

5. Implication Chains. In order to coordinate the action of nodes working for the same (densely distributed) requirement of type 1 or 2 so that iterated limits will exist, we will have to force extensions of certain nodes to follow specified paths, so that we can form implication chains. This will allow us to show that the nodes work together to specify the same outcome for their axioms. Suppose that σ and $\hat{\sigma}$ are two such nodes. If σ and $\hat{\sigma}$ are incomparable, then the notion of *control* defined in Section 6 allows us to prevent the node which is off the true path from declaring too many axioms. So we restrict our attention in this section to the case where σ and $\hat{\sigma}$ are comparable. We try to arrange that, whenever possible, either both σ and $\hat{\sigma}$ are activated or both σ and $\hat{\sigma}$ are validated. (Such attempts begin on $T^{\dim(\sigma)-1}$, as the notion of control is used to coordinate action taken by the construction for this requirement at nodes on trees T^k for $k < \dim(\sigma)-1$, allowing us to verify the existence of iterated limits, except for the outermost iteration.) Whenever faced with a path along which this is not the case, we try to force an extension of paths which causes one of these two nodes to switch before declaring any new axioms. As we also want these nodes to act in accordance with the validity of the sentences which generate their action, we try to construct implication chains between nodes which yield implications either from M_σ to $M_{\hat{\sigma}}$ or from $M_{\hat{\sigma}}$ to M_σ (see Definitions 2.9 and 2.10). These implication chains are carried down to T^0 , where decisions on action can be made effectively, based on the truth of the sentences.

We now describe the construction of implication chains in more detail. Fix $\Lambda^0 \in [T^0]$ and assume that Λ^0 is the true path for the construction. For all $i \leq n$, let $\Lambda^i = \lambda^i(\Lambda^0)$. Suppose that we have $\sigma^r \subset \hat{\sigma}^r \subset \Lambda^r$ such that $r = \dim(\sigma^r)-1$, $\text{up}(\sigma^r) \neq \text{up}(\hat{\sigma}^r)$, and σ^r and $\hat{\sigma}^r$ are working for the same densely distributed requirement R . Then at most one of σ^r and $\hat{\sigma}^r$ will have all of its antiderivatives on T^i lying along Λ^i for all $i \in [r, n]$, but we will not be able to recursively identify if either of these nodes has this property, and if so, which one has the property. We may then be forced to define infinitely many axioms for R for derivatives of both σ^r and $\hat{\sigma}^r$. Such axioms have value determined by the prediction of the truth of certain sentences derived from the sentence assigned to R . However, we can only show that these predictions are correct, and hence that the proper value is specified, when all antiderivatives of the node lie on the true path. Thus the values produced by derivatives of $\hat{\sigma}^r$ and derivatives of σ^r may be different, preventing us from computing limits needed to satisfy R . We must therefore try to coordinate the actions taken for σ^r and $\hat{\sigma}^r$.

The same sentence, $M_{\text{up}(\sigma^r)}$, will be assigned to both $\text{up}(\sigma^r)$ and $\text{up}(\hat{\sigma}^r)$. The sentences M_{σ^r} and $M_{\hat{\sigma}^r}$ assigned to σ^r and $\hat{\sigma}^r$, respectively, will be obtained by bounding all quantifiers in the first quantifier block of $M_{\text{up}(\sigma^r)}$ by numbers $\text{wt}(\sigma^r)$ and $\text{wt}(\hat{\sigma}^r)$, respectively, where $\text{wt}(\sigma^r) < \text{wt}(\hat{\sigma}^r)$. If the quantifier block is a block of universal

quantifiers, then M_{σ^r} will formally imply $M_{\hat{\sigma}^r}$, and if it is a block of existential quantifiers, then $M_{\hat{\sigma}^r}$ will formally imply M_{σ^r} . Assume the latter, and so, that r is even, for concreteness.

The coordination problem arises when we reach τ^r such that $(\tau^r)^- = \hat{\sigma}^r$, and σ^r has finite outcome along τ^r iff $\hat{\sigma}^r$ has infinite outcome along τ^r . We briefly describe the attempt to coordinate action. There are two cases to consider, depending on whether σ^r has finite or infinite outcome along τ^r .

Case 1: First suppose that σ^r has finite outcome along τ^r , and so, that $\hat{\sigma}^r$ has infinite outcome along τ^r . Recall that $\sigma^r \subset \hat{\sigma}^r \subset \tau^r$. We will only need to follow this case if $\text{up}(\sigma^r) \subset \text{up}(\hat{\sigma}^r)$, so we assume that this latter condition holds. (If this is not the case, then we will be able to show that there are too few conflicting axioms to prevent the existence of iterated limits.) The sentences M_{σ^r} and $M_{\hat{\sigma}^r}$ assigned to σ^r and $\hat{\sigma}^r$ are obtained by bounding the leading unbounded quantifier block (a block of existential quantifiers) in $M_{\text{up}(\sigma^r)} = M_{\text{up}(\hat{\sigma}^r)}$ by numbers $\text{wt}(\sigma^r) < \text{wt}(\hat{\sigma}^r)$, respectively. As σ^r has finite outcome along τ^r , we are predicting that M_{σ^r} is false, so do not have a formal implication from the truth of M_{σ^r} to the truth of $M_{\hat{\sigma}^r}$. But if it were the case that $\text{wt}(\sigma^r) \geq \text{wt}(\hat{\sigma}^r)$ and as we are predicting that $M_{\hat{\sigma}^r}$ is true, $M_{\hat{\sigma}^r}$ would formally imply M_{σ^r} . We thus try to create an implication between sentences by replacing σ^r with a derivative $\tilde{\sigma}^r$ of $\text{up}(\sigma^r)$ which extends τ^r . (The process of obtaining $\tilde{\sigma}^r$ will require us to switch certain nodes which are the ends of primary links, or which caused other implication chains to be created. We may need to iterate this process down to T^0 , and nodes of T^1 which are switched will place elements into sets, so could injure the truth of the instance of $M_{\hat{\sigma}^r}$ on T^1 . We will be able to check to see if this is the case, and will show that it will be unnecessary to pass from τ^r to $\tilde{\sigma}^r$ in this situation, as the construction will resolve conflicts between axioms declared by derivatives of σ^r and axioms declared by derivatives of $\hat{\sigma}^r$ automatically. We will try to provide more intuition as to how this occurs later.) Suppose that we decide to extend τ^r to $\tilde{\sigma}^r$. (In this case, we say that τ^r *requires extension* for σ^r .) We now look at $\tilde{\tau}^r$ such that $(\tilde{\tau}^r)^- = \tilde{\sigma}^r$. If $\tilde{\sigma}^r$ has finite outcome along $\tilde{\tau}^r$, we proceed as in Case 2 below (with $\tilde{\sigma}^r$ in place of $\hat{\sigma}^r$ and $\hat{\sigma}^r$ in place of σ^r), where it is assumed that σ^r has infinite outcome along τ^r . If $\tilde{\sigma}^r$ has infinite outcome along $\tilde{\tau}^r$, then we will have switched the outcome of $\text{up}(\sigma^r)$, thus forcing $\text{up}(\hat{\sigma}^r)$ off the true path, and will have prevented derivatives of $\hat{\sigma}^r$ from defining any axioms which might prevent the computation of an iterated limit, as we have delayed the declaration of axioms by derivatives of $\hat{\sigma}^r$.

Case 2: Suppose that σ^r has infinite outcome along τ^r . We now have a formal implication from M_{σ^r} which seems to be true, to $M_{\hat{\sigma}^r}$ which seems to be false. (We will not

allow this to happen for $r = 0$.) If the immediate successor of σ^r along τ^r does not require extension, then we call $\hat{\sigma}^r$ a *pseudocompletion* of σ^r . We form an r -implication chain $\langle\langle\sigma^r, \hat{\sigma}^r, \tau^r\rangle\rangle$, to try to resolve this discrepancy on T^{r-1} . This discrepancy is first observed at $\tilde{\tau}^{r-1} = \text{out}(\tau^r)$ along Λ^{r-1} . We will then have $\tilde{\sigma}^{r-1} \subset \hat{\sigma}^{r-1} \subset \tilde{\tau}^{r-1}$ such that $\tilde{\sigma}^{r-1}$ and $\hat{\sigma}^{r-1}$ are, respectively, the principal derivatives of σ^r and $\hat{\sigma}^r$ along $\tilde{\tau}^{r-1}$, and $(\tilde{\tau}^{r-1})^- = \hat{\sigma}^{r-1}$. Furthermore, $\tilde{\sigma}^{r-1}$ has finite outcome along $\tilde{\tau}^{r-1}$ and $\hat{\sigma}^{r-1}$ has infinite outcome $\hat{\tau}^{r-1}$. We now have the situation for $r-1$ which we discussed in Case 1 for r . If $\tilde{\tau}^{r-1}$ requires extension for $\tilde{\sigma}^{r-1}$, then the r -implication chain $\langle\langle\sigma^r, \hat{\sigma}^r, \tau^r\rangle\rangle$ will be called *amenable*, and we will either be able to extend our implications between sentences to level $r-1$ and build an amenable $(r-1)$ -implication chain, or will switch paths as described above.

Once we have an $(r-1)$ -implication chain, we repeat this process. There are three possibilities. Either we eventually switch σ^r , thus removing $\hat{\sigma}^r$ from the current path. Or we switch $\hat{\sigma}^r$ (this can occur when we try to build a j -implication chain for j even), thus resolving the conflict by forcing derivatives of σ^r and $\hat{\sigma}^r$ to define axioms with identical outputs (no axioms are defined by $\hat{\sigma}^r$ while we are resolving the conflict), or we reach T^0 and do not allow the construction of a 0-implication chain. We show that the action of the construction on T^0 is still in accordance with the potential truth of the sentences described.

The process of defining implication chains requires us to define several notions by simultaneous induction on $\text{lh}(\eta)$ for $\eta \in T^0$. We begin by defining η^k *requires extension for* v^k , where $\eta^k = \lambda^k(\eta)$. (In Case 1 of our intuitive remarks, η^k corresponds to τ^r and v^k to σ^r .) When η^k *requires extension for* v^k , then either $k = \dim(v^k)-1$ and we will be beginning an attempt to construct a k -implication chain, or $k < \dim(v^k)-1$ and we will be attempting to extend a $(k+1)$ -implication chain which has been defined by the time η^k is reached, to a k -implication chain. If η^k *requires extension for* v^k , then we will begin a process of defining the k -completion of η^k for v^k . (The k -completion will correspond to the node $\tilde{\sigma}^r$ when $k = r$ in Case 1). We may need to switch nodes while constructing a k -completion, and may thereby discover a new node which requires extension, and so wants to find a j -completion. In order to resolve potential conflicts about which completion to pursue, we stipulate that we obtain the j -completion of the new node before continuing with the process of finding the k -completion of the original node. (We will show that this process is finitary.) Nodes which are first encountered during the process of finding a k -completion will not be *implication-free*, and so will not be allowed to control the declaration of axioms. The decision as to whether η^k requires extension for v^k will depend on the elements in $\text{PL}(\text{up}(v^k), \lambda(\eta^k))$, a set of ends of primary links along $\lambda(\eta^k)$ which restrain $\text{up}(v^k)$, and nodes extending $\text{up}(v^k)$ which caused implication chains to be created.

These are nodes which will have to be switched in order to obtain the k -completion of η^k , and in the iterative process of finding a 0-implication chain, could place elements into sets which might destroy the truth of the instance M_{σ^r} of the sentence whose truth at a given stage caused us to try to construct the implication chain. (We note that this can only occur for requirements of type 1.) Should such a destruction occur, then η^k will not require extension for v^k ; we will show that if v^k really is on the true path for the construction, then any way of returning v^k to the true path will cause such a destruction, and that this destruction will also allow us to correct axioms. The *amenable* implication chains are those which give rise to sets $PL(\text{up}(v^k), \lambda(\eta^k))$ for which no such destruction will occur.

There are five conditions which must be satisfied in order for a node to *require extension*. Fix nodes $v^k \subset \delta^k \subset \eta^k$ (the nodes corresponding to σ^r , $\hat{\sigma}^r$ and τ^r , respectively, in Case 1 of our intuitive remarks), and let ξ^k be the immediate successor of v^k along η^k . Condition (5.1) requires that, if $k = r$, then for all $i \leq k$, the principal derivatives of v^k along $\text{out}^i(\xi^k)$ and δ^k along $\text{out}^i(\eta^k)$ are *implication-free* (see Definition 5.7). This will correspond to assuming that all action to find j -completions for $j \geq k$ which was started before $\text{out}^0(\eta^k)$ has been completed, so we are free to try to resolve the current conflict between sentences. (If $k < r$, then we must try to build completions even when a node is not implication-free as part of the process for finding completions for other nodes.) We also require that $\text{out}^0(\xi^k)$ is pseudotrue; should this condition fail, then ξ^k will not be allowed to define axioms. Condition (5.2) implies that two nodes disagree about the value to be assigned to a newly declared axiom, but there is no implication between the sentences. This corresponds to Case 1 of our intuitive remarks, and v^k in (5.2) corresponds to σ^r in Case 1. By (5.2) and Lemma 4.3(i)(a) (Link Analysis), condition (5.3) will imply that $\text{up}(v^k) \subset \text{up}(\delta^k)$; and the failure of (5.3) will imply that $\text{up}(v^k)$ and $\text{up}(\delta^k)$ are incomparable, so by (2.6), no derivatives of $\text{up}(v^k)$ can extend δ^k . In the latter case, it is impossible to carry out the extension process needed to find a completion. (5.4) is the condition which determines if any node which must be switched during the iteration process for finding completions will place elements into the restraint set for the sentence whose apparent truth caused us to want to act; the condition requires that such nodes do not exist. This will always be the case for requirements of type 2, so (5.4) only applies to requirements of type 1. (5.5)(i) is the condition required to start building an r -implication chain, and condition (5.5)(ii) describes the situation which arises in extending a $(k+1)$ -implication chain to a k -implication chain.

Definition 5.1: Suppose that $k \leq r < n$ and $v^k \subset \xi^k \subseteq \delta^k \subset \eta^k \in T^k$ are given such that $(\eta^k)^- = \delta^k$, $(\xi^k)^- = v^k$, and $r = \dim(v^k) - 1$. We say that η^k *requires extension for v^k* if v^k is the shortest node for which the following conditions hold:

- (5.1) If $k = r$, then for all $i \leq r$, the principal derivatives of v^r along $\text{out}^i(\xi^r)$ and δ^r along $\text{out}^i(\eta^r)$ are implication-free (see Definition 5.7), and $\text{out}^0(\xi^r)$ is pseudotrue (see Definition 5.9).
- (5.2) $\text{tp}(v^k) \in \{1, 2\}$, $v^k \equiv \delta^k$, $\text{up}(\delta^k) \neq \text{up}(v^k)$, δ^k has infinite outcome along η^k , v^k is the principal derivative of $\text{up}(v^k)$ along η^k , and v^k has finite outcome along η^k (so v^k is the initial derivative of $\text{up}(v^k)$ along η^k).
- (5.3) There is no primary η^k -link which restrains v^k .
- (5.4) If $k = r$ and $\text{tp}(\delta^k) = 1$, then for every $\pi^{k+1} \in \text{PL}(\text{up}(v^k), \lambda(\eta^k))$, $\text{TS}(\pi^{k+1}) \cap \text{RS}(\delta^k) = \emptyset$ (see Definition 5.3 for the definition of PL sets).
- (5.5) One of the following conditions holds:
- (i) $r = k$.
 - (ii) There is an amenable $(k+1)$ -implication chain $\langle \langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle : r \geq j \geq k+1 \rangle$ along $\lambda(\eta^k)$ such that $\eta^k = \text{out}(\tau^{k+1})$, and δ^k (v^k , resp.) is the principal derivative of $\hat{\sigma}^{k+1}$ (σ^{k+1} , resp.) along η^k . (See Definitions 5.4 and 5.2 for the definitions of *amenable* and *implication chain*.)

We say that η^k *requires extension* if η^k requires extension for some v^k . \square

Implication chains keep track of the implications between sentences for a requirement. The first and second coordinates of the triple at a given level of the implication chain determine the nodes which are potentially responsible for defining axioms for the requirement. The third coordinate keeps track of the conflicting outcomes of the first and second coordinates. The k -implication chain follows the implications of sentences from the starting level, T^r , down to T^k . The conditions mentioned in Definition 5.2 are described in the motivation at the beginning of the section. In addition, we require the principal derivatives of σ^r along $\text{out}^i(\bar{\tau}^r)$ and $\hat{\sigma}^r$ along $\text{out}^i(\tau^r)$ to be implication-free for all $i \leq r$ (condition (5.10)). This will correspond to assuming that all action to find j -completions for $j \geq k$ which was started before $\text{out}^i(\bar{\tau}^r)$ or $\text{out}^i(\tau^r)$ was completed before that node is reached, so we are free to try to resolve the current conflict between sentences. If $k < r$, then we have already begun building the implication chain, and must continue to extend it within other implication chains; thus the principal derivatives of σ^k and $\hat{\sigma}^k$ along $\text{out}^i(\tau^k)$ need not be implication-free. (We note that condition (5.6) below allows $\text{up}(\sigma^r) \upharpoonright \text{up}(\hat{\sigma}^r)$.)

