weakly controlling a space passes down weak control of a thick subset of that space to its derivatives.

□

Lemma 6.3 (Thick Control Lemma): Fix an admissible $\Lambda^0 \in [T^0]$, and for all $u \leq n$, let $\Lambda^u = \lambda^u(\Lambda^0)$. Fix $k < n$, and suppose that $\nu^{k+1} \subset \Lambda^{k+1}$ weakly controls the space X along Λ^{k+1}. Then:

(i) If ν^{k+1} is validated along Λ^{k+1}, then $VAL(\nu^{k+1},\Lambda^k,X)$ is a thick subset of X.

(ii) If ν^{k+1} is activated along Λ^{k+1}, then $ACT(\nu^{k+1},\Lambda^k,X)$ is a thick subset of X.

Proof: If $tp(\nu^{k+1}) = 0$ and $dim(\nu^{k+1}) = k+1$, then ν^{k+1} is the unique controller for X on T^{k+1}, and its immediate successor δ^{k+1} along Λ^{k+1} is the unique initiator for X at any node extending δ^{k+1}. Thus let $\eta^{k+1} = \delta^{k+1}$ in this case. Otherwise, we note that as X is weakly controlled along Λ^{k+1}, $dim(\nu^{k+1}) > k+1$. By (6.5)(ii), (6.7), (2.4), and Lemma 3.1 (Limit Path), we can fix the shortest $\eta^{k+1} \subset \Lambda^{k+1}$ such that $wt(\eta^{k+1}) > wt(X)$ and ν^{k+1} is the controller for X with fixed initiator δ^{k+1} at all γ^{k+1} such that $\eta^{k+1} \subseteq \gamma^{k+1} \subset \Lambda^{k+1}$. Note that as $wt(\eta^{k+1}) > wt(X)$, it follows from (6.4) that there is no initiator for X (along any path through T^{k+1}) which extends η^{k+1}. In both cases, let $\eta^k = out(\eta^{k+1})$. By Lemma 3.2(i) (Out), $\lambda(\eta^k) = \eta^{k+1}$.

We first show that for all $i \geq wt(\eta^k)$, the controller of $X^{[i]}$ along Λ^k is a derivative of ν^{k+1}. By Definition 6.3, for all $i \geq wt(\eta^k)$, $X^{[i]}$ will have a controller ν^k at η^k, and ν^k will be a derivative of ν^{k+1}; furthermore, by Lemma 3.1(ii) (Limit Path), $\nu^k \supseteq \pi^k$, where π^k is the principal derivative of ν^{k+1} along Λ^k. By (4.1), the initiator corresponding to ν^k is restrained by a primary link along Λ^k iff it is restrained by that same link at η^k. Also by Lemma 6.2(ii) (Terminator) and Definition 6.3, if $i \geq wt(\eta^k)$, $\eta^k \subset \delta^k \subset \Lambda^k$ and δ^k is an initiator for $X^{[i]}$ at δ^k, then either δ^{k+1} is the initiator for X at $\lambda(\delta^k)$, or $tp(\nu^{k+1}) = 1$ and there is a terminator for δ^k and $X^{[i]}$ along Λ^k. Thus the controller of $X^{[i]}$ along Λ^k must be a derivative of ν^{k+1}.

Fix $i \geq wt(\eta^k)$. By (6.7), $X^{[i]}$ is weakly controlled along Λ^k. If π^k has infinite outcome along Λ^k, then by (2.8), π^k weakly controls $X^{[i]}$ along Λ^k. And if π^k has finite outcome along Λ^k, then every derivative of ν^{k+1} along Λ^k has finite outcome along Λ^k; so if ν^k weakly controls $X^{[i]}$ along Λ^k, then ν^k has finite outcome along Λ^k. (i) and (ii) now follow, as by Definition 2.1, ν^k is validated along Λ^k iff ν^{k+1} is validated along Λ^{k+1}. $\blacksquare$

The next two lemmas combine to show that if a space X is not weakly controlled along Λ^{k+1}, then either a thick subset of X is weakly controlled along Λ^k, or cofinitely many sections of X of dimension k have only a thin subset weakly controlled along $\Lambda^{k\pm1}$. Also, if X is weakly influenced along Λ^{k+1}, then a thick subset of X is weakly controlled along Λ^k.

Lemma 6.4 (Indirect Control Lemma): Fix $k < n$ and an admissible $\Lambda^k \in [T^k]$, and let $\Lambda^{k+1} = \lambda(\Lambda^k)$. Let X be a section of the space assigned to the requirement R of dimension r, where $r \geq k+1$ if $tp(R) = 0$, and $r > k+1$ if $tp(R) \in \{1,2\}$. Suppose that X is not weakly controlled along Λ^{k+1}, but that $X^{[i]}$ is weakly controlled along Λ^k for infinitely many i. Then there is a $\nu^k \subset \Lambda^k$ such that ν^k weakly controls a thick subset of X along Λ^k. In particular, this will be the case if X is weakly influenced along Λ^{k+1}.

Proof: First suppose that $k+1 = \dim(R)$, and so, that $tp(R) = 0$. By hypothesis, X is not weakly controlled along Λ^{k+1}, and we note that as $tp(R) = 0$, there is at most one controller for X on T^{k+1} and there are no terminators for X along Λ^{k+1}. Hence if there is a controller for X on T^{k+1}, then that controller is not $\subset \Lambda^{k+1}$. It thus follows from Lemma 3.1(ii) (Limit Path) that there is a $\xi^k \subset \Lambda^k$ such that for all $\bar{\xi}^k \supseteq \xi^k$, if $\bar{\xi}^k \subset \Lambda^k$, then $\lambda(\bar{\xi}^k)$ does not extend an initiator for X.

Suppose that $k+1 < \dim(R)$. By Lemma 6.1(iii) (Finite Control), we can fix a finite subset F of T^{k+1} such that for all $\Lambda \in [T^{k+1}]$, X is weakly controlled along Λ iff Λ extends some element of F which does not have a terminator along Λ. As X is not weakly controlled along Λ^{k+1}, it follows from the finiteness of F and Lemma 3.1(ii) (Limit Path) that there is a $\xi^k \subset \Lambda^k$ such that for all $\bar{\xi}^k \supseteq \xi^k$, if $\bar{\xi}^k \subset \Lambda^k$ and $\lambda(\bar{\xi}^k)$ extends an element $\delta^{k+1} \in F$, then $\delta^{k+1} \subset \Lambda^{k+1}$ and both $\lambda(\bar{\xi}^k)$ and Λ^{k+1} properly extend the same terminator for δ^{k+1} and X along Λ^{k+1}.

In either case, we conclude that there are only finitely many initiators for sections of X along Λ^k. As infinitely many sections of X are weakly controlled along Λ^k, there must be a $\delta^k \subset \Lambda^k$ such that $\lambda(\delta^k)$ extends an element of F, some $\nu^k \subset \delta^k$ weakly controls a section of X at δ^k, and δ^k is not restrained by any ν^k-correcting primary Λ^k-link. By choice of ξ^k, $\delta^k \subseteq \xi^k$ for each such δ^k. Fix the longest such δ^k, and the unique ν^k for δ^k. By Definition 6.3 and (6.7), ν^k will weakly control all but finitely many sections of X along Λ^k.

We now note that if X is weakly influenced along Λ^{k+1}, then X has a controller v^{k+1} and an initiator δ^{k+1} along Λ^{k+1}. By Lemma 3.1(i) (Limit Path), v^{k+1} will have a derivative $\nu^k \subset \Lambda^k$ and $\delta^k = out(\delta^{k+1})$ is an initiator for a section of X at δ^k. Furthermore, $tp(v^{k+1}) = 2$, so there will be no terminators for sections of X along Λ^k. Thus by Definitions 6.3 and 6.4, δ^k will will witness the fact that infinitely many sections of X are weakly controlled along Λ^k. The last sentence of the lemma now follows from the first part of the lemma. n

The next lemma shows that if X is a space which is not weakly controlled along Λ^{k+1} and no section Y of X is weakly controlled along Λ^k, then for cofinitely many sections

Y of X, there is very little weak control of sections of Y along $\Lambda^{k\pm1}$. More precisely, for cofinitely many sections Y of X, the number of sections of Y which are weakly controlled at some node along $\Lambda^{k\pm1}$ is finite, and if X is assigned to a requirement of type 0 or 2, then this number is 0 (so no section of Y is weakly controlled along $\Lambda^{k\pm1}$). (Because of the definition of terminators, the set of sections of X weakly controlled along $\Lambda^{k\pm1}$ will be a (possibly proper) subset of the set of sections of X weakly controlled at some $\gamma^{k\pm1} \subset \Lambda^{k\pm1}$.)

Lemma 6.5 (Non-Control Lemma): Fix an admissible $\Lambda^0 \in [T^0]$, and for all $u \leq n$, let $\Lambda^u = \lambda^u(\Lambda^0)$. Fix $k \in (0,n\text{-}1)$ and a requirement R of dimension r and type j, where $r \geq k+1$ if $j = 0$, and $r > k+1$ if $j \in \{1,2\}$. Let X be a section of a space assigned to R which is not weakly controlled along Λ^{k+1}. Suppose that $X^{[i]}$ is weakly controlled along Λ^k for at most finitely many $i \in \mathbf{N}$. Then:

(i) For all $i \in \mathbf{N}$, either $\{u: (X^{[i]})^{[u]}$ is weakly controlled along $\Lambda^{k\pm1}\}$ is cofinite, or $\{u: (X^{[i]})^{[u]}$ is weakly controlled at some $\gamma^{k\pm1} \subset \Lambda^{k\pm1}\}$ is finite.

(ii) For cofinitely many $i \in \mathbf{N}$, $\{u: (X^{[i]})^{[u]}$ is weakly controlled at some $\gamma^{k\pm1} \subset \Lambda^{k\pm1}\}$ is finite.

(iii) If $j \in \{0,2\}$, then for cofinitely many $i \in \mathbf{N}$, $\{u: (X^{[i]})^{[u]}$ is weakly controlled at some $\gamma^{k\pm1} \subset \Lambda^{k\pm1}\} = \emptyset$.

Proof: By Lemma 3.7 (Infinite Injury), Lemma 6.1(iii) (Finite Control), and as, if $j = 0$ and $r = k+1$, then there can be no controller for X along Λ^{k+1} and there is at most one controller for X on T^{k+1}, we can choose $\tilde{\eta}^{k\pm1} \subset \Lambda^{k\pm1}$ such that for all initiators $\rho^{k+1} \in T^{k+1}$ for X such that $\rho^{k+1} \not\subset \Lambda^{k+1}$ and all $\xi^{k\pm1}$, if $\tilde{\eta}^{k\pm1} \subseteq \xi^{k\pm1} \subset \Lambda^{k\pm1}$ then $\lambda^{k+1}(\xi^{k\pm1}) \not\supseteq \rho^{k+1}$. By hypothesis, the preceding sentence, and Lemma 6.1(iii) (Finite Control), we can fix $\eta^{k+1} \subset \Lambda^{k+1}$ such that for all initiators $\rho^{k+1} \subset \Lambda^{k+1}$ for X, there is a terminator for ρ^{k+1} and X along η^{k+1}. Let $\eta^k = \mathrm{out}(\eta^{k+1})$ and $\eta^{k\pm1} = \mathrm{out}(\eta^k)$, and note that by (2.5), $\eta^k \subset \Lambda^k$ and $\eta^{k\pm1} \subset \Lambda^{k\pm1}$. Without loss of generality, we may assume that $\eta^{k\pm1} \supseteq \tilde{\eta}^{k\pm1}$. By (2.5) and (2.6), for all $\xi^{k\pm1}$ such that $\eta^{k\pm1} \subseteq \xi^{k\pm1} \subset \Lambda^{k\pm1}$, $\lambda(\xi^{k\pm1}) \supseteq \eta^k$.

By (2.5), $\lambda(\eta^k) = \eta^{k+1}$. Now $(\eta^k)^- = (\mathrm{out}(\eta^{k+1}))^-$ is the principal derivative of $(\eta^{k+1})^-$ along Λ^k, so by Lemma 4.3(i)(c) (Link Analysis), there is no primary Λ^k-link which restrains $(\eta^k)^-$. Hence by hypothesis, there is no initiator δ^k for any section of X at $(\eta^k)^-$, else by (4.1) and Lemma 4.4 (Free Implies True Path), δ^k would have no terminator along Λ^k, so by Definition 6.3, cofinitely many sections of X would have initiators along Λ^k. But then by Definition 6.4, infinitely many sections of X would be weakly controlled along Λ^k, contrary to hypothesis. Furthermore, η^k cannot be an initiator for a section of X, else either X would be weakly controlled at $\eta^{k+1} = \lambda(\eta^k)$, or some section of X would have an initiator at $(\eta^k)^-$, neither of which is possible. Hence there is no initiator for any section of X at η^k. Also, $\lambda(\eta^{k-1}) = \eta^k$ and $(\eta^{k-1})^- = (out(\eta^k))^-$ is the principal derivative of $(\eta^k)^-$ along $\Lambda^{k\pm1}$, so again by Lemma 4.3(i)(c) (Link Analysis), there is no primary $\Lambda^{k\pm1}$-link which restrains $(\eta^{k-1})^-$.

Fix i. First assume that $i < wt(\eta^k)$. By (6.4) and (2.1), there is no initiator $\delta^k \supseteq \eta^k$ for $X^{[i]}$. Hence as there is no initiator for $X^{[i]}$ at η^k, if $\delta^{k\pm1} \subset \Lambda^{k\pm1}$ is first defined to be an initiator for a section of $X^{[i]}$ by Case 1 or Case 2 of Definition 6.3, then $\delta^{k\pm1} \subset \eta^{k-1}$. Also, as there is no primary $\Lambda^{k\pm1}$-link which restrains $(\eta^{k-1})^-$, if $\delta^{k\pm1} \subset \Lambda^{k\pm1}$ is first defined to be an initiator for a section of $X^{[i]}$ by Case 3 of Definition 6.3, then $\delta^{k\pm1} \subset \eta^{k-1}$. We conclude that if $\delta^{k\pm1}$ is an initiator for a section of $X^{[i]}$ at any $\xi^{k\pm1} \subset \Lambda^{k\pm1}$, then $\delta^{k\pm1} \subset \eta^{k-1}$. Now if there is a $\delta^{k\pm1} \subset \Lambda^{k\pm1}$ such that $\delta^{k\pm1}$ is an initiator for a section of $X^{[i]}$ and there is no terminator for $\delta^{k\pm1}$ along $\Lambda^{k\pm1}$, then by Definition 6.3, infinitely many sections of X will have initiators along $\Lambda^{k\pm1}$, so (i) follows for i from (6.7) and Definition 6.4. Otherwise, as there is no primary $\Lambda^{k\pm1}$-link which restrains $(\eta^{k-1})^-$, each initiator $\delta^{k\pm1} \subset \Lambda^{k\pm1}$ for a section of $X^{[i]}$ has a terminator $\tau^{k\pm1} \subset \eta^{k-1}$, so by Definition 6.4, for all $u \geq wt(\eta^{k-1})$, $(X^{[i]})^{[u]}$ is not weakly controlled at any $\xi^{k\pm1} \subset \Lambda^{k\pm1}$, and again, (i) follows for this i.

Suppose that $i \geq wt(\eta^k)$. As there is no initiator δ^k for $X^{[i]}$ at η^k and $\lambda(\eta^{k-1}) = \eta^k$, η^{k-1} cannot be an initiator for a section of $X^{[i]}$. Furthermore, for any $\xi^{k\pm1} \subset \eta^{k-1}$, it follows from (2.4) that $\lambda(\xi^{k\pm1}) \neq \lambda(\eta^{k-1})$, so by (2.11) and (6.6), $X^{[i]}$ is not weakly controlled at $\lambda(\xi^{k\pm1})$. Hence any initiator for a section of $X^{[i]}$ at some $\xi^{k\pm1} \subset \Lambda^{k\pm1}$ must properly extend η^{k-1}.

The broad outline of the verification of (ii) in this case is as follows. We first show that if $\delta^{k\pm1}$ is an initiator for a section of $X^{[i]}$ at some $\xi^{k\pm1} \subset \Lambda^{k\pm1}$, then $\lambda(\delta^{k\pm1})$ extends an initiator for $X^{[i]}$ which, in turn, extends a node which switches a terminator for X along Λ^{k+1}. We then show that the node on T^k which switched the terminator must have its immediate predecessor switched back by a node on $T^{k\pm1}$ in order to return the terminator for X to Λ^{k+1}, and that this switching process can be characterized in terms of PL sets, in a way to ensure correction of axioms. The switching process will ensure that $\delta^{k\pm1}$ has a terminator along $\Lambda^{k\pm1}$, so only finitely many sections of $X^{[i]}$ are weakly controlled along $\Lambda^{k\pm1}$. Furthermore, we will be able to obtain a uniform bound on these terminators, so (ii) will follow.

Suppose that $\delta^{k\pm1} \subset \Lambda^{k\pm1}$ is an initiator for a section of $X^{[i]}$. We have shown that $\delta^{k\pm1} \supset \eta^{k-1}$, so $\lambda(\delta^{k\pm1}) \supseteq \eta^k$. By Definition 6.3, there must be an initiator $\delta^k \subseteq \lambda(\delta^{k\pm1})$ for $X^{[i]}$ at $\lambda(\delta^{k\pm1})$, and again by the second paragraph of the proof and (6.4), $\delta^k \supset \eta^k$. By Definition 6.3, there is an initiator δ^{k+1} for X at $\lambda(\delta^k)$ with corresponding initiator ν^{k+1}.

But by (2.5), $\delta^{k\pm1} \supseteq \mathrm{out}(\delta^k) \supset \mathrm{out}(\eta^k) = \eta^{k-1}$ and by Lemma 3.2(i) (Out), $\lambda^{k+1}(\mathrm{out}(\delta^k)) = \lambda(\delta^k)$, so by choice of η^{k-1}, $\delta^{k+1} \subset \Lambda^{k+1}$ and δ^{k+1} has a terminator $\tau^{k+1} \subset \eta^{k+1} \subset \Lambda^{k+1}$. Fix $\tilde{\tau}^{k+1} \subseteq \eta^{k+1}$ such that $(\tilde{\tau}^{k+1})^- = \tau^{k+1}$, and let $\tilde{\tau}^k = \mathrm{out}(\tilde{\tau}^{k+1})$. By Definition 6.2 and Case 3 of Definition 6.3, there is a $\zeta^{k+1} \in \overline{\mathrm{PL}}(\tilde{\tau}^{k+1})$ such that $\mathrm{OS}(\nu^{k+1}) \subseteq \mathrm{TS}(\zeta^{k+1})$.

We now note that τ^{k+1} has infinite outcome along $\tilde{\tau}^{k+1} = \lambda(\tilde{\tau}^k)$, and if $\tau^{k+1} \subset \lambda(\delta^k)$, then τ^{k+1} does not have infinite outcome along $\lambda(\delta^k)$. Furthermore, $\tilde{\tau}^k \subseteq \eta^k \subset \delta^k \subseteq \lambda(\delta^{k\pm1})$, so by (2.4), if τ^{k+1} were to have infinite outcome along $\lambda^{k+1}(\delta^{k\pm1})$, then that outcome would be the same as the outcome of τ^{k+1} along $\tilde{\tau}^{k+1} = \lambda(\tilde{\tau}^k)$, and by (2.6), τ^{k+1} would have that outcome along $\lambda(\gamma^k)$ for all $\gamma^k \in [\tilde{\tau}^k,\lambda(\delta^{k\pm1})]$. In particular, τ^{k+1} would have that same infinite outcome along $\lambda(\delta^k)$, which we have shown not to be the case. Hence τ^{k+1} does not have infinite outcome along $\lambda^{k+1}(\delta^{k\pm1})$. As $\lambda(\delta^k) \supseteq \delta^{k+1}$ and there is a primary $\lambda(\eta^k)$-link $[\mu^{k+1},\tau^{k+1}]$ which restrains δ^{k+1} with $\mu^{k+1} \subset \delta^{k+1}$, it follows from (2.10) that there is a node $\hat{\tau}^k$ such that $\tilde{\tau}^k \subseteq \eta^k \subset \hat{\tau}^k \subseteq \delta^k$ and $\hat{\tau}^k$ switches τ^{k+1}. ((2.10) implies that a node can be switched only when it is free; and by (2.6), $\delta^{k+1} \subseteq \lambda(\alpha^k)$ for all α^k such that $\tilde{\tau}^k \subseteq \alpha^k \subseteq \delta^k$. So no node $\subset \tau^{k+1}$ can be switched by such an $\alpha^k \supset \tilde{\tau}^k$ until τ^{k+1} is switched.) Let $\bar{\tau}^k = (\hat{\tau}^k)^-$, and let $\tau^k = (\tilde{\tau}^k)^-$. Then $[\tau^k,\bar{\tau}^k]$ is a primary $\lambda(\delta^{k\pm1})$-link, and $\mathrm{up}(\bar{\tau}^k) = \tau^{k+1}$.

As $\eta^k \subseteq \bar{\tau}^k = (\hat{\tau}^k)^- \subset \hat{\tau}^k \subseteq \delta^k$, it follows from (2.5) and Lemma 3.1 (Limit Path) that $\bar{\tau}^k$ has an initial derivative $\bar{\tau}^{k\pm1}$ such that $\eta^{k\pm1} \subseteq \bar{\tau}^{k\pm1} \subset \mathrm{out}(\delta^k) \subseteq \delta^{k\pm1}$; fix $\hat{\tau}^{k\pm1} \subseteq \delta^{k\pm1}$ such that $(\hat{\tau}^{k\pm1})^- = \bar{\tau}^{k\pm1}$. Now $\tilde{\tau}^k = \mathrm{out}(\tilde{\tau}^{k+1})$, so $\mathrm{up}(\tau^k) = \tau^{k+1}$, and τ^k is the principal derivative of τ^{k+1} along both η^k and Λ^k. Furthermore, by (2.10) and as η^k is Λ^k-free and $\tau^k \subset \eta^k \subseteq \bar{\tau}^k$, $\bar{\tau}^k$ must be switched by some proper extension $\tilde{\tau}^{k\pm1}$ of $\delta^{k\pm1}$ along $\Lambda^{k\pm1}$. Let $\tau^{k\pm1} = (\tilde{\tau}^{k\pm1})^-$, and note that $\tau^{k\pm1}$ is the principal derivative of $\bar{\tau}^k$ along $\Lambda^{k\pm1}$, so $[\bar{\tau}^{k\pm1},\tau^{k\pm1}]$ is a primary $\Lambda^{k\pm1}$-link with $\bar{\tau}^{k\pm1} \subset \delta^{k\pm1} \subseteq \tau^{k\pm1}$.

We now show that $\tau^{k\pm1}$ is a terminator for $\delta^{k\pm1}$ along $\Lambda^{k\pm1}$. First assume that τ^{k+1} is not a primary completion. Then $\overline{\mathrm{PL}}(\tilde{\tau}^{k+1}) = \{\tau^{k+1}\}$, $\tau^{k\pm1} \in \overline{\mathrm{PL}}(\tilde{\tau}^{k\pm1})$, and $\mathrm{up}^{k+1}(\tau^{k\pm1}) = \tau^{k+1}$. Hence $\tau^{k\pm1}$ is a terminator for $\delta^{k\pm1}$ along $\Lambda^{k\pm1}$.