We will also need to describe the situation when the first triple of an implication

chain can be formed by taking an immediate extension of a node $\hat{\sigma}^r$ in the absence of a requires extension configuration; such a $\hat{\sigma}^r$ will be called a *pseudocompletion*.

Definition 5.2: Fix $k \leq r \leq n$. A *k-implication chain* is a sequence $\langle\langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle\rangle: r \geq j \geq k$ such that:

$$(5.6) \quad \sigma^r \equiv \hat{\sigma}^r \text{ and } \text{up}(\sigma^r) \neq \text{up}(\hat{\sigma}^r).$$

$$(5.7) \quad \text{tp}(\sigma^r) \in \{1, 2\}, \dim(\sigma^r) = r+1.$$

$$(5.8) \quad \begin{array}{l} \text{(i)} \quad \sigma^k \subset \hat{\sigma}^k. \\ \text{(ii)} \quad \hat{\sigma}^k = (\tau^k)^- \subset \tau^k. \end{array}$$

$$(5.9) \quad \text{If } k < r, \text{ then } \text{up}(\sigma^k) = \hat{\sigma}^{k+1} \text{ and } \text{up}(\hat{\sigma}^k) = \sigma^{k+1}.$$

$$(5.10) \quad \begin{array}{l} \text{(i)} \quad \text{Fix } \bar{\tau}^r \subseteq \hat{\sigma}^r \text{ such that } (\bar{\tau}^r)^- = \sigma^r. \text{ Then for all } i \leq r, \text{ the principal derivative of } \\ \sigma^r \text{ along } \text{out}^i(\bar{\tau}^r) \text{ is implication-free (see Definition 5.7), and } \hat{\sigma}^r \text{ is implication-} \\ \text{free.} \\ \text{(ii)} \quad \text{For all } i \leq r, \text{ the principal derivative of } \hat{\sigma}^r \text{ along } \text{out}^i(\tau^r) \text{ is implication-free.} \end{array}$$

$$(5.11) \quad \begin{array}{l} \text{(i)} \quad \sigma^k \text{ has infinite outcome along } \hat{\sigma}^k. \\ \text{(ii)} \quad \hat{\sigma}^k \text{ has finite outcome along } \tau^k. \end{array}$$

$$(5.12) \quad \text{If } k < r, \text{ then } \langle\langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle\rangle: r \geq j \geq k+1 \text{ is a } (k+1)\text{-implication chain along } \tau^{k+1} \text{ and } \text{out}(\tau^{k+1}) \subset \tau^k.$$

We say that this implication chain is *along* $\rho^k \in T^k$ ($\Lambda^k \in [T^k]$, resp.) if $\tau^k \subseteq \rho^k$ ($\tau^k \subset \Lambda^k$, resp.).

Suppose that $k = r$, conditions (5.6), (5.7), (5.8)(i), (5.9), (5.10)(i), and (5.11)(i) hold, and $\hat{\sigma}^r$ is an initial derivative. In this case, we call $\hat{\sigma}^r$ a *pseudocompletion* of σ^r . $\hat{\sigma}^r$ is a *pseudocompletion* if it is a pseudocompletion of some node. \square

The process of building a new r -implication chain, or of extending a $(k+1)$ -implication chain to a k -implication chain, will require us to build completions. We will define PL sets, which keep track of the antiderivatives of those nodes of T^1 which will eventually have to be switched (and might thereby injure restraint sets), should we need to pull the implication chains down to T^0 during the process of building completions. Consider the situation wherein a node of T^k requires extension. Thus assume that we have

$v^k \subset \delta^k \subset \eta^k \in T^k$ such that $(\eta^k)^- = \delta^k$ and η^k requires extension for v^k . We wish to construct a $\kappa^k \supset \eta^k$ such that $\text{up}(\kappa^k) = \text{up}(v^k) = v^{k+1}$. By (2.10), this requires taking extensions of η^k with the goal of making v^{k+1} a $\lambda(\kappa^k)$ -free node. Thus we must eliminate the links which restrain v^{k+1} .

Let $\eta^u = \lambda^u(\eta^k)$ for all $u \in [k, n]$. We will show later that, in this situation, there is an η^{k+1} -link which restrains v^{k+1} and $v^{k+1} \subset \delta^{k+1} = \text{up}(\delta^k)$. By Lemma 4.1 (Nesting), there will be an η^{k+1} -link $[\mu^{k+1}, \pi^{k+1}]$ which restrains v^{k+1} and contains all η^{k+1} -links which restrain v^{k+1} , and π^{k+1} will be η^{k+1} -free. By (2.10), we must eliminate this link in order to make v^{k+1} free; this is done as follows. Let $[\mu^{k+1}, \pi^{k+1}]$ be derived from the primary η^j -link $[\mu^j, \pi^j]$ (we allow $j = k+1$). By Lemma 3.5 (Nonswitching Extension) and since all blocks defined in Section 2 are finite, we will be able to find a nonswitching extension $\hat{\eta}^k$ of η^k such that $\text{up}^j(\hat{\eta}^k) = \pi^j$ and $\hat{\eta}^k$ is an initial derivative of $\text{up}^{j \pm 1}(\hat{\eta}^k) \supset \eta^{j \pm 1}$. By Lemma 3.6 (Switching), we can find $\hat{\eta}^k$ such that $(\hat{\eta}^k)^- = \tilde{\eta}^k$, $\lambda^i(\hat{\eta}^k) \supset \lambda^i(\tilde{\eta}^k)$ for all $i < j$, and $\hat{\eta}^k$ switches π^j . $[\mu^{k+1}, \pi^{k+1}]$ will not be a $\lambda^{k+1}(\hat{\eta}^k)$ -link, and every $\lambda^{k+1}(\hat{\eta}^k)$ -link which restrains v^{k+1} will be properly contained in the interval $[\mu^{k+1}, \pi^{k+1}]$. Hence barring other considerations, we can repeat this process for the longest $\lambda^{k+1}(\hat{\eta}^k)$ -link which restrains v^{k+1} , and eventually find a new derivative κ^k of v^{k+1} on T^k . (There may be additional considerations, but for this paragraph, assume that there are none.) This procedure will be induced by taking extensions of nodes on T^0 which will be nonswitching except when needed to switch one of the above nodes ending a link. $\text{out}^0(\kappa^k)$ will act according to the validity of its sentence unless $k = 0$, in which case we force κ^0 to have infinite outcome, and show that this action is in accordance with the validity of the sentence assigned to κ^0 . If $k > 0$ and the action of $\text{out}^0(\kappa^k)$ produces an immediate successor $\bar{\eta}$ of $\text{out}^0(\kappa^k)$ such that κ^k has infinite outcome along $\lambda^k(\bar{\eta})$, then the process halts since we will then have switched v^{k+1} , so will have forced δ^{k+1} not to lie along Λ^{k+1} . Otherwise, we will have constructed a k -implication chain, and $\lambda^{k+1}(\bar{\eta})$ will require extension, so we can repeat this process. π^j is placed in $\text{PL}(v^{k+1}, \eta^{k+1})$ via (5.13) whenever $j = k+1$, i.e., whenever $[\mu^{k+1}, \pi^{k+1}]$ is a primary η^{k+1} -link. Each such π^j will be the last node of a primary η^{k+1} -link which restrains v^{k+1} . The nodes in $\text{PL}(v^{k+1}, \eta^{k+1})$ are those which cause a small element to be placed in a set when we carry out the backtracking process for $k = 0$, and may thereby injure the oracle of the computation which has generated the implication chain. We will check to see, for all nodes in $\text{PL}(v^{k+1}, \eta^{k+1})$, whether this action causes an element to be placed into this oracle. If not, then the implication chains constructed during this process are called *amenable* (see Definition 5.4). The derivative operation will provide a one-one correspondence between $\text{PL}(v^{k+1}, \eta^{k+1})$ and $\text{PL}(\delta^k, \lambda^k(\bar{\eta}))$, so it will suffice to consider only the nodes in PL sets.

There are additional considerations which we need to take into account. Our proof requires that we follow the backtracking process for a node whenever that node requires

extension. In the preceding paragraph, $\lambda^{j\pm 1}(\hat{\eta}^k)$ will have infinite outcome along $\lambda^{j\pm 1}(\hat{\eta}^k)$. It is thus possible that $\lambda^{j\pm 1}(\hat{\eta}^k)$ will require extension for some $\gamma^{j\pm 1}$. Furthermore, it is possible that for such a $\gamma^{j\pm 1}$, if $\gamma^j = \text{up}(\gamma^{j\pm 1})$, then there is a $\lambda^j(\hat{\eta}^k)$ -link which restrains γ^j , but no $\lambda^{k+1}(\hat{\eta}^k)$ -link derived from this $\lambda^j(\hat{\eta}^k)$ -link restrains v^{k+1} , so this situation is not covered by (5.13).

Suppose that $\lambda^{j\pm 1}(\hat{\eta}^k)$ requires extension for $\gamma^{j\pm 1}$. By (5.1) and since $\lambda^{j\pm 1}(\hat{\eta}^k)$ is implication-restrained, $\dim(\gamma^j) > j$. In order to make our construction cohere, we must perform the backtracking process for $\lambda^{j\pm 1}(\hat{\eta}^k)$ (which entails removing *all* links around γ^j) before proceeding as in the preceding paragraph for the next link which restrains v^{k+1} . This may require us to switch additional primary links, say $[\rho^t, \hat{\tau}^t]$ on T^t for $t \geq j$, with $v^{k+1} \subset \text{out}^{k+1}(\rho^t) \subset \delta^{k+1}$. Also, once we have found a new derivative $\tau^{j\pm 1}$ of γ^j , we must force it to have infinite outcome along its immediate extension in order to preclude the existence of a $(j-1)$ -implication chain along the true path. In either case, we have to switch nodes on T^{k+1} if $t = k+1$ or $j = k+1$, respectively, until we complete the backtracking process for $\lambda^{j\pm 1}(\hat{\eta}^k)$, i.e., until we reach the primary completion of $\lambda^{j\pm 1}(\hat{\eta}^k)$. For the first case, we put all nodes $\tau^{k+1} \in \text{PL}(\gamma^{k+1}, \eta^{k+1})$ into $\text{PL}(v^{k+1}, \eta^{k+1})$ via (5.14)(ii), as these nodes have to be switched in order to backtrack $\lambda(\hat{\eta}^k)$, and call $\text{PL}(\gamma^{k+1}, \eta^{k+1})$ a *component* of $\text{PL}(v^{k+1}, \eta^{k+1})$. For the second case, we put γ^{k+1} into $\text{PL}(v^{k+1}, \eta^{k+1})$ via (5.14)(i).

In the preceding paragraphs, we have tried to motivate the definition of $\text{PL}(v^{k+1}, \eta^{k+1})$ by looking ahead to some $\hat{\eta}^k \supset \text{out}(\eta^{k+1})$, and seeing which nodes $\subseteq \eta^{k+1}$ need to be switched in order to carry out the backtracking process beginning at $\hat{\eta}^k$. However, in the definition of PL sets below, we will want to inductively describe this set in advance, as we pass from v^{k+1} to η^{k+1} , in anticipation of later finding $\hat{\eta}^k$ and having to carry out the corresponding backtracking process. When we wanted to place an element τ^{k+1} into $\text{PL}(v^{k+1}, \eta^{k+1})$ through (5.13), it was the case that τ^{k+1} was the end of a primary η^{k+1} -link restraining v^{k+1} , so these nodes are readily identified in advance. We will show that the other case, described in the preceding paragraph and specified in (5.14), corresponds precisely to a reversal of a backtracking process beginning at a node δ^{k+1} which requires extension for some $\mu^{k+1} \subset v^{k+1}$ with $v^{k+1} \subset (\delta^{k+1})^-$, and so we can again identify these nodes in advance. ($(\delta^{k+1})^-$ will be the γ^{k+1} of the preceding paragraph.) Once we complete the backtracking process for δ^{k+1} , i.e., once we find a primary completion κ^{k+1} of δ^{k+1} , the component corresponding to action for δ^{k+1} does not place elements $\supset \kappa^{k+1}$ into $\text{PL}(v^{k+1}, \eta^{k+1})$. Thus the node ξ^{k+1} in (5.14) (for $j = k+1$) must satisfy $\xi^{k+1} \subseteq \kappa^{k+1}$.

The *backtracking process* is induced by the process described above, starting at $\text{out}^0(\eta^k)$ and ending at $\text{out}^0(\kappa^k)$. Thus we begin at $\text{out}^0(\eta^k)$, and proceed as described above by taking extensions on T^0 which are never j -switching for any $j \leq k$, until we reach a node $\kappa^0 \supset \text{out}^0(\eta^k)$ which has the properties of $\text{out}^0(\kappa^k)$. As activated and validated outcomes are unique on T^0 , there will be a unique way to carry out the backtracking

process, as long as we decide to follow activated outcomes of nodes when not otherwise specified. Assume that κ^k has been defined in this way. For all $i \leq k$, $\text{out}^i(\kappa^k)$ will be called the i -completion of η^k for v^k , and will be defined in Definition 5.6. In the definition of the PL sets, which we now present, it would be helpful for the reader to think of j as the $k+1$ of the preceding remarks. The definition is an inductive definition, proceeding by induction on $n-j$ and then by induction on $\text{lh}(\eta^j)$ - $\text{lh}(v^j)$.

Definition 5.3: Fix $j < n$ and $v^j \subset \eta^j \in T^j$. We place $\tau^j \subset \eta^j$ into $\text{PL}(v^j, \eta^j)$ if one of the following conditions holds:

(5.13) There is a μ^j such that $\mu^j \subseteq v^j \subset \tau^j$ and $[\mu^j, \tau^j]$ is a primary η^j -link.

(5.14) There are μ^j , δ^j , and ξ^j such that $\mu^j \subset v^j \subset (\delta^j)^- \subset \delta^j \subseteq \xi^j \subseteq \eta^j$, δ^j requires extension for μ^j and has no j -completion with infinite outcome along ξ^j , and either:

- (i) $\tau^j = (\delta^j)^-$; or
- (ii) $\tau^j \in \text{PL}((\delta^j)^-, \xi^j)$.

If nodes satisfying the hypotheses of (5.14) exist, then we call $\text{PL}((\delta^j)^-, \xi^j)$ a *component* of $\text{PL}(v^j, \eta^j)$. \square

Lemma 5.1 (PL Analysis Lemma): Fix $j \leq n$ and $v^j \subseteq \bar{\rho}^j \subset \sigma^j \subseteq \eta^j \in T^j$ such that $(\sigma^j)^- = \bar{\rho}^j$. Then:

- (i) $\text{PL}(v^j, \bar{\rho}^j) \subseteq \text{PL}(v^j, \sigma^j)$.
- (ii) $\text{PL}(v^j, \sigma^j) \setminus \text{PL}(v^j, \bar{\rho}^j) \subseteq \{\bar{\rho}^j\}$.
- (iii) If $\text{PL}(v^j, \sigma^j) \setminus \text{PL}(v^j, \bar{\rho}^j) \neq \emptyset$, then either $\text{PL}(v^j, \sigma^j) \setminus \text{PL}(v^j, \bar{\rho}^j) = \{\bar{\rho}^j\}$ and $\bar{\rho}^j$ is the last node of a primary σ^j -link, or σ^j requires extension.
- (iv) If $\bar{\rho}^j$ has finite outcome along σ^j then $\text{PL}(v^j, \sigma^j) = \text{PL}(v^j, \bar{\rho}^j)$.
- (v) If $\bar{\rho}^j$ is η^j -free and for every δ^j and μ^j such that δ^j requires extension for μ^j and $\mu^j \subset v^j \subset (\delta^j)^- \subset \delta^j \subseteq \eta^j$, it is the case that there is a $\kappa^j \subset \eta^j$ such that κ^j is the j -completion of δ^j and κ^j has infinite outcome along η^j , then $\text{PL}(v^j, \sigma^j) = \text{PL}(v^j, \eta^j)$.
- (vi) If $\xi^j \subseteq \eta^j$ and $\text{PL}(\bar{\rho}^j, \xi^j)$ is a component of $\text{PL}(v^j, \eta^j)$, then $\text{PL}(\bar{\rho}^j, \xi^j) \cup \{\bar{\rho}^j\} \subseteq \text{PL}(v^j, \eta^j)$.
- (vii) If $\text{PL}(\bar{\rho}^j, \eta^j)$ is a component of $\text{PL}(v^j, \eta^j)$ and $(\eta^j)^- \in \text{PL}(v^j, \eta^j)$, then $(\eta^j)^- \in \text{PL}(\bar{\rho}^j, \eta^j)$ or $(\eta^j)^- = \bar{\rho}^j$.
- (viii) If every $\delta^j \subseteq \bar{\rho}^j$ which requires extension has a j -completion $\subseteq \bar{\rho}^j$ with infinite outcome along σ^j , then $\text{PL}(v^j, \eta^j) \subseteq \text{PL}(v^j, \bar{\rho}^j) \cup \text{PL}(\bar{\rho}^j, \eta^j) \cup \{\bar{\rho}^j\}$.