Now assume that τ^{k+1} is a primary completion of some ρ^{k+1}, which we fix, and let $\sigma^{k+1} = (\rho^{k+1})^-$. As τ^{k+1} has infinite outcome along $\tilde{\tau}^{k+1}$ but finite outcome along $\lambda(\hat{\tau}^k)$, it follows from Lemma 5.1(i),(ii) (PL Analysis) and Definition 5.3 that

$$\overline{\mathrm{PL}}(\tilde{\tau}^{k+1}) = \mathrm{PL}(\sigma^{k+1},\tilde{\tau}^{k+1}) \cup \{\sigma^{k+1}\} = \mathrm{PL}(\sigma^{k+1},\tau^{k+1}) \cup \{\tau^{k+1}\} \cup \{\sigma^{k+1}\},$$

and by Lemma 5.1(iv) (PL Analysis),

$$PL(\sigma^{k+1},\tau^{k+1}) = PL(\sigma^{k+1},\lambda(\hat{\tau}^{k})).$$

By Lemma 5.3(ii) (Implication Chain), Lemma 5.2 (Requires Extension), and (5.5)(ii), $\hat{\tau}^k$ requires extension for some derivative σ^k of σ^{k+1}. As Λ^0 is admissible, and, by (2.5), out($\hat{\tau}^k$) $\subset \Lambda^{k\pm1}$, it follows from (5.27), Lemma 5.15(ii) (Admissibility), and Lemma 5.4 (Compatibility) that $\hat{\tau}^k$ has a (k-1)-completion $\beta^{k\pm1} \subset \Lambda^{k\pm1}$, and that κ^k = up($\beta^{k\pm1}$) is the primary completion of $\hat{\tau}^k$. Furthermore, by Lemma 5.12(ii) (PL),

$$\{up(\zeta^k): \zeta^k \in PL(\bar{\tau}^k,\kappa^k)\} = PL(\sigma^{k+1},\lambda(\hat{\tau}^k)).$$

Fix $\tilde{\beta}^{k\pm1} \subset \Lambda^{k\pm1}$ such that $(\tilde{\beta}^{k\pm1})^- = \beta^{k\pm1}$, let $\tilde{\kappa}^k = \lambda(\tilde{\beta}^{k\pm1})$, and note that since $\beta^{k\pm1}$ the initial derivative of κ^k, follows from (2.4) that $(\tilde{\kappa}^k)^- = \kappa^k$. Now $\sigma^{k+1} \subset \eta^{k+1}$ and by (5.2), σ^k is an initial derivative of σ^{k+1}; hence by Lemma 3.1(i) (Limit Path), $\sigma^k \subset \eta^k \subset \hat{\tau}^k \subset \kappa^k$. We now recall that there is no primary Λ^k-link which restrains $(\eta^k)^-$. Thus there must be a $\bar{\kappa}^{k\pm1} \subset \Lambda^{k\pm1}$ such that up($\bar{\kappa}^{k\pm1}$) = κ^k and $\bar{\kappa}^{k\pm1}$ has infinite outcome along $\Lambda^{k\pm1}$, else by (2.6) and (2.10), $[\sigma^k,\kappa^k]$ would be a primary Λ^k-link restraining $(\eta^k)^-$. Fix $\hat{\kappa}^{k\pm1} \subset \Lambda^{k\pm1}$ such that $(\hat{\kappa}^{k\pm1})^- = \bar{\kappa}^{k\pm1}$. By Lemma 5.1(iv) (PL Analysis),

$$PL(\bar{\tau}^k,\lambda(\hat{\kappa}^{k\pm1})) = PL(\bar{\tau}^k,\kappa^k).$$

As $\bar{\tau}^{k\pm1}$ is the initial derivative of $\bar{\tau}^k$ along $\hat{\kappa}^{k\pm1}$, it follows from Lemma 5.3(ii) (Implication Chain) and Lemma 5.2 (Requires Extension) that $\hat{\kappa}^{k\pm1}$ requires extension for $\bar{\tau}^{k\pm1}$. As Λ^0 is admissible, it follows from Lemma 5.15(i),(ii) (Admissibility) and Lemma 5.3(ii) (Implication Chain) that there are $\pi^{k\pm1} \subset \tilde{\pi}^{k\pm1} \subset \Lambda^{k\pm1}$ such that $\pi^{k\pm1}$ is the primary completion of $\hat{\kappa}^{k\pm1}$, $(\tilde{\pi}^{k\pm1})^- = \pi^{k\pm1}$, and $\pi^{k\pm1}$ has infinite outcome along $\tilde{\pi}^{k\pm1}$. By (5.19), up($\pi^{k\pm1}$) = $\bar{\tau}^k$, so by (2.8), $\pi^{k\pm1} = \tau^{k\pm1}$ and $\tilde{\pi}^{k\pm1} = \tilde{\tau}^{k\pm1}$. By Lemma 5.12(ii) (PL),

$$PL(\bar{\tau}^k,\lambda(\hat{\kappa}^{k\pm1})) = \{up(\zeta^{k\pm1}): \zeta^{k\pm1} \in PL(\bar{\kappa}^{k\pm1},\pi^{k\pm1})\}.$$

Now by Lemma 5.1(i),(ii) (PL Analysis) and Definitions 5.3 and 6.3,

$$\overline{PL}(\tilde{\pi}^{k\pm1}) = PL(\bar{\kappa}^{k\pm1},\tilde{\pi}^{k\pm1})\cup\{\bar{\kappa}^{k\pm1}\} = PL(\bar{\kappa}^{k\pm1},\pi^{k\pm1})\cup\{\pi^{k\pm1}\}\cup\{\bar{\kappa}^{k\pm1}\}.$$

Furthermore, up($\pi^{k\pm1}$) = $\bar{\tau}^k$, up($\bar{\kappa}^{k\pm1}$) = κ^k, up($\bar{\tau}^k$) = τ^{k+1}, and up(κ^k) = σ^{k+1}. Hence

$$\{up^{k+1}(\zeta^{k\pm1}) : \zeta^{k\pm1} \in \overline{PL}(\tilde{\pi}^{k\pm1})\} = \overline{PL}(\tilde{\tau}^{k+1}),$$

so $\pi^{k\pm1} = \tau^{k\pm1}$ is a terminator for $\delta^{k\pm1}$ along $\Lambda^{k\pm1}$.

We now verify (ii) by showing that only finitely many sections of $X^{[i]}$ are weakly controlled at nodes $\subset \Lambda^{k\pm1}$. By (2.11) and Lemma 3.1 (Limit Path), fix $\alpha^{k\pm1} \subset \Lambda^{k\pm1}$ such that $wt(\lambda(\alpha^{k\pm1})) > i$, $\alpha^{k\pm1} \supset \eta^{k\pm1}$, and $\lambda(\alpha^{k\pm1}) \subset \Lambda^k$. By Lemma 6.1(iii) (Finite Control), there are only finitely many initiators for $X^{[i]}$ on T^k; since $X^{[i]}$ is not weakly controlled along Λ^k, we can assume without loss of generality that every initiator for $X^{[i]}$ at some node along Λ^k has a terminator $\subset \lambda(\alpha^{k\pm1})$. Furthermore, by (2.4) and Lemma 3.1 (Limit Path), we can assume that for all $\xi^{k\pm1}$ such that $\alpha^{k\pm1} \subseteq \xi^{k\pm1} \subset \Lambda^{k\pm1}$, if $\lambda(\xi^{k\pm1})$ extends an initiator for $X^{[i]}$, then that initiator lies along $\Lambda^{k\pm1}$. Suppose that $\tilde{\alpha}^{k\pm1}$ is given such that $\alpha^{k\pm1} \subseteq \tilde{\alpha}^{k\pm1} \subset \Lambda^{k\pm1}$. By (2.4) and (2.6), $\lambda(\tilde{\alpha}^{k\pm1}) \supseteq \lambda(\alpha^{k\pm1})$. As $wt(\lambda(\alpha^{k\pm1})) > i$, it follows from the choice of $\alpha^{k\pm1}$, (2.1), and (6.4) that there is no initiator for $X^{[i]}$ at $\lambda(\tilde{\alpha}^{k\pm1})$, so $X^{[i]}$ is not weakly controlled at $\lambda(\tilde{\alpha}^{k\pm1})$. Hence $\tilde{\alpha}^{k\pm1}$ cannot be an initiator for a section of $X^{[i]}$ at any node. Thus there are only finitely many initiators for sections of $X^{[i]}$ along $\Lambda^{k\pm1}$. By the preceding paragraph, every initiator along $\Lambda^{k\pm1}$ for a section of $X^{[i]}$ has a terminator along $\Lambda^{k\pm1}$, so we can fix $\xi^{k\pm1} \subset \Lambda^{k\pm1}$ such that each such terminator is $\subset \xi^{k\pm1}$. It now follows from Definition 6.4 that if $u \geq wt(\xi^{k\pm1})$, then $(X^{[i]})^{[u]}$ is not weakly controlled at any node $\subset \Lambda^{k\pm1}$, so (ii) follows.

Fix i and u and assume that $j \in \{0,2\}$. Then there are no terminators for sections of $X^{[i]}$ along $\Lambda^{k\pm1}$. Hence if $(X^{[i]})^{[u]}$ is weakly controlled at some $\gamma^{k\pm1} \subset \Lambda^{k\pm1}$, then by Definition 6.4, there is an initiator for $(X^{[i]})^{[u]}$ which has no terminator along $\Lambda^{k\pm1}$. By Definition 6.3, for all $v \geq u$, $(X^{[i]})^{[v]}$ will have an initiator along $\Lambda^{k\pm1}$ which has no terminator along $\Lambda^{k\pm1}$, so by Definition 6.4, $(X^{[i]})^{[v]}$ will be weakly controlled along $\Lambda^{k\pm1}$. (iii) now follows from (ii). ∎

As we extend nodes along Λ^k, the path approximation to Λ^{k+1} via the function λ will occasionally switch paths. We show that for requirements of types 0 and 2, the choice of initiators is invariant under switches of paths, as long as the initiator remains on the switched path and no terminators are eliminated.

Lemma 6.6 (Constancy of Initiator Lemma): Fix $k \leq n$ and $\eta^k \in T^k$. Let S be a space associated with the requirement R of dimension r and type $j \in \{0,2\}$, and assume that $k \leq r-1$ if $j = 0$, and $k < r-1$ if $j = 2$. Suppose that S is weakly controlled at $\lambda((\eta^k)^-)$ with initiator δ^{k+1}, and that $\lambda(\eta^k) \supseteq \delta^{k+1}$. Then δ^{k+1} is the initiator for S at $\lambda(\eta^k)$.

Proof: First assume that $j = 0$ and $k = r-1$. Let ν^{k+1} be the controller for S at $\lambda(\eta^k)$. Then ν^{k+1} is the only controller for S on T^{k+1}, and the initiator for S along any path properly extending ν^{k+1} is the immediate successor of ν^{k+1} along that path. The lemma now follows in this case.

Suppose that $k < r-1$. Let $\rho^{k+1} = \lambda(\eta^k)\wedge\lambda((\eta^k)^-)$, and note, by hypothesis, that $\rho^{k+1} \supseteq \delta^{k+1}$. We assume that $\rho^{k+1} \neq \lambda(\eta^k)$, else by (2.4), $\lambda(\eta^k) = \lambda((\eta^k)^-)$. Under this assumption, it follows from (2.4) that $(\lambda(\eta^k))^- = \rho^{k+1}$. As S is weakly controlled along $\lambda((\eta^k)^-)$, it follows from (6.6) and (2.11) that $wt(S) \leq wt(\lambda((\eta^k)^-)) < wt(\lambda(\eta^k))$, so by (6.4), $\lambda(\eta^k)$ cannot be an initiator for S, and by Case 3 of Definition 6.3, ρ^{k+1} cannot be a terminator for S at $\lambda(\eta^k)$. Hence as $\rho^{k+1} = (\lambda(\eta^k))^-$, all terminators for S along $\lambda(\eta^k)$ are $\subset \rho^{k+1}$. By (6.7), δ^{k+1} is the longest initiator for S along $\lambda((\eta^k)^-)$ which has no terminator along $\lambda((\eta^k)^-)$, so as $\delta^{k+1} \subseteq \rho^{k+1} \subseteq (\lambda(\eta^k))^-$, δ^{k+1} is the longest initiator for S along $\lambda(\eta^k)$ which has no terminator along $\lambda(\eta^k)$. By (6.7), δ^{k+1} is the initiator for S at $\lambda(\eta^k)$. ∎

In order to show later that the functionals which we define are total on certain oracles, we want to show that for requirements of types 0 and 2, if a space is weakly controlled along an approximation to Λ^1 but not along a later approximation, then that space is never weakly controlled again. This will fail to be the case only when a terminator is switched. As the proof does not depend on Λ^1, we prove the general case.

Lemma 6.7 (Loss of Control Lemma): Fix $k < n$, a space S for a requirement R of type 0 or 2 with $k+1 < \dim(R)$, and $\eta^k \in T^k$ such that $wt(S) \leq wt(\lambda((\eta^k)^-))$. Suppose that S has no initiator at $\lambda((\eta^k)^-)$. Then S has no initiator at $\lambda(\eta^k)$.

Proof: Suppose that S has an initiator δ^{k+1} at $\lambda(\eta^k)$ in order to obtain a contradiction. By (2.4), $(\lambda(\eta^k))^- \subseteq \lambda((\eta^k)^-)$. As $wt(S) \leq wt(\lambda((\eta^k)^-))$, either $\lambda((\eta^k)^-) = \lambda(\eta^k)$, or by (2.11), $wt(S) < wt(\lambda(\eta^k))$; and in the latter case, it follows from (6.4) that $\lambda(\eta^k)$ is not an initiator for S at any node. Hence $\delta^{k+1} \subseteq \lambda((\eta^k)^-)$. By Case 3 of Definition 6.3, the immediate successor ρ^{k+1} of any terminator for S along $\lambda((\eta^k)^-)$ is an initiator for S at ρ^{k+1}; hence the longest node which is an initiator for S at some node $\subset \lambda((\eta^k)^-)$ can have no terminator along $\lambda((\eta^k)^-)$. As $\delta^{k+1} \subseteq \lambda((\eta^k)^-)$, it follows that there is an initiator for S at $\lambda((\eta^k)^-)$, contrary to hypothesis. ∎

When a node ν^1 relinquishes control of a space to a node $\hat{\nu}^1$, we will need to know that, often enough, the axioms which were defined by derivatives of ν^1 are either the same axioms that would have been defined by derivatives of $\hat{\nu}^1$, or are corrected. The next lemma is a key ingredient in showing that this happens. It allows us to trace the process of switching initiators, and will be used to show that we can correct axioms for requirements of types 0 and 2. We consider the case where η switches $\kappa^1 \in T^1$, causing weak control of a space S to pass from a node ν^1 to a node $\hat{\nu}^1$. We will show that this can occur only when $\hat{\delta}^1 \subseteq \kappa^1 \subset \delta^1$, where δ^1 and $\hat{\delta}^1$ are, respectively, the initiators for ν^1 at $\lambda(\eta^-)$ and $\hat{\nu}^1$ at $\lambda(\eta)$. By Lemma 3.3 (λ-Behavior), for all $t \geq 1$, η will switch $up^t(\kappa^1) = \kappa^t$. We try to carry this situation up to successive trees, by showing that $up^t(\nu^1)$ weakly controls $up^t(S)$ along $\lambda^t(\eta^-)$ with some initiator δ^t, $up^t(\hat{\nu}^1)$ weakly controls $up^t(S)$ along $\lambda^t(\eta)$ with some initiator $\hat{\delta}^t$, and $\delta^t \wedge \hat{\delta}^t \subseteq \kappa^t \subset \delta^t \vee \hat{\delta}^t$. Furthermore, the shortest element of $\{\delta^t, \hat{\delta}^t\}$ will alternate by level, i.e., $\delta^t \subset \hat{\delta}^t$ iff $\hat{\delta}^{t+1} \subset \delta^{t+1}$. We will be able to carry this alternation up inductively through T^p where p+1 is the smallest j such that $\nu^t = \hat{\nu}^t$, and in some cases, to T^{p+1}. (In the other cases for t = p+1, we will have to resort to a different proof, as some of the arguments will fail.) The remaining lemmas of this section will then enable us to show, in the next section, that we can correct axioms when necessary.

Lemma 6.8 (Alternating Initiator Lemma): Fix $\eta \in T^0$ and let S be a section of a space assigned to the requirement R of dimension $r \geq 2$ and type 0 or 2. Suppose that S is weakly controlled by ν^1 at $\lambda(\eta^-)$ with initiator δ^1, S is weakly controlled by $\hat{\nu}^1$ at $\lambda(\eta)$ with initiator $\hat{\delta}^1$, and $\delta^1 \neq \hat{\delta}^1$. Let p be the smallest t such that $up^{t+1}(\nu^1) = up^{t+1}(\hat{\nu}^1)$ if such a t exists, and let p = r-1 otherwise. (Note that, if tp(R) = 0, then t must exist by the definition of $\equiv$ for type 0 nodes.) Then for all $t \in [1,p]$, there are $\nu^t \subset \delta^t \subseteq \lambda^t(\eta^-)$, $\hat{\nu}^t \subset \hat{\delta}^t \subseteq \lambda^t(\eta)$, $\kappa^t = \lambda^t(\eta^-) \wedge \lambda^t(\eta)$, and a space S^t such that $\nu^t = up^t(\nu^1)$, $\hat{\nu}^t = up^t(\hat{\nu}^1)$, $S \subseteq S^t$, and:

(6.8) ν^t weakly controls S^t at $\lambda^t(\eta^-)$ with initiator δ^t, and if $t > 1$, then $\lambda(\delta^{t\pm 1}) \supseteq \delta^t$.

(6.9) $\hat{\nu}^t$ weakly controls S^t at $\lambda^t(\eta)$ with initiator $\hat{\delta}^t$, and if $t > 1$, then $\lambda(\hat{\delta}^{t\pm 1}) \supseteq \hat{\delta}^t$.

(6.10) $\delta^t \subseteq \kappa^t \subset \hat{\delta}^t$ if t is even, and $\hat{\delta}^t \subseteq \kappa^t \subset \delta^t$ if t is odd.

Furthermore, if $t \in [2,p]$, then by (6.8) inductively and Definitions 6.3 and 6.4, ν^t weakly controls S^t at $\lambda(\delta^{t\pm 1})$, so we can fix the initiator $\tilde{\delta}^t \subseteq \lambda(\delta^{t\pm 1})$ such that ν^t weakly controls S^t at $\lambda(\delta^{t\pm 1})$ with initiator $\tilde{\delta}^t$. Similarly, by (6.9) inductively and Definitions 6.3 and 6.4, $\hat{\nu}^t$ weakly controls S^t at $\lambda(\hat{\delta}^{t\pm 1})$, so we can fix the initiator $\bar{\delta}^t \subseteq \lambda(\hat{\delta}^{t\pm 1})$ such that $\hat{\nu}^t$ weakly controls S^t at $\lambda(\hat{\delta}^{t\pm 1})$ with initiator $\bar{\delta}^t$. (We need to introduce $\tilde{\delta}^t$ and $\bar{\delta}^t$ here, as the initiators for S^t at $\lambda^t(\eta^-)$ and $\lambda^t(\eta)$ may differ from those at $\lambda(\delta^{t\pm 1})$ and $\lambda(\hat{\delta}^{t\pm 1})$, respectively). Let $\rho^t = \lambda(\delta^{t\pm1}) \wedge \lambda(\hat{\delta}^{t\pm1})$. Then for all $t \in [2,p]$:

(6.11) (i) $\tilde{\delta}^t \subseteq \rho^t \subset \bar{\delta}^t$ if t is even and $\bar{\delta}^t \subseteq \rho^t \subset \tilde{\delta}^t$ if t is odd.

(ii) $\tilde{\delta}^t = \delta^t$ if t is even, and $\bar{\delta}^t = \hat{\delta}^t$ if t is odd.

In addition:

(6.12) (6.8)-(6.11) will hold for $t = p+1$ unless either:

(i) $p+1 = r$; or

(ii) ν^p has finite outcome along $\lambda^p(\eta^-)$ iff $\hat{\nu}^p$ has finite outcome along $\lambda^p(\eta)$.

Proof: First assume that $t = 1$. Then (6.8) and (6.9) follow by hypothesis. As $\delta^1 \neq \hat{\delta}^1$, it follows from (6.7) and Definition 6.7 that $\lambda(\eta)|\lambda(\eta^-)$, so by Lemma 3.3 ($\lambda$-Behavior), $\lambda(\eta)^- \subset \lambda(\eta^-)$. Thus by Lemma 6.6 (Constancy of Initiator), $\lambda(\eta^-) \wedge \lambda(\eta) \subset \delta^1$. By (6.4), (6.6), and (2.11), $wt(\hat{\delta}^1) \leq wt(S) \leq wt(\lambda(\eta^-)) < wt(\lambda(\eta))$, so $\hat{\delta}^1 \neq \lambda(\eta)$. Hence $\hat{\delta}^1 \subseteq \lambda(\eta)^- = \lambda(\eta^-) \wedge \lambda(\eta)$, and (6.10) holds.

Suppose that $t \geq 2$. We first verify (6.11)(i), assuming that t is odd. (An analogous argument gives the proof for even t by interchanging the hatted and unhatted nodes, the nodes with bars and tildes, η and η^-, and odd and even in the proof below.) By (6.10) inductively, $\delta^{t\pm1} \subset \hat{\delta}^{t\pm1}$.

Case 1: $\lambda(\delta^{t\pm1})|\lambda(\hat{\delta}^{t\pm1})$. We begin with the proof that $\bar{\delta}^t \subseteq \rho^t$. By (2.4) and Lemma 3.1(ii) (Limit Path), there must be a $\xi^{t\pm1}$ such that $\delta^{t\pm1} \subset \xi^{t\pm1} \subseteq \hat{\delta}^{t\pm1}$, $\lambda(\delta^{t\pm1})|\lambda(\xi^{t\pm1})$, $(\xi^{t\pm1})^-$ is a derivative of ρ^t, $\lambda(\xi^{t\pm1}) \subseteq \lambda(\hat{\delta}^{t\pm1})$, and $(\lambda(\xi^{t\pm1}))^- = \rho^t$. As $\rho^t, \bar{\delta}^t \subseteq \lambda(\hat{\delta}^{t\pm1})$ by (6.2), ρ^t and $\bar{\delta}^t$ are comparable. Suppose that $\rho^t \subset \bar{\delta}^t$ in order to obtain a contradiction. Then $\lambda(\xi^{t\pm1}) \subseteq \bar{\delta}^t$. By (6.4) and Definition 6.7,

(6.13) $wt(\tilde{\delta}^t) \leq wt(S^t) \leq wt(\lambda(\delta^{t\pm1}))$

and

(6.14) $wt(\bar{\delta}^t) \leq wt(S^t) \leq wt(\lambda(\hat{\delta}^{t\pm1}))$.

(Note that (6.13) and (6.14) do not make sense when $t = r$.) As $\lambda(\delta^{t\pm1}) \neq \lambda(\xi^{t\pm1})$, it follows from (6.13), (2.11), (2.1), and (6.14) that

$$wt(S^t) \leq wt(\lambda(\delta^{t\pm1})) < wt(\lambda(\xi^{t\pm1})) \leq wt(\bar{\delta}^t) \leq wt(S^t),$$

a contradiction. Hence $\bar{\delta}^t \subseteq \rho^t$.

We complete the proof of (6.11)(i) for Case 1 by showing that $\rho^t \subset \tilde{\delta}^t$. By (6.2), $\rho^t, \tilde{\delta}^t \subseteq \lambda(\delta^{t\pm1})$, so ρ^t and $\tilde{\delta}^t$ must be comparable. It suffices to assume that $\tilde{\delta}^t \subseteq \rho^t$, and show that t = p+1 and (6.12)(ii) holds. By (6.10), $\delta^{t\pm1} \subset \hat{\delta}^{t\pm1}$, so iterating Lemma 6.6 (Constancy of Initiator) for $\delta^{t\pm1} \subset \hat{\delta}^{t\pm1}$, we see that $\tilde{\delta}^t = \bar{\delta}^t$; thus by (6.3), $\nu^t = \hat{\nu}^t$. Hence t = p+1. There are two cases to consider.

First consider the case in which ν^{p+1} has infinite outcome along $\tilde{\delta}^{p+1} = \bar{\delta}^{p+1}$. Then ν^{p+1} has infinite outcome along both $\lambda(\delta^p) \supseteq \tilde{\delta}^{p+1}$ and $\lambda(\widehat{\delta}^p) \supseteq \bar{\delta}^{p+1}$, so all derivatives of ν^{p+1} along δ^p ($\widehat{\delta}^p$, resp.) have finite outcome along δ^p ($\widehat{\delta}^p$, resp.). In particular, by (6.2) and inductively by (6.8) and (6.9), ν^p has finite outcome along $\lambda^p(\eta^-) \supseteq \delta^p$ and $\hat{\nu}^p$ has finite outcome along $\lambda^p(\eta^-) \supseteq \widehat{\delta}^p$, so (6.12)(ii) holds.

Now consider the case in which ν^{p+1} has finite outcome γ^p along $\tilde{\delta}^{p+1} = \bar{\delta}^{p+1}$. Then ν^{p+1} has outcome γ^p along both $\lambda(\delta^p) \supseteq \tilde{\delta}^{p+1}$ and $\lambda(\widehat{\delta}^p) \supseteq \bar{\delta}^{p+1}$, so by (2.5), $\gamma^p \subseteq \delta^p, \widehat{\delta}^p$. By (2.4) and (2.8), $(\gamma^p)^-$ has infinite outcome along γ^p and is the longest (and principal) derivative of ν^{p+1} along either δ^p or $\widehat{\delta}^p$. Now by Lemma 4.3(i)(c)(a), any primary δ^p-link ($\widehat{\delta}^p$-link, resp.) which restrains $(\gamma^p)^-$ restrains all derivatives of ν^{p+1} along δ^p ($\widehat{\delta}^p$, resp.). Hence by Definitions 6.3 and 6.4, the controllers for $up^p(S)$ corresponding to δ^p and $\widehat{\delta}^p$, respectively, are the longest derivatives of ν^{p+1} properly contained in δ^p and $\widehat{\delta}^p$, respectively, so $\nu^p = \hat{\nu}^p = (\gamma^p)^-$ and (6.12)(ii) holds. Thus $\tilde{\delta}^t \supset \rho^t$ unless (6.12)(i) or (ii) holds, concluding the proof of (6.11)(i) for this case.

Case 2: $\lambda(\delta^{t\pm1})$ and $\lambda(\hat{\delta}^{t\pm1})$ are comparable. By Definition 6.7, $wt(\lambda(\delta^{t\pm1})), wt(\lambda(\hat{\delta}^{t\pm1})) \geq wt(S^t)$, so by Case 3 of Definition 6.3 and (6.4), $\tilde{\delta}^t$ ($\bar{\delta}^t$, resp.) has a terminator along $\lambda(\delta^{t\pm1})$ iff $\tilde{\delta}^t$ ($\bar{\delta}^t$, resp.) has a terminator along $\lambda(\hat{\delta}^{t\pm1})$. Thus by Definitions 6.3 and 6.4, $\tilde{\delta}^t = \bar{\delta}^t$, so by (6.3), $\nu^t = \hat{\nu}^t$. Hence t = p+1. We now proceed as in the preceding two paragraphs, showing that (6.12)(ii) holds, and thus that this case is contrary to hypothesis, and concluding the proof of Case 2.

We now verify (6.8)-(6.10) and (6.11)(ii). Assume that t is odd. (If t is even, then an analogous proof is obtained by interchanging the hatted and unhatted nodes, the nodes with bars and tildes, η and η^-, odd and even, and (6.8) and (6.9).) We begin by showing that $\bar{\delta}^t \subseteq \kappa^t$ (a portion of (6.10)) by eliminating the other possibilities. Let $\zeta^t = \kappa^t \wedge \bar{\delta}^t$, and assume that $\zeta^t \subset \bar{\delta}^t$ in order to obtain a contradiction. First suppose that ζ^t has finite outcome $\zeta^{t\pm1}$ along $\bar{\delta}^t$, and so that $(\zeta^{t\pm1})^-$ has infinite outcome along $\zeta^{t\pm1}$ and $\mathrm{up}((\zeta^{t\pm1})^-) = \zeta^t$. As $\bar{\delta}^t \subset \tilde{\delta}^t \subseteq \lambda(\delta^{t\pm1})$ by (6.11)(i) and the definition of $\tilde{\delta}^t$, it follows from (2.5) that $\zeta^{t\pm1} \subset \delta^{t\pm1}$. By (6.10) inductively, $\zeta^{t\pm1} \subset \lambda^{t\pm1}(\eta^-) \wedge \lambda^{t\pm1}(\eta) = \kappa^{t\pm1}$. Hence by (2.4) and Lemma 3.1 (Limit Path), $\kappa^t, \bar{\delta}^t \supseteq \zeta^t \frown \langle \zeta^{t\pm1} \rangle$, contrary to the choice of ζ^t.

Suppose that ζ^t has infinite outcome $\hat{\xi}^{t\pm1}$ along $\bar{\delta}^t$. By Lemma 3.3 (λ-Behavior) and as t is odd and η switches κ^1, κ^t has finite outcome $\beta^{t\pm1}$ along $\lambda^t(\eta)$. Now it cannot be the case that $\kappa^t \subset \bar{\delta}^t$, else as $\kappa^{t\pm1} \subset \beta^{t\pm1} \subseteq \hat{\delta}^{t\pm1} \subseteq \lambda^{t\pm1}(\eta)$ by (6.10), it would follow from (2.4) that $\kappa^t = \zeta^t$ has finite outcome along $\bar{\delta}^t$, contrary to our assumption. Hence as $\zeta^t \subset \bar{\delta}^t$, $\kappa^t | \bar{\delta}^t$. As $\lambda(\hat{\delta}^{t\pm1}) \supseteq \bar{\delta}^t \supset \zeta^t$, we have $\hat{\xi}^{t\pm1} \subseteq \hat{\delta}^{t\pm1}$ by (2.5), and so $(\hat{\xi}^{t\pm1})^-$ is the initial derivative of ζ^t along $\hat{\delta}^{t\pm1}$. By Lemma 3.1 (Limit Path) and as $\zeta^t \subset \kappa^t$, ζ^t has an initial derivative along $\kappa^{t\pm1}$; and by (6.10) inductively, $\hat{\delta}^{t\pm1}$ and $\kappa^{t\pm1}$ are comparable; hence this initial derivative must also be $(\hat{\xi}^{t\pm1})^-$. As $\zeta^t = \kappa^t \wedge \bar{\delta}^t$ and $\kappa^t | \bar{\delta}^t$, it follows from (2.4) that ζ^t must have finite outcome $\zeta^{t\pm1}$ along κ^t, so by (2.7), $\zeta^{t\pm1} \subseteq \kappa^{t\pm1}$. By (6.10) inductively, $\kappa^{t\pm1} \subset \hat{\delta}^{t\pm1}$, so $\zeta^t \frown \langle \zeta^{t\pm1} \rangle \subseteq \lambda(\hat{\delta}^{t\pm1})$ by (2.4). Thus $\zeta^t = \kappa^t \wedge \lambda(\hat{\delta}^{t\pm1})$ and $\zeta^t \frown \langle \zeta^{t\pm1} \rangle \subseteq \kappa^t, \lambda(\hat{\delta}^{t\pm1})$, a contradiction. We thus conclude that $\bar{\delta}^t \subseteq \kappa^t$.

We next verify (6.11)(ii) and (6.9). By Definition 6.4, we noted in the hypothesis of the lemma that $\hat{\nu}^t$ weakly controls S^t at $\lambda(\hat{\delta}^{t\pm1})$ with initiator $\bar{\delta}^t$, and $\lambda(\hat{\delta}^{t\pm1}) \supseteq \bar{\delta}^t$. By (6.9) inductively, $\hat{\delta}^{t\pm1} \subseteq \lambda^{t\pm1}(\eta)$. As S^t is weakly controlled at $\lambda(\hat{\delta}^{t\pm1})$, $\mathrm{wt}(S^t) \leq \mathrm{wt}(\lambda(\hat{\delta}^{t\pm1}))$ by Definition 6.7. By (2.11), for all $\mu^{t\pm1}$ such that $\hat{\delta}^{t\pm1} \subset \mu^{t\pm1} \subseteq \lambda^{t\pm1}(\eta)$ and $\lambda(\hat{\delta}^{t\pm1}) \neq \lambda(\mu^{t\pm1})$, $\mathrm{wt}(\lambda(\mu^{t\pm1})) > \mathrm{wt}(\lambda(\hat{\delta}^{t\pm1})) \geq \mathrm{wt}(S^t)$, so by (6.4), $\lambda(\mu^{t\pm1})$ cannot be an initiator for S^t, and $(\lambda(\mu^{t\pm1}))^-$ cannot be a terminator for S^t along $\lambda(\mu^{t\pm1})$. Hence by (6.5)(ii), we have $\hat{\delta}^t = \bar{\delta}^t$, verifying (6.11)(ii). Also note, by (6.7), that $\hat{\delta}^t$ is the longest initiator for S^t at $\lambda^t(\eta)$ which has no terminator along $\lambda^t(\eta)$. (6.9) now follows from Definition 6.3 and (6.3).

We now verify (6.8). By (6.8) inductively, $\delta^{t\pm1} \subseteq \lambda^{t\pm1}(\eta^-)$, so by Definition 6.7 and (2.11), $\mathrm{wt}(S^t) \leq \mathrm{wt}(\lambda(\delta^{t\pm1})) \leq \mathrm{wt}(\lambda^t(\eta^-))$. As $\hat{\delta}^t = \bar{\delta}^t \subseteq \kappa^t \subseteq \lambda^t(\eta^-)$ and $\hat{\delta}^t$ is the initiator for S^t at $\lambda^t(\eta) \supseteq \kappa^t$, it follows from Definition 6.3 that $\hat{\delta}^t$ has no terminator along κ^t; and as κ^t is $\lambda^t(\eta^-)$-free by (2.10), there is no primary $\lambda^t(\eta^-)$-link which restrains κ^t. It thus follows from (6.5) that there is an initiator δ^t for S^t at $\lambda^t(\eta^-)$. Hence by Definition 6.4

and as $wt(S^t) \le wt(\lambda^t(\eta^-))$, S^t is weakly controlled at $\lambda^t(\eta^-)$. We complete the proof that (6.8) holds by showing that $\lambda(\delta^{t\pm1}) \supseteq \delta^t$. Assume to the contrary, i.e., that $\delta^t \not\subseteq \lambda(\delta^{t\pm1})$, in order to obtain a contradiction. As $\delta^{t\pm1} \subseteq \lambda^{t\pm1}(\eta^-)$, $\delta^t \not\subseteq \lambda(\delta^{t\pm1})$, and $\delta^t \subseteq \lambda^t(\eta^-)$, it follows from Lemma 3.1 (Limit Path) that there must be a $\mu^{t\pm1}$ such that $\delta^{t\pm1} \subset \mu^{t\pm1} \subseteq \lambda^{t\pm1}(\eta^-)$ and $\lambda(\mu^{t\pm1}) = \delta^t$. But then by Definition 6.7, (2.11), and (6.4), $wt(S^t) \le wt(\lambda(\delta^{t\pm1})) < wt(\lambda(\mu^{t\pm1})) = wt(\delta^t) \le wt(S^t)$, a contradiction. Hence (6.8) holds.

Finally, we complete the verification of (6.10). Since we have already shown that $\hat{\delta}^t = \bar{\delta}^t \subseteq \kappa^t$, it remains only to show that $\kappa^t \subset \delta^t$. As δ^t is an initiator at $\lambda^t(\eta^-) \supseteq \kappa^t$, it follows from (6.2) that κ^t and δ^t are comparable. We assume that $\delta^t \subseteq \kappa^t$, and obtain a contradiction. By (6.7), the initiator for a space at a node γ is the longest initiator for that space at any node $\alpha \subseteq \gamma$ which has no terminator along γ. We showed earlier that $\hat{\delta}^t = \bar{\delta}^t \subseteq \kappa^t$. Now δ^t, $\hat{\delta}^t \subseteq \kappa^t = \lambda^t(\eta^-)\wedge\lambda^t(\eta)$, δ^t is the initiator for S^t at $\lambda^t(\eta^-)$, and $\hat{\delta}^t$ is the initiator for S^t at $\lambda^t(\eta)$. By (2.10) or Lemma 4.5 (Free Extension), any terminator γ^t for δ^t along $\lambda^t(\eta^-)$ ($\hat{\delta}^t$ along $\lambda^t(\eta)$, resp.) must be $\subseteq \kappa^t$. If $\gamma^t = \kappa^t$, then by Definition 6.3, the immediate successor β^t of κ^t along $\lambda^t(\eta^-)$ ($\lambda^t(\eta)$, resp.) must be an initiator for S^t at β^t, so by (6.7), must have a terminator along $\lambda^t(\eta^-)$ ($\lambda^t(\eta)$, resp.). But this would imply that there is a primary $\lambda^t(\eta^-)$-link ($\lambda^t(\eta)$-link, resp.) restraining κ^t, contradicting (2.10) or Lemma 4.5 (Free Extension). Hence, $\gamma^t \subset \kappa^t$, so γ^t is the terminator for δ^t ($\hat{\delta}^t$, resp.) along both $\lambda^t(\eta^-)$ and $\lambda^t(\eta)$. By (6.7), it must then be the case that $\delta^t = \hat{\delta}^t$. But then by (6.3), $\nu^t = \hat{\nu}^t$, so $t = p+1$ and (6.12)(ii) follows from (6.2).

As we have noted above throughout the proof, (6.12) also holds. ∎

We now show that, under the hypotheses and notation of the Alternating Initiator Lemma, activation (validation, resp.) for ν^t along $\lambda^t(\eta^-)$ corresponds to activation (validation, resp.) for ν^{t+1} along $\lambda^{t+1}(\eta^-)$ for $t \in [1,p]$; and activation (validation, resp.) for $\hat{\nu}^t$ along $\lambda^t(\eta)$ corresponds to activation (validation, resp.) for $\hat{\nu}^{t+1}$ along $\lambda^{t+1}(\eta)$ for $t \in [1,p]$. Furthermore, the same will be true for $t = p+1$ if ν^p is activated along $\lambda^p(\eta^-)$ iff $\hat{\nu}^p$ is validated along $\lambda^p(\eta)$ and $up(\nu^p) = up(\hat{\nu}^p)$. (If the latter fails, then we will not need the lemma, as correction of axioms will be unnecessary.) We need to add the hypothesis that no $\xi^1 \in T^1$ such that $\xi^1 \equiv \nu^1$ is switched at η; if some $\xi^1 \equiv \nu^1$ is switched at η, axioms which are newly weakly controlled by $\hat{\nu}^1$ at η are corrected, so we will not have to use the Outcome Lemma below. For requirements of type 0, we only need the simpler, but equivalent condition that no ξ such that $up^t(\xi^1) = up^t(\nu^1)$ for some $t \le dim(\nu^1)$ is switched at η. The more general condition is needed for requirements of type 2.

Lemma 6.9 (Outcome Lemma): Fix $\eta \in T^0$. Suppose that S is weakly controlled by ν^1 at $\lambda(\eta^-)$ with initiator δ^1, S is weakly controlled by $\hat{\nu}^1$ at $\lambda(\eta)$ with initiator $\hat{\delta}^1$, no $\xi^1 \in T^1$ such that $\xi^1 \equiv \nu^1$ is switched at η, $\delta^1 \neq \hat{\delta}^1$, and $tp(\nu^1) \in \{0,2\}$. For all $t \in [1,n]$, let $\nu^t = up^t(\nu^1)$ and $\hat{\nu}^t = up^t(\hat{\nu}^1)$. Let p be the smallest t such that $\nu^{t+1} = \hat{\nu}^{t+1}$ if such a t exists, and let $p = \dim(\nu^1)$-1 otherwise. Then for all $t \in [1,p]$, ν^t is activated along $\lambda^t(\eta^-)$ iff ν^1 is activated along $\lambda(\eta^-)$; and $\hat{\nu}^t$ is activated along $\lambda^t(\eta)$ iff $\hat{\nu}^1$ is activated along $\lambda(\eta)$. If, furthermore, ν^p is activated along $\lambda^p(\eta^-)$ iff $\hat{\nu}^p$ is validated along $\lambda^p(\eta)$ and $\nu^{p+1} = \hat{\nu}^{p+1}$, then ν^{p+1} is activated along $\lambda^{p+1}(\eta^-)$ iff ν^1 is activated along $\lambda(\eta^-)$, and $\hat{\nu}^{p+1}$ is activated along $\lambda^{p+1}(\eta)$ iff $\hat{\nu}^1$ is activated along $\lambda(\eta)$.

Proof: We proceed by induction on t. We will prove the lemma for ν^t only (a similar argument yields a proof for $\hat{\nu}^t$). The lemma is vacuous for t = 1. Fix notation as in Lemma 6.8 (Alternating Initiator). Let q = p+1. As the Alternating Initiator Lemma cannot be applied if $t = q = \dim(\nu^1)$, we first prove a weak version, (6.15), of (6.8) to cover the case in which $t = q = \dim(\nu^1)$, $\nu^q = up(\nu^p) = up(\hat{\nu}^p) = \hat{\nu}^q$, and ν^p is activated along $\lambda^p(\eta^-)$ iff $\hat{\nu}^p$ is validated along $\lambda^p(\eta)$. This weak version of (6.8) will suffice for this case. (Note that a similar proof will also yield a weak version, (6.16), of (6.9).) By hypothesis, (6.12)(ii) will not preclude the use of Lemma 6.8 (Alternating Initiator).

Suppose that $t = q = p+1 = \dim(\nu^1)$ and $\nu^q = \hat{\nu}^q$. By hypothesis, $\nu^q \subset \lambda^q(\eta^-),\lambda^q(\eta)$. Fix $\delta^q \subseteq \lambda^q(\eta^-)$ such that $(\delta^q)^- = \nu^q$ and $\hat{\delta}^q \subseteq \lambda^q(\eta)$ such that $(\hat{\delta}^q)^- = \hat{\nu}^q$. We will show that:

(6.15) If ν^q has finite outcome along $\lambda^q(\eta^-)$, then $\nu^q \subset \delta^q \subseteq \lambda(\delta^{q\pm1})$.

We leave it to the reader to verify with a similar proof that:

(6.16) If $\hat{\nu}^q$ has finite outcome along $\lambda^q(\eta)$, then $\hat{\nu}^q \subset \hat{\delta}^q \subseteq \lambda(\hat{\delta}^{q\pm1})$.

We have noted that:

(6.17) $\nu^q \subseteq \kappa^q = \lambda^q(\eta^-)\wedge\lambda^q(\eta)$.

We note that, in the notation of Lemma 6.8 (Alternating Initiator), η switches $\kappa^1 = \lambda(\eta^-)\wedge\lambda(\eta)$. By hypothesis, $(\hat{\delta}^q)^- = \hat{\nu}^q = \nu^q = (\delta^q)^-$. By (6.17), $\lambda^q(\eta^-)\wedge\lambda^q(\eta) = \kappa^q \supseteq \nu^q$. Fix $\beta^{q\pm1}$ such that that ν^q has finite outcome $\beta^{q\pm1}$ along $\lambda^q(\eta^-)$, let $\pi^{q\pm1} = (\beta^{q\pm1})^-$, and let

$\mu^{q\pm1}$ be the initial derivative of ν^q along $\beta^{q\pm1}$. As $\delta^q \subseteq \lambda^q(\eta^-)$ and $(\delta^q)^- = \nu^q$, it follows from (2.4) that $\nu^q{}^\wedge\langle\beta^{q\pm1}\rangle = \delta^q$. By Definition 2.1, $\pi^{q\pm1}$ has infinite outcome along $\beta^{q\pm1} \subseteq \lambda^{q\pm1}(\eta^-)$, and by (6.2) and (6.10), $\mu^{q\pm1} \subset \kappa^{q\pm1}$. As $\beta^{q\pm1},\kappa^{q\pm1} \subseteq \lambda^{q\pm1}(\eta^-)$, $\beta^{q\pm1}$ and $\kappa^{q\pm1}$ are comparable. It cannot be the case that $\pi^{q\pm1} \supset \kappa^{q\pm1}$, else $[\mu^{q\pm1},\pi^{q\pm1}]$ would be a primary $\lambda^{q\pm1}(\eta^-)$-link restraining $\kappa^{q\pm1}$, so by (2.10), η could not switch κ, contrary to assumption. By hypothesis, η does not switch any node $\equiv \nu^1$, so $\pi^{q\pm1} \neq \kappa^{q\pm1}$. Hence $\pi^{q\pm1} \subset \kappa^{q\pm1}$ and so $\beta^{q\pm1} \subseteq \kappa^{q\pm1}$. By (2.8), $\pi^{q\pm1}$ is the longest derivative of ν^q along $\lambda^{q\pm1}(\eta^-)$, so all initiators for $S^{q\pm1}$ at nodes $\subseteq \lambda^{q\pm1}(\eta^-)$ whose corresponding controller is a derivative of ν^q are $\subseteq \kappa^{q\pm1}$. Now no initiator for $S^{q\pm1}$ at any node along $\lambda^{q\pm1}(\eta^-)$ can have $\pi^{q\pm1}$ as its controller via Case 3 of Definition 6.3 unless there is a shorter initiator for $S^{q\pm1}$ which has $\pi^{q\pm1}$ as its controller via Subcase 1.1 of Definition 6.3 ; and by Case 1 of Definition 6.3, that shorter initiator must be $\beta^{q\pm1}$. As $\nu^{q\pm1}$ is activated along $\lambda^{q\pm1}(\eta^-)$ iff $\hat{\nu}^{q\pm1}$ is validated along $\lambda^{q\pm1}(\eta)$ and $\nu^{q\pm1}$ and $\hat{\nu}^{q\pm1}$ are derivatives of ν^q and are controllers for sections of $S^{q\pm1}$, it must therefore be the case that $\beta^{q\pm1}$ is an initiator for $S^{q\pm1}$ at $\beta^{q\pm1}$. As $\lambda^{q\pm1}(\eta^-) \supseteq \kappa^{q\pm1} \supseteq \beta^{q\pm1}$, we have $\delta^{q\pm1} \supseteq \beta^{q\pm1}$ by Definition 6.3. So as $\nu^q \subseteq \lambda(\delta^{q\pm1})$, $up(\nu^{q\pm1}) = \nu^q$ and $(\beta^{q\pm1})^-$ has infinite outcome along $\beta^{q\pm1}$, it follows that $\lambda(\delta^{q\pm1}) \supseteq \nu^q{}^\wedge\langle\beta^{q\pm1}\rangle = \delta^q$, so (6.15) holds.