- (ix) Given $\tilde{\rho}^j$ such that $PL(\tilde{\rho}^j, \eta^j)$ is a component of $PL(\bar{\rho}^j, \eta^j)$ and $PL(\bar{\rho}^j, \eta^j)$ is a component of $PL(v^j, \eta^j)$, then $PL(\tilde{\rho}^j, \eta^j) \cup \{\tilde{\rho}^j\} \subseteq PL(v^j, \eta^j)$.

Proof: (i): By definition.

(ii),(iii): Any primary σ^j -link which is not a primary $\bar{\rho}^j$ -link has $\bar{\rho}^j$ as its last element. And new components can first appear at σ^j only if σ^j requires extension. Hence (iii) holds. (ii) now follows from (5.13), (5.14)(i), and induction on $lh(\eta^j)$ - $lh(v^j)$ for (5.14)(ii).

(iv): If $\bar{\rho}^j$ has finite outcome along σ^j , then $\bar{\rho}^j$ is not the last element of a primary σ^j -link. By (5.2), if σ^j requires extension, then $\bar{\rho}^j$ has infinite outcome along σ^j . (iv) now follows from (i) and (iii).

(v): As $\bar{\rho}^j$ is η^j -free, it follows from (4.1) that the primary η^j -links which restrain v^j coincide with the primary σ^j -links restraining v^j . Hence all nodes placed in $PL(v^j, \eta^j)$ via (5.13) are already in $PL(v^j, \sigma^j)$. Suppose that $v^j \subset (\delta^j)^- \subset \delta^j \subseteq \eta^j$ and δ^j requires extension for $\mu^j \subset v^j$. By the hypothesis of (v), there is a κ^j such that $[\mu^j, \kappa^j]$ is a primary η^j -link which restrains v^j . As $\bar{\rho}^j$ is η^j -free and $v^j \subseteq \bar{\rho}^j$, $\kappa^j \subseteq \bar{\rho}^j = (\sigma^j)^-$. Hence by (5.14), all elements placed in $PL(v^j, \eta^j)$ via (5.14) are already in $PL(v^j, \sigma^j)$, so (v) follows.

(vi): Immediate from (5.14).

(vii): We proceed by induction on $lh(\bar{\rho}^j)$ - $lh(v^j)$. By definition, if $PL(\bar{\rho}^j, \eta^j)$ is a component of $PL(v^j, \eta^j)$, then $v^j \subset \bar{\rho}^j$ and there is a $\bar{\mu}^j \subset v^j$ such that σ^j requires extension for $\bar{\mu}^j$. Hence if $(\eta^j)^-$ enters $PL(v^j, \eta^j)$ via (5.13), then the corresponding primary link $[\mu^j, (\eta^j)^-]$ also restrains $\bar{\rho}^j$. Thus $(\eta^j)^- \in PL(\bar{\rho}^j, \eta^j)$ as desired.

Suppose that $(\eta^j)^-$ enters $PL(v^j, \eta^j)$ via (5.14). Then there are $\delta^j \subset \tau^j \subseteq \eta^j$ such that $v^j \subset \delta^j = (\tau^j)^-$, τ^j requires extension for some $\mu^j \subset v^j$, $PL(\delta^j, \eta^j)$ is a component of $PL(v^j, \eta^j)$, and either $(\eta^j)^- = \delta^j$ or $(\eta^j)^- \in PL(\delta^j, \eta^j)$. If $\bar{\rho}^j \subset \delta^j$, then as $\mu^j \subset v^j \subset \bar{\rho}^j$, $PL(\delta^j, \eta^j)$ is a component of $PL(\bar{\rho}^j, \eta^j)$, so (vii) follows from (5.14). If $\bar{\rho}^j = \delta^j$, then (vii) is immediate. Otherwise, as $\delta^j, \bar{\rho}^j \subset \eta^j$, it follows that $\delta^j \subset \bar{\rho}^j \subset \eta^j$; hence as $\bar{\mu}^j \subset v^j \subset \delta^j$, $PL(\bar{\rho}^j, \eta^j)$ is a component of $PL(\delta^j, \eta^j)$. Now $lh(\bar{\rho}^j)$ - $lh(\delta^j) < lh(\bar{\rho}^j)$ - $lh(v^j)$ and $(\eta^j)^- \neq \delta^j$, so by induction, $(\eta^j)^- \in PL(\delta^j, \eta^j)$. But $PL(\delta^j, \eta^j)$ is a component of $PL(v^j, \eta^j)$ and $lh(\delta^j)$ - $lh(v^j) < lh(\bar{\rho}^j)$ - $lh(v^j)$, so by induction, either $(\eta^j)^- = \bar{\rho}^j$ or $(\eta^j)^- \in PL(\bar{\rho}^j, \eta^j)$.

(viii): Suppose that $\tau^j \in PL(v^j, \eta^j)$. First assume that (5.13) holds for τ^j . Then there is a $\mu^j \subset \tau^j$ such that $[\mu^j, \tau^j]$ is a primary η^j -link restraining v^j . If $\tau^j \subset \bar{\rho}^j$, then τ^j is placed into $PL(v^j, \bar{\rho}^j)$ by (5.13). And if $\tau^j \supset \bar{\rho}^j$, then τ^j is placed into $PL(\bar{\rho}^j, \eta^j)$ by (5.13).

Next assume that τ^j is placed into $PL(v^j, \eta^j)$ by (5.14), because of the component $PL(\delta^j, \xi^j)$ associated with some $\delta^j \supset v^j$ which requires extension for some $\mu^j \subset v^j$. If $\delta^j \subseteq \bar{\rho}^j$, then by hypothesis, δ^j has a j -completion $\kappa^j \subseteq \bar{\rho}^j$ which has infinite outcome along $\bar{\rho}^j$, so by the properties of ξ^j in (5.14), $\tau^j \subset \xi^j \subseteq \bar{\rho}^j$. Hence either $\tau^j = \bar{\rho}^j$, or τ^j is placed into $PL(v^j, \bar{\rho}^j)$ by (5.14). Otherwise, $\bar{\rho}^j \subset \delta^j$. As $\mu^j \subset v^j \subseteq \bar{\rho}^j$, $PL(\delta^j, \xi^j)$ is a component of

$PL(\bar{\rho}^j, \eta^j)$, so τ^j is placed into $PL(\bar{\rho}^j, \eta^j)$ by (5.14).

(ix): Immediate from (vi). \square

As mentioned earlier, the process of extending a k -implication chain to a 0 -implication chain may injure the validity of a sentence whose truth we are trying to preserve. When this happens, we will not act to extend the k -implication chain. Our next definition allows us to differentiate between the k -implication chains which we want to extend (the *amenable* implication chains), and those which we do not want to extend (the *nonamenable* implication chains). Condition (5.15) applies when $up(\hat{\sigma}^r)$ has an initial derivative $\subset \sigma^r$, specifying that in this case, when we first observe the $(k+1)$ -implication chain along a path of T^{k+1} generated by a node on T^k , then we have a configuration of nodes on T^k which gives rise to a *requires extension* situation, so condition (5.4) will be applicable. Condition (5.16) imposes a restriction similar to that imposed by (5.4) when $up(\hat{\sigma}^r)$ does not have an initial derivative $\subset \sigma^r$. This restriction requires the ability to preserve certain computations while the backtracking process is carried out. (Note that at the beginning level r for an implication chain, it is possible to have an implication chain which arises without a *requires extension* situation, if, for example, $\hat{\sigma}^r$ is an initial derivative.) We will show later that similar restrictions are automatically carried down to lower levels. A similar restriction needs to apply to separate the pseudocompletions which potentially give rise to amenable implication chains from those which do not. Thus we also define amenable pseudocompletions.

Definition 5.4: Suppose that $k = r$ and that $\hat{\sigma}^r$ is a pseudocompletion of σ^r . We say that $\hat{\sigma}^r$ is an *amenable pseudocompletion* of σ^r if either $tp(\sigma^r) \neq 1$, or for every $\pi^r \in PL(\sigma^r, \hat{\sigma}^r)$, $TS(\pi^r) \cap RS(\sigma^r) = \emptyset$. $\square \square \square \square \square \square \square \square \square \square \square \square$

Now suppose that $\langle \langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle : r \geq j \geq k \rangle$ is a k -implication chain along ρ^k , and for each $j \in [k, r]$, fix $\bar{\tau}^j \subset \tau^j$ such that $(\bar{\tau}^j)^- = \sigma^j$. Let v^k be the principal derivative of $up(\hat{\sigma}^k)$ along τ^k . We say that $\langle \langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle : r \geq j \geq k \rangle$ is *amenable* if one of conditions (5.15) and (5.16) below holds, and if $k = r$, then σ^k is the shortest string satisfying (5.6)-(5.11), and (5.15) or (5.16) for $\hat{\sigma}^k$ and τ^k .

(5.15) $\bar{\tau}^k$ requires extension for v^k and $\hat{\sigma}^k$ is the primary k -completion of τ^k . (See Definition 5.6 for the definition of k -completion.)

(5.16) $k = r$ and $\hat{\sigma}^r$ is an amenable pseudocompletion of σ^r .

A *nonamenable* implication chain is an implication chain which is not amenable. \square

The backtracking process requires us to keep track, along each path, of the nodes which require extension but have no 0-completion along the path, and to find 0-completions for these nodes in reverse order of the order in which we discover that they require extension. This ordering is defined in Definition 5.8, and depends on the definition of completions (Definition 5.6). In order to show that backtracking can be carried out, we require that the final paths through T^0 be admissible (see Definition 5.9). There is a potential circularity here, which we avoid by requiring these nodes to be *preadmissible*. Preadmissibility ensures that nodes will only be switched when they do not interfere with the backtracking process; and in the course of finding completions, nodes will be switched only as required by the backtracking process. Completions are then defined as the nodes reached when the backtracking process has been completed.

Because of the interdependence of the next five definitions, we will explain some of the terminology used in the next definition. A node ρ will be *completion-respecting* if for all $j \leq n$, any node along $\lambda^j(\rho)$ which requires extension has a completion along $\lambda^j(\rho)$. ρ is *completion-consistent* via the sequence S if the paths determined by ρ are compatible with primary completions of all nodes of S , where the nodes in S are those which require extension but have not yet found primary completions, and the order in which the completions are to be found is the reverse of the ordering of S . ρ is *implication-free* if ρ is not a derivative of any node which is captured in the backtracking process, and is *implication-restrained* otherwise. Implication-restrained nodes will not define too many axioms during the construction, so there is no harm in forcing their outcomes.

The clauses of Definition 5.5 spell out the extensions which allow us to maintain compatibility with the backtracking process. Condition (5.17)(i) requires that we take switching extensions of primary 0-completions and pseudocompletions on T^0 , and condition (5.17)(ii) requires that nonswitching extensions be taken for all nodes which are not captured during a backtracking process but are derivatives of captured nodes. Condition (5.18) covers extensions taken during the backtracking process. Clause (i) requires that we take $(k+1)$ -switching extensions of nodes of T^k which are primary completions. This condition is needed to maintain compatibility with all completions which are forced to be taken during the construction. Clause (ii) requires us to switch outcomes of primary links in a minimal way, in order to return a designated node to the true path. Clause (iii) specifies that no other nodes captured by the backtracking process have switching extensions. (The reader may want to refer back to the remarks following Definition 5.2 for intuition.)

Definition 5.5: Fix $\sigma \in T^0$. If $\text{lh}(\sigma) > 0$, let $\rho = \sigma^-$, and assume that ρ is completion-consistent via some sequence $S = \langle \eta_i; i < m \rangle$ for some $m \geq 0$ (see Definition 5.8), and for each $i < m$, fix $k(i)$ such that $\eta_i \in T^{k(i)}$ and v_i such that η_i requires extension for v_i . We say that σ is *preadmissible* if either $\sigma = \langle \rangle$, or $\sigma \neq \langle \rangle$, ρ is admissible (see Definition 5.9),

and the following conditions hold:

- (5.17) (i) If either ρ is a primary 0-completion or an amenable pseudocompletion, then ρ has infinite outcome along σ .
(ii) If the hypotheses of (i) fail, $S = \langle \rangle$, and ρ is implication-restrained (see Definition 5.7), then σ is a nonswitching extension of ρ .
- (5.18) If $S \neq \langle \rangle$, then one of the following conditions holds:
(i) (a) There are $k(m)$ and $\eta_m \in T^{k(m)}$ such that ρ^- is completion-consistent via $\langle \eta_i; i \leq m \rangle$, ρ is a 0-completion of η_m ; and
(b) σ is a $(k(m)+1)$ -switching extension of ρ .
(ii) (a) (i)(a) fails, and there is a $j > k(m-1)$ and a $\lambda^{k(m\pm 1)+1}(\rho)$ -link $[\mu^{k(m\pm 1)+1}, \pi^{k(m\pm 1)+1}]$ restraining $\text{up}(v_{m\pm 1})$ of shortest length which is derived from a primary $\lambda^j(\rho)$ -link $[\mu^j, \pi^j]$ such that ρ is the initial derivative of $\text{up}^{j\pm 1}(\rho)$ along ρ and $\text{up}^{j\pm 1}(\rho)$ is a derivative of π^j ; and
(b) σ is a j -switching extension of ρ .
(iii) (i)(a) and (ii)(a) fail, σ is a nonswitching extension of ρ , and if there are two nonswitching immediate extensions of ρ , then ρ is activated along σ .

(We note that the extensions specified by (5.18) are unique.) \square

We described the role of completions earlier. We will need to show later that completions never require extension. This will follow from our requirement that completions be nonswitching extensions.

Definition 5.6: Fix $k \leq n$ and $\kappa^k \in T^k$. We say that κ^k is the k -completion if $\text{out}^0(\kappa^k)$ is nonswitching and either:

- (5.19) There are $m \geq 0$, $\gamma^k \subset \rho^k \subset \kappa^k$ and a sequence $S = \langle \eta_i; i \leq m \rangle$ such that $\eta_m = \rho^k$ requires extension for γ^k , $\text{up}(\gamma^k) = \text{up}(\kappa^k)$, both $\text{out}^0(\rho^k)$ and $\text{out}^0((\kappa^k)^-)$ are completion-consistent via S (see Definition 5.8), and there is no k -completion $\tilde{\kappa}^k$ of ρ^k such that $\tilde{\kappa}^k \subset \kappa^k$ (in this case, we say that κ^k is the *primary k -completion of ρ^k (for γ^k)*); or:
(5.20) There is a $j > k$ and a $\kappa^j \in T^j$ such that κ^j is a primary j -completion of some ρ^j and κ^k is an initial derivative of κ^j . (In this case, we say that κ^k is the *k -completion of ρ^j* .)

We say that κ^k is a *completion* if κ^k is a k -completion of some ρ^j . (We note that if κ is a 0-completion, then it must be a 0-completion of the last element, ρ^k , of the sequence via which κ is completion-consistent, and cannot be the 0-completion of any other node. It also follows from (5.18) that for all $j \leq k$, there is at most one j -completion of ρ^k .) \square

The process of finding a 0-completion of η^k may force paths to follow nodes on T^j for all $j \leq n$ which were not previously followed. For $j \leq k$, the new nodes will be those in the interval $[\text{out}^j(\eta^k), \kappa^j]$, where κ^j is the j -completion of η^k . We will not want to switch any of these nodes except for κ^j , unless we are forced to do so during the backtracking process (it is here that we need to add condition (5.14) to the definition of PL). Thus we call nodes in this interval *primarily implication-restrained* (condition (5.21)) if $j = k$ and *hereditarily implication-restrained* (condition (5.22)) if $j < k$. In addition, we do not want derivatives of implication-restrained nodes to be switched, unless we are forced to switch these derivatives during the backtracking process; so we specify that all derivatives of implication-restrained nodes are also *implication-restrained* (condition (5.23)).

Definition 5.7: A node $\xi^k \in T^k$ is *primarily implication-restrained* if:

(5.21) There is an $\eta^k \subseteq \xi^k$ which requires extension, but there is no k -completion $\kappa^k \subseteq \xi^k$ of η^k .

ξ^k is *hereditarily implication-restrained* if:

(5.22) There are $j > k$ and η^j such that $\text{out}^k(\eta^j) \subseteq \xi^k$, η^j requires extension, and there is no k -completion $\kappa^k \subseteq \xi^k$ of η^j .

ξ^k is *inductively implication-restrained* if the following condition holds:

(5.23) $\text{up}^j(\xi^k)$ is implication-restrained for some $j \in (k, n]$.

ξ^k is *implication-restrained* if ξ^k is either primarily, hereditarily, or inductively implication-restrained. ξ^k is *implication-free* if ξ^k is not implication-restrained. (By Definition 2.1, the implication-restrained nodes can be recursively recognized.) \square

Suppose that $\xi^k \in T^k$. ξ^k is *completion-respecting* if for all $j \in [k, n]$ and all $\rho^j \subseteq \lambda^j(\xi^k)$, if ρ^j requires extension, then ρ^j has a j -completion along $\lambda^j(\xi^k)$. It is possible for such a node $\rho^j \subseteq \lambda^j(\xi^k)$ to have a k -completion along ξ^k but not to have a j -completion along $\lambda^j(\xi^k)$. This will happen only during an iteration of the backtracking process, and in this case, ρ^j will have an i -completion along $\lambda^i(\delta^k)$ for all $i \in [k, j]$. Such a ρ^j has already

found a j -completion, and does not need to find another one; in fact, an attempt to maintain compatibility with its j -completion may conflict with being able to carry out a finitary backtracking process. Thus we will need to determine the nodes $\rho^j \subseteq \lambda^j(\xi^k)$ which require extension but do not have k -completions along ξ^k . These are the nodes for which we need to find k -completions, and are placed in the *completion-deficient* set at ξ^k . These nodes are ordered into a sequence by the order of the appearance of their images under out^k on the path of T^k under construction. This ordering is *completion-consistent* if it respects the dimension ordering of the trees on which the nodes appear, refined by the length of nodes on trees of the same dimension. We will show that the backtracking process produces completions in the reverse order to the completion-consistent ordering, if paths through trees are *admissible*, as defined in Definition 5.9.

Definition 5.8: Fix $k \leq n$, $\xi^k \in T^k$ and a set S of nodes of $\cup \{\rho^j \subseteq \lambda^j(\xi^k) : k \leq j \leq n\}$. We say that ξ^k is *completion-deficient for S* if the following condition holds:

(5.24) For all $j \in [k, n]$ and $\rho^j \subseteq \lambda^j(\xi^k)$, $\rho^j \in S$ iff ρ^j requires extension and has no k -completion $\subseteq \xi^k$.

ξ^k is *completion-respecting* if for all $j \in [k, n]$ and $\rho^j \subseteq \lambda^j(\xi^k)$, if ρ^j requires extension, then there is a j -completion $\kappa^j \subseteq \lambda^j(\xi^k)$ of ρ^j .

Given S such that ξ^k is completion-deficient for S , let $\bar{S} = \langle \eta_i : i < m \rangle$ be the linear ordering of S induced by the inclusion ordering on $\text{out}^k(v)$ for $v \in S$. By (2.5) and Lemma 5.6 (Uniqueness of Requiring Extension), this ordering will be well-defined. For all $i < m$, fix $k(i)$ such that $\eta_i \in T^{k(i)}$. (Note that, by Lemma 3.2(ii) (Out) and Lemma 3.1(ii) (Limit Path), this ordering will be independent of k as long as $k \leq k(i)$ for all $i < m$.) We say that ξ^k is *completion-consistent via \bar{S}* if the following conditions hold:

(5.25) If $i < j < m$, then $k(i) \leq k(j)$.

(5.26) If $i < j < m$ and $k(i) = k(j)$, then $\eta_i \subset \eta_j$.

ξ^k is *hereditarily completion-consistent* if every $\rho^k \subseteq \xi^k$ is completion-consistent. \square

Admissible nodes, as defined below, are nodes which are preadmissible, hereditarily completion-consistent in a uniform manner as specified by condition (5.27), act in a way to preclude the existence of amenable implication chains along the final paths through the trees as specified in (5.28), and preserve a correspondence between PL sets on consecutive trees, as specified in (5.29)(i)-(iii). (5.29)(i) specifies that when the extension of a path on T^k causes the path on T^{k+1} to switch and a node to leave a viable PL set on T^{k+1} , then a derivative of that node enters a corresponding PL set on T^k . If the above

happens during the backtracking process for a node, then (5.29)(ii) specifies that immediately at the end of that process, the PL set on T^k for the predecessor of the node requiring extension consists exactly of derivatives of all nodes in a corresponding PL set on T^{k+1} at the beginning of the backtracking process. Furthermore, if no additional nodes need to go through the backtracking process at this point, then (5.29)(iii) specifies that the node completing the backtracking process is implication-free. *Pseudotrue* nodes are nodes which are not involved in the backtracking process, so action of the construction at these nodes is according to the truth of the sentences generating action.

Definition 5.9: Fix $k \leq n$ and $\sigma^k \in T^k$, and let $\sigma = \text{out}^0(\sigma^k)$. We say that σ^k is *k-completion-free* if for every $j \in [k, n]$, $\lambda^j(\sigma)$ is not a primary completion, and if $k = 0$, we say that $\sigma = \sigma^k$ is *completion-free* if σ is 0-completion-free. We say that σ is *pseudotrue* if σ is preadmissible, completion-consistent via $\langle \rangle$, and completion-free. We say that σ is *admissible* if σ is preadmissible, hereditarily completion-consistent, completion-consistent via a sequence S , and the following conditions hold:

(5.27) If $\xi \subset \sigma$ is completion-consistent via \tilde{S} and $\eta \in \tilde{S}$, then either η has a 0-completion $\kappa \subseteq \sigma$, or $\eta \in S$.

(5.28) If $\eta \subseteq \sigma$ is pseudotrue, then there is no amenable j -implication chain along $\lambda^j(\eta)$ for any $j \leq n$.

(5.29) (i) For all $k < n$ and $\mu^k \subset \nu^k \subset \eta^k \subseteq \lambda^k(\sigma) \in T^k$, if $\text{up}(\mu^k) \subset \text{up}(\nu^k), \lambda(\eta^k)$ and ν^k is implication-free, then

$$\text{PL}(\text{up}(\mu^k), \text{up}(\nu^k)) \subseteq \{\text{up}(\xi^k): \xi^k \in \text{PL}(\nu^k, \eta^k)\} \cup \text{PL}(\text{up}(\mu^k), \lambda(\eta^k)).$$

(ii) For all $k < n$ and $\mu^k \subset \nu^k = (\eta^k)^- \subset \eta^k \subset \kappa^k \subseteq \lambda^k(\sigma) \in T^k$, if η^k requires extension for μ^k and κ^k is the primary completion of η^k , then

$$\text{PL}(\text{up}(\mu^k), \lambda(\eta^k)) = \{\text{up}(\xi^k): \xi^k \in \text{PL}(\nu^k, \kappa^k)\}.$$

(iii) If $\eta \subseteq \sigma$ is completion-consistent via $\langle \rangle$ and η is a 0-completion, then η is implication-free.

$\Lambda^0 \in [T^0]$ is *admissible* if every $\sigma \subset \Lambda^0$ is admissible. $\Lambda^k \in [T^k]$ is *admissible* if $\Lambda^k = \lambda^k(\Lambda^0)$ for some admissible $\Lambda^0 \in [T^0]$. η

We now show that an amenable k -implication chain gives rise to a node on T^{k+1} which requires extension.

Lemma 5.2 (Requires Extension Lemma): Fix k such that $0 < k < n$ and fix $\sigma^k \in T^k$. Let $r = \dim(\sigma^k) - 1$, and assume that $k \leq r$. Suppose that $\langle \langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle : r \geq j \geq k \rangle$ is an amenable k -implication chain. Let $\eta^{k\pm 1} = \text{out}(\tau^k)$, let $\nu^{k\pm 1}$ be the principal derivative of σ^k along $\eta^{k\pm 1}$. Assume that $\eta = \text{out}^0(\eta^{k\pm 1})$ is preadmissible. Then $\eta^{k\pm 1}$ requires extension for $\nu^{k\pm 1}$.

Proof: Let $\delta^{k\pm 1} = (\eta^{k\pm 1})^-$. As $\eta^{k\pm 1} = \text{out}(\tau^k)$ and as, by (5.8)(ii), $\hat{\sigma}^k = (\tau^k)^-$, $\delta^{k\pm 1}$ is the principal derivative of $\hat{\sigma}^k$ along $\eta^{k\pm 1}$. We verify (5.1)-(5.5).

(5.1) is vacuous. By (5.6) and (5.9), $\sigma^k \equiv \hat{\sigma}^k \equiv \sigma^r$, so by (5.7), $\text{tp}(\nu^{k\pm 1}) \in \{1, 2\}$. Furthermore, $\nu^{k\pm 1}$ and $\delta^{k\pm 1}$ are, respectively, the principal derivatives of σ^k and $\hat{\sigma}^k$ along η^{k-1} , so $\nu^{k\pm 1} \equiv \delta^{k\pm 1}$. By (5.6) and (5.9), $\text{up}^{r+1}(\nu^{k\pm 1}) \neq \text{up}^{r+1}(\delta^{k\pm 1})$, so $\text{up}(\nu^{k\pm 1}) \neq \text{up}(\delta^{k\pm 1})$. By (5.11), $\hat{\sigma}^k$ has finite outcome along τ^k and σ^k has infinite outcome along τ^k , so by (2.4) and as $\nu^{k\pm 1}$ and $\delta^{k\pm 1}$ are, respectively, the principal derivatives of σ^k and $\hat{\sigma}^k$ along $\eta^{k\pm 1}$, $\nu^{k\pm 1}$ has finite outcome along $\eta^{k\pm 1}$ and $\delta^{k\pm 1}$ has infinite outcome along $\eta^{k\pm 1}$. Hence (5.2) holds.

By Lemma 3.2(i) (Out) and hypothesis, $\lambda(\eta^{k\pm 1}) = \tau^k \supset \sigma^k = \text{up}(\nu^{k\pm 1})$. By (5.2), $\nu^{k\pm 1}$ must be both the initial and principal derivative of $\text{up}(\nu^{k\pm 1})$ along $\lambda(\eta^{k\pm 1})$, so cannot be the first node in a primary $\lambda(\eta^{k\pm 1})$ -link. (5.3) now follows from Lemma 4.3(i)(d) (Link Analysis). (5.4) is vacuous as $k-1 < r$. (5.5) follows from the hypothesis. The minimality of $\text{lh}(\nu^{k\pm 1})$ follows from the uniqueness of $\hat{\sigma}^r$ for $\hat{\sigma}^r$ given by Definition 5.4, if $k = r$. And if $k < r$, then the minimality of $\text{lh}(\nu^{k\pm 1})$ follows from (5.15) and the fact that, by Definition 5.6, a primary completion along a preadmissible path is the primary completion of exactly one node. \square

Suppose that η^k requires extension for ν^k , κ^k is the k -completion of η^k , and $(\xi^k)^- = \kappa^k$. If κ^k has finite outcome along ξ^k , then a k -implication chain will have been formed along ξ^k . Otherwise, we show that $[\nu^k, \kappa^k]$ is a primary ξ^k -link.

Lemma 5.3 (Implication Chain Lemma): Fix $k \leq r < n$ and $\nu^k \subset \delta^k \subset \eta^k \subset \kappa^k \subset \xi^k \in T^k$ such that $k < \dim(\nu^k) = r+1$, $(\eta^k)^- = \delta^k$, $(\xi^k)^- = \kappa^k$, and $\text{out}^0(\xi^k)$ is preadmissible. Assume that η^k requires extension for ν^k , and that κ^k is the k -completion of η^k for ν^k . Then:

- (i) If κ^k has infinite outcome along ξ^k , then $[\nu^k, \kappa^k]$ is a primary ξ^k -link.
- (ii) If κ^k has finite outcome along ξ^k , then there is an amenable k -implication chain $\langle \langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle : r \geq j \geq k \rangle$ such that $\tau^k = \xi^k$, $\hat{\sigma}^k = \kappa^k$, and $\sigma^k = \delta^k$.

Now fix $\bar{\delta}^k \subset \bar{\kappa}^k \subset \bar{\xi}^k \in T^k$ such that $(\bar{\xi}^k)^- = \bar{\kappa}^k$, $\bar{\kappa}^k$ has finite outcome along $\bar{\xi}^k$, $\bar{\kappa}^k$ is an

amenable pseudocompletion of $\bar{\delta}^k$, and for all $i \leq k$, the principal derivative of $\bar{\kappa}^k$ along $\text{out}^i(\bar{\xi}^k)$ is implication-free. Then:

(iii) $\langle\langle \bar{\delta}^k, \bar{\kappa}^k, \bar{\xi}^k \rangle\rangle$ is an amenable k -implication chain.

Proof: We proceed by induction on $n-k$, and then by induction on $\text{lh}(\kappa^k)$.

(i): By (5.19), $\text{up}(v^k) = \text{up}(\kappa^k)$, and by (5.2), v^k is the initial derivative of $\text{up}(v^k)$ along ξ^k . Since κ^k has infinite outcome along ξ^k , $[v^k, \kappa^k]$ is a primary ξ^k -link.

(ii): We first show that (5.6)-(5.12) hold. By (5.19), $\text{up}(v^k) = \text{up}(\kappa^k)$. Hence (5.6) follows from (5.2) if $k = r$, and from (5.5)(ii) and (5.6) inductively if $k < r$. (5.7) follows from (5.2), the definition of r , and (5.5)(i) if $k = r$, and (5.5)(ii) and (5.7) inductively if $k < r$. (5.8) follows from the definitions of σ^k , $\hat{\sigma}^k$, and τ^k in (ii). (5.9) and (5.12) follow from (5.5)(ii) and (5.19). (5.10) follows from the definitions of σ^k , $\hat{\sigma}^k$, and τ^k in (ii), (5.1), and hypothesis if $k = r$, and by (5.5)(ii) and (5.10) inductively if $k < r$. (5.11) follows from the definitions of σ^k , $\hat{\sigma}^k$, and τ^k in (ii), (5.2), and hypothesis. Hence $\langle\langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle\rangle: r \geq j \geq k$ is a k -implication chain.

(5.15) follows from hypothesis, so this k -implication chain is amenable. We complete the proof of (ii) by verifying the minimality condition for the case $k = r$. By the minimality of the choice of v^r in Definition 5.1 and by (5.15), the minimality condition can only fail if there is a $\mu^r \subset v^r$ such that $\langle\langle \mu^r, \kappa^r, \xi^r \rangle\rangle$ is an amenable r -implication chain, which we assume in order to obtain a contradiction. Now by (5.2), v^r is an initial derivative, and by Definition 5.6, $\text{up}(v^r) = \text{up}(\kappa^r)$ and κ^r is not an initial derivative. Hence by Definition 5.2, κ^r cannot be a pseudocompletion. Thus (5.16) must fail for $\langle\langle \mu^r, \kappa^r, \xi^r \rangle\rangle$. By Definition 5.6, κ^r is the primary completion of only one node, and by hypothesis, that node must be η^r . Hence by (5.15) for $\langle\langle \mu^r, \kappa^r, \xi^r \rangle\rangle$, η^r must require extension for μ^r . By Definition 5.1, η^r requires extension for at most one node. But η^r requires extension for v^r , so $v^r = \mu^r$, yielding a contradiction.

(iii): Immediate from hypothesis and the definition of amenable pseudocompletions (Definitions 5.2 and 5.4). \square

We will need to know that admissible paths are always compatible with completions, except when we are iterating the backtracking process to try to eliminate an amenable implication chain. In the latter case, by (5.18) and (5.24), the only completions which may be incompatible with the path under construction are the primary completions.

Lemma 5.4 (Compatibility Lemma): Fix $\rho \in T^0$ such that ρ is preadmissible. Fix $i \leq n$, $\beta \subseteq \rho$, and $\eta^i \subseteq \lambda^i(\beta)$ such that η^i requires extension, and suppose that $\kappa \subseteq \rho$ is the 0-completion of η^i . Fix $v < i$, let $\eta^v = \text{out}^v(\eta^i)$, and suppose that $\eta^v \subseteq \lambda^v(\rho) = \rho^v$. Then for all $j \leq v$, $\rho^j = \lambda^j(\rho) \supseteq \lambda^j(\kappa) = \kappa^j$.

Proof: We proceed by induction on $j \leq v$, noting that, by hypothesis, the lemma holds for $j = 0$. Assume that $j > 0$. As $\eta^v \subseteq \rho^v$, it follows from (2.5) that $\eta^q = \text{out}^q(\eta^i) \subseteq \rho^q$ for all $q \leq j$. By (5.22), every ξ^j such that $\eta^j \subseteq \xi^j \subseteq \kappa^j$ is implication-restrained. We note, by (5.18) and (5.25), that if $u < n$, $\sigma^u \in T^u$ requires extension and has u -completion τ^u , $\sigma = \text{out}^0(\sigma^u)$, τ is the 0-completion of σ^u and is preadmissible, and $\sigma \subseteq \delta \subseteq \tau$, then δ cannot be t -switching for any $t \leq u$.