Now consider any t such that $2 \leq t \leq q$. First consider the case where ν^t has infinite outcome along $\lambda^t(\eta^-)$. Then all derivatives of ν^t along $\lambda^{t\pm1}(\eta^-)$ must have finite outcome along $\lambda^{t\pm1}(\eta^-)$. In particular, $\nu^{t\pm1}$ has finite outcome along $\lambda^{t\pm1}(\eta^-)$, so the lemma follows by induction in this case.

Next consider the case where ν^t has finite outcome $\beta^{t\pm1}$ along $\lambda^t(\eta^-)$. By (2.5), $\beta^{t\pm1} \subseteq \lambda^{t\pm1}(\eta^-)$, so by (2.8), $(\beta^{t\pm1})^-$ is the longest (and principal) derivative of ν^t along $\lambda^{t\pm1}(\eta^-)$; hence $\nu^{t\pm1} \subseteq (\beta^{t\pm1})^-$. As δ^t is an initiator at $\lambda^t(\eta^-)$ if $t < q$, and by choice of δ^t if $t = q$, $\nu^t \subset \delta^t \subseteq \lambda^t(\eta^-)$, so $\nu^t{}^\wedge\langle\beta^{t\pm1}\rangle \subseteq \delta^t$. By (6.8) or (6.15), $\delta^t \subseteq \lambda(\delta^{t\pm1})$, so by (2.5), $\beta^{t\pm1} \subseteq \delta^{t\pm1}$. Now $\delta^{t\pm1}$ is an initiator for $S^{t\pm1}$ and $\nu^{t\pm1}$ at $\lambda^{t\pm1}(\eta^-)$, $\nu^{t\pm1} \subseteq (\beta^{t\pm1})^-$, and $up((\beta^{t\pm1})^-) = up(\nu^{t\pm1})$. By Lemma 4.3(i)(c),(a), $(\beta^{t\pm1})^-$ is restrained by a primary $\delta^{t\pm1}$-link iff every derivative of ν^t is restrained by the same primary $\delta^{t\pm1}$-link; hence by Definition 6.3, the controller $\nu^{t\pm1}$ chosen for the initiator $\delta^{t\pm1}$ is the longest derivative of ν^t along $\delta^{t\pm1}$, so $\nu^{t\pm1} = (\beta^{t\pm1})^-$. Thus the lemma follows by induction. ∎

We now want to show that when the controlling node on T^1 is changed, then either the new controller inherits axioms with the value it desires, or the axioms are corrected, allowing the new controller to redefine those axioms. The situation differs with the type of the requirement, so we prove different lemmas for each type.

We begin with a requirement R of type 0. The situation will be as follows. η will be 1-switching, causing weak control of a space to pass from ν^1 to $\hat{\nu}^1$ on T^1. If $\eta^- \equiv \nu^1$, then η^- will cause something to be placed into the oracle set for the axioms newly weakly controlled by $\hat{\nu}^1$, thus allowing $\hat{\nu}^1$ to correct the axioms to the value which it predicts. Otherwise, we will show that both ν^1 and $\hat{\nu}^1$ predict the same value for those axioms, so no correction is necessary. To show that the predictions by ν^1 and $\hat{\nu}^1$ agree, we need to go up to the smallest q such that $up^q(\nu^1) = up^q(\hat{\nu}^1)$. An analysis of the situation on T^q will enable us to go down to $T^{q\pm 1}$ and show that $up^{q\pm 1}(\nu^1)$ is activated along $\lambda^{q\pm 1}(\eta^-)$ iff $up^{q\pm 1}(\hat{\nu}^1)$ is activated along $\lambda^{q\pm 1}(\eta)$. It will then follow from Lemma 6.9 (Outcome) that ν^1 is activated along $\lambda(\eta^-)$ iff $\hat{\nu}^1$ is activated along $\lambda(\eta)$.

Lemma 6.10 (0-Correction Lemma): Fix $\eta \in T^0$. Suppose that S is weakly controlled by ν^1 at $\lambda(\eta^-)$ with initiator δ^1, S is weakly controlled by $\hat{\nu}^1$ at $\lambda(\eta)$ with initiator $\hat{\delta}^1$, $\delta^1 \neq \hat{\delta}^1$, and $tp(\nu^1) = 0$. Let $\kappa^1 = \lambda(\eta^-)\wedge\lambda(\eta)$. Then one of the following holds:

(i) ν^1 is activated along δ^1 iff $\hat{\nu}^1$ is activated along $\hat{\delta}^1$.

(ii) η switches $\kappa^1 \subset \delta^1$ and $\kappa^1 \equiv \nu^1$.

Proof: Fix notation as in Lemma 6.8 (Alternating Initiator), and fix the least q such that $\nu^q = \hat{\nu}^q$. If q = 1, then by hypothesis, either (i) holds, or η must switch ν^1 and (ii) will hold. So we may assume that q > 1. Let p = q-1.

If ν^p is activated along $\lambda^p(\eta^-)$ iff $\hat{\nu}^p$ is activated along $\lambda^p(\eta)$, then (i) follows from Lemma 6.9 (Outcome) if $\eta^- \not\equiv \nu^1$, and (ii) follows if $\eta^- \equiv \nu^1$. So we assume that ν^p is activated along $\lambda^p(\eta^-)$ iff $\hat{\nu}^p$ is validated along $\lambda^p(\eta)$.

Suppose that $q < \dim(\nu^1)$. By our assumptions, the conditions of (6.12) fail, so we can apply Lemma 6.8 (Alternating Initiator) with t = q. By (6.10), $\kappa^1 \subset \delta^1$. As $\nu^q = \hat{\nu}^q$, it follows from (6.10) that $\nu^q = \nu^q\wedge\hat{\nu}^q \subset \delta^q\wedge\hat{\delta}^q \subseteq \kappa^q = \lambda^q(\eta^-)\wedge\lambda^q(\eta)$. Thus the outcome of ν^q along $\lambda^q(\eta^-)$ is the same as the outcome of $\hat{\nu}^q$ along $\lambda^q(\eta)$, so ν^q is activated along along $\lambda^q(\eta^-)$ iff $\hat{\nu}^q$ is activated along $\lambda^q(\eta)$. (i) now follows from the second conclusion of Lemma 6.9 (Outcome) if $\eta^- \not\equiv \nu^1$, and (ii) follows if $\eta^- \equiv \nu^1$.

Suppose that $q = \dim(\nu^1)$. By (6.10), $\delta^p\wedge\hat{\delta}^p \subseteq \kappa^p \subset \delta^p\vee\hat{\delta}^p$. Furthermore, as $tp(\nu^1) = 0$, Subcase 1.2 or Case 2 of Definition 6.3 must be followed to define controllers and initiators on T^p, so $\nu^p = (\delta^p)^-$ and $\hat{\nu}^p = (\hat{\delta}^p)^-$. (We note that if Subcase 1.2 is followed, then as, by Subcase 1.1, all initiators for $up(\nu^p) = up(\hat{\nu}^p)$ are immediate successors of $up(\hat{\nu}^p)$, it follows from Lemma 3.1 (Limit Path) and Lemma 3.3 (λ-Behavior) that $\nu^p = (\delta^p)^-$ and $\hat{\nu}^p = (\hat{\delta}^p)^-$.) Thus $\nu^p\wedge\hat{\nu}^p \subset \kappa^p$. It cannot be the case that $\nu^p\vee\hat{\nu}^p \supset \kappa^p$, else $\nu^p\vee\hat{\nu}^p$

would be the last node of a primary $\lambda^p(\eta^-)$-link or $\lambda^p(\eta)$-link which restrains κ^p, contrary to (2.10) or Lemma 4.5 (Free Derivative). It cannot be the case that $\nu^p v \hat{\nu}^p \subset \kappa^p$, else $\delta^p v \hat{\delta}^p \subseteq \kappa^p$. Hence $\nu^p v \hat{\nu}^p = \kappa^p$, and (ii) holds. ∎

Suppose R is a requirement of dimension r and type 1 and that the space X is assigned to R. Control of sections of X along a path $\Lambda^{r\pm1}$ is divided among derivatives of many different nodes of T^n. The following lemma, together with the requirement that the construction of Section 7 respect implication chains, will ensure that all but finitely many of these sections are controlled by nodes which are activated along $\Lambda^{r\pm1}$, or all but finitely many of these sections are controlled by nodes which are validated along $\Lambda^{r\pm1}$. The lemma will be used to analyze the situations which can occur when control of a space is relinquished by $\hat{\sigma}^{r\pm1}$ to $\sigma^{r\pm1}$. Condition (i) says that both $\hat{\sigma}^{r\pm1}$ and $\sigma^{r\pm1}$ want to declare axioms with the same value, so the axioms declared by derivatives of $\sigma^{r\pm1}$ are safe for $\hat{\sigma}^{r\pm1}$. Conditions (ii) and (iv) will be used to show that enough of the axioms declared by derivatives of $\sigma^{r\pm1}$ are corrected when control is interchanged. And condition (iii) will allow us to show that the set of conflicting axioms is sufficiently thin, and so will not interfere with the existence of the desired limit. The hypotheses placed on the lemma are chosen to capture exactly the cases for which the lemma is used.

Lemma 6.11 (1-Similarity Lemma): Fix an admissible $\Lambda^0 \in [T^0]$ and for all $t \leq n$, let $\Lambda^t = \lambda^t(\Lambda^0)$. Fix $r \leq n$ and $\sigma^{r\pm1} \subset \hat{\sigma}^{r\pm1} \subset \tau^{r\pm1} \subset \Lambda^{r\pm1}$, such that $\sigma^{r\pm1} \equiv \hat{\sigma}^{r\pm1}$, $up(\sigma^{r\pm1}) \neq up(\hat{\sigma}^{r\pm1})$, $(\tau^{r\pm1})^- = \hat{\sigma}^{r\pm1}$, $tp(\sigma^{r\pm1}) = 1$, $dim(\sigma^{r\pm1}) = r$, and $\sigma^{r\pm1}$ and $\hat{\sigma}^{r\pm1}$ control (different) sections of a space X at $\tau^{r\pm1}$. Fix $\bar{\tau}^{r\pm1} \subset \tau^{r\pm1}$ such that $(\bar{\tau}^{r\pm1})^- = \sigma^{r\pm1}$ and assume that if $\sigma^{r\pm1}$ has infinite outcome along $\bar{\tau}^{r\pm1}$, then there is no derivative of $up(\hat{\sigma}^{r\pm1})$ along $\sigma^{r\pm1}$. Then one of the following conditions holds:

(i) $\sigma^{r\pm1}$ has finite outcome along $\tau^{r\pm1}$ iff $\hat{\sigma}^{r\pm1}$ has finite outcome along $\tau^{r\pm1}$.

(ii) $\sigma^{r\pm1}$ has infinite outcome along $\tau^{r\pm1}$, $\hat{\sigma}^{r\pm1}$ has finite outcome along $\tau^{r\pm1}$, and there is a $\sigma^{r\pm1}$-injurious primary $\tau^{r\pm1}$-link $[\mu^{r\pm1},\pi^{r\pm1}]$ such that $\pi^{r\pm1} \in PL(\sigma^{r\pm1},\tau^{r\pm1})$.

(iii) $\sigma^{r\pm1}$ has finite outcome along $\tau^{r\pm1}$, $\hat{\sigma}^{r\pm1}$ has infinite outcome along $\tau^{r\pm1}$, and there is a primary $\tau^{r\pm1}$-link which restrains $\sigma^{r\pm1}$.

(iv) $\sigma^{r\pm1}$ has finite outcome along $\tau^{r\pm1}$, $\hat{\sigma}^{r\pm1}$ has infinite outcome along $\tau^{r\pm1}$, $up(\sigma^{r\pm1}) \subset up(\hat{\sigma}^{r\pm1})$, there is no primary $\tau^{r\pm1}$-link which restrains $\sigma^{r\pm1}$, but there is a $\pi^r \in PL(up(\sigma^{r\pm1}),\lambda(\tau^{r\pm1}))$ such that $OS(\sigma^{r\pm1}) \subseteq TS(\pi^r)$.

Proof: Suppose that (i)-(iv) fail, in order to obtain a contradiction. By choice of r, as $\sigma^{r\pm1}$ and $\hat{\sigma}^{r\pm1}$ control spaces at $\tau^{r\pm1}$, and by Subcase 1.1 of Definition 6.3, for all $i \le r-1$, the principal derivatives of $\sigma^{r\pm1}$ along $out^i(\tau^{r\pm1})$ and $\hat{\sigma}^{r\pm1}$ along $out^i(\tau^{r\pm1})$ must be implication-free.

First suppose that $\hat{\sigma}^{r\pm1}$ has finite outcome along $\tau^{r\pm1}$. We can assume, without loss of generality, that $\sigma^{r\pm1}$ is the shortest string satisfying the hypotheses, but not the conclusion of the lemma for $\hat{\sigma}^{r\pm1}$. By the failure of (i), $\sigma^{r\pm1}$ has infinite outcome along $\tau^{r\pm1}$. As $\sigma^{r\pm1}$ controls a space at $\tau^{r\pm1}$, it follows from Definitions 6.3 and 6.4 that $\sigma^{r\pm1}$ controls a space at $\bar{\tau}^{r\pm1}$, and so that $out^0(\bar{\tau}^{r\pm1})$ is pseudotrue. Thus $\bar{\tau}^{r\pm1}$ must be implication-free, and cannot require extension.

Let $\tilde{\sigma}^{r\pm1}$ be the initial derivative of $up(\hat{\sigma}^{r\pm1})$ along $\tau^{r\pm1}$, and let $\tilde{\tau}^{r\pm1}$ be the immediate successor of $\tilde{\sigma}^{r\pm1}$ along $\tau^{r\pm1}$. We show that $\tilde{\sigma}^{r\pm1}$ controls a section of X at $\tilde{\tau}^{r\pm1}$. If $\tilde{\sigma}^{r\pm1} = \hat{\sigma}^{r\pm1}$, then this follows by hypothesis. Otherwise, it follows from (2.8) that $\tilde{\sigma}^{r\pm1}$ has finite outcome along $\tau^{r\pm1}$. Now by Lemma 4.5 (Free Extension), $up(\tilde{\sigma}^{r\pm1}) = up(\hat{\sigma}^{r\pm1}) \subset \lambda(\tau^{r\pm1})$ and $up(\hat{\sigma}^{r\pm1})$ is $\lambda(\tau^{r\pm1})$-free. Furthermore, $up(\hat{\sigma}^{r\pm1})$ must be implication-free, else by (5.23), $\hat{\sigma}^{r\pm1}$ would not be implication-free and would not control a section of X at $\tau^{r\pm1}$. Hence by Lemma 5.16(iv) (Implication-Freeness), $out^0(\tilde{\tau}^{r\pm1})$ is pseudotrue. Now by Lemma 4.5 (Free Extension), $up^n(\tilde{\sigma}^{r\pm1}) \subset \lambda^n(\tilde{\tau}^{r\pm1})$ must be $\lambda^n(\tilde{\tau}^{r\pm1})$-free, and by (2.9) and $\tilde{\sigma}^{r\pm1}$ is both the initial and principal derivative of $up^n(\tilde{\sigma}^{r\pm1})$ along $\tilde{\tau}^{r\pm1}$. By Lemma 5.17(iii) (Assignment), $\tilde{\sigma}^{r\pm1}$ is $\tilde{\tau}^{r\pm1}$-free and implication-free. Now iterating Lemma 4.6(i) (Free Derivative) and Lemma 5.16(ii) (Implication-Freeness), we see that for all $i \le r-1$, the principal derivative of $\tilde{\sigma}^{r\pm1}$ along $\tilde{\tau}^{r\pm1}$ is implication-free. It follows from Definitions 6.3 and 6.4 that $\tilde{\sigma}^{r\pm1}$ controls a section of X at $\tilde{\tau}^{r\pm1}$. Hence without loss of generality, we may assume that $\hat{\sigma}^{r\pm1} = \tilde{\sigma}^{r\pm1}$.

As $up(\hat{\sigma}^{r\pm1})$ has no derivative along $\sigma^{r\pm1}$ and (ii) fails, (5.16) holds; hence as $\sigma^{r\pm1}$ controls a space at $\tau^{r\pm1}$, it follows from Definition 5.2 and Subcase 1.1 of Definition 6.3 that for some $\bar{\sigma}^{r\pm1} \subseteq \sigma^{r\pm1}$, $\langle\langle \bar{\sigma}^{r\pm1}, \hat{\sigma}^{r\pm1}, \tau^{r\pm1} \rangle\rangle$ is an amenable (r-1)-implication chain along $\Lambda^{r\pm1}$. But this contradicts Lemma 5.15(i) (Admissibility).

Now suppose that $\hat{\sigma}^{r\pm1}$ has infinite outcome along $\tau^{r\pm1}$. As (i) fails, $\sigma^{r\pm1}$ has finite outcome along $\tau^{r\pm1}$. As (iii) fails, it follows from Lemma 4.3(i)(a) (Link Analysis) that $up(\sigma^{r\pm1}) \subseteq \lambda(\tau^{r\pm1})$. As $\hat{\sigma}^{r\pm1} = (\tau^{r\pm1})^-$, it follows from Lemma 4.5 (Free Extension) that $up(\hat{\sigma}^{r\pm1}) \subseteq \lambda(\tau^{r\pm1})$. Hence $up(\hat{\sigma}^{r\pm1})$ and $up(\sigma^{r\pm1})$ are comparable. Now $\hat{\sigma}^{r\pm1}$ has infinite outcome along $\tau^{r\pm1}$, so $\hat{\sigma}^{r\pm1}$ is the principal derivative of $up(\hat{\sigma}^{r\pm1})$ along $\tau^{r\pm1}$. It cannot be the case that $up(\hat{\sigma}^{r\pm1}) \subset up(\sigma^{r\pm1})$, else by Lemma 3.1 (Limit Path), there would be no derivative of $up(\sigma^{r\pm1})$ which is $\subset \hat{\sigma}^{r\pm1}$, contrary to the hypothesis that $\sigma^{r\pm1} \subset \hat{\sigma}^{r\pm1}$. Thus by the above, $up(\sigma^{r\pm1}) \subset up(\hat{\sigma}^{r\pm1})$.

We now show that $\tau^{r\pm1}$ requires extension for $\sigma^{r\pm1}$. (5.1), (5.2), and (5.5)(i) follow easily from hypothesis and the observations already made. The failure of (iii) implies (5.3). We noted, following Definition 6.2, that every $\sigma^{r\pm1}$-injurious primary $\lambda(\tau^{r\pm1})$-link $[\mu^r,\pi^r]$ is $\sigma^{r\pm1}$-correcting. Suppose that $\gamma^r \in PL(up(\sigma^{r\pm1}),\lambda(\tau^{r\pm1}))$ and $TS(\gamma^r)\cap RS(\sigma^{r\pm1}) \neq \emptyset$, in order to obtain a contradiction. First suppose that (5.13) causes γ^r to enter $PL(up(\sigma^{r\pm1}),\lambda(\tau^{r\pm1}))$. Then there is a μ^r such that $[\mu^r,\gamma^r]$ is a primary $\lambda(\tau^{r\pm1})$-link restraining $up(\sigma^{r\pm1})$. Let ξ^r be the immediate successor of γ^r along $\lambda(\tau^{r\pm1})$. Then $\gamma^r \in \overline{PL}(\xi^r)$, so $[\mu^r,\gamma^r]$ is $up(\sigma^{r\pm1})$-injurious and restrains $up(\sigma^{r\pm1})$. But then $[\mu^r,\gamma^r]$ is $up(\sigma^{r\pm1})$-correcting, contrary to our assumption that (iv) fails.

Now suppose that (5.14) causes γ^r to enter $PL(up(\sigma^{r\pm1}),\lambda(\tau^{r\pm1}))$, but (5.13) does not. Then there are $\mu^r \subset up(\sigma^{r\pm1}) \subset \delta^r = (\sigma^r)^- \subset \sigma^r \subseteq \xi^r$ such that σ^r requires extension but has no primary completion with infinite outcome along ξ^r, and as (5.13) did not apply, $\gamma^r \in PL(\delta^r,\xi^r)\cup\{\delta^r\}$. As $out^0(\tau^{r\pm1})$ is pseudotrue, it follows from Lemma 5.5(ii) (Completion-Respecting) that σ^r has a primary completion κ^r along $\lambda(\tau^{r\pm1})$ which has infinite outcome along $\lambda(\tau^{r\pm1})$. Fix $\alpha^r \subseteq \lambda(\tau^{r\pm1})$ such that $(\alpha^r)^- = \kappa^r$. By Definition 5.3 and Lemma 5.1(i) (PL Analysis), $\gamma^r \in PL(\delta^r,\kappa^r)\cup\{\delta^r\} \subseteq \overline{PL}(\xi^r)$. Thus $[\mu^r,\kappa^r]$ is $up(\sigma^{r\pm1})$-injurious and restrains $up(\sigma^{r\pm1})$. But then $[\mu^r,\kappa^r]$ is $up(\sigma^{r\pm1})$-correcting, contrary to our assumption that (iv) fails.

We conclude that (5.4) holds, and so that $\tau^{r\pm1}$ requires extension for some $\bar{\sigma}^{r\pm1} \subseteq \sigma^{r\pm1}$. By Definition 5.6, $\tau^{r\pm1}$ is not the completion of $\tau^{r\pm1}$ for $\bar{\sigma}^{r\pm1}$. Hence by (5.21), $\tau^{r\pm1}$ is implication-restrained, and so $out^0(\tau^{r\pm1})$ is not pseudotrue. But then by Subcase 1.1 of Definition 6.3, $\hat{\sigma}^{r\pm1}$ is not a controller for a section of X at $\tau^{r\pm1}$, contrary to hypothesis. ∎

Because of the finiteness of the number of initiators for a given space X, we can settle on an initiator which will control a given space along a path. However, it is possible to have comparable initiators along a given path, each determining control of sections of the same space at infinitely many nodes along the approximation to the path. The switching of control is determined by the terminators. The next lemma will allow us to show that all but finitely many axioms declared for a space controlled by a node of type 1 along Λ^1 will have the correct value.

Lemma 6.12 (1-Correction Lemma): Fix an admissible $\Lambda^0 \in T^0$ and let $\Lambda^1 = \lambda(\Lambda^0)$. Suppose that $\nu^1 \subset \Lambda^1$ controls the space S along Λ^1 with initiator δ^1, and that $tp(\nu^1) = 1$. Assume that $\eta \subset \kappa \subset \Lambda^0$, $wt(\eta^\pm) \geq wt(S)$, δ^1 is the initiator for S at $\lambda(\eta^\pm)$ and at $\lambda(\kappa)$, but not at any $\lambda(\gamma)$ such that $\eta \subseteq \gamma \subset \kappa$. Then there is a μ^1 such that for all $\gamma \in [\eta,\kappa)$, $[\mu^1,(\lambda(\eta))^-]$ is a ν^1-correcting primary $\lambda(\gamma)$-link with $\mu^1 \subset \delta^1 \subseteq (\lambda(\eta))^-$, and κ switches

$(\lambda(\eta))^-$. Furthermore, if ξ is the shortest pseudotrue node such that $\kappa \subseteq \xi \subset \Lambda^0$, then for every node $\beta^1 \in \overline{PL}(\lambda(\eta))$, there is a β such that $\kappa \subseteq \beta \subseteq \xi$ and β switches β^1.

Proof: By hypothesis, for all γ such that $\eta \subseteq \gamma \subset \kappa$, $\lambda((\eta)^-) \neq \lambda(\gamma)$. As $wt(\eta^\pm) \geq wt(S)$, it follows from (2.11) that $wt(\lambda(\gamma)) > wt(S)$ for all γ such that $\eta \subseteq \gamma \subset \kappa$. Thus Case 3 of Definition 6.3 must be followed at $\lambda(\eta)$ to define $(\lambda(\eta))^-$ as a terminator for δ^1, so there is a ν^1-correcting primary $\lambda(\eta)$-link $[\mu^1,(\lambda(\eta))^-]$ with $\mu^1 \subset \delta^1 \subseteq (\lambda(\eta))^-$, and $(\lambda(\eta))^-$ has infinite outcome along $\lambda(\eta)$.

As $\delta^1 \subset \Lambda^1$, it follows from (2.6) that no γ such that $\eta \subseteq \gamma \subset \kappa$ can switch any $\rho^1 \subset \delta^1$. By (2.10), no such γ can switch any ρ^1 such that $\mu^1 \subseteq \rho^1 \subset (\lambda(\eta))^-$. Hence by (2.10), $\lambda(\kappa)$ and $(\lambda(\eta))^-$ must be comparable. Also, no γ such that $\eta \subseteq \gamma \subset \kappa$ can switch $(\lambda(\eta))^-$, else by Lemma 3.3 (λ-Behavior), $(\lambda(\gamma))^- = (\lambda(\eta))^-$ and $(\lambda(\gamma))^-$ would have finite outcome along $\lambda(\gamma)$, so δ^1 would be the initiator for S at $\lambda(\gamma)$. Hence for all γ such that $\eta \subseteq \gamma \subset \kappa$, $[\mu^1,(\lambda(\eta))^-]$ is a ν^1-correcting primary $\lambda(\gamma)$-link which restrains δ^1.

Now as κ cannot switch any $\rho^1 \subset (\lambda(\eta))^-$, as $\lambda(\kappa)$ and $(\lambda(\eta))^-$ are comparable, and as δ^1 is the initiator for S at $\lambda(\kappa)$, $[\mu^1,(\lambda(\eta))^-]$ cannot be a primary $\lambda(\kappa)$-link, so κ must switch $(\lambda(\eta))^-$. If $(\lambda(\eta))^-$ is not a primary completion, then $\overline{PL}(\lambda(\eta)) = \{(\lambda(\eta))^-\}$. Otherwise, let $(\lambda(\eta))^-$ be the primary completion of the immediate successor γ^1 of a node σ^1 along $\lambda(\eta)$. Then $\overline{PL}(\lambda(\eta)) = PL(\sigma^1,\lambda(\eta)) \cup \{\sigma^1\}$. By Lemma 5.3(ii) (Implication Chain) and Lemma 5.2 (Requires Extension), κ must require extension for a derivative of σ^1, and so as ξ is pseudotrue, it follows from Lemma 5.5(ii) (Completion-Respecting) that κ must have a primary completion $\bar{\kappa} \subset \xi$ which has infinite outcome along ξ. By (5.19), $up(\bar{\kappa}) = \sigma^1$. Hence the immediate successor of $\bar{\kappa}$ along ξ switches σ^1. By Lemma 5.1(ii) (PL Analysis), $PL(\sigma^1,\lambda(\eta)) \subseteq PL(\sigma^1,(\lambda(\eta))^-) \cup \{(\lambda(\eta))^-\}$. As κ switches $(\lambda(\eta))^-$, it follows from (2.4) that $\lambda(\kappa) = (\lambda(\eta))^- \hat{}\langle\kappa\rangle$, and that $(\lambda(\eta))^-$ has finite outcome along $\lambda(\kappa)$. Hence by Lemma 5.1(iv) (PL Analysis), $PL(\sigma^1,\lambda(\eta)) = PL(\sigma^1,(\lambda(\eta))^-)$. It now follows from Lemma 5.12(i) (PL) and as κ switches $(\lambda(\eta))^-$ that every node in $PL(\sigma^1,\lambda(\eta))$ must be switched by some node in $[\kappa,\bar{\kappa}]$. n

Suppose that X is a space assigned to a requirement of dimension r and type 2. When k = r-1, control of sections of X along a path $\Lambda^{r\pm1}$ is divided among derivatives of many different nodes of T^n. The following lemma will allow us to use implication chains to ensure that all but finitely many of these sections are controlled by nodes which are activated along $\Lambda^{r\pm1}$, or all but finitely many of these sections are controlled by nodes which are validated along $\Lambda^{r\pm1}$.

Lemma 6.13 (2-Similarity Lemma): Fix an admissible $\eta \in T^0$, and $\sigma^{r\pm1} \subset \bar{\tau}^{r\pm1} \subseteq \hat{\sigma}^{r\pm1} \subset \tau^{r\pm1} \subseteq \lambda^{r\pm1}(\eta)$ such that $\sigma^{r\pm1}$ and $\hat{\sigma}^{r\pm1}$ are nodes to which the requirement R of dimension r and type 2 has been assigned. Assume that $\sigma^{r\pm1} \equiv \hat{\sigma}^{r\pm1}$, $up(\sigma^{r\pm1}) \neq up(\hat{\sigma}^{r\pm1})$, $(\tau^{r\pm1})^- = \hat{\sigma}^{r\pm1}$, $(\bar{\tau}^{r\pm1})^- = \sigma^{r\pm1}$, and $\sigma^{r\pm1}$ and $\hat{\sigma}^{r\pm1}$ are controllers at $\bar{\tau}^{r\pm1}$ and $\tau^{r\pm1}$, respectively. Then one of the following conditions holds:

(i) $\sigma^{r\pm1}$ has finite outcome along $\tau^{r\pm1}$ iff $\hat{\sigma}^{r\pm1}$ has finite outcome along $\tau^{r\pm1}$.

(ii) $\sigma^{r\pm1}$ has finite outcome along $\tau^{r\pm1}$, $\hat{\sigma}^{r\pm1}$ has infinite outcome along $\tau^{r\pm1}$, and there is a primary $\tau^{r\pm1}$-link which restrains $\sigma^{r\pm1}$.

Proof: We assume that (i) and (ii) fail, and obtain a contradiction. We will be showing, under additional assumptions, either that $\langle\langle\sigma^{r\pm1},\hat{\sigma}^{r\pm1},\tau^{r\pm1}\rangle\rangle$ is an amenable implication chain, or that $\tau^{r\pm1}$ requires extension for $\sigma^{r\pm1}$. We begin by showing that certain clauses from (5.1)-(5.12), (5.15) and (5.16) hold without any additional assumptions. (5.5)-(5.9) and (5.12) follow from hypothesis.

As $\sigma^{r\pm1}$ and $\hat{\sigma}^{r\pm1}$ are controllers at $\bar{\tau}^{r\pm1}$ and $\tau^{r\pm1}$, respectively, it follows from Subcase 1.1 of Definition 6.3 that for all $i \leq r-1$, the principal derivatives of $\sigma^{r\pm1}$ along $out^i(\bar{\tau}^{r\pm1})$ and $\hat{\sigma}^{r\pm1}$ along $out^i(\tau^{r\pm1})$, are implication-free, and that $out^0(\bar{\tau}^{r\pm1})$ and $out^0(\tau^{r\pm1})$ are pseudotrue. Hence (5.1) and (5.10) hold.

We next show that we may assume, without loss of generality, that $\sigma^{r\pm1}$ ($\hat{\sigma}^{r\pm1}$, resp.) is the principal derivative of $up(\sigma^{r\pm1})$ ($up(\hat{\sigma}^{r\pm1})$, resp.) along $\bar{\tau}^{r\pm1}$ ($\tau^{r\pm1}$, resp.). This is clearly the case if $\sigma^{r\pm1}$ ($\hat{\sigma}^{r\pm1}$, resp.) has infinite outcome along $\bar{\tau}^{r\pm1}$ ($\tau^{r\pm1}$, resp.). Suppose that $\sigma^{r\pm1}$ ($\hat{\sigma}^{r\pm1}$, resp.) has finite outcome along $\bar{\tau}^{r\pm1}$ ($\tau^{r\pm1}$, resp.), and let $\tilde{\sigma}^{r\pm1}$ be the initial derivative of $up(\sigma^{r\pm1})$ ($up(\hat{\sigma}^{r\pm1})$, resp.) along $\bar{\tau}^{r\pm1}$ ($\tau^{r\pm1}$, resp.). By Lemma 5.15(iv) (Implication-Freeness), one of the conclusions of the lemma must hold for $\tilde{\sigma}^{r\pm1}$ in place of $\sigma^{r\pm1}$ ($\hat{\sigma}^{r\pm1}$, resp.). If (i) holds for $\tilde{\sigma}^{r\pm1}$, then (i) also holds for $\sigma^{r\pm1}$ ($\hat{\sigma}^{r\pm1}$, resp.). Suppose that (ii) holds for $\tilde{\sigma}^{r\pm1}$, and let $[\mu^{r\pm1},\pi^{r\pm1}]$ be the associated primary $\tau^{r\pm1}$-link. If $up(\tilde{\sigma}^{r\pm1}) \not\subseteq \lambda(\tau^{r\pm1})$, then by Lemma 4.3(i)(a), (Link Analysis) $[\mu^{r\pm1},\pi^{r\pm1}]$ restrains $\sigma^{r\pm1}$ ($\hat{\sigma}^{r\pm1}$, resp.). Otherwise, by Lemma 4.3(i)(d) (Link Analysis), $\mu^{r\pm1} = \tilde{\sigma}^{r\pm1}$, so by (2.8), $[\mu^{r\pm1},\pi^{r\pm1}]$ restrains $\sigma^{r\pm1}$ ($\hat{\sigma}^{r\pm1}$, resp.).

We next note that $tp(\sigma^{r\pm1}) = tp(\hat{\sigma}^{r\pm1}) = 2$, so (5.4) holds, and if $\hat{\sigma}^{r\pm1}$ is a pseudocompletion of $\sigma^{r\pm1}$, then $\hat{\sigma}^{r\pm1}$ is an amenable pseudocompletion of $\sigma^{r\pm1}$, so (5.16) will follow once the appropriate clauses of (5.6)-(5.12) are verified.

We now proceed by cases.

Case 1: $\hat{\sigma}^{r\pm1}$ has finite outcome along $\tau^{r\pm1}$. Then by the failure of (i), $\sigma^{r\pm1}$ has infinite outcome along $\tau^{r\pm1}$. We will obtain a contradiction in this case, so may assume without loss of generality that $\sigma^{r\pm1}$ has shortest possible length satisfying the properties of the lemma. (5.11) follows from the case assumption, so $\langle\langle \sigma^{r\pm1},\bar{\sigma}^{r\pm1},\bar{\tau}^{r\pm1}\rangle\rangle$ is an implication chain. Now we have assumed that $\hat{\sigma}^{r\pm1}$ is the principal derivative of up($\hat{\sigma}^{r\pm1}$) along $\tau^{r\pm1}$, so as $\hat{\sigma}^{r\pm1}$ has finite outcome along $\tau^{r\pm1}$, $\hat{\sigma}^{r\pm1}$ is an initial derivative. Hence up($\hat{\sigma}^{r\pm1}$) has no derivative $\subset \sigma^{r\pm1}$. We have already noted that (5.16) holds, so $\langle\langle \sigma^{r\pm1},\hat{\sigma}^{r\pm1},\tau^{r\pm1}\rangle\rangle$ is an amenable (r-1)-implication chain. But then by Lemma 5.2 (Requires Extension), out($\tau^{r\pm1}$) requires extension, so out^0($\tau^{r\pm1}$) is implication-restrained, hence cannot be pseudotrue, yielding a contradiction.

Case 2: $\hat{\sigma}^{r\pm1}$ has infinite outcome along $\tau^{r\pm1}$. We first show that $\tau^{r\pm1}$ requires extension for some $\bar{\sigma}^{r\pm1} \subseteq \sigma^{r\pm1}$, by showing that (5.1)-(5.5) hold for $\sigma^{r\pm1}$ in place of ν^k, $\hat{\sigma}^{r\pm1}$ in place of δ^k, and $\tau^{r\pm1}$ in place of η^k. (5.2) and (5.5)(i) follow easily from hypothesis. (5.1) follows from Case 1 of Definition 6.3 and the comments at the beginning of the proof. (5.3) follows from the failure of (ii). And we have already noted that (5.4) holds. Thus $\tau^{r\pm1}$ requires extension for some $\bar{\sigma}^{r\pm1} \subseteq \sigma^{r\pm1}$. But then, by Definition 5.6, $\tau^{r\pm1}$ is implication-restrained, so out^0($\tau^{r\pm1}$) cannot be pseudotrue. Hence by Subcase 1.1 of Definition 6.3, $\hat{\sigma}^{r\pm1}$ cannot be a controller at $\tau^{r\pm1}$, contradicting our assumption. ∎

The next lemma will be used to show that whenever necessary, axioms for type 2 requirements which need to be corrected when control is changed, will be corrected.

Lemma 6.14 (2-Correction Lemma): Fix an admissible $\eta \in T^0$. Suppose that S is weakly controlled by ν^1 at $\lambda(\eta^-)$ with initiator δ^1, S is weakly controlled by $\hat{\nu}^1$ at $\lambda(\eta)$ with initiator $\hat{\delta}^1$, $\delta^1 \neq \hat{\delta}^1$, and tp($\nu^1$) = 2. Let κ^1 = up(η^-). Then one of the following holds:

(i) ν^1 is activated along δ^1 iff $\hat{\nu}^1$ is activated along $\hat{\delta}^1$.

(ii) η switches $\kappa^1 \subset \delta^1$ and dim(κ^1) ≥ dim(ν^1).

Proof: Let r = dim(ν^1). Fix notation as in Lemma 6.8 (Alternating Initiator). If $\nu^{r\pm1} = \hat{\nu}^{r\pm1}$, then the proof follows as in the third paragraph of the proof of Lemma 6.10 (0-Correction). Suppose that $\nu^{r\pm1} \neq \hat{\nu}^{r\pm1}$. We assume that (i) and (ii) fail, and derive a contradiction. As (i) and (ii) fail, it follows from Lemma 6.9 (Outcome) that $\nu^{r\pm1}$ is activated along iff $\hat{\nu}^{r\pm1}$ is validated along $\eta^{r\pm1}$.

We assume that r is even. A similar proof holds when r is odd. By (6.10) and Definition 6.3, $\nu^{r\pm1}\wedge\hat{\nu}^{r\pm1} \subset \nu^{r\pm1}\wedge\hat{\nu}^{r\pm1} \subset \lambda^{r\pm1}(\eta)$. Fix $\bar{\tau}^{r\pm1},\tau^{r\pm1} \subseteq \lambda^{r\pm1}(\eta)$ such that $(\bar{\tau}^{r\pm1})^- = \nu^{r\pm1}\wedge\hat{\nu}^{r\pm1}$, and $(\tau^{r\pm1})^- = \nu^{r\pm1}\vee\hat{\nu}^{r\pm1}$. It follows by an easy induction that $\bar{\tau}^{r\pm1}$ and $\tau^{r\pm1}$ are initiators for up$^{r\pm1}$(S), else either $\nu^{r\pm1}\wedge\hat{\nu}^{r\pm1}$ or $\nu^{r\pm1}\vee\hat{\nu}^{r\pm1}$ would not be a controller for up$^{r\pm1}$(S). There are two cases.

Case 1: $\nu^r \neq \hat{\nu}^r$. By the preceding paragraph, we can apply Lemma 6.13 (2-Similarity), to conclude that there is a primary $\lambda^{r\pm1}(\eta)$-link $[\mu^{r\pm1},\pi^{r\pm1}]$ restraining $\nu^{r\pm1}\wedge\hat{\nu}^{r\pm1}$. By (2.10) and Lemma 4.5 (Free Derivative), $\kappa^{r\pm1}$ is both $\lambda^{r\pm1}(\eta^-)$-free and $\lambda^{r\pm1}(\eta)$-free; and by (6.10), $\nu^{r\pm1}\wedge\hat{\nu}^{r\pm1} \subset \kappa^{r\pm1}$. Hence $\pi^{r\pm1} \subseteq \kappa^{r\pm1}$.

By (6.10), $\delta^{r\pm1}\vee\hat{\delta}^{r\pm1} \supset \kappa^{r\pm1}$, so by (2.1), $wt(\kappa^{r\pm1}) < wt(\delta^{r\pm1}\vee\hat{\delta}^{r\pm1})$. Now $\pi^{r\pm1} \not\subset \kappa^{r\pm1}$, else $\pi^{r\pm1}$ would be a terminator for $\bar{\tau}^{r\pm1}$ along both $\lambda^{r\pm1}(\eta^-)$ and $\lambda^{r\pm1}(\eta)$, so by (6.19) $\nu^{r\pm1}\wedge\hat{\nu}^{r\pm1}$ could not be a controller at either of these nodes. Thus $\pi^{r\pm1} = \kappa^{r\pm1}$, so by Lemma 3.3 ($\lambda$-Behavior), η switches $\pi^{r\pm1}$. But then $\pi^{r\pm1}$ is not an initial derivative, so by (2.9), $\dim(\pi^{r\pm1}) > r-1$; so (ii) must hold, yielding a contradiction.

Case 2: $\nu^r = \hat{\nu}^r$. By (6.10), $\nu^{r\pm1}\wedge\hat{\nu}^{r\pm1} \subset \delta^{r\pm1}\wedge\hat{\delta}^{r\pm1} \subseteq \kappa^{r\pm1} \subset \delta^{r\pm1}\vee\hat{\delta}^{r\pm1}$. By the case assumption and as (i) fails, $[\nu^{r\pm1}\wedge\hat{\nu}^{r\pm1},\nu^{r\pm1}\vee\hat{\nu}^{r\pm1}]$ must form a primary $(\delta^{r\pm1}\vee\hat{\delta}^{r\pm1})$-link, so by (2.10) or Lemma 4.5 (Free Extension), $\kappa^{r\pm1} \supseteq \nu^{r\pm1}\vee\hat{\nu}^{r\pm1}$. We now set $\pi^{r\pm1} = \nu^{r\pm1}\vee\hat{\nu}^{r\pm1}$, and proceed as in the last paragraph of Case 1. ∎

Our final lemma shows that nodes coming from the true path of the construction control spaces.

Lemma 6.15 (Initial Control Lemma): Fix an admissible $\Lambda^0 \in [T^0]$ and for all $k \leq n$, let $\Lambda^k = \lambda^k(\Lambda^0)$. Fix $\zeta^n \subset \Lambda^n$ and $r \leq n$ such that $\dim(\zeta^n) = r$ and $tp(\zeta^n) \in \{1,2\}$, let $\zeta^{r\pm1}$ be the principal derivative of ζ^n along $\Lambda^{r\pm1}$, and let $\zeta^r = up(\zeta^{r\pm1})$. Let S be the space, S_{ζ^r}, assigned to $up(\zeta^{r\pm1})$, and fix $\delta^{r\pm1} \subset \Lambda^{r\pm1}$ such that $(\delta^{r\pm1})^- = \zeta^{r\pm1}$. Then:

(i) $\zeta^{r\pm1}$ controls $S^{[wt(\delta^{r\pm1})]}$ along $\Lambda^{r\pm1}$ with initiator $\delta^{r\pm1}$.

(ii) If ζ^r has infinite outcome along Λ^r, then infinitely many derivatives of ζ^r control spaces along $\Lambda^{r\pm1}$.

Proof: By Lemma 5.17(ii),(iii) (Assignment), ζ^r and $\zeta^{r\pm1}$ are implication-free, ζ^r is Λ^r-free, and $\zeta^{r\pm1}$ is $\Lambda^{r\pm1}$-free.

By Lemma 4.6(ii) (Free Derivative) and Lemma 5.16(ii) (Implication-Freeness), we see that if ζ^r has infinite outcome along Λ^r, then ζ^r has infinitely many implication-free derivatives which are $\Lambda^{r\pm1}$-free. Fix a $\Lambda^{r\pm1}$-free and implication-free derivative $\bar{\zeta}^{r\pm1}$ of ζ^r along $\Lambda^{r\pm1}$, and fix $\xi^{r\pm1} \subset \Lambda^{r\pm1}$ such that $(\xi^{r\pm1})^- = \bar{\zeta}^{r\pm1}$. Note that, by definition, for all $i \leq r-1$, the principal derivative of $\bar{\zeta}^{r\pm1}$ along Λ^i is $\bar{\zeta}^i = (out^i(\xi^{r\pm1}))^-$. By repeated applications of Lemma 4.6(i) (Free Derivative) and Lemma 5.16(ii) (Implication-Freeness), we see that for all $i \leq r-1$, $\bar{\zeta}^i$ is Λ^i-free and implication-free. By Lemma 5.17(iv) (Assignment), $\xi =$

$out^0(\xi^{r\pm1})$ is pseudotrue.