As ρ is preadmissible, ρ^- is admissible and thus hereditarily completion-consistent. Fix ξ^j such that $\eta^j \subseteq \xi^j \subseteq \kappa^j$. As $j < i$, it follows from the above paragraph that ξ^j is not $(j+1)$ -switching, so if ξ^j is the principal derivative of $\text{up}(\xi^j)$ along κ^j , then ξ^j must be the initial derivative of $\text{up}(\xi^j)$ along κ^j ; thus there is no μ^j such $[\mu^j, \xi^j]$ is a primary κ^j -link. By (5.19) and (5.25) and as $j < i$, ξ^j is not a primary completion or an amenable pseudocompletion. Fix δ such that $\kappa \subseteq \delta \subseteq \rho$ and $\text{up}^j(\delta^-) = \xi^j$. If δ^- is primarily or hereditarily implication-restrained, then by (5.18), δ will not switch ξ^j . Otherwise, δ^- will be inductively implication-restrained. We will show that δ^- is neither a primary 0-completion nor an amenable pseudocompletion. It will then follow from (5.17)(ii) that δ does not switch ξ^j . Thus as $\kappa^{j\pm 1} \subseteq \rho^{j\pm 1}$ by induction, it follows from (2.4) that $\rho^j \supseteq \kappa^j$.

We complete the proof of the lemma by assuming that δ^- is either a primary 0-completion or an amenable pseudocompletion, and obtaining a contradiction. First assume that $\dim(\xi^j) > j$. If j is even, then by repeated applications of (5.5)(ii), (5.9) and (5.15) ((5.16) cannot apply at any $t < j$), it follows that ξ^j is a primary completion or an amenable pseudocompletion, contrary to the preceding paragraph. Suppose that j is odd. By repeated applications of (5.5)(ii), (5.9) and (5.15) ((5.16) cannot apply at any $t < j$), it follows that $\text{up}^{j-1}(\delta^-)$ is a primary completion, and that the immediate successor of ξ^j along κ^j requires extension, contrary to (5.25) which would require $j \geq i$. Thus in either case, we have a contradiction.

Now suppose that $\dim(\xi^j) \leq j$. By Lemma 3.1(i) (Limit Path), ξ^j has an initial derivative $\xi^{j-1} \subseteq \kappa^{j\pm 1}$, and as $\eta^{j-1} = \text{out}(\eta^j)$ and $\eta^j \subseteq \xi^j$, it follows from (2.5) and Lemma 3.1(i) (Limit Path) that $\eta^{j-1} \subseteq \xi^{j-1}$. If $\dim(\xi^j) < j$, then by (2.9), ξ^{j-1} is the only derivative of ξ^j along $\kappa^{j\pm 1}$, so it follows by induction that δ^- is neither a primary 0-completion nor an amenable pseudocompletion. Suppose that $\dim(\xi^j) = j$. By (5.9), (5.1), and (5.10), ξ^{j-1} would have to be implication-free. But by Lemma 3.1(i) (Limit Path), $\xi^{j-1} \in [\eta^{j-1}, \kappa^{j\pm 1}]$, so is hereditarily implication-restrained, yielding the desired contradiction. \square

One consequence of the next lemma, is that if η is admissible and pseudotrue, then for all $j \leq n$, if $\rho^j \subseteq \lambda^j(\eta)$ requires extension, then ρ^j has a primary completion along $\lambda^j(\eta)$. Hence for pseudotrue nodes, completion-respecting and completion-consistent via $\langle \rangle$

coincide. We will need a somewhat more general statement.

Lemma 5.5 (Completion-Respecting Lemma):

- (i) Fix $k < n$ and $\delta^k \subset \xi^k \subseteq \kappa^k \in T^k$ such that $\kappa = \text{out}^0(\kappa^k)$ is admissible, δ^k and ξ^k both require extension, and κ^k is the primary completion of δ^k . Then ξ^k has a primary completion $\tau^k \subset \kappa^k$, and τ^k has infinite outcome along κ^k .
- (ii) Fix $\eta \in T^0$ such that η is preadmissible and completion-consistent via $\langle \rangle$. Suppose that $\rho^j \subseteq \lambda^j(\eta)$ requires extension for v^j and $\gamma^j = (\rho^j)^-$. If η is the 0-completion corresponding to a primary k -completion $\hat{\sigma}^k$ and $\hat{\sigma}^k$ is the primary completion of the immediate successor of σ^k along $\hat{\sigma}^k$, then assume further that it is neither the case that $j \geq k$, $j-k$ is odd and $\text{up}^j(\hat{\sigma}^k) = \gamma^j$, nor the case that $j \geq k$, $j-k$ is even and $\text{up}^j(\sigma^k) = \gamma^j$. Then ρ^j has a primary completion $\kappa^j \subset \lambda^j(\eta)$ which has infinite outcome along $\lambda^j(\eta)$.
- (iii) Fix $\xi, \eta \in T^0$ such that η and ξ are preadmissible and completion-consistent via $S = \langle \rangle$, and $\xi^- = \eta$. Suppose that $\rho^j \subseteq \lambda^j(\eta)$ requires extension. Then ρ^j has a primary completion $\kappa^j \subset \lambda^j(\xi)$ which has infinite outcome along $\lambda^j(\xi)$.

Proof: (i): By (5.26) and Definition 5.6, ξ^k has a primary completion $\tau^k \subset \kappa^k$. By (5.18)(i) and as, by (5.18), (5.24), and (5.25), if κ is the 0-completion corresponding to κ^k , then no node in $(\text{out}^0(\delta^k), \kappa]$ can be v -switching for any $v \leq k$, τ^k has infinite outcome along κ^k .

(ii),(iii): We prove (ii), and indicate the modifications needed for (iii) in parentheses. We assume that ρ^j satisfies the hypotheses of (ii) or (iii), and either ρ^j has no primary completion $\kappa^j \subset \lambda^j(\eta)$ ($\lambda^j(\xi)$, resp.), or that κ^j exists and has finite outcome along $\lambda^j(\eta)$ ($\lambda^j(\xi)$, resp.), and derive a contradiction under the assumption, in the proof of (ii), that the exclusionary conditions in (ii) fail. (For (iii), fix v^j and γ^j such that ρ^j requires extension for v^j and $\gamma^j = (\rho^j)^-$.) Without loss of generality, we may assume that j is the smallest number for which the conclusion fails for some $\rho^j \subseteq \lambda^j(\eta)$ ($\lambda^j(\xi)$, resp.) satisfying the hypothesis of (ii) ((iii), resp.). As η is completion-consistent via $\langle \rangle$, ρ^j has a 0-completion $\kappa \subseteq \eta$. Now for (ii), $\kappa \neq \eta$, else η would be the 0-completion corresponding to the primary j -completion of ρ^j and so $k = j$, $\hat{\sigma}^k = \kappa^j = \text{up}^j(\eta)$, and $\sigma^k = \gamma^j$, contrary to hypothesis. Hence $\kappa \subset \eta$ ($\kappa \subset \xi$, resp.). If $j = 0$, then by (5.17)(i) or (5.18)(i), the immediate successor of κ along η (ξ , resp.) switches $\text{up}^{j+1}(\kappa)$; so κ has infinite outcome along η . Hence $j > 0$.

By Lemma 5.4 (Compatibility), ρ^j has a $(j-1)$ -completion $\kappa^{j\pm 1} \subseteq \eta^{j\pm 1} = \lambda^{j\pm 1}(\eta)$, and by Definition 5.6, $\kappa^{j\pm 1}$ is an initial derivative of the primary completion κ^j of ρ^j , and κ is an initial derivative of $\kappa^{j\pm 1}$. As $\eta \supset \kappa$ ($\xi \supset \kappa$, resp.), it follows from (2.4) that $\kappa^{j\pm 1} \subset \eta^{j\pm 1}$

($\kappa^{j\pm 1} \subset \xi^{j\pm 1} = \lambda^{j\pm 1}(\xi)$, resp.). Fix $\tau^{j\pm 1} \subseteq \eta^{j\pm 1}$ ($\xi^{j\pm 1}$, resp.) such that $(\tau^{j\pm 1})^- = \kappa^{j\pm 1}$. By Lemma 3.1(ii) (Limit Path), $(\lambda(\tau^{j\pm 1}))^- = \kappa^j$.

We assume that all derivatives of κ^j along $\eta^{j\pm 1}$ ($\xi^{j\pm 1}$, resp.) have finite outcome along $\eta^{j\pm 1}$ ($\xi^{j\pm 1}$, resp.), and derive a contradiction. Under this assumption and by (5.19), there is a primary $\lambda(\tau^{j\pm 1})$ -link $[\mu^j, \kappa^j]$ which restrains ρ^j with $\mu^j \subset \rho^j \subset \kappa^j$. As $\rho^j \subseteq \lambda^j(\eta)$ ($\lambda^j(\xi)$, resp.), it follows from (2.6) and since $\tau^{j\pm 1} \subseteq \eta^{j\pm 1}$ ($\xi^{j\pm 1}$, resp.) that $[\mu^j, \kappa^j]$ is a $\lambda(\beta^{j\pm 1})$ -link for all $\beta^{j\pm 1}$ such that $\tau^{j\pm 1} \subseteq \beta^{j\pm 1} \subseteq \eta^{j\pm 1}$ ($\xi^{j\pm 1}$, resp.), so $[\mu^j, \kappa^j]$ is a $\lambda^j(\eta)$ -link ($\lambda^j(\xi)$ -link, resp.). But then $\kappa^j \subset \lambda^j(\eta)$ ($\lambda^j(\xi)$, resp.), and by (2.4), κ^j has infinite outcome along $\lambda^j(\eta)$ ($\lambda^j(\xi)$, resp.), contrary to the choice of j .

We conclude that there is a derivative $\bar{\kappa}^{j\pm 1} \subset \eta^{j\pm 1}$ ($\xi^{j\pm 1}$, resp.) of κ^j which has infinite outcome along $\eta^{j\pm 1}$ ($\xi^{j\pm 1}$, resp.). Fix $\bar{\tau}^{j\pm 1} \subseteq \eta^{j\pm 1}$ ($\xi^{j\pm 1}$, resp.) such that $(\bar{\tau}^{j\pm 1})^- = \bar{\kappa}^{j\pm 1}$. By Lemma 3.1(ii) (Limit Path), $(\lambda(\bar{\tau}^{j\pm 1}))^- = \kappa^j$ and κ^j has finite outcome along $\lambda(\bar{\tau}^{j\pm 1})$. Hence by Lemma 5.3(ii) (Implication Chain) and Lemma 5.2 (Requires Extension), $\bar{\tau}^{j\pm 1}$ requires extension for some derivative $\gamma^{j\pm 1}$ of γ^j . We assume that one of the exclusionary conditions of (ii) holds for $\bar{\tau}^{j\pm 1}$, and derive a contradiction. By (5.9), $\dim(\eta) > j$, so (5.5)(ii) must hold for the immediate successor of σ^k along $\hat{\sigma}^k$; fix the corresponding $(k+1)$ -implication chain $\langle\langle \sigma^i, \hat{\sigma}^i, \alpha^i \rangle\rangle: r \geq i \geq k+1$. First suppose that $k \leq j-1$, $(j-1)-k$ is odd, and $\text{up}^{j\pm 1}(\hat{\sigma}^k) = \bar{\kappa}^{j\pm 1}$. By (5.9), $\text{up}^{j\pm 1}(\hat{\sigma}^k) = \sigma^{j\pm 1}$ and so $\text{up}^j(\hat{\sigma}^k) = \hat{\sigma}^j = \kappa^j$. Thus by (5.5), the failure of (5.16), and our assumptions, $\hat{\sigma}^j = \kappa^j$ is the primary completion both of the immediate successor of γ^j along $\hat{\sigma}^j$, and of the immediate successor of σ^j along $\hat{\sigma}^j$, so $\gamma^j = \sigma^j = \text{up}^j(\sigma^k)$, $k \leq j$, and $k-j$ is even, and the exclusionary conditions of (ii) hold for ρ^j , contrary to our assumption. Finally, suppose that $k \leq j-1$, $k-(j-1)$ is even, and $\text{up}^{j\pm 1}(\sigma^k) = \bar{\kappa}^{j\pm 1}$. By (5.9), $\text{up}^{j\pm 1}(\sigma^k) = \sigma^{j\pm 1}$ and so $\text{up}^j(\sigma^k) = \hat{\sigma}^j = \kappa^j$. Thus by (5.5), the failure of (5.16), and our assumptions, $\hat{\sigma}^j = \kappa^j$ is the primary completion both of the immediate successor of γ^j along $\hat{\sigma}^j$, and of the immediate successor of σ^j along $\hat{\sigma}^j$, so $\gamma^j = \sigma^j = \text{up}^j(\hat{\sigma}^k)$, $k \leq j$, and $k-j$ is odd, and the exclusionary conditions of (ii) hold for ρ^j , contrary to our assumption.

By the minimality of the choice of j and as $\bar{\tau}^{j\pm 1}$ requires extension for some derivative $\gamma^{j\pm 1}$ of γ^j , it follows that $\bar{\tau}^{j\pm 1}$ has a primary completion $\bar{\gamma}^{j\pm 1}$ which has infinite outcome along $\eta^{j\pm 1}$ ($\xi^{j\pm 1}$, resp.). By (5.19), $\text{up}(\bar{\gamma}^{j\pm 1}) = \gamma^j$, so by (2.4) and as $\gamma^j \subseteq \lambda^j(\eta)$, γ^j has finite outcome along $\lambda^j(\eta)$ ($\lambda^j(\xi)$, resp.). But by (5.2), γ^j has infinite outcome along $\rho^j \subseteq \lambda^j(\eta)$ ($\lambda^j(\xi)$, resp.), a contradiction. \square

We now show that at most one new string requires extension at any admissible η .

Lemma 5.6 (Uniqueness of Requiring Extension Lemma): Fix $\eta \in T^0$ such that η is preadmissible and $\text{lh}(\eta) > 0$. Let η and η^- be completion-consistent via S and \tilde{S} , respectively. Suppose that $i \leq j$ and $\lambda^i(\eta), \lambda^j(\eta) \in S\tilde{S}$. Then $i = j$.

Proof: For all $u \leq n$, let $\eta^u = \lambda^u(\eta)$. Fix p and s as in Lemma 3.3 (λ -Behavior) for η . First assume that $j > s$, in order to obtain a contradiction. Then by Lemma 3.3 (λ -Behavior), there will be a $\xi \subset \eta$ such that $\lambda^j(\xi) = \eta^j$. By (5.27) for the admissible node η^- and as $\eta^j \notin \tilde{S}$, there will be a 0-completion κ of η^j such that $\xi \subset \kappa \subseteq \eta^-$. Hence by Definition 5.8, $\eta^j \notin S$, contrary to hypothesis.

We conclude that $j \leq s$, and hence, by Lemma 3.3 (λ -Behavior), that $(\eta^j)^- = \text{up}^j((\eta^i)^-)$. We assume that $j > i$ and derive a contradiction. Let $\tau^i = \eta^i$. By (5.5)(ii), there is an $r > i$ and an amenable $(i+1)$ -implication chain $\langle\langle \sigma^t, \hat{\sigma}^t, \tau^t \rangle: r \geq t \geq i+1 \rangle$ such that $\tau^i = \text{out}(\tau^{i+1})$. By Lemma 3.2(i) (Out), $\tau^{i+1} = \lambda(\text{out}(\tau^{i+1})) = \lambda(\tau^i) = \eta^{i+1}$.

We now claim that if $i < t \leq j$ and $t-i$ is odd then:

$$(5.30) \quad \tau^t \subseteq \eta^t.$$

By the preceding paragraph, (5.30) is true for $t = i+1$. We proceed by induction, assuming that (5.30) holds, and verifying (5.30) with $t+2$ in place of t , under the assumption that $t+2 \leq j$. Let $\rho^t = (\tau^t)^-$, $\rho^{t+1} = \text{up}(\rho^t)$ and $\rho^i = (\tau^i)^-$. By (5.8)(ii), $\rho^t = \hat{\sigma}^t$, and by (5.9), $\rho^t = \text{up}^t(\rho^i)$ and $\rho^{t+1} = \text{up}^{t+1}(\rho^i) = \sigma^{t+1}$. By Lemma 4.5 (Free Extension), ρ^i is τ^i -free. Hence ρ^{t+1} is η^{t+1} -free. By (5.30) and (2.4), $\rho^{t+1} \subset \lambda^{t+1}(\eta)$. Hence we can fix $\bar{\tau}^{t+1} \subseteq \eta^{t+1}$ such that $(\bar{\tau}^{t+1})^- = \rho^{t+1}$.