By Lemma 6.1(iv), (Finite Control), $\xi^{r\pm1}$ is an initiator for $S^{[wt(\xi^{r\pm1})]}$ at $\xi^{r\pm1}$, with corresponding controller $\bar{\zeta}^{r\pm1}$. As ξ is pseudotrue, it follows from Definition 6.4 that $\bar{\zeta}^{r\pm1}$ controls $S^{[wt(\xi^{r\pm1})]}$ at $\xi^{r\pm1}$ with initiator $\xi^{r\pm1}$. As $\bar{\zeta}^{r\pm1}$ is $\Lambda^{r\pm1}$-free and $(\xi^{r\pm1})^- = \bar{\zeta}^{r\pm1}$, $\xi^{r\pm1}$ cannot have a terminator along $\Lambda^{r\pm1}$, else $\bar{\zeta}^{r\pm1}$ would be restrained by a primary $\Lambda^{r\pm1}$-link. Hence by Definition 6.4, $\bar{\zeta}^{r\pm1}$ controls $S^{[wt(\xi^{r\pm1})]}$ along $\Lambda^{r\pm1}$. ∎

7. Construction and Proof. Fix $k \le n$ and $\langle b,c\rangle \in Z_{0,k}$. In order to show that $A_c^{(k\pm1)} \not\le_T A_b^{(k\pm1)}$, we wish to define a partial recursive functional $\Delta_{b,c}^{0,k}$ which is total on domain $\mathbf{N}^k$ from oracle A_c such that for each $e \in \mathbf{N}$, there is an x such that $\lim_{\bar{u}}\Phi_e(A_b;\bar{u},x) \neq \lim_{\bar{v}} \Delta_{b,c}^{0,k}(A_c;\bar{v},x)$, and for all y, $\lim_{\bar{v}} \Delta_{b,c}^{0,k}(A_c;\bar{v},y)$ exists. $\Delta_{b,c}^{0,k}(A_c;\bar{v},x)$ will be the value defined by some ζ controlling $\langle \bar{v},s,x,\zeta\rangle$ along Λ^0 for some s whenever such a ζ exists, where $R_\zeta = R_{e,b,c}^{0,k}$ for some e. (We recall that there is an additional limit which enters into the computation, namely, the limit over *stages* at which we place elements into A_c and declare axioms, which we must also take into account.) Thus all axioms declared for such Δ_ζ will be axioms for $\Delta_{b,c}^{0,k}$. We will take additional steps to ensure that $\Delta_{b,c}^{0,k}$ is total on oracle A_c by defining this functional on arguments which are not in spaces being controlled, and will prove that $\Delta_{b,c}^{0,k}$ is a well-defined partial recursive functional and $\Delta_{b,c}^{0,k}(A_c)$ is total in Lemma 7.2 (Well-Definedness and Totality). Similarly, for $j \in \{1,2\}$, the requirement $R_{e,b,c}^{j,k}$ requires us to define a functional $\Delta_{b,c}^{j,k}$ for each $\langle b,c\rangle \in Z_{j,k}$, uniformly in e. We define this function to contain the union of all functionals Δ_ζ such that ζ deals with a requirement for this fixed $\langle b,c\rangle \in Z_{j,k}$, and take additional steps to ensure that $\Delta_{b,c}^{j,k}$ is total on oracle A_c by defining this functional on arguments which are not in spaces being controlled. We identify Δ_η with Δ_ξ whenever Δ_η and Δ_ξ are components of the same functional $\Delta_{b,c}^{j,k}$. (Thus if η defines an axiom for Δ_η, then that axiom is in existence for Δ_ξ as well.)

The decision about the action taken for a requirement associated with $\eta \in T^0$ is based on our ability to force M_η to be true. M_η will be equivalent to a Π_1-sentence with a single unbounded (universal) quantifier which will be part of a quantifier block $\exists s{\le}wt(\eta)\forall t{\ge}s$, which is equivalent to $\forall t{\ge}wt(\eta)$. (This quantifier will range over stages.)

Definition 7.1: For $\eta \in T^0$, we say that M_η is *potentially true* if the sentence $M_\eta^{[wt(\eta)]}$, obtained from M_η by dropping the quantifier block $\exists s{\le}wt(\eta)\forall t{\ge}s$ and replacing all occurrences of s and t with wt(η), is true.

The Construction

We define an admissible path $\Lambda^0 \in [T^0]$ by induction on lh(η) for $\eta \subset \Lambda^0$. We begin by specifying that $\langle\,\rangle \subset \Lambda^0$. Fix $\eta \subset \Lambda^0$. If lh(η) = 0, then no axioms are declared and all sets A^s are empty for s ≤ wt(η). Assume that lh(η) ≥ 0. We assume, by induction, that η is admissible and completion-consistent via ⟨ ⟩. In Step 1, we will determine an admissible node $\widehat{\eta}$ such that $\eta \subset \widehat{\eta} \subset \Lambda^0$. We begin, in Step 1.1, by determining an immediate successor β of η. There will be three cases to the definition of β, designed to ensure that β is preadmissible. If β is completion-consistent via ⟨ ⟩, then we will set $\widehat{\eta} = \beta$. Otherwise, $\lambda^k(\eta)$ will require extension for a unique k, and we will define $\widehat{\eta}$ to be the 0-completion of β in Step 1.2. We will determine which elements are placed into sets in Step 2, and this will depend on the path chosen in Step 1. New axioms for our functionals are declared in Step 3.

Step 1: (Path Definition.) We note, by induction, that η is admissible and completion-consistent via ⟨ ⟩.

Step 1.1: There are three cases.

Case 1: η is a primary 0-completion or a pseudocompletion. Set $\beta = \eta\hat{}\langle\infty\rangle$.

Case 2: The previous case is not followed and η is implication-restrained. Let β be a nonswitching extension of η. (We take the activated extension if both possible extensions are nonswitching, in order to satisfy (5.17)(ii).)

Case 3: Otherwise. Set $\beta = \eta\hat{}\langle\infty\rangle \subseteq \Lambda^0$ if M_η is potentially true, and $\beta = \eta\hat{}\langle 0\rangle \subseteq \Lambda^0$ otherwise.

It follows from (5.17) and (5.18) that β is preadmissible, and from Lemma 5.8 (Completion-Respecting Admissible Extension) that β is admissible. If β is completion-consistent via ⟨ ⟩, then the induction hypothesis holds at β, and we set $\widehat{\eta} = \beta$ and go to Step 2. Otherwise, by Lemma 5.8 (Completion-Respecting Admissible Extension) and Lemma 5.6 (Uniqueness of Requiring Extension), there is a unique k, which we fix, such that $\lambda^k(\beta)$ requires extension. We now go to Step 1.2.

Step 1.2: By Lemma 5.14 (Completion) we can effectively obtain the 0-completion $\widehat{\eta}$ of $\lambda^k(\beta)$. By (5.19) and Lemma 5.14 (Completion), $\widehat{\eta}$ is admissible and completion-consistent via $\langle\ \rangle$, so the induction condition holds. Now go to Step 2.

Step 2: (Set Definition.) For each node π such that $\eta \subseteq \pi \subset \widehat{\eta}$, π is validated along $\widehat{\eta}$, and π is not the initial derivative of up(π) along $\widehat{\eta}$, place wt(up(π)) into $A^{wt(\pi)+1}$ for all $A \in TS(\pi)$. For each set A and all s such that $wt(\eta) < s \leq wt(\widehat{\eta})$, we let $A^s = A^{wt(\eta)} \cup \{x$: x is placed in $A^{wt(\pi)+1}$ for some π such that $\eta \subseteq \pi \subset \widehat{\eta}$ and $wt(\pi) < s\}$.

Step 3: (Declaration of Axioms.) We carry out this step only if $\widehat{\eta}$ is pseudotrue. Let $\alpha = \widehat{\eta}$. This step is carried out for each functional $\Delta = \Delta^{j,k}_{b,c}$ and each $\langle \bar{x},s,x\rangle$ which is potentially in the domain of Δ such that $x < wt(\lambda(\alpha))$, and $x_i < wt(\lambda(\alpha))$ for all coordinates x_i of $\bar{x}$. (Note that we identify functionals whose last coordinates are $\equiv$, so choose to ignore the last coordinate. If such an $\langle \bar{x},t,x\rangle$ is not controlled at α for any t and $tp(R) \in \{0,2\}$, then we will show in Lemma 7.2 (Well-Definedness and Totality) that $\langle \bar{x},t,x\rangle$ will not be controlled at any $\rho \subset \Lambda^0$ for any t; hence it is safe to declare an axiom $\Delta_{wt(\alpha)}(A^{wt(\alpha)};\bar{x},x) = 0$, and we do so in Case 3.3. And if tp(R) = 1, then terminators will let us correct such axioms as required.) Let $A = A_c$ be the oracle for Δ.

Case 1: $\Delta_{wt(\gamma)}(A^{wt(\alpha)};\bar{x},x)\downarrow = q$ for some q and $\gamma \subset \alpha$. Set $\Delta_t(A^{wt(\alpha)};\bar{x},x) = \Delta_{wt(\gamma)}(A^{wt(\alpha)};\bar{x},x)$ for all t such that $wt(\gamma) < t \leq wt(\alpha)$. The use of all such axioms is the use of the axiom $\Delta_{wt(\gamma)}(A^{wt(\alpha)};\bar{x},x) = q$.

Case 2: Case 1 does not apply, and there is a $t < wt(\alpha)$ such that $\langle \bar{x},t,x\rangle$ is in the space controlled at α. (Note that we identify functionals whose last coordinates are $\equiv$, so choose to ignore the last coordinate.) Fix the largest such t, and let $\langle \bar{x},t,x\rangle$ be in the space controlled by ν at α with initiator δ. We *declare* the axiom $\Delta_{wt(\alpha)}(A^{wt(\alpha)};\bar{x},x) = 1$ if $\delta \supseteq \nu\hat{}\langle\infty\rangle$ and $\Delta_{wt(\alpha)}(A^{wt(\alpha)};\bar{x},x) = 0$ if $\delta \supseteq \nu\hat{}\langle 0\rangle$. The use of each axiom so defined is $wt(\lambda(\alpha))$-1.

Case 3: Otherwise. *Declare* the axiom $\Delta_{wt(\alpha)}(A^{wt(\alpha)};\bar{x},x) = 0$ with use $wt(\lambda(\alpha))$-1.

The construction is now complete. For all $r \leq n$, let $\Lambda^r = \lambda^r(\Lambda^0)$. We note that as the induction hypotheses are satisfied, Λ^0 is admissible. ∎

Our first lemma provides upper and lower bounds on the use of any axiom on a point controlled by some $\xi \in T^0$. The upper bound is used to prove that all functionals are total on the required oracles. The lower bound is obtained only if $tp(\xi) \in \{0,2\}$, and is used to show that axioms are corrected when necessary. (Recall that correction of axioms is unnecessary on a thin subspace of the space assigned to a requirement of type 1, so a lower bound is unnecessary in that case.)

Lemma 7.1 (Use Lemma): Let $\xi \subset \Lambda^0$ be given such that ξ is pseudotrue, and let $s = wt(\xi)$. Let $\Delta = \Delta^{j,k}_{b,c}$ be a functional, and fix $\langle \bar{x}, wt(\xi), x \rangle$ potentially in the domain of Δ such that $x < wt(\lambda(\xi))$ and for all coordinates x_i of $\bar{x}$, $x_i < wt(\lambda(\xi))$. Then:

(i) $\Delta_s(A^s_c;\bar{x},x)$ converges with some use $u < wt(\lambda(\xi))$.

(ii) If $\lambda(\xi) \subset \lambda(\Lambda^0)$, then $A_c \restriction wt(\lambda(\xi)) = A^s_c \restriction wt(\lambda(\xi))$.

(iii) If $j \in \{0,2\}$, $\nu^1 \subset \delta^1 \subseteq \lambda(\xi)$ and $\langle \bar{x}, s, x \rangle$ is in the space S such that ν^1 controls S at $\lambda(\xi)$ with initiator δ^1, then $wt(\nu^1) < wt(\delta^1) \leq u$, where u is the use determined in (i).

Proof: (i): By (2.11) and Step 3 of the construction, $\Delta_s(A^s_c;\bar{x},x)\downarrow$ with some use $u < wt(\lambda(\xi))$.

(ii): By Step 2 of the construction, if z enters A_c, there is a $\pi \subset \Lambda^0$ such that $z \in A^{wt(\pi)}_c \backslash A^{wt(\pi^{\pm})}_c$, π^- is validated along π, and $z = wt(up(\pi^-))$. If $wt(up(\pi^-)) < wt(\lambda(\xi))$, then as $\lambda(\xi) \subset \lambda(\Lambda^0)$, it follows from (2.1), (2.4), and (2.6) that $\pi^- \subset \xi$ and so that $wt(\pi^-) < wt(\xi)$. Hence $z \in A^{wt(\xi)}_c \restriction wt(\lambda(\xi))$.

(iii): Suppose that $\nu^1 \subset \delta^1 \subseteq \lambda(\xi)$ and $\langle \bar{x}, s, x \rangle$ is in the space S such that ν^1 controls S at $\lambda(\xi)$ with initiator δ^1. (Note that we identify functionals whose last coordinates are $\equiv$, so choose to ignore the last coordinate.) By (2.1), $wt(\nu^1) < wt(\delta^1)$. Let $y = x$ if $k = dim(\nu^1) = 1$ and $j = tp(\nu^1) = 0$, and let $y = x_{k\pm1}$ if $k = dim(\nu^1) > 1$. By (6.4) and (6.6), $wt(\delta^1) \leq y \leq wt(\lambda(\xi))$. By Step 3 of the construction, $\Delta_t(A^t_c;\bar{x},x)$ diverges unless $t \geq wt(\mu)$ for some $\mu \subset \Lambda^0$ such that $wt(\lambda(\mu)) > y$. Hence by (2.11) and Step 3 of the construction, all axioms $\Delta_t(A^t_c;\bar{x},x) = q$ which are ever declared have use $u \geq wt(\lambda(\mu))-1$ for some such μ, so $u \geq y \geq wt(\delta^1)$. ∎

We now begin to show that all requirements are satisfied. We first show that the functionals which we define are partial recursive, total on the appropriate oracles, and well-defined.

Lemma 7.2 (Well-Definedness and Totality Lemma): For all $j\square\leq 2$, $k \leq n$ and $\langle b,c\rangle \in Z_{j,k}$, $\Delta^{j,k}_{b,c}(A_c)$ is total and $\Delta^{j,k}_{b,c}$ is a well-defined partial recursive functional.

Proof: By Step 3 of the construction, all functionals are partial recursive, and new axioms are not defined when an axiom from an oracle compatible with A_c already exists, so $\Delta^{j,k}_{b,c}(A_c)$ is well-defined. Fix x and $\bar{x}$. Any axiom $\Delta^{j,k}_{b,c}(A_c;\bar{x},x) = q$ which is ever declared at $\pi \subset \Lambda^0$ has use $< wt(\lambda(\pi))$, and furthermore, $wt(\lambda(\pi)) > x$ and $wt(\lambda(\pi)) > x_i$ for all coordinates x_i of $\bar{x}$. By Lemma 5.17(v) (Assignment), there are infinitely many nodes $\pi \subset \Lambda^0$ such that π is Λ^0-true and pseudotrue, $x < wt(\lambda(\pi))$, and $x_i < wt(\lambda(\pi))$ for all coordinates x_i of $\bar{x}$. By Lemma 7.1(ii) (Use), $A_c \upharpoonright wt(\lambda(\pi)) = A_c^{wt(\pi)} \upharpoonright wt(\lambda(\pi))$, so as the use of $\Delta^{j,k}_{b,c}(A_c;\bar{x},x) = q$ is $< wt(\lambda(\pi))$, $\Delta^{j,k}_{b,c}(A_c;\bar{x},x) = \Delta^{j,k}_{b,c}(A_c^{wt(\pi)};\bar{x},x)$. Thus $\Delta^{j,k}_{b,c}(A_c)$ is total. n

The next lemma establishes the existence of all (iterated) limits except for the outermost limit, and relates the limiting value to the outcome of a controller, should the latter exist.

Lemma 7.3 (Convergence and Correctness Lemma): Fix a requirement $R = R^{j,r}_{e,b,c}$, and let $\Delta = \Delta^{j,r}_{b,c}$ be the functional associated with R. Fix $k \in [1,r\text{-}1]$. (Thus we explicitly exclude the case where $\dim(R) = 1$.) Let $p = r\text{-}k\text{+}1$. Fix $u_1,\ldots,u_{p-1},x \in \mathbf{N}$, and let $S = \{\langle u_1,\ldots,u_{p\pm1}\rangle\}\times\mathbf{N}^k\times\{\langle x\rangle\}$ if $j \in \{0,2\}$ and $S = \{\langle u_1,\ldots,u_{p\pm1}\rangle\}\times\mathbf{N}^{k+1}\times\{\langle x\rangle\}$ if $j = 1$. (Note that we use identification of axioms here, so that $S = \{\langle u_1,\ldots,u_{p\pm1}\rangle\}\times\mathbf{N}^k\times\{\langle x,\xi\rangle\}$ or $\{\langle u_1,\ldots,u_{p\pm1}\rangle\}\times\mathbf{N}^{k+1}\times\{\langle x,\xi\rangle\}$ for some ξ.) Then:

(i) If $tp(R) \in \{0,2\}$, then $\lim_{u_p}\ldots\lim_{u_{r\pm1}}\Delta(A_c;u_1,\ldots,u_{r\pm1},x)\downarrow \in \{0,1\}$; and if $tp(R) = 1$, then $\lim_{u_p}\ldots\lim_{u_r}\Delta(A_c;u_1,\ldots,u_r,x)\downarrow \in \{0,1\}$. In both cases, define this value to be $L(u_1,\ldots,u_{p\pm1},x)$.

(ii) If ν^k controls S along Λ^k, then $L(u_1,\ldots,u_{p\pm1},x) = 1$ iff ν^k is validated along Λ^k.

(iii) If S is not controlled along Λ^k and only finitely many sections of S are controlled along $\Lambda^{k\pm1}$, then $L(u_1,\ldots,u_{p\pm1},x) = 0$.

Proof: We proceed by induction on k, considering various cases.

Case 1: $k = 1$ (so $p = r$).

Subcase 1.1: $j = 1$. By clause (iii) of Lemma 6.1 (Finite Control), there are only finitely many initiators for S on T^1. Suppose first that S is controlled along Λ^1. By (6.7), we can fix $\nu^1 \subset \delta^1 \subseteq \tau^1 \subset \Lambda^1$ such that for all $\rho^1 \subset \Lambda^1$ with $\rho^1 \supseteq \tau^1$, S has controller ν^1 and initiator δ^1 at ρ^1. By Lemma 3.1 (Limit Path), we can fix $\eta \subset \Lambda^0$ such that $\lambda(\eta) = \tau^1$. Suppose that $u_r \geq wt(\eta)$ and an axiom $\Delta_{wt(\xi)}(A_c^{wt(\xi)};u_1,\ldots,u_r,x) = q$ is declared at ξ where $\eta \subseteq \xi \subset \Lambda^0$. If S has controller ν^1 and initiator δ^1 at $\lambda(\xi)$, then we set q = 0 if ν^1 is activated along δ^1, and q = 1 if ν^1 is validated along δ^1.

If the controller of S at $\lambda(\xi)$ is not ν^1 or the initiator for S at $\lambda(\xi)$ is not δ^1, then by Lemma 6.12 (1-Correction), there is a ν^1-correcting $\lambda(\xi)$-link $[\mu^1,\pi^1]$ such that $\mu^1 \subset \delta^1 \subseteq \pi^1$. By the construction and (2.1), any axiom $\Delta_{wt(\xi)}(A_c^{wt(\xi)};u_1,\ldots,u_r,x) = q$ declared at ξ (but not in existence at $\xi^{\pm}$) has use $wt(\lambda(\xi))-1 \geq wt(\pi^1)$. As S is controlled by ν^1 with initiator δ^1 along Λ^1, it follows from Lemma 3.1 (Limit Path) that there is a shortest $\rho \supset \xi$ such that S is controlled by ν^1 with initiator δ^1 at $\lambda(\rho)$, and note ρ that is pseudotrue. By Lemma 6.12 (1-Correction) and the construction, as $[\mu^1,\pi^1]$ is a primary ν^1-correcting link, there is a $\beta^1 \subseteq \pi^1$ such that $A_c \in TS(\beta^1)$ and $wt(\beta^1)$ is placed in A_c at some γ such that $\xi \subset \gamma \subseteq \rho$. Furthermore, when axioms are changed on a fixed argument at any node $\tilde{\eta} \subset \Lambda^0$, the use of the axiom declared at $\tilde{\eta}$ is $wt(\lambda(\tilde{\eta})-1)$, so by (2.11) and (2.1), if an axiom $\Delta_{wt(\xi)}(A_c^{wt(\xi)};u_1,\ldots,u_r,x) = q$ is in existence at $\gamma^{\pm}$, then it has use $\geq wt(\pi^1) \geq wt(\beta^1)$. But this allows us to define a new axiom $\Delta_{wt(\rho)}(A_c^{wt(\rho)};u_1,\ldots,u_r,x) = q$, where q = 0 if ν^1 is activated along δ^1, and q = 1 if ν^1 is validated along δ^1. By Lemma 7.2 (Well-Definedness and Totality), we see that (i) and (ii) hold in this case.

Suppose that S is not controlled along Λ^1 and only finitely many sections of S are controlled along Λ^0. We note that by Lemma 5.17(v) (Assignment), there are infinitely many pseudotrue nodes $\subset \Lambda^0$. By Lemma 6.1(iii) (Finite Control), there are only finitely many initiators for S on T^1, and as S is not controlled along Λ^1, every initiator for S at some node $\subset \Lambda^1$ has a terminator along Λ^1. Thus there is an $\eta \subset \Lambda^0$ such that for all $\alpha \subset \Lambda^0$ such that $\alpha \supset \eta$, S has no controller at $\lambda(\alpha)$; so every initiator $\bar{\delta} \subset \Lambda^0$ for a section of S at any node along Λ^0 must satisfy $\bar{\delta} \subseteq \eta$. As only finitely many sections of S are controlled along Λ^0 and there are infinitely many pseudotrue nodes along Λ^0, each such $\bar{\delta}$ has a terminator along Λ^0. If $\eta \subset \bar{\eta} \subset \Lambda^0$ and $\bar{\eta}$ properly extends each such terminator, then no section of S is controlled at any node along Λ^0 which extends $\bar{\eta}$, so by (6.6), if $S^{[i]}$ is controlled along Λ^0, then $i \leq wt(\bar{\eta})$. (i) and (iii) now follow from Case 3 of Step 3 of the construction.

Suppose that S is not controlled along Λ^1 but infinitely many sections of S are controlled along Λ^0. As in the preceding paragraph, we see that there are only finitely many initiators for sections of X along Λ^0. As infinitely many sections of S are controlled along Λ^0, there is a longest initiator, for a section of S, along Λ^0 which has no terminator along along Λ^0. Let ν be the controller corresponding to this initiator. Then by (6.7), for all but finitely many sections Y of S, ν will control Y at all sufficiently long pseudotrue $\rho \subset \Lambda^0$. So for all but finitely many u_r, the axioms $\Delta_{wt(\xi)}(A_c^{wt(\xi)};u_1,\dots,u_r,x) = q$ which are declared have value q determined by the outcome of ν along Λ^0. (i) now follows.