By (5.2), ρ^i has infinite outcome along τ^i , so as $\tau^i = \eta^i$ and ρ^i is τ^i -free, it follows from (2.4) that $\rho^{i+1} = \text{up}(\rho^i)$ has finite outcome along η^{i+1} ; thus by (5.11) and (5.30), ρ^t has finite outcome along η^t . As ρ^i is τ^i -free, $\rho^t = \text{up}^t(\rho^i)$ must be η^t -free. So as $\tau^t \subseteq \eta^t$, it follows from the definition of links that all derivatives of ρ^{t+1} along η^t have finite outcome along η^t . Hence by (2.4), $\bar{\tau}^{t+1} = \rho^{t+1} \wedge \langle \gamma^t \rangle \subseteq \eta^{t+1}$, where $\gamma^t \subseteq \eta^t$ and $(\gamma^t)^-$ is the initial derivative of ρ^{t+1} along η^t . Thus $(\gamma^t)^- \subseteq \rho^t$, and so by (5.30), $\gamma^t \subseteq \tau^t$. By (5.12) and (5.30), $\text{out}(\tau^{t+1}) \subseteq \tau^t \subseteq \eta^t$ and $\rho^{t+1} \subset \tau^{t+1}$. Now all derivatives of ρ^{t+1} have finite outcome along $\text{out}(\tau^{t+1}) \subseteq \eta^t$. Hence by (2.4), $\bar{\tau}^{t+1} \subseteq \tau^{t+1}$. Now by (5.12), $\text{out}(\tau^{t+2}) \subseteq \tau^{t+1}$, and by (5.5)(ii), $(\text{out}(\tau^{t+2}))^- = \rho^{t+1}$. So as $(\bar{\tau}^{t+1})^- = \rho^{t+1}$ and $\bar{\tau}^{t+1}, \text{out}(\tau^{t+2}) \subseteq \tau^{t+1}$, we have $\bar{\tau}^{t+1} = \text{out}(\tau^{t+2})$. As $\rho^{t+2} = \text{up}(\rho^{t+1})$ is η^{t+2} -free and ρ^{t+1} has infinite outcome along $\bar{\tau}^{t+1} \subseteq \tau^{t+1} \wedge \eta^{t+1}$, it follows from (2.4) that $\tau^{t+2} = \rho^{t+2} \wedge \langle \bar{\tau}^{t+1} \rangle \subseteq \eta^{t+2}$, verifying (5.30) with $t+2$ in place of t . Furthermore, we note that ρ^{t+2} has finite outcome along τ^{t+2} , and by (5.8) and (5.9), $\rho^{t+2} = \text{up}(\rho^{t+1}) = \text{up}^{t+2}(\rho^i)$. Hence since $\text{up}^j((\eta^i)^-) = (\eta^j)^-$ and by (5.2) and (5.30), $j > t+2$.

We conclude that $j-i$ is even, and that (5.30) holds for $t = j-1$. By (5.12) and (5.30), $\text{out}(\tau^j) \subset \tau^{j-1} \subseteq \eta^{j-1}$. Iterating (5.5)(ii) and recalling that $(\eta^j)^- = \text{up}^j((\eta^i)^-)$ and that $j-i$ is even, we see that $(\eta^j)^- = \sigma^j \subset \hat{\sigma}^j \subset \tau^j$; hence $\eta^j \not\supseteq \tau^j$ and so by (2.4) and as $\text{out}(\tau^j) \subset \eta^{j-1}$

$\subseteq \text{out}(\eta^j)$, it must be the case that $\eta^j \not\subset \tau^j$ and so that $\eta^j \wedge \tau^j = \sigma^j$. By (5.2), σ^j has infinite outcome $\beta^{j\pm 1}$ along η^j , so by (2.4), all derivatives of σ^j which are $\subset \eta^{j\pm 1}$ must have finite outcome along $\eta^{j\pm 1}$, and $(\beta^{j\pm 1})^-$ is the initial derivative of σ^j along $\eta^{j\pm 1}$. As $\tau^{j\pm 1} \subseteq \eta^{j\pm 1}$, all derivatives of σ which are $\subset \tau^{j\pm 1}$ must have finite outcome along $\tau^{j\pm 1}$. By Lemma 3.1(ii) (Limit Path), $\beta^{j\pm 1} \subseteq \tau^{j\pm 1}$ and by (2.4), $(\beta^{j\pm 1})^-$ is the principal derivative of σ^j along $\tau^{j\pm 1}$. Hence since $\text{out}(\tau^j) \subset \tau^{j\pm 1}$ and by (2.4), $\sigma^{j\wedge} \langle \beta^{j\pm 1} \rangle \subseteq \tau^j$, so $\sigma^{j\wedge} \langle \beta^{j\pm 1} \rangle \subseteq \tau^j \wedge \eta^j$, contradicting the fact that $\eta^j \wedge \tau^j = \sigma^j$. \square

In order to show that the backtracking process is finitary, we will need to know that if a node requires extension, then its immediate predecessor is not a primary j -completion.

Lemma 5.7 (Primary Completion Lemma): Fix $j \leq n$ and $\eta^j \in T^j$ such that η^j is preadmissible and requires extension, and let $\eta = \text{out}^0(\eta^j)$ and $\delta^j = (\eta^j)^-$. Then:

- (i) δ^j is not a primary j -completion or an amenable pseudocompletion.
- (ii) If $\eta^j \neq \lambda^j(\eta^\pm)$, then either η is switching or η^\pm is not primarily or hereditarily implication-restrained; hence η is not a 0-completion.

Proof: We prove (i) and (ii) simultaneously by induction on $r-j$. For all $i \leq n$, let $\eta^i = \lambda^i(\eta)$.

(i): Let $r = \dim(\delta^j) - 1$. Let η^j require extension for μ^j . To see that δ^j is not a primary j -completion or a pseudocompletion, we proceed by induction on $r-j$ and then by induction on $\text{lh}(\eta^j)$, assuming to the contrary and deriving a contradiction. Let η^j require extension for μ^j . There are several cases.

Case 1: $j = r$. There are two subcases, depending on whether we assume that δ^r is a primary completion or a pseudocompletion.

Subcase 1.1: δ^r is a primary completion of some ρ^r which requires extension for some ν^r . By Definition 5.6, $\text{up}(\nu^r) = \text{up}(\delta^r)$; and by (5.2) and the hypothesis of the lemma, δ^r has infinite outcome along η^r . Hence $[\nu^r, \delta^r]$ is a primary η^r -link. By (5.2), $\text{up}(\mu^r) \neq \text{up}(\delta^r)$, so $\mu^r \neq \nu^r$. By (5.3), it now follows that $\mu^r \subset \nu^r$. We show that (5.1)-(5.5) hold for $\mu^r \subset \gamma^r = (\rho^r)^- \subset \rho^r$, and thus contradict the minimality of $\text{lh}(\nu^r)$ for ρ^r in Definition 5.1.

By Definition 5.1 and the preceding paragraph, $\mu^r \subset \nu^r \subset \rho^r \subset \delta^r$, and by (5.2), $\gamma^r = (\rho^r)^-$ has infinite outcome along η^r . Hence $\text{up}(\gamma^r) \neq \text{up}(\mu^r)$, else by (5.2) and the preceding paragraph, $[\nu^r, \delta^r]$ and $[\mu^r, \gamma^r]$ would be primary η^r -links, contradicting Lemma 4.1 (Nesting). (5.1)-(5.3) and (5.5) can now be routinely verified, using those same conditions and the assumptions that $\text{up}(\nu^r) = \text{up}(\delta^r)$, η^r requires extension for μ^r , and ρ^r

requires extension for v^r . By (2.7), (5.3), and Lemma 4.3(i)(a) (Link Analysis), $\text{up}(\mu^r) \subseteq \lambda(\mu^r), \lambda(\eta^r)$, so by (2.6), $\text{up}(\mu^r) \subseteq \lambda(\rho^r)$; and by (5.3) and Lemma 4.3(i)(a) (Link Analysis), $\text{up}(v^r) \subseteq \lambda(\rho^r)$. Thus $\text{up}(\mu^r)$ and $\text{up}(v^r)$ are comparable. By (5.2), μ^r and v^r are both initial derivatives, so by Lemma 5.1(i) (Limit Path) and as $\mu^r \subseteq v^r$, it follows that $\text{up}(\mu^r) \subseteq \text{up}(v^r)$. As $(\eta^r)^- = \delta^r$ and $\text{up}(v^r) = \text{up}(\delta^r)$, it follows from Lemma 4.5 (Free Extension) that $\text{up}(v^r) \subseteq \lambda(\eta^r)$. By (5.2), v^r has finite outcome along ρ^r and is the principal derivative of $\text{up}(v^r)$ along ρ^r ; so by (2.4), $\text{up}(v^r)$ has infinite outcome along $\lambda(\rho^r)$. Let β^r be the immediate successor of v^r along ρ^r . By (2.4), $\lambda(\beta^r)$ is the immediate successor of $\text{up}(v^r)$ along $\lambda(\rho^r)$, and by (5.1), $\text{out}^0(\beta^r) = \text{out}^0(\lambda(\beta^r))$ is pseudotru. By (5.1), v^r is implication-free, so by (5.23), $\text{up}(v^r)$ is implication-free. Hence by Lemma 5.5(iii) (Completion-Respecting) and Lemma 5.1(viii),(i) (PL Analysis),

$$\text{PL}(\text{up}(\mu^r), \lambda(\rho^r)) \subseteq \text{PL}(\text{up}(\mu^r), \text{up}(v^r)) \cup \{\text{up}(v^r)\} \cup \text{PL}(\text{up}(v^r), \lambda(\rho^r)) \subseteq \\ \text{PL}(\text{up}(\mu^r), \lambda(\eta^r)) \cup \{\text{up}(v^r)\} \cup \text{PL}(\text{up}(v^r), \lambda(\rho^r)).$$

Thus (5.4) for $\mu^r \subseteq \gamma^r \subseteq \rho^r$ follows from Lemma 2.2(i) (Interaction) and (5.4) for $\mu^r \subseteq \delta^r \subseteq \eta^r$ and for $v^r \subseteq \gamma^r \subseteq \rho^r$, contradicting the minimality of $\text{lh}(v^r)$ for ρ^r in Definition 5.1.

Subcase 1.2: δ^r is an amenable pseudocompletion. Let δ^r be a pseudocompletion of v^r . By (5.2) and (5.11)(i), μ^r has finite outcome along η^r and v^r has infinite outcome along η^r , so $v^r \neq \mu^r$. We compare the locations of μ^r and v^r .

Subcase 1.2.1: $v^r \subseteq \mu^r$. Let τ^r be the immediate successor of μ^r along η^r . By (5.2), μ^r is an initial derivative, so $\text{up}(\mu^r) \neq \text{up}(v^r)$. (5.6)-(5.12) are routinely verified for $\langle\langle v^r, \mu^r, \tau^r \rangle\rangle$, using the conditions obtained from the assumptions that η^r requires extension for μ^r and that δ^r is a pseudocompletion of v^r . As $\mu^r \subseteq \delta^r$, it follows from Lemma 5.1(i) (PL Analysis) that $\text{PL}(v^r, \mu^r) \subseteq \text{PL}(v^r, \delta^r)$, so (5.16) follows from the amenability condition for pseudocompletions. Thus $\langle\langle v^r, \mu^r, \tau^r \rangle\rangle$ is an amenable implication chain along τ^r . But by (5.1), $\text{out}^0(\tau^r)$ is pseudotru, so we have contradicted (5.28).

Subcase 1.2.2: $\mu^r \subseteq v^r$. Let ξ^r be the immediate successor of v^r along η^r . Recall that v^r has infinite outcome along η^r , and by (5.2), μ^r is the principal derivative of $\text{up}(\mu^r)$ along η^r , so $\text{up}(\mu^r) \neq \text{up}(v^r)$. Conditions (5.1)-(5.3) and (5.5) are now routinely verified for $\mu^r \subseteq v^r \subseteq \xi^r$, using the conditions obtained from the assumptions that η^r requires extension for μ^r and that δ^r is a pseudocompletion of v^r . Recall that v^r has infinite outcome along η^r , hence along ξ^r , so by Lemma 3.3 (λ -Behavior), $\text{up}(v^r) = (\lambda(\xi^r))^-$ and $\text{up}(v^r)$ has finite outcome along $\lambda(\xi^r)$. By (5.2), μ^r is an initial derivative, so by Lemma 3.1(i) (Limit

Path), $\text{up}(\mu^r) \subset \text{up}(\nu^r)$. Hence by Lemma 5.1(iv) (PL Analysis),

$$\text{PL}(\text{up}(\mu^r), \lambda(\xi^r)) = \text{PL}(\text{up}(\mu^r), \text{up}(\nu^r)).$$

Hence by (5.29)(i),

$$\text{PL}(\text{up}(\mu^r), \text{up}(\nu^r)) \subseteq \{\text{up}(\gamma^r) : \gamma^r \in \text{PL}(\nu^r, \eta^r)\} \cup \text{PL}(\text{up}(\mu^r), \lambda(\eta^r)).$$

Now by Lemma 5.1(ii) (PL Analysis),

$$\text{PL}(\nu^r, \eta^r) \subseteq \text{PL}(\nu^r, \delta^r) \cup \{\delta^r\}.$$

Now $\delta^r \equiv \mu^r$ by (5.2), so by (5.4) for μ^r and η^r , Definition 5.4 for ν^r and δ^r , and Lemma 2.2(i) (Interaction), for all $\pi \in \text{PL}(\text{up}(\mu^r), \lambda(\eta^r)) \cup \text{PL}(\nu^r, \delta^r) \cup \{\delta^r\}$, $\text{TS}(\pi) \cap \text{RS}(\mu^r) = \emptyset$. Thus for all $\pi \in \text{PL}(\text{up}(\mu^r), \lambda(\xi^r))$, $\text{TS}(\pi) \cap \text{RS}(\mu^r) = \emptyset$, so (5.4) holds for $\mu^r \subset \nu^r \subset \xi^r$. Thus ξ^r requires extension for some $\alpha^r \subseteq \mu^r$. As $\nu^r \subset \delta^r$, it follows that $\xi^r \subseteq \delta^r$. Now $\delta = \eta^\pm$ is the principal derivative of δ^r along η , and by (5.1), δ is implication-free. So as δ is admissible, it follows from (5.27) that δ is completion-consistent via $\langle \rangle$. Furthermore, $\text{up}^r(\delta) = \delta^r \supset \nu^r$. Hence by Lemma 5.5(ii) (Completion-Respecting), ξ^r has a primary completion $\kappa^r \subset \delta^r$ which has infinite outcome along $\delta^r \subset \eta^r$. Thus $[\alpha^r, \kappa^r]$ is a primary η^r -link restraining μ^r , contradicting (5.3) for $\mu^r \subset \delta^r \subset \eta^r$.

Case 2: $j = r-1$. By case assumption, $\delta^{r\pm 1}$ must be a primary completion; fix $\rho^{r\pm 1}$ such that $\delta^{r\pm 1}$ is a primary completion of $\rho^{r\pm 1}$. By (5.5)(ii), there is an amenable r -implication-chain $\langle\langle \sigma^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ such that $\rho^{r-1} = \text{out}(\tau^r)$. As η^{r-1} requires extension, it follows from (5.5)(ii) that there is an amenable r -implication-chain $\langle\langle \bar{\sigma}^r, \tilde{\sigma}^r, \bar{\tau}^r \rangle\rangle$ such that $\eta^{r-1} = \text{out}(\bar{\tau}^r)$. By Definition 5.6 and (5.5)(ii), $\sigma^r = \text{up}(\delta^{r-1}) = \tilde{\sigma}^r$, so by (5.8)(i), $\bar{\sigma}^r \subset \tilde{\sigma}^r = \sigma^r \subset \hat{\sigma}^r$. We show that $\langle\langle \bar{\sigma}^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ satisfies (5.6)-(5.12) and (5.15) or (5.16), contradicting the minimality of $\text{lh}(\sigma^r)$ for $\langle\langle \sigma^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ in Definition 5.4.

(5.6)-(5.12) for $\langle\langle \bar{\sigma}^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ follow routinely from (5.6)-(5.12) for $\langle\langle \bar{\sigma}^r, \tilde{\sigma}^r, \bar{\tau}^r \rangle\rangle$ and $\langle\langle \sigma^r, \hat{\sigma}^r, \tau^r \rangle\rangle$, once we recall that $\bar{\sigma}^r \subset \tilde{\sigma}^r \subset \hat{\sigma}^r$, and note that $\bar{\sigma}^r$ has infinite outcome along $\tilde{\sigma}^r$ by (5.11)(i), so $\text{up}(\bar{\sigma}^r) \neq \text{up}(\hat{\sigma}^r)$ by (2.8). Let μ^r be the initial derivative of $\text{up}(\hat{\sigma}^r)$ along $\hat{\sigma}^r$, and let $\tilde{\mu}^r$ be the initial derivative of $\text{up}(\tilde{\sigma}^r)$ along $\tilde{\sigma}^r$. By (5.6), $\mu^r \neq \sigma^r$ and $\tilde{\mu}^r \neq \bar{\sigma}^r$.