Subcase 1.2: $j \in \{0,2\}$. (Note that no limit is being computed, and $L(u_1,\dots,u_{r\pm1},x)$ just gives the value of an axiom.) Recall that, by (6.7), a space is controlled by a node along a path iff it is controlled by that node at all sufficiently long pseudotrue nodes along the path. If S is not controlled along Λ^1 and no section of S is controlled along Λ^0, then as controllers are never terminated along Λ^0, all axioms $\Delta_{wt(\xi)}(A_c^{wt(\xi)};u_1,\dots,u_{r\pm1},x) = q$ will be declared in Case 3 of Step 3 of the construction and will set q = 0, so (i) and (iii) follow from Lemma 7.2 (Well-Definedness and Totality). If S is not controlled along Λ^1 but some section of S is controlled along Λ^0, then (i) follows from Lemma 7.2 (Well-Definedness and Totality). As controllers are never terminated along Λ^0, infinitely many sections of S will be controlled along Λ^0, so the hypothesis of (iii) fails.

In order to complete the verification of (i) and (ii) for $j \neq 1$, it suffices to verify the following condition, under the assumption that S is controlled along Λ^1:

(7.1) For all η and ν^1, if $\eta \subset \Lambda^0$ is pseudotrue and ν^1 controls S at $\lambda(\eta)$, then $\Delta_{wt(\eta)}(A_c^{wt(\eta)};u_1,\dots,u_{r\pm1},x) = 1$ iff ν^1 is validated along Λ^1.

We proceed by induction on $lh(\eta)$ for η pseudotrue. Given $u_{r\pm1}$, let η_0 be the shortest string for which $\Delta_{wt(\eta_0)}(A_c^{wt(\eta_0)};u_1,\dots,u_{r-1},x)\downarrow$, and note that by Step 3 of the construction, η_0 is pseudotrue. If $\eta = \eta_0$, then by the construction, we define $\Delta_{wt(\eta)}(A_c^{wt(\eta)};u_1,\dots,u_{r-1},x) = q$ for some q, and the value chosen for q is the one satisfying (7.1) if there is a ν^1 which controls S at $\lambda(\eta)$. Suppose, by induction on $lh(\eta)$ with η pseudotrue, that (7.1) holds for ρ, where $\eta_0 \subseteq \rho$ and ρ is the longest pseudotrue node $\subset \eta$. By Lemma 6.7 (Loss of Control), (7.1) will hold at η through the absence of a controller, unless there is a controller ν^1 and initiator δ^1 for S at $\lambda(\rho)$; so we may fix such ν^1 and δ^1. Let u be the use of the axiom $\Delta_{wt(\rho)}(A_c^{wt(\rho)};u_1,\dots,u_{r\pm1},x) = q$, and note that by Lemma 7.1(iii) (Use), $wt(\delta^1) \leq u$.

If $\delta^1 \subseteq \lambda(\eta)$, then by Lemma 6.6 (Constancy of Initiator), ν^1 controls S at $\lambda(\eta)$ with initiator δ^1, so (7.1) follows by induction. Suppose that $\delta^1 \not\subseteq \lambda(\eta)$, and fix the shortest β such that $\rho \subset \beta \subseteq \eta$ and $\delta^1 \not\subseteq \lambda(\beta)$, and fix κ^1 such that β switches κ^1. By Lemma 6.7 (Loss of Control), (7.1) will hold for η if S does not have an initiator and controller at $\lambda(\beta)$; thus we may fix an initiator $\widehat{\delta}^1$ and controller $\widehat{\nu}^1$ for S at $\lambda(\beta)$, and note that, by our assumption, $\delta^1 \neq \widehat{\delta}^1$. By (6.10), $\widehat{\delta}^1 \subseteq \lambda(\beta)^- \subset \lambda(\beta^\pm)$, so $\widehat{\delta}^1 \subseteq \kappa^1 \subset \delta^1$. Hence we may apply Lemmas 6.10 or 6.14 (Correction).

If conclusion (i) of the relevant Correction Lemma holds, then (7.1) follows by induction. If conclusion (ii) of the Correction Lemma holds and $tp(\nu^1) = 0$, then $\beta^\pm \equiv \nu^1$ and we place $wt(\kappa^1) \in A_c^{wt(\beta)} \backslash A_c^{wt(\beta^\pm)}$. And if conclusion (ii) of the Correction Lemma holds and $tp(\nu^1) = 2$, then $\dim(\beta^\pm) \geq \dim(\nu^1)$ and by Lemma 2.2(iv) (Interaction), we place $wt(\kappa^1) \in A_c^{wt(\beta)} \backslash A_c^{wt(\beta^\pm)}$. As $\kappa^1 \subset \delta^1$, it follows from (2.1) that $wt(\kappa^1) < wt(\delta^1) \leq u$, and so that $A_c^{wt(\eta)} \upharpoonright u \neq A_c^{wt(\rho)} \upharpoonright u$. Now axioms are only defined at pseudotrue nodes, so the construction declares a new axiom $\Delta_{wt(\eta)}(A_c^{wt(\eta)};u_1,\ldots,u_{r\pm1},x)\downarrow$ to satisfy (7.1).

Case 2: $k > 1$. By induction, the lemma holds for k-1.

Subcase 2.1: S is controlled by ν^k along Λ^k. By Lemma 6.3 (Thick Control), a thick subset of S is controlled along $\Lambda^{k\pm1}$ by derivatives of ν^k which are validated along $\Lambda^{k\pm1}$ if ν^k is validated along Λ^k, and are activated along $\Lambda^{k\pm1}$ if ν^k is activated along Λ^k. (i) and (ii) now follow by induction.

Subcase 2.2: S is not controlled along Λ^k and only finitely many sections of S are controlled along $\Lambda^{k\pm1}$. By Lemma 6.5(iii) (Non-Control), there are only finitely many i such that a section of $S^{[i]}$ is controlled along $\Lambda^{k\pm2}$. (i) and (iii) now follow inductively from (i) and (iii) for k-1.

Subcase 2.3: S is not controlled along Λ^k, but infinitely many sections of S are controlled along $\Lambda^{k\pm1}$. By Lemma 6.4 (Indirect Control), all but finitely many sections of S are controlled by a fixed node along $\Lambda^{k\pm1}$. (i) now follows from (i) and (ii) inductively. ∎

The next lemma relates the outcomes of nodes which are critical for axiom definition, to the truth of the sentences assigned to those nodes.

Lemma 7.4 (Accuracy Lemma): Fix $k \leq n$ and $\xi^k \subset \Lambda^k$ such that $k \leq \dim(\xi^k)$ and ξ^k is Λ^k-free and implication-free. Then ξ^k is validated along Λ^k iff M_{ξ^k} is true.

Proof: Case 1: $k = 0$. Let $\xi = \xi^0$. Recall that M_ξ is a Π_1-sentence beginning with a block of bounded quantifiers and followed by $\exists s \leq wt(\eta^1) \forall t \geq s\, S$, where S is quantifier-free and $\eta^1 = \mathrm{up}(\xi)$.

Case 1.1: ξ is validated along Λ^0. We first show that M_ξ is potentially true, and all uses in M_ξ are $< \mathrm{wt}(\xi)$, under the weaker assumption that $\xi \subset \Lambda^0$ is implication-free. We proceed by induction on $\mathrm{lh}(\xi)$. There are two cases.

Case 1.1.1: ξ is not a primary 0-completion or an amenable pseudocompletion. Then by the construction, M_ξ is potentially true, and by (0.1), all uses in M_ξ are $< \mathrm{wt}(\xi)$.

Case 1.1.2: ξ is a primary 0-completion or an amenable pseudocompletion. Thus $\mathrm{tp}(\xi) \in \{1,2\}$. If ξ is a primary completion, fix η such that ξ is a primary completion of η, and let $\gamma = \eta^{\pm}$. And if ξ is a pseudocompletion, fix the shortest γ such that ξ is a pseudocompletion of γ, and fix $\eta \subseteq \xi$ such that $\eta^{\pm} = \gamma$. By (5.5)(ii) and Lemma 5.13 (Amenable Implication Chain) if $\dim(\xi) > 1$ and by (5.1) or (5.10)(i) if $\dim(\xi) = 1$, γ is implication-free.

By (5.2) or (5.11)(i), γ is validated along ξ, so it follows by induction that M_γ is potentially true, and by (0.1) and (2.1), all uses in M_γ are $\leq \mathrm{wt}(\gamma) < \mathrm{wt}(\xi)$. First suppose that ξ is a primary 0-completion. By Lemma 5.12(i) (PL) and (5.19), all nodes β^1 of T^1 which are switched by nodes in $(\eta,\xi]$ are in $\mathrm{PL}(\mathrm{up}(\xi),\lambda(\eta))$. If $\dim(\xi) = 1$ (and hence $\mathrm{tp}(\xi) = 1$), it follows from (5.4) that $\mathrm{TS}(\beta^1) \cap \mathrm{RS}(\xi) = \emptyset$ for each such β^1. Suppose that $\dim(\xi) = r > 1$. Then by (5.5)(ii), there is an amenable 1-implication chain $\langle\langle \sigma^j, \hat{\sigma}^j, \tau^j\rangle\ r-1 \geq j \geq 1\rangle$ such that $\mathrm{out}(\tau^1) = \eta$. By Lemma 5.12(ii),(iii) (PL), $\{\mathrm{up}^{r\pm1}(\beta^1)\colon \beta^1 \in \mathrm{PL}(\mathrm{up}(\xi),\lambda(\eta))\} = \mathrm{PL}(\sigma^{r\pm1},\tau^{r\pm1})$, and if we fix $\bar{\tau}^{r\pm1} \subset \tau^{r\pm1}$ such that $(\bar{\tau}^{r\pm1})^- = \sigma^{r\pm1}$, then either $\hat{\sigma}^{r\pm1}$ is a pseudocompletion of $\sigma^{r\pm1}$, or $\bar{\tau}^{r\pm1}$ requires extension. If $\bar{\tau}^{r\pm1}$ requires extension, then by (5.11)(ii), $\hat{\sigma}^{r\pm1}$ has finite outcome along $\tau^{r\pm1}$, so by Lemma 5.1(iv) (PL Analysis) and Lemma 5.12(ii) (PL) and (5.19), $\{\mathrm{up}(\pi^{r\pm1})\colon \pi^{r\pm1} \in \mathrm{PL}(\sigma^{r\pm1},\tau^{r\pm1})\} = \{\mathrm{up}(\pi^{r\pm1})\colon \pi^{r\pm1} \in \mathrm{PL}(\sigma^{r\pm1},\hat{\sigma}^{r\pm1})\} = \mathrm{PL}(\mathrm{up}(\hat{\sigma}^{r\pm1}),\lambda(\bar{\tau}^{r\pm1}))$. Hence by Definition 5.4 if $\hat{\sigma}^{r\pm1}$ is an amenable pseudocompletion and by (5.4) otherwise, $\mathrm{TS}(\beta^1) \cap \mathrm{RS}(\xi) = \emptyset$ for each such β^1. Thus by Lemma 2.2(i) (Interaction) and the construction, M_ξ must be potentially true, and all uses in M_ξ are $< \mathrm{wt}(\xi)$.

Now suppose that ξ is an amenable pseudocompletion. By (5.11)(i), γ is the principal derivative of $\mathrm{up}(\gamma)$ along ξ. Hence by (2.11), (2.2), and (2.4), any element $\leq \mathrm{wt}(\gamma)$ placed in a set at any $\pi \in (\eta,\xi]$ is of the form $\mathrm{wt}(\mathrm{up}(\pi))$ with $\mathrm{up}(\pi) \subset \mathrm{up}(\gamma)$, and π is validated along ξ. By Lemma 3.1(i) (Limit Path), there must be a $\mu \subset \xi$ such that $[\mu,\pi]$ is a primary ξ-link which restrains γ. As ξ is an amenable pseudocompletion of γ,

$TS(\pi) \cap RS(\xi) = \emptyset$ for each such π. Thus by Lemma 2.2(i) (Interaction) and the construction, M_ξ must be potentially true, and all uses in M_ξ are < wt(ξ).

For both Subcase 1.1.1 and Subcase 1.1.2, we note that elements placed into sets are of the form z = wt(up(ν)) for $\nu \subset \Lambda^0$, and z is first placed in a set A^{s+1} when s = wt(δ) and up(ν) is validated along $\lambda(\delta)$ but not along $\lambda(\delta^{\pm})$. Hence by Lemma 3.1 (Limit Path), M_ξ will be true if no element < wt(ξ) is first placed in any $A \in RS(\xi)$ by any $\nu \supseteq \xi$ such that $\nu \subset \Lambda^0$. By Lemma 2.2(i) (Interaction), ξ does not place elements into any set in RS(ξ). Fix $\pi \subset \Lambda^0$ such that $\pi^{\pm} = \xi$. By Lemma 3.1 (Limit Path), it follows that that $\lambda(\pi)^- = \text{up}(\xi)$ and for all ν such that $\pi \subseteq \nu \subset \Lambda^0$, $\lambda(\nu) \supseteq \lambda(\pi)$. Hence the elements placed into sets by $\nu \supset \xi$ are of the form wt(α), where up(ν) = $\alpha \supseteq \lambda(\pi)$. By (2.1) and (2.2), wt(up(ν)) ≥ wt($\lambda(\pi)$) > wt(out($\lambda(\pi)$)) = wt(π) > wt(ξ). Hence M_ξ is true.

Case 1.2: k = 0 and ξ is activated along Λ^0. M_ξ cannot be potentially true, else the action taken for ξ would force ξ to be validated along Λ^0. Hence M_ξ cannot be true.

Case 2: k > 0. By induction, we may assume that the lemma holds for k-1. Let ν be the principal derivative of ξ along $\Lambda^{k\pm1}$. It follows from Lemma 4.6 (Free Derivative) and Lemma 5.16(ii) (Implication-Freeness), that ν is $\Lambda^{k\pm1}$-free and implication-free, and if ξ has infinite outcome along Λ^k, then ξ has infinitely many $\Lambda^{k\pm1}$-free, implication-free derivatives μ along $\Lambda^{k\pm1}$.

Suppose that k is odd. By Definitions 2.9 and 2.10, M_ξ is a sentence of the form $Q_1y_1\ldots Q_py_p\exists\bar{z}P(\bar{y},\bar{z})$ where P is Π_k, and the Q_j are bounded quantifiers, and M_ν is $Q_1y_1\ldots Q_py_p\exists\bar{z}\leq \text{wt}(\nu)P(\bar{y},\bar{z})$. If M_ν is true, then M_ξ is true. But then by induction, ν is validated along $\Lambda^{k\pm1}$, i.e., ν has infinite outcome along $\Lambda^{k\pm1}$, so by the definition of the function λ, ξ has finite outcome along Λ^k and ξ is validated along Λ^k. If M_ν is not true, then as ν is the principal derivative of ξ along $\Lambda^{k\pm1}$, it follows from (2.4) that all derivatives of ξ along $\Lambda^{k\pm1}$ are activated along $\Lambda^{k\pm1}$, i.e., have finite outcome along $\Lambda^{k\pm1}$. Hence by induction, for every derivative μ of ξ along $\Lambda^{k\pm1}$ which is $\Lambda^{k\pm1}$-free and implication-free, M_μ is not true. For each such μ, M_μ is $Q_1y_1\ldots Q_py_p\exists\bar{z}\leq \text{wt}(\mu)P(\bar{y},\bar{z})$. As there are infinitely many such μ, wt(μ) is unbounded as we range over these μ. Thus M_ξ is not true. By induction, μ has finite outcome along $\Lambda^{k\pm1}$ for each such μ, so by the definition of the function λ, ξ has infinite outcome along Λ^k, so ξ is activated along Λ^k.

Suppose that k is even. We proceed as in the preceding paragraph, interchanging universal and existential quantifiers, Π and Σ, and true and not true. ∎

We now show that all requirements are satisfied.

Lemma 7.5 (0-Satisfaction Lemma): Every requirement of type 0 is satisfied.

Proof: Fix a requirement $R = R^{0,r}_{e,b,c}$ of type 0, and let $\Delta = \Delta^{0,r}_{b,c}$ be the functional for the requirement R as described at the beginning of this section. By Lemma 5.17(i),(ii),(iv) (Assignment), R is assigned to a unique $\sigma^r \subset \Lambda^r$ such that σ^r is Λ^r-free and implication-free, and that if τ^r is the immediate successor of σ^r along Λ^r, then $out^0(\tau^r)$ is pseudotrue.

First assume that r = 1. Let $x = wt(\sigma^1)$. By Lemma 7.2 (Well-Definedness and Totality), we can fix q such that $\Delta(A_c;x) = q$. Let ν (π, resp.) be the initial (principal, resp.) derivative of σ^1 along Λ^0 and let β (δ, resp.) be the immediate successor of ν (π, resp.) along Λ^0. By Lemma 5.17(iv) (Assignment), δ is pseudotrue, and by Lemma 5.17(iii) (Assignment), π is δ-free and implication-free. By Lemma 5.16(iv) (Implication-Freeness), β is pseudotrue and ν is implication-free, and by Lemma 4.5 (Free Extension), ν is β-free. By Definition 6.4 and the construction, we declare an axiom $\Delta_{wt(\beta)}(A_c^{wt(\beta)};x) = z$ for some $z \in \{0,1\}$ with use $wt(\lambda(\beta))$ -1, where z = 0 iff ν is activated along β. As $\sigma^1 \subset \Lambda^1$, it follows from (2.6) that no α such that $\beta \subset \alpha \subset \Lambda^0$ can switch any $\rho^1 \subset \sigma^1$. Hence by Lemma 7.1(ii) (Use) and (2.1), $\Delta(A_c;x) = z$ unless $\pi \supset \nu$, i.e., $\lambda(\beta) \not\subset \Lambda^1$. Suppose this to be the case. Then the construction places $wt(\sigma^1)$ into $A_c^{wt(\delta)}$. By (2.1), $wt(\sigma^1) \leq wt(\lambda(\beta))$ -1, so we define a new axiom $\Delta_{wt(\delta)}(A_c^{wt(\delta)};x) = 1$ with use $wt(\lambda(\delta))$ -1, and ν is activated along $\delta \subset \Lambda^1$. As $\sigma^1 \subset \Lambda^1$, it follows from (2.8) and (2.6) that no α such that $\delta \subset \alpha \subset \Lambda^0$ can switch any $\rho^1 \subseteq \sigma^1$, so $\lambda(\delta) \subset \Lambda^1$. Hence by Lemma 7.1(ii) (Use) and (2.1), $\Delta(A_c;x) = 1$. Hence σ^1 is activated along Λ^1 if z = 0, and σ^1 is validated along Λ^1 if z = 1. By Lemma 7.4 (Accuracy), σ^1 is validated along Λ^1 iff M_{σ_1} is true. Hence if M_{σ_1} is true then z = 1, and if M_{σ_1} is not true then z = 0. Thus R is satisfied in this case.

Now assume that r > 1. Fix a space $S = N^r \times \{x\}$ in the domain of the functional Δ. First suppose that S is not controlled along Λ^r. If infinitely many sections of S are controlled along $\Lambda^{r\pm1}$, then by Lemma 6.4 (Indirect Control), cofinitely many sections of S are controlled along $\Lambda^{r\pm1}$ by the same node $\nu^{r\pm1}$, so by Lemma 7.3(i),(ii) (Convergence and Correctness) applied separately to each section of S, $\lim_{u_1}\ldots\lim_{u_{r\pm1}}\Delta(A_c;u_1,\ldots,u_{r\pm1},x) = L$ exists, L = 0 if $\nu^{r\pm1}$ is activated along $\Lambda^{r\pm1}$, and L = 1 if $\nu^{r\pm1}$ is validated along $\Lambda^{r\pm1}$. Otherwise, by Lemma 6.5(iii), (Non-Control) and Lemma 7.3(iii) (Convergence and Correctness) applied separately to each section of S, $\lim_{u_1}\ldots\lim_{u_{r\pm1}}\Delta(A_c;u_1,\ldots,u_{r\pm1},x) = 0$.

Now suppose that $S = S_{\gamma^r}$ for some $\gamma^r \subset \Lambda^r$ associated with Δ such that γ^r controls

S along Λ^r. Then by Lemma 6.3 (Thick Control) either cofinitely many sections of S are controlled, along $\Lambda^{r\pm1}$, by derivatives of γ^r which are activated along $\Lambda^{r\pm1}$, or cofinitely many sections of S are controlled, along $\Lambda^{r\pm1}$, by derivatives of γ^r which are validated along $\Lambda^{r\pm1}$. It now follows from Lemma 7.3(i),(ii) (Convergence and Correctness) applied separately to each section X of S_{γ^r} that $\lim_{u_1}\ldots\lim_{u_{r\pm1}}\Delta(A_c;u_1,\ldots,u_{r\pm1},wt(\gamma^r)) = L(wt(\gamma^r))$ exists, and that γ^r is validated along Λ^r iff $L(wt(\gamma^r)) = 1$.

Recall that R is assigned to a $\sigma^r \subset \Lambda^r$ such that σ^r is Λ^r-free and implication-free, and that if τ^r is the immediate successor of σ^r along Λ^r, then $out^0(\tau^r)$ is pseudotrue. Hence by Definition 6.4, σ^r controls S along Λ^r. By the preceding paragraph, $\lim_{u_1}\ldots\lim_{u_{r\pm1}}\Delta(A_c;u_1,\ldots,u_{r\pm1},wt(\sigma^r)) = L(wt(\sigma^r))$ exists, and σ^r is validated along Λ^r iff $L(wt(\sigma^r)) = 1$. By Lemma 7.4 (Accuracy), σ^r is validated along Λ^r iff M_{σ^r} is true. Hence if M_{σ^r} is true then $L(wt(\sigma^r)) = 1$, and if M_{σ^r} is not true then $L(wt(\sigma^r)) = 0$. Thus R is satisfied. n

Lemma 7.6 (1-Satisfaction Lemma): Every requirement of type 1 is satisfied.

Proof: Fix a requirement $R = R^{1,r}_{e,b,c}$ of type 1, and let $\Delta = \Delta^{1,r}_{b,c}$ be the functional for the requirement R as described at the beginning of this section. By Lemma 7.3(i) (Convergence and Correctness) for $r > 1$, $L(i,e) = \lim_{u_2}\ldots\lim_{u_r}\Delta(A_c;i,u_2,\ldots,u_r,e)$ exists and takes a value in $\{0,1\}$ for all $e,i \in \mathbf{N}$.

By Lemma 5.17(i),(ii) (Assignment), R is assigned to a unique $\kappa^r \subset \Lambda^r$ such that κ^r is Λ^r-free and implication free. Let $\nu^{r\pm1}$ be the principal derivative of κ^r along $\Lambda^{r\pm1}$, and fix $\delta^{r\pm1} \subset \Lambda^{r\pm1}$ such that $(\delta^{r\pm1})^- = \nu^{r\pm1}$. By Lemma 6.15(i) (Initial Control), $\nu^{r\pm1}$ controls $\{wt(\delta^{r\pm1})\}\times\mathbf{N}^r\times\{e\}$ with initiator $\delta^{r\pm1}$ along $\Lambda^{r\pm1}$. By Case 1.1 of Definition 6.3, $\delta^{r\pm1}$ is also the initiator for $\{i\}\times\mathbf{N}^r\times\{e\}$ at $\delta^{r\pm1}$ for all $i \geq wt(\delta^{r\pm1})$. Now if $i \geq wt(\delta^{r\pm1})$, then $\{i\}\times\mathbf{N}^r\times\{e\}$ is controlled along $\Lambda^{r\pm1}$ iff there is an initiator $\gamma^{r\pm1} \subset \Lambda^{r\pm1}$ for $\{i\}\times\mathbf{N}^r\times\{e\}$ such that there is no $\nu^{r\pm1}$-correcting primary $\Lambda^{r\pm1}$-link $[\mu^{r\pm1},\pi^{r\pm1}]$ with $\mu^{r\pm1} \subset \gamma^{r\pm1} \subseteq \pi^{r\pm1}$; and by (6.7), if $\{i\}\times\mathbf{N}^r\times\{e\}$ is controlled along $\Lambda^{r\pm1}$, then the initiator for $\{i\}\times\mathbf{N}^r\times\{e\}$ along $\Lambda^{r\pm1}$ is the longest such $\gamma^{r\pm1}$. As $\nu^{r\pm1}$ is $\Lambda^{r\pm1}$-free, $\delta^{r\pm1}$ is such a $\gamma^{r\pm1}$. Hence for all $i \geq wt(\delta^{r\pm1})$, $\{i\}\times\mathbf{N}^r\times\{e\}$ is controlled along $\Lambda^{r\pm1}$, and if $\{i\}\times\mathbf{N}^r\times\{e\}$ is controlled at any $\gamma^{r\pm1} \subset \Lambda^{r\pm1}$ with initiator $\delta^{r\pm1}_i$, then $\delta^{r\pm1}_i \supseteq \delta^{r\pm1}$.

Fix i and $\delta^{r\pm1}_i$ as in the preceding paragraph such that $\delta^{r\pm1}_i$ has no terminator along $\Lambda^{r\pm1}$. Let $\nu^{r\pm1}_i$ be the controller corresponding to $\delta^{r\pm1}_i$. As $\nu^{r\pm1}$ is $\Lambda^{r\pm1}$-free and implication-free, it follows from (4.1) and Case 3 of Definition 6.3 that $\nu^{r\pm1}_i \supseteq \nu^{r\pm1}$.

First suppose that $\nu^{r\pm1}$ has infinite outcome along $\Lambda^{r\pm1}$. We note that as $\nu^{r\pm1}$ is $\Lambda^{r\pm1}$-free, there is no primary $\Lambda^{r\pm1}$-link restraining $\nu^{r\pm1}$. Furthermore, by Lemma 5.17(v)

(Assignment), there are infinitely many $\tau^{r\pm1} \subset \Lambda^{r\pm1}$, such that $out^0(\tau^{r\pm1})$ is pseudotrue, so by (5.28), every node along $\Lambda^{r\pm1}$ which requires extension has a primary completion along $\Lambda^{r\pm1}$ which has infinite outcome along $\Lambda^{r\pm1}$; hence every component of $PL(\nu^{r\pm1},\xi^{r\pm1})$ for some $\xi^{r\pm1} \subset \Lambda^{r\pm1}$ gives rise to a primary $\Lambda^{r\pm1}$-link which restrains $\nu^{r\pm1}$, so no such component can exist. If $up(\nu^{r\pm1}) = up(\nu_i^{r\pm1})$, then by (2.8), $\nu^{r\pm1} = \nu_i^{r\pm1}$; so $\nu_i^{r\pm1}$ has infinite outcome along $\Lambda^{r\pm1}$. And if $up(\nu^{r\pm1}) \neq up(\nu_i^{r\pm1})$, then as $\nu^{r\pm1}$ is $\Lambda^{r\pm1}$-free, it follows from Lemma 6.11 (1-Similarity, with $\sigma^{r\pm1} = \nu^{r\pm1}$ and $\hat{\sigma}^{r\pm1} = \nu_i^{r\pm1}$) that $\nu_i^{r\pm1}$ has infinite outcome along $\Lambda^{r\pm1}$.

Suppose that $\nu^{r\pm1}$ has finite outcome along $\Lambda^{r\pm1}$. If $up(\nu^{r\pm1}) = up(\nu_i^{r\pm1})$, then as $\nu^{r\pm1}$ is the principal derivative of κ^r along $\Lambda^{r\pm1}$, it follows from (2.4) that $\nu_i^{r\pm1}$ has finite outcome along $\Lambda^{r\pm1}$. Suppose that $up(\nu^{r\pm1}) \neq up(\nu_i^{r\pm1})$, and let $\eta_i^{r\pm1}$ be the immediate successor of $\nu_i^{r\pm1}$ along $\Lambda^{r\pm1}$. By Subcase 1.2 of Definition 6.3, $out^0(\eta_i^{r\pm1})$ must be pseudotrue, else $\nu_i^{r\pm1}$ would not be a controller at $\eta_i^{r\pm1}$, so could not be a controller at any node extending $\eta_i^{r\pm1}$. We note that as $\nu^{r\pm1}$ is $\Lambda^{r\pm1}$-free, there is no primary $\Lambda^{r\pm1}$-link restraining $\nu^{r\pm1}$. Furthermore, by Lemma 5.17(v) (Assignment), there is a $\Lambda^{r\pm1}$-free node $\xi^{r\pm1} \subset \Lambda^{r\pm1}$ such that $out^0(\xi^{r\pm1})$ is pseudotrue and $\eta_i^{r\pm1} \subseteq \xi^{r\pm1}$. Fix the shortest such $\xi^{r\pm1}$. We show that there is no $\rho^r \in PL(up(\nu^{r\pm1}),\lambda(\eta_i^{r\pm1}))$ such that $OS(\nu^{r\pm1}) \subseteq TS(\rho^r)$. For suppose that such a ρ^r exists, in order to obtain a contradiction. By hypothesis, $\nu^{r\pm1}$ is $\Lambda^{r\pm1}$-free, so $up(\nu^{r\pm1})$ is Λ^r-free. By (4.1) and Lemma 4.3(iii) (Link Analysis), there are no primary $\lambda(\xi^{r\pm1})$-links restraining $up(\nu^{r\pm1})$. Hence we may apply Lemma 5.18(ii) (Nonamenable Backtracking) (with $\xi^k = \xi^{r\pm1}$, $(\eta^k)^- = \nu_i^{r\pm1}$, $\eta^{k+1} = \lambda(\eta_i^{r\pm1})$, $\delta^{k+1} = up(\nu^{r\pm1})$, and $\eta^k = \eta_i^{r\pm1}$) to conclude that $PL(up(\nu^{r\pm1}),\lambda(\eta_i^{r\pm1})) \subseteq \{up(\gamma^{r\pm1})\colon \gamma^{r\pm1} \in PL(\nu_i^{r\pm1},\xi^{r\pm1})\}$. Hence we may fix $\rho^{r\pm1} \in PL(\nu_i^{r\pm1},\xi^{r\pm1})$ such that $up(\rho^{r\pm1}) = \rho^r$.

As $out^0(\xi^{r\pm1})$ is pseudotrue and by Definition 5.3, there are $\mu_i^{r\pm1} \subset \rho^{r\pm1} \subseteq \pi_i^{r\pm1} \subset \beta_i^{r\pm1} \subseteq \xi^{r\pm1}$ such that $(\beta_i^{r\pm1})^- = \pi_i^{r\pm1}$, $[\mu_i^{r\pm1},\pi_i^{r\pm1}]$ is a primary $\xi^{r\pm1}$-link, and $\rho^{r\pm1} \in \overline{PL}(\beta_i^{r\pm1}) \subseteq PL(\nu_i^{r\pm1},\xi^{r\pm1})$. Furthermore, either $\overline{PL}(\beta_i^{r\pm1}) = \{\pi_i^{r\pm1}\}$ and $[\mu_i^{r\pm1},\pi_i^{r\pm1}]$ restrains $\nu_i^{r\pm1}$, or by Definitions 5.3 and 6.2, $\pi_i^{r\pm1}$ is the primary completion of some node for $\mu_i^{r\pm1}$ and $\mu_i^{r\pm1} \subset \nu_i^{r\pm1} \subset \pi_i^{r\pm1}$. As $\nu_i^{r\pm1}$ is a principal derivative along $\xi^{r\pm1}$, it follows that $\mu_i^{r\pm1} \subset \nu_i^{r\pm1}$ in both cases. Hence as $OS(\nu^{r\pm1}) = OS(\nu_i^{r\pm1}) \subseteq TS(\rho^r) = TS(\rho^{r\pm1})$, $[\mu_i^{r\pm1},\pi_i^{r\pm1}]$ is a $\nu_i^{r\pm1}$-injurious link. By the comments following Definition 6.2, $[\mu_i^{r\pm1},\pi_i^{r\pm1}]$ is a $\nu_i^{r\pm1}$-correcting link.

Recall that $\eta_i^{r\pm1}$ is the immediate successor of $\nu_i^{r\pm1}$ along $\xi^{r\pm1}$. Now $\pi_i^{r\pm1}$ is a terminator for $\eta_i^{r\pm1}$ along $\xi^{r\pm1}$. By Case 3 of Definition 6.3, when a terminator for $\eta_i^{r\pm1}$ is found at $\alpha^{r\pm1} \subset \Lambda^{r\pm1}$, it is a terminator for all initiators for $\nu_i^{r\pm1}$ which are $\subset \alpha^{r\pm1}$, and so $\nu_i^{r\pm1}$ cannot be a controller at any $\tilde{\alpha}^{r\pm1}$ such that $\alpha^{r\pm1} \subseteq \tilde{\alpha}^{r\pm1} \subset \Lambda^{r\pm1}$. Thus by Case 3 of Definition 6.3, $\nu_i^{r\pm1}$ cannot control a space along $\Lambda^{r\pm1}$, contrary to assumption. This contradiction shows that there is no $\rho^r \in PL(up(\nu^{r\pm1}),\lambda(\eta_i^{r\pm1}))$ such that $OS(\nu^{r\pm1}) \subseteq TS(\rho^r)$.

It now follows from Lemma 6.11 (1-Similarity, with $\sigma^{r\pm1} = \nu^{r\pm1}$ and $\hat{\sigma}^{r\pm1} = \nu_i^{r\pm1}$) that $\nu_i^{r\pm1}$ has finite outcome along $\Lambda^{r\pm1}$. We thus conclude that for all $i \geq wt(\eta_i^{r\pm1})$, $\nu_i^{r\pm1}$ is validated along $\Lambda^{r\pm1}$ iff $\nu^{r\pm1}$ is validated along $\Lambda^{r\pm1}$. There are two cases:

Case 1: $r > 1$. By Lemma 7.3(ii) (Convergence and Correctness), $\nu_i^{r\pm1}$ is validated along $\Lambda^{r\pm1}$ iff $L(i,e) = 1$. But as $\nu^{r\pm1}$ is the principal derivative of κ^r along $\Lambda^{r\pm1}$, it follows from (2.4) that $\nu^{r\pm1}$ is validated along $\Lambda^{r\pm1}$ iff κ^r is validated along Λ^r. By Lemma 7.4 (Accuracy), κ^r is validated along Λ^r iff M_{κ^r} is true. Hence if M_{κ^r} is true then $L(i,e) = 1$ for cofinitely many i, and if M_{κ^r} is not true then $L(i,e) = 0$ for cofinitely many i. Thus R is satisfied.

Case 2: $r = 1$. First suppose that M_{κ^1} is true. For all $\sigma^1,\tau^1 \in T^1$, if $\sigma^1 \equiv \tau^1 \equiv \kappa^1$ then $M_{\sigma^1} = M_{\tau^1}$. Hence for all sufficiently long $\xi \subset \Lambda^0$, if $\xi \equiv \kappa^1$ then M_ξ is potentially true, so $\Delta(A_c;i,e) = 1$ for cofinitely many i.

Suppose that M_{κ^1} is not true. By Lemma 7.4 (Accuracy), $\nu = \nu^0$ is the initial derivative of κ^1 along Λ^0 and ν has finite outcome along Λ^0. By the last sentence of the paragraph preceding Case 1, $\nu_i = \nu_i^0$ has finite outcome along Λ^0. But then by the construction, $\Delta(A_c;i,e) = 0$ for cofinitely many i, and R is satisfied. ∎

Lemma 7.7 (2-Satisfaction Lemma): Every requirement of type 2 is satisfied.

Proof: Fix a requirement $R = R_{e,1,c}^{2,r}$ of type 2, and let $\Delta = \Delta_{1,c}^{2,r}$ be the functional for the requirement R as described at the beginning of this section. By Lemma 7.3(i) (Convergence and Correctness), $L(i,e) = \lim_{u_2}\ldots\lim_{u_{r\pm1}}\Delta(A_c;i,u_2,\ldots,u_{r\pm1},e)$ exists and takes a value in $\{0,1\}$ for all $i \in \mathbf{N}$.

By Lemma 5.17(i),(ii) (Assignment), R is assigned to a unique $\kappa^r \subset \Lambda^r$ such that κ^r is Λ^r-free and implication-free. Let $\nu^{r\pm1}$ be the principal derivative of κ^r along $\Lambda^{r\pm1}$, and fix $\delta^{r\pm1} \subset \Lambda^{r\pm1}$ such that $(\delta^{r\pm1})^- = \nu^{r\pm1}$. By Lemma 6.15(i) (Initial Control) and Definition 6.3, $\nu^{r\pm1}$ controls $\{wt(\delta^{r\pm1})\}\times\mathbf{N}^{r\pm1}\times\{e\}$ with initiator $\delta^{r\pm1}$ along $\Lambda^{r\pm1}$, and $\delta^{r\pm1}$ is also the initiator for $\{i\}\times\mathbf{N}^{r\pm1}\times\{e\}$ at $\delta^{r\pm1}$ for all $i \geq wt(\delta)$. Now if $i \geq wt(\delta^{r\pm1})$, then $\{i\}\times\mathbf{N}^{r\pm1}\times\{e\}$ is controlled along $\Lambda^{r\pm1}$ iff there is an initiator $\gamma^{r\pm1} \subset \Lambda^{r\pm1}$ for $\{i\}\times\mathbf{N}^{r\pm1}\times\{e\}$ such that there is no primary $\Lambda^{r\pm1}$-link $[\mu^{r\pm1},\pi^{r\pm1}]$ with $\mu^{r\pm1} \subset \gamma^{r\pm1} \subseteq \pi^{r\pm1}$; and if $\alpha^{r\pm1} \subset \Lambda^{r\pm1}$ and $(\alpha^{r\pm1})^- = \pi^{r\pm1}$, then $wt(\alpha^{r\pm1}) \leq i$. (By Lemma 2.2(iv) (Interaction), every primary $\Lambda^{r\pm1}$-link is $\nu^{r\pm1}$-correcting.) Also, if $\{i\}\times\mathbf{N}^{r\pm1}\times\{e\}$ is controlled along $\Lambda^{r\pm1}$, then the initiator for

$\{i\}\times\mathbf{N}^{r\pm1}\times\{e\}$ along $\Lambda^{r\pm1}$ is the longest such $\gamma^{r\pm1}$. As $\nu^{r\pm1}$ is $\Lambda^{r\pm1}$-free, $\delta^{r\pm1}$ is such a $\gamma^{r\pm1}$. Hence for all $i \geq wt(\delta^{r\pm1})$, $\{i\}\times\mathbf{N}^{r\pm1}\times\{e\}$ is controlled along $\Lambda^{r\pm1}$, and if $\{i\}\times\mathbf{N}^{r\pm1}\times\{e\}$ is controlled at any $\gamma^{r\pm1} \subset \Lambda^{r\pm1}$ with initiator $\delta_i^{r\pm1}$, then $\delta_i^{r\pm1} \supseteq \delta^{r\pm1}$. Fix such an i and let $\nu_i^{r\pm1}$ be the controller corresponding to the initiator $\delta_i^{r\pm1}$ for $\{i\}\times\mathbf{N}^{r\pm1}\times\{e\}$ at $\gamma^{r\pm1}$. As $\nu^{r\pm1}$ is $\Lambda^{r\pm1}$-free, it follows from Case 3 of Definition 6.3 that $\nu_i^{r\pm1} \supseteq \nu^{r\pm1}$.

If $up(\nu^{r\pm1}) = up(\nu_i^{r\pm1})$, then as $\nu^{r\pm1}$ is the principal derivative of κ^r along $\Lambda^{r\pm1}$ and $\nu_i^{r\pm1} \supseteq \nu^{r\pm1}$, it follows from (2.8) and (2.4) that $\nu_i^{r\pm1}$ has finite outcome along $\Lambda^{r\pm1}$ iff $\nu^{r\pm1}$ has finite outcome along $\Lambda^{r\pm1}$. And if $up(\nu^{r\pm1}) \neq up(\nu_i^{r\pm1})$, then we note that as $\nu^{r\pm1}$ is $\Lambda^{r\pm1}$-free, there is no primary $\Lambda^{r\pm1}$-link restraining $\nu^{r\pm1}$; hence by Lemma 6.13 (2-Similarity, with $\sigma^{r\pm1} = \nu^{r\pm1}$ and $\hat{\sigma}^{r\pm1} = \nu_i^{r\pm1}$), $\nu_i^{r\pm1}$ has finite outcome along $\Lambda^{r\pm1}$ iff $\nu^{r\pm1}$ has finite outcome along $\Lambda^{r\pm1}$. Thus for all $i \geq wt(\delta^{r\pm1})$, $\nu_i^{r\pm1}$ is validated along $\Lambda^{r\pm1}$ iff $\nu^{r\pm1}$ is validated along $\Lambda^{r\pm1}$. By Lemma 7.3(ii) (Convergence and Correctness), $\nu_i^{r\pm1}$ is validated along $\Lambda^{r\pm1}$ iff $L(i,e) = 1$. But as $\nu^{r\pm1}$ is the principal derivative of κ^r along $\Lambda^{r\pm1}$, it follows from (2.4) that $\nu^{r\pm1}$ is validated along $\Lambda^{r\pm1}$ iff κ^r is validated along Λ^r. By Lemma 7.4 (Accuracy), κ^r is validated along Λ^r iff M_{κ^r} is true. Hence if M_{κ^r} is true then $L(i,e) = 1$ for cofinitely many i, and if M_{κ^r} is not true then $L(i,e) = 0$ for cofinitely many i. Thus R is satisfied. n

Our main theorem is now immediate from the definition of the functionals $\Delta_{b,c}^{j,k}$, Lemmas 1.1, 2.1, Lemma 7.2 (Well-Definedness and Totality), and Lemmas 7.5-7.7 (j-Satisfaction for $j \leq 2$).

Theorem 7.8: Fix $m \in \mathbf{N}$, and let $\mathcal{P} = \langle P_0,\leq_0,P_1,\leq_1,f_1,\ldots,P_m,\leq_m,f_m\rangle$ be a finite m-jump poset such that P_0 has least element 0 and greatest element 1. Then there is a finite set $\mathbf{G}_0$ of r.e. degrees, and there are finite sets $\mathbf{G}_k = \{\mathbf{d}: \exists \mathbf{a} \in \mathbf{G}_0\ (\mathbf{a}^{(k)}\Box{=}\Box\mathbf{d})\}$ for each $k \in [1,m]$ such that the following diagram commutes.

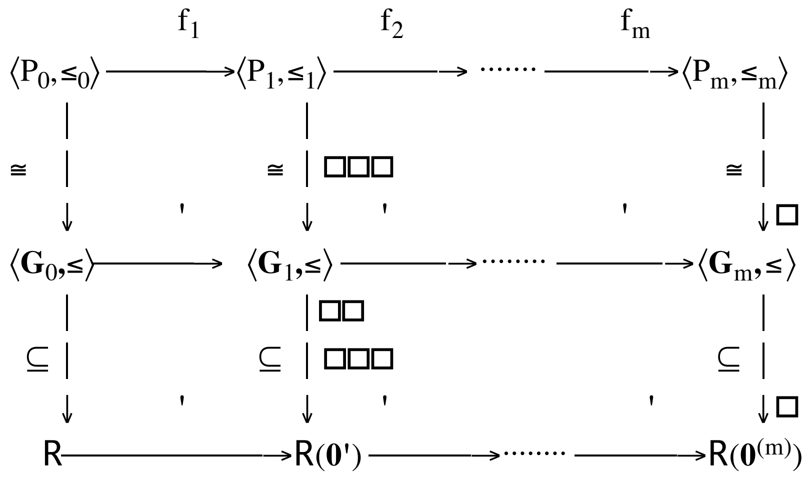

Figure 7.1

Furthermore, the embedding maps $0 \in P_0$ to $\mathbf{0}$ and $1 \in P_0$ to $\mathbf{0'}$. n

We have the following corollary, as proved in the introduction.

Corollary 7.9: The existential theory of $R^{(<\omega)} = \langle \mathbf{R},\mathbf{0},\mathbf{0'},\leq,\leq_1 \ldots,\leq_n,\ldots\rangle$ is decidable. n

If J is any recursively presented <ω-jump-poset, then we can modify our construction to embed J into $R^{(<\omega)}$. Requirements are listed as before, and form a recursive list. Each requirement has a well-defined dimension. We assign a given requirement to a tree of the correct dimension. As only finitely many trees will have been defined at any stage of the construction, and when a new tree T^{k+1} is needed, we assign the finitely many requirements already assigned to T^k and which need to be assigned to T^{k+1} in the same order that the requirements were assigned to T^k. All lemmas now can be proved as before. It is also not difficult to show that there is a countable universal recursively-presented <ω-jump poset. Hence:

Theorem 7.10: Let $P = \langle P_0,\leq_0,P_1,\leq_1,f_1,\ldots,P_m,\leq_m,f_m,\ldots\rangle$ be a countable <ω-jump poset such that P_0 has least element 0 and greatest element 1. Then for all m, $\langle P_m,\leq_m\rangle$ can be embedded isomorphically into $R[\mathbf{0}^{(m)},\mathbf{0}^{(m+1)}]$ so that Figure 7.1 commutes for all $m \in \mathbf{N}$. Furthermore, the embedding maps $0 \in P_0$ to $\mathbf{0}$ and $1 \in P_0$ to $\mathbf{0'}$. n

Slaman and Sui have noted that the methods of proof of Theorem 7.8 should work for <ω-jump usls in place of posets, and that we can add joins at all levels to our language and decide the corresponding ∃-theory if 1 is removed from the language. The construction need not be modified. The fact that the target sets are complements of prime ideals suffices to show that joins are preserved.

The methods presented in this paper will carry over to other priority arguments, if certain basic properties are satisfied. One can weaken the requirement assignment process to simultaneously assign requirements, and their derivatives, to the trees at all levels. Each requirement will have a basic module on each tree, which will be a segment of the tree of finite height. This assignment should provide the sentences generating action at each node of each tree. To study the interaction between requirements, an injury analysis similar to that provided by Lemma 2.2 (Interaction) is needed. A notion of control, different for each requirement, will be needed to determine how axioms are to be declared and elements placed into sets, and implication chains will be needed whenever a requirement needs to act off the true path. One can isolate a *guiding principle* for the definition of implication chains. Thus implication chains are to be built (and control relinquished) when there is a primary link which, if later switched, corrects any action for the requirement.

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