Subcase 2.1: $\mu^r \subset \sigma^r$. Then (5.15) must hold for $\langle\langle \sigma^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ and so if we fix $\tilde{\tau}^r \subseteq \hat{\sigma}^r$ such that $(\tilde{\tau}^r)^- = \sigma^r$, then $\tilde{\tau}^r$ requires extension for μ^r . But $\sigma^r = \tilde{\sigma}^r$, so by (5.15) or (5.16) for $\langle\langle \bar{\sigma}^r, \tilde{\sigma}^r, \bar{\tau}^r \rangle\rangle$, σ^r is either a primary completion or an amenable pseudocompletion.

As $r > j$, we have contradicted (i) by induction.

Subcase 2.2: $\sigma^r \subset \mu^r$. We show that $\langle\langle \bar{\sigma}^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ is an amenable implication chain, contradicting the minimality condition in Definition 5.4 as $\bar{\sigma}^r \subset \sigma^r$.

It suffices to show that $\hat{\sigma}^r$ is an amenable pseudocompletion of $\bar{\sigma}^r$. The relevant conditions from (5.6)-(5.11) follow easily from our assumptions that $\langle\langle \bar{\sigma}^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ and $\langle\langle \sigma^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ are amenable implication chains. Fix $\tilde{\tau}^r \subseteq \hat{\sigma}^r$ such that $(\tilde{\tau}^r)^- = \sigma^r$. Then $\hat{\sigma}^r$ is a pseudocompletion of $\bar{\sigma}^r$, and by (5.10)(i), σ^r and $(\text{out}^0(\tilde{\tau}^r))^-$ are implication-free. As any implication-free node on T^0 must be completion-consistent via $\langle \rangle$ (else (5.21) or (5.22) would cause it to be implication-restrained), $(\text{out}^0(\tilde{\tau}^r))^-$ is completion-consistent via $\langle \rangle$. Suppose that $\xi^r \subseteq \sigma^r$ requires extension. If $\xi^r \subset \sigma^r$, then as $(\text{out}^0(\tilde{\tau}^r))^-$ is a derivative of σ^r , the exclusionary conditions of Lemma 5.5(ii) (Completion-Respecting) cannot hold unless σ^r is the primary completion of ξ^r , so ξ^r has a primary completion with infinite outcome along σ^r . And if σ^r is the primary completion of ξ^r , then σ^r has infinite outcome along $\tilde{\tau}^r$. Hence Lemma 5.1(viii) (PL Analysis) can be applied (for $\tilde{\tau}^r$ as the σ^j of the lemma).

Subcase 2.2.1: $\tilde{\mu}^r \subset \bar{\sigma}^r$. Then (5.15) must hold for $\langle\langle \bar{\sigma}^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ and so if we fix $\hat{\tau}^r \subseteq \hat{\sigma}^r$ such that $(\hat{\tau}^r)^- = \bar{\sigma}^r$, then $\hat{\tau}^r$ requires extension for $\tilde{\mu}^r$. But then by (5.15) for $\langle\langle \bar{\sigma}^r, \hat{\sigma}^r, \tau^r \rangle\rangle$, $\bar{\sigma}^r = \sigma^r$ is the primary completion of $\hat{\tau}^r$. By Lemma 5.1(viii) (PL Analysis),

$$\text{PL}(\bar{\sigma}^r, \hat{\sigma}^r) \subseteq \text{PL}(\bar{\sigma}^r, \sigma^r) \cup \{\sigma^r\} \cup \text{UPL}(\sigma^r, \hat{\sigma}^r).$$

By (5.29)(ii),

$$\{\text{up}(\xi^r): \xi^r \in \text{PL}(\bar{\sigma}^r, \sigma^r)\} = \text{PL}(\text{up}(\tilde{\mu}^r), \lambda(\hat{\tau}^r)).$$

Hence by (5.4) for $\tilde{\mu}^r \subset \bar{\sigma}^r \subset \hat{\tau}^r$, Lemma 2.2(i) (Interaction) and Definition 5.4 for $\sigma^r \subset \hat{\sigma}^r$, for all $\pi \in \text{PL}(\bar{\sigma}^r, \sigma^r) \cup \{\sigma^r\} \cup \text{UPL}(\sigma^r, \hat{\sigma}^r)$, $\text{TS}(\pi) \cap \text{RS}(\bar{\sigma}^r) = \emptyset$, so $\hat{\sigma}^r$ is an amenable pseudocompletion of $\bar{\sigma}^r$. As $r > j$, we have contradicted (i) by induction.

Subcase 2.2.2: $\tilde{\mu}^r \supset \bar{\sigma}^r$. Then (5.16) must hold for both $\langle\langle \bar{\sigma}^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ and $\langle\langle \sigma^r, \hat{\sigma}^r, \tau^r \rangle\rangle$. By Lemma 5.1(viii) (PL Analysis), $\text{PL}(\bar{\sigma}^r, \tau^r) \subseteq \text{PL}(\bar{\sigma}^r, \sigma^r) \cup \text{UPL}(\sigma^r, \tau^r) \cup \{\sigma^r\}$. (5.16) for $\langle\langle \bar{\sigma}^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ now follows from (5.16) for $\langle\langle \bar{\sigma}^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ and $\langle\langle \sigma^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ and Lemma 2.2(i) (Interaction). Thus $\langle\langle \bar{\sigma}^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ satisfies (5.6)-(5.12) and (5.16), contradicting the minimality of $\text{lh}(\sigma^r)$ for $\langle\langle \sigma^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ in Definition 5.4.

Case 3: $j < r-1$. Let δ^j be the j -completion of v^j and let v^j require extension for ρ^j . By (5.5)(ii), there is an amenable r -implication-chain $\langle\langle \sigma^i, \widehat{\sigma}^i, \tau^i \rangle\rangle: r \geq i \geq j+1$ such that $v^j = \text{out}(\tau^{j+1})$. As η^j requires extension, it follows from (5.5)(ii) that there is an amenable r -implication-chain $\langle\langle \bar{\sigma}^i, \widetilde{\sigma}^i, \bar{\tau}^i \rangle\rangle: r \geq i \geq j+1$ such that $\eta^j = \text{out}(\bar{\tau}^{j+1})$. By (5.5)(ii) and (5.9), $\widetilde{\sigma}^{j+1} = \text{up}(\delta^j) = \sigma^{j+1}$. As $j+1 < r$, the conditions of (5.16) at $j+1$ are not satisfied by either amenable implication chain, so (5.15) must hold at $j+1$ for both implication chains. By (5.5)(ii) for $\langle\langle \bar{\sigma}^i, \widetilde{\sigma}^i, \bar{\tau}^i \rangle\rangle: r \geq i \geq j+1$, $\widetilde{\sigma}^{j+1}$ is a primary completion. By (5.5)(ii) for $\langle\langle \sigma^i, \widehat{\sigma}^i, \tau^i \rangle\rangle: r \geq i \geq j+1$, if $\tilde{\tau}^{j+1}$ is the immediate successor of σ^{j+1} along $\widehat{\sigma}^{j+1}$, then $\tilde{\tau}^{j+1}$ requires extension. But $\widetilde{\sigma}^{j+1} = \sigma^{j+1}$, so we have contradicted (i) inductively.

(ii): Let $r = \dim(\eta^\pm) - 1$. We assume that η is nonswitching and η^\pm is primarily or hereditarily implication-restrained, and derive a contradiction. By hypothesis, η^j requires extension. As $\eta^j \neq \lambda^j(\eta^\pm)$ and η is nonswitching, it follows from Lemma 3.3 (λ -Behavior) that $(\eta^j)^- = \text{up}^j(\eta^\pm) = \lambda^j(\eta^\pm)$, so η^\pm is the principal derivative of $(\eta^j)^-$ along $\eta = \text{out}^0(\eta^j)$; and by (5.2), $\lambda^j(\eta^\pm)$ has infinite outcome along η^j . Now by Definition 5.1, $j \leq r$. By (2.4), $\lambda^j(\eta^\pm)$ is the principal derivative of $\text{up}(\lambda^j(\eta^\pm))$ along η^j , so as η is nonswitching, $(\eta^{j+1})^- = \text{up}^{j+1}(\eta^\pm) = \lambda^{j+1}(\eta^\pm)$. By (5.1), (5.10)(ii) and as η^\pm is implication-restrained, $j+1 < r$, so by (5.5)(ii), $\text{up}^{j+1}(\eta^\pm)$ is a primary completion. But then as η^\pm is implication-restrained, it follows from (5.18)(i) that η is switching, contrary to hypothesis.

If η is a 0-completion, then by Definition 5.6, η is nonswitching and η^\pm is implication-restrained. (ii) now follows. \square

In order to show that k -completions exist, it will be necessary for the paths constructed to be admissible. We thus need to analyze the process of constructing paths, and to show that we can construct admissible paths. The proof will proceed by induction on $n-k$, and then by induction on $\text{lh}(\eta^k)$ for $\eta^k \in T^k$. There are some induction hypotheses that will also need to be verified. We will need to know that admissible nodes are completion-consistent for some set. And we will need to show a relationship between certain PL sets on T^k at η^k and corresponding PL sets on T^{k+1} at $\lambda(\eta^k)$ whenever η^k is not completion-respecting. We prove several lemmas which will give us the desired information. The first lemma treats the case where the node to be extended is completion-consistent via $\langle \rangle$. We treat the case where extensions are taken during the backtracking process in Lemmas 5.9-5.14.

Lemma 5.8 (Completion-Respecting Admissible Extension Lemma): Fix $\eta, \xi \in T^0$ such that $\xi^- = \eta$, ξ is preadmissible, and η is completion-consistent via $\langle \rangle$. Then ξ is admissible and either ξ is completion-consistent via $S = \langle \rangle$, or ξ is completion-consistent via $S = \langle \lambda^j(\xi) \rangle$ for some $j \leq n$.

Proof: As η is admissible and completion-consistent via $S = \langle \rangle$, (5.27) is vacuous. We now verify (5.28). As η is admissible, (5.28) follows by induction if ξ is not pseudotrue. Hence we may suppose that ξ is pseudotrue for the sake of proving (5.28). It then follows that $\lambda^j(\xi)$ is not a primary completion for any $j \leq n$.

Suppose that $\langle \langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle : r \geq j \geq k \rangle$ is an amenable k -implication chain with $\tau^k \subseteq \lambda^k(\xi)$ in order to obtain a contradiction. Then by (5.15) or (5.16), $\hat{\sigma}^k$ is a primary completion or an amenable pseudocompletion, and by (5.11)(ii), $\hat{\sigma}^k$ has finite outcome along τ^k . Thus $k > 0$, else by (5.17)(i) or (5.18)(i), $\hat{\sigma}^k$ would have infinite outcome along τ^k . By Lemma 5.2 (Requires Extension), $\text{out}(\tau^k)$ requires extension; so as η and ξ are completion-consistent via $\langle \rangle$, it follows from Lemma 5.5(iii) (Completion-Respecting) that $\text{out}(\tau^k)$ has a primary completion κ^{k+1} which has infinite outcome along its immediate successor $\beta^{k+1} \subseteq \lambda^{k+1}(\xi)$. But now by (5.19) and (5.5)(ii), β^{k+1} switches σ^k , so $\tau^k \not\subseteq \lambda^k(\xi)$, a contradiction. Hence (5.28) holds.

We now verify (5.29)(i)-(iii). For all $i \leq n$, let $\xi^i = \lambda^i(\xi)$, $\eta^i = \lambda^i(\eta)$, and $\bar{\eta}^i = \text{up}^i(\eta)$. By (2.5) and Lemma 3.1 (Limit Path), for any $\gamma^k \in T^k$, if $\gamma \in T^0$ is an initial derivative of γ^k , then for all β such that $\text{out}^0(\gamma^k) \subseteq \beta \subseteq \gamma$, $\lambda^k(\beta) = \gamma^k$; and by (2.4) and Lemma 5.4 (Compatibility), if $\alpha \supset \gamma$, $\gamma^k \subseteq \lambda^k(\alpha)$, and γ^k is a k -completion, then $\gamma^k \subseteq \lambda^k(\alpha)$. Hence if ξ^k were a k -completion, then $\text{out}^0(\xi^k) \subseteq \xi$ and either ξ would be a 0-completion or there would be no 0-completion corresponding to ξ^k along ξ ; in either case, it follows from Definition 5.6, that $\eta = \xi^-$ must be primarily or hereditarily implication-restrained so cannot be completion-consistent via $\langle \rangle$, contrary to our hypothesis. (5.29)(ii) and (5.29)(iii) now follow from (5.27) by induction.

We now note that if η is implication-restrained, then ξ does not switch η . For by (5.17), if ξ were to switch η and η were implication-restrained, then η would have to be either a primary completion or an amenable pseudocompletion. If η is an amenable pseudocompletion, then $\dim(\eta) = 1$ and this is impossible by (5.10); and if η is a primary completion, then $\dim(\eta) > 1$ and η must be a 0-completion contrary to (5.29)(iii). Hence by (5.23), if ξ is switching, then $\bar{\eta}^i$ is implication-free for all $i \leq n$.

We now verify (5.29)(i). Fix $k < n$ and $\mu^k \subset \nu^k \subset \xi^k$ such that ν^k is implication-free and $\text{up}(\mu^k) \subset \text{up}(\nu^k), \xi^{k+1}$, and fix p and s for ξ as in Lemma 3.3 (λ -Behavior). Note that η is admissible, so (5.29)(i) holds for all $\gamma \subseteq \eta$.

Case 1: $k+1 \leq p$. Then by Lemma 3.3 (λ -Behavior), $\nu^k \subseteq \eta^k = (\xi^k)^-$ and $\text{up}(\nu^k) \subseteq \eta^{k+1}$, so as $\text{up}(\mu^k) \subset \text{up}(\nu^k)$, it follows that $\text{up}(\mu^k) \subset \eta^{k+1}$. Hence by (5.29)(i) for η if $\nu^k \subset \eta^k$, and by (2.7) and Lemma 5.1(i) (PL Analysis) if $\nu^k = \eta^k$,

$$\text{PL}(\text{up}(\mu^k), \text{up}(\nu^k)) \subseteq \{\text{up}(\alpha^k): \alpha^k \in \text{PL}(\nu^k, \eta^k)\} \cup \text{PL}(\text{up}(\mu^k), \eta^{k+1}).$$

Again by Lemma 3.3 (λ -Behavior), $\eta^{k+1} \subset \xi^{k+1}$, so by Lemma 5.1(i) (PL Analysis), $\text{PL}(\nu^k, \eta^k) \subseteq \text{PL}(\nu^k, \xi^k)$ and $\text{PL}(\text{up}(\mu^k), \eta^{k+1}) \subseteq \text{PL}(\text{up}(\mu^k), \xi^{k+1})$, so (5.29)(i) holds for k .

Case 2: $k+1 > s$. By Lemma 3.3 (λ -Behavior), there are $i < k+1$ and $\sigma^i = (\xi^i)^-$ such that for all $q \in [i+1, n]$, $\xi^q = \lambda^q(\sigma^i) = \sigma^q$. As $\nu^k \subset \xi^k$, it follows that $\nu^k \subseteq \sigma^k$. Let $\sigma = \text{out}^0(\sigma^i)$, and note that by (2.5), $\sigma \subset \xi$. Thus by (2.5) and (5.29)(i) for σ if $\nu^k \subset \sigma^k$, and by (2.7) and Lemma 5.1(i) (PL Analysis) if $\nu^k = \sigma^k$,

$$\text{PL}(\text{up}(\mu^k), \text{up}(\nu^k)) \subseteq \{\text{up}(\alpha^k): \alpha^k \in \text{PL}(\nu^k, \sigma^k)\} \cup \text{PL}(\text{up}(\mu^k), \sigma^{k+1}).$$

We have noted that $\sigma^{k+1} = \xi^{k+1}$, and that $\sigma^k \subseteq \xi^k$; hence by Lemma 5.1(i) (PL Analysis), $\text{PL}(\nu^k, \sigma^k) \subseteq \text{PL}(\nu^k, \xi^k)$, so (5.29)(i) holds for k .

Case 3: $p < k+1 \leq s$. By Lemma 3.3 (λ -Behavior), $\bar{\eta}^k = (\xi^k)^-$ and ξ switches η^{k+1} . As $\nu^k \subset \xi^k$, it follows that $\nu^k \subseteq \bar{\eta}^k$. Let β be the initial derivative of $\bar{\eta}^k$ along ξ . By (2.7), $\text{up}(\nu^k) \subseteq \lambda(\nu^k)$. Hence by (2.4), (2.6), and as $\text{up}(\mu^k) \subset \text{up}(\nu^k), \xi^{k+1}$, it follows that $\text{up}(\mu^k) \subset \lambda(\bar{\eta}^k)$. Hence by (5.29)(i) for β if $\nu^k \subset \bar{\eta}^k$, and by Lemma 5.1(i) (PL Analysis) if $\nu^k = \bar{\eta}^k$,

$$\text{PL}(\text{up}(\mu^k), \text{up}(\nu^k)) \subseteq \{\text{up}(\alpha^k): \alpha^k \in \text{PL}(\nu^k, \bar{\eta}^k)\} \cup \text{PL}(\text{up}(\mu^k), \lambda(\bar{\eta}^k)).$$

Suppose that $\rho^{k+1} \in (\text{PL}(\text{up}(\mu^k), \text{up}(\nu^k)) \cap \text{PL}(\text{up}(\mu^k), \lambda(\bar{\eta}^k))) \setminus \text{PL}(\text{up}(\mu^k), \xi^{k+1})$. As ξ is q -switching for some $q \leq k+1$, it follows from an earlier observation that $\bar{\eta}^k$ is implication-free. Furthermore, by Lemma 3.3 (λ -Behavior), $(\xi^k)^- = \bar{\eta}^k$ and $(\xi^{k+1})^- = \bar{\eta}^{k+1} = \text{up}(\bar{\eta}^k)$.

First suppose that (5.13) places ρ^{k+1} into $\text{PL}(\text{up}(\mu^k), \text{up}(\nu^k))$. Then there is a γ^{k+1} such that $[\gamma^{k+1}, \rho^{k+1}]$ is a primary $\text{up}(\nu^k)$ -link restraining $\text{up}(\mu^k)$, so $\rho^{k+1} \subset \text{up}(\nu^k)$. By Lemma 3.1(i) (Limit Path), ρ^{k+1} has an initial derivative $\rho^k \subset \nu^k$. By (2.10) and as $\rho^{k+1} \in \text{PL}(\text{up}(\mu^k), \lambda(\bar{\eta}^k)) \setminus \text{PL}(\text{up}(\mu^k), \xi^{k+1})$ and ξ switches $\bar{\eta}^{k+1}$, $\bar{\eta}^{k+1} = \rho^{k+1}$, and ρ^{k+1} has infinite outcome along $\lambda(\bar{\eta}^k)$ but finite outcome along ξ^{k+1} ; hence by Lemma 3.3 (λ -Behavior), $\bar{\eta}^k$ has infinite outcome along ξ^k and $[\rho^k, \bar{\eta}^k]$ is a primary ξ^k -link. Now $\nu^k \neq \bar{\eta}^k$, else $\text{up}(\nu^k) = \bar{\eta}^{k+1} = \rho^{k+1}$, so $\rho^{k+1} \notin \text{PL}(\text{up}(\mu^k), \text{up}(\nu^k))$, contrary to our assumption. Hence $[\rho^k, \bar{\eta}^k]$ is a primary ξ^k -link restraining ν^k . But then (5.13) places $\bar{\eta}^k$ into $\text{PL}(\nu^k, \xi^k)$, as required by (5.29)(i).

Suppose that (5.14) places ρ^{k+1} into $\text{PL}(\text{up}(\mu^k), \text{up}(\nu^k))$, but (5.13) does not. Now $\bar{\eta}^{k+1} = \text{up}^{k+1}(\eta)$, and we have noted that $\bar{\eta}^{k+1}$ is implication-free. Let $\text{PL}(\gamma^{k+1}, \pi^{k+1})$ be a component of $\text{PL}(\text{up}(\mu^k), \text{up}(\nu^k))$ which causes ρ^{k+1} to be placed into $\text{PL}(\text{up}(\mu^k), \text{up}(\nu^k))$, with π^{k+1} as long as possible. It follows from Definition 5.3 that if $\pi^{k+1} \subset \text{up}(\nu^k)$, then π^{k+1} has infinite outcome along $\text{up}(\nu^k)$. As ν^k is implication-free, it follows from (5.23) that $\text{up}(\nu^k)$ is implication-free; so by (5.21) and Definition 5.3, π^{k+1} must be the primary completion of the immediate successor δ^{k+1} of γ^{k+1} along π^{k+1} for some $\bar{\mu}^{k+1} \subset \text{up}(\mu^k)$. By Definition 5.6, $\delta^{k+1} \subset \rho^{k+1}$, so as $\rho^{k+1} \in \text{PL}(\text{up}(\mu^k), \lambda(\bar{\eta}^k))$, $\delta^{k+1} \subset \rho^{k+1} \subseteq \lambda(\bar{\eta}^k)$. By Lemma 5.5(ii) (Completion-Respecting), either $[\bar{\mu}^{k+1}, \pi^{k+1}]$ is a primary $\lambda(\bar{\eta}^k)$ -link restraining $\text{up}(\mu^k)$, or $\bar{\eta}^{k+1} = \pi^{k+1}$ or $\bar{\eta}^{k+1} = \gamma^{k+1}$.

Subcase 3.1: $[\bar{\mu}^{k+1}, \pi^{k+1}]$ is a primary $\lambda(\bar{\eta}^k)$ -link restraining $\text{up}(\mu^k)$. Now $\pi^{k+1} \subset \lambda(\bar{\eta}^k)$, and by (2.7), $\bar{\eta}^{k+1} \subseteq \lambda(\bar{\eta}^k)$; hence π^{k+1} and $\bar{\eta}^{k+1}$ are comparable. By (2.10), $\bar{\eta}^{k+1} \notin [\bar{\mu}^{k+1}, \pi^{k+1}]$. Also, $\bar{\eta}^{k+1} \not\subset \bar{\mu}^{k+1}$, else as $\mu^{k+1} \supset \bar{\mu}^{k+1}$ and ξ switches $\bar{\eta}^{k+1}$, we would not have $\mu^{k+1} \subset \xi^{k+1}$. Hence $\pi^{k+1} \subseteq \bar{\eta}^{k+1} \subset \xi^{k+1}$, and so $\text{PL}(\gamma^{k+1}, \pi^{k+1})$ is a component of $\text{PL}(\text{up}(\mu^k), \xi^{k+1})$. But then $\rho^{k+1} \in \text{PL}(\text{up}(\mu^k), \xi^{k+1})$, a contradiction.

Subcase 3.2: $\bar{\eta}^{k+1} = \pi^{k+1}$. Proceed as in the last two sentences of Subcase 3.1.

Subcase 3.3: $\bar{\eta}^{k+1} = \gamma^{k+1}$. Recall that $\delta^{k+1} \subseteq \text{up}(\nu^k) \subseteq \eta^{k+1}$ requires extension. As η is completion-consistent via $\langle \rangle$, δ^{k+1} has a 0-completion $\pi^0 \subseteq \eta$. And as $\delta^{k+1} \subseteq \eta^{k+1}$, it follows from Lemma 5.4 (Compatibility) that for all $i \leq k$, δ^{k+1} has an i -completion $\pi^i \subseteq \eta^i$; and by Definition 5.6, $\text{up}(\pi^i) = \pi^{i+1}$ for all $i \leq k$. Now $\text{up}^{k+1}(\pi^0) = \pi^{k+1} \neq \gamma^{k+1} = \text{up}^{k+1}(\eta)$, so $\pi^0 \neq \eta$. Hence $\pi^0 \subset \eta$. Thus by induction using (2.4), $\pi^i \subset \eta^i$ for all $i \leq k$, so $\pi^k \subset \eta^k$.

First suppose that all derivatives of π^{k+1} along η^k have finite outcome along η^k . Then by (2.4), $[\bar{\mu}^{k+1}, \pi^{k+1}]$ is a primary η^{k+1} -link restraining $\gamma^{k+1} \supset \text{up}(\mu^k)$. But then by (2.10), ξ could not switch $\bar{\eta}^{k+1} = \gamma^{k+1}$, a contradiction.

We conclude that there is a derivative $\bar{\pi}^k$ of π^{k+1} which has infinite outcome along η^k . Let σ^k be the immediate successor of $\bar{\pi}^k$ along η^k . By (2.4), π^{k+1} has finite outcome along $\lambda(\sigma^k)$, so by Lemma 5.3(ii) (Implication Chain) and Lemma 5.2 (Requires Extension), σ^k requires extension for the initial derivative γ^k of γ^{k+1} along η^k . As η is completion-consistent via $\langle \rangle$, σ^k must have a 0-completion $\kappa \subseteq \eta$. First suppose that η is a 0-completion corresponding to a primary i -completion. As $\text{up}^{k+1}(\eta) = \bar{\eta}^{k+1} = \gamma^{k+1}$ and, by Lemma 5.7(i) (Primary Completion), γ^{k+1} is not a primary completion, it follows from (5.5)(ii), (5.9) and (5.12) that $k+1-i$ is odd. Hence $k-i$ is even and by (5.5)(ii), (5.9) and

(5.12), $\bar{\eta}^k$ is the primary completion of σ^k . Otherwise, by Lemma 5.5(ii) (Completion-Respecting), σ^k must have a primary completion $\kappa^k \subseteq \eta^k$. Now $\gamma^{k+1} \subset \text{up}(v^k)$, so by Lemma 3.1(i) (Limit Path) $\gamma^k \subset v^k$. Thus $\text{PL}(\bar{\pi}^k, \kappa^k)$ is a component of $\text{PL}(v^k, \xi^k)$ which places $\bar{\pi}^k$ into $\text{PL}(v^k, \xi^k)$ via (5.14)(i), completing the proof for the case in which $\rho^{k+1} = \gamma^{k+1}$. Furthermore, by (5.29)(ii) and (5.14)(ii), $\text{PL}(\gamma^{k+1}, \pi^{k+1}) \subseteq \{\text{up}(\alpha^k): \alpha^k \in \text{PL}(\bar{\pi}^k, \kappa^k)\} \subseteq \{\text{up}(\alpha^k): \alpha^k \in \text{PL}(v^k, \xi^k)\}$, so (5.29)(i) holds in this case.

As η is completion-consistent via $S = \langle \rangle$, it follows from Lemma 3.1(i) (Limit Path) that any $\rho^j \subset \lambda^j(\xi)$ which requires extension satisfies $\rho^j = \lambda^j(\gamma)$ for some $\gamma \subseteq \eta$, so has a 0-completion $\subseteq \eta$ by (5.27). Hence if ξ is completion-deficient via some $Z \neq \langle \rangle$, then all elements of Z are of the form $\lambda^j(\xi)$ for some j . The last conclusion of the lemma now follows from Lemma 5.6 (Uniqueness of Requiring Extension), so ξ is admissible. \square

When an admissible $\eta \in T^0$ requires extension, we will need to find an admissible 0-completion κ of η . The process of constructing κ is called *backtracking*. The next lemma indicates the manner in which backtracking preserves completion-consistency.

Lemma 5.9 (Completion-Consistency Lemma): Fix $m \geq 0$ and $\rho, \sigma \in T^0$ such that σ is preadmissible, and $\sigma^- = \rho$. Assume that ρ is completion-consistent via $S = \langle \eta_i: i \leq m \rangle$ and that σ is completion-deficient for V . Let U be the sequence obtained by ordering V according to the inclusion relation induced by out^0 on the elements of V . (Note that by Lemma 5.6 (Uniqueness of Requiring Extension), a linear ordering is obtained in this way.) For all $j \leq n$, let $\sigma^j = \lambda^j(\sigma)$, $\rho^j = \lambda^j(\rho)$, and $\bar{\rho}^j = \text{up}^j(\rho)$. Then σ is completion-consistent via U and:

- (i) If σ is a 0-completion, then $U = \langle \eta_i: i < m \rangle$.
- (ii) If (5.18)(ii)(a) holds for ρ , j is defined as in (5.18)(ii)(a), s is defined as in Lemma 3.3 (λ -Behavior), $\bar{\rho}^{t+1}$ is a primary completion for some t such that $j-1 \leq t \leq s$, and $\bar{\rho}^t$ has infinite outcome along σ^t , then $t = j-1$, $U = S \wedge \langle \sigma^t \rangle$ and σ^t requires extension.
- (iii) If the hypotheses of (i) or (ii) are not satisfied, then $U = S$.
- (iv) σ satisfies (5.27) and (5.28).

Proof: For each $i \leq m$, fix $k(i)$ such that $\eta_i \in T^{k(i)}$. If σ^j requires extension for some j such that $\sigma^j \neq \rho^j$, then by Lemma 5.6 (Uniqueness of Requiring Extension), we let $k(m+1)$ be the unique such j and let $\eta_{m+1} = \sigma^{k(m+1)}$.

Fix $u \leq n$, and $\eta^u \subset \sigma^u$ such that η^u requires extension and $\eta^u \notin S$. By (2.5), $\eta = \text{out}^0(\eta^u) \subset \sigma$, so $\eta \subseteq \rho$. Furthermore, if \bar{S} is the set via which η is completion-consistent,

then $\eta^u \in \bar{S}$. As $\eta^u \notin S$, it follows from (5.27) and the admissibility of ρ that η^u must have a 0-completion along $\rho \subset \sigma$. Hence $\eta^u \notin U$. We conclude that $U \setminus S \subseteq \{\sigma^u: u \leq n \text{ \& \; } \text{out}^0(\sigma^u) \not\subseteq \rho\}$. By the preceding paragraph, $U \setminus S$ has at most one element.

Suppose that σ is a 0-completion. By Definition 5.6, σ is nonswitching, so $\rho^j \subseteq \sigma^j$ for all $j \leq n$. By Lemma 5.7(ii) (Primary Completion), σ^j cannot require extension for any $j \leq n$. Thus $U \subseteq S$ (as sets) by the preceding paragraph. By Definition 5.6, σ must be a 0-completion of η_m and, as noted in Definition 5.6, cannot be a 0-completion of η_i for any $i < m$. Thus (i) follows.

Assume that σ is not a 0-completion. (We note that this will be the case if ρ satisfies (5.18)(ii)(a), as then by (5.18)(ii)(b), σ would be switching, and 0-completions are nonswitching.) We first show that $S \subseteq U$ as sets. Suppose that $\eta_i \in S$. By Definition 5.8, η_i has no 0-completion along ρ . As σ is switching, it follows from Definition 5.6 that σ is not a 0-completion of η_i . Hence η_i has no 0-completion along σ . By (5.18) and (5.25), σ is not u -switching for any $u \leq k(i)$, so $\eta_i \subseteq \rho^{k(i)} \subseteq \sigma^{k(i)}$. Hence $\eta_i \in U$.

Suppose that the hypotheses of (ii) hold. Then by (5.18)(ii)(b), Lemma 5.2 (Requires Extension), and Lemma 5.3(ii) (Implication Chain), $\sigma^t \in U$. Now by (2.5), $\text{out}^0(\sigma^t) = \sigma$, so σ^t is the last element of U . By (5.18)(ii)(b), $t \geq k(m)$; so it follows by induction, our characterization of $U \setminus S$, and as $\sigma^t = \lambda^t(\sigma)$ that (5.25) and (5.26) hold.

In order to complete the proof of (ii), we must show that if j is defined as in (5.18)(ii)(a), then $t = j-1$. By Lemma 3.3 (λ -Behavior), it must be the case that $(\sigma^t)^- = \bar{\rho}^t$. As $S \neq \langle \rangle$, it follows from (5.21) or (5.22) that ρ is not implication-free; hence by (5.1), $u < \dim((\sigma^t)^-)-1$. Now $\rho^{t+1} \neq \bar{\rho}^{t+1}$, else by (5.5)(ii) and (5.15), $\rho^{t+1} = \bar{\rho}^{t+1}$ would be a primary completion, hence (5.18)(i)(a) would hold, excluding the possibility that (5.18)(ii)(a) holds. Thus as σ is j -switching, $j \leq t+1$. Fix p and s as in Lemma 3.3 (λ -Behavior) for σ . As σ is j -switching, $j = p+1$, and as σ^t requires extension and $\sigma = \text{out}^0(\sigma^t)$ (else $\sigma \in S$), $t \leq s$. By choice of j in (5.18)(ii)(a), $\bar{\rho}^j$ is the end of a primary ρ^j -link, so $\bar{\rho}^j$ has infinite outcome along ρ^j and is not an initial derivative; hence by (2.4) and Lemma 3.3 (λ -Behavior), $s = j$ and $\bar{\rho}^j$ has finite outcome along σ^j . Hence $j \leq t+1 \leq s+1 \leq j+1$, so $t \leq j \leq t+1$. By hypothesis, $\bar{\rho}^t$ has infinite outcome along σ^t , so $t \neq j$. Thus $t = j-1$, completing the proof of (ii).

We now complete the proof of (iii). Suppose that $u \leq n$ and $\sigma^u \in U \setminus S \subseteq \{\gamma^u: u \leq n \text{ \& \; } \text{out}^0(\gamma^u) \not\subseteq \rho\}$, in order to obtain a contradiction. By Lemma 3.3 (λ -Behavior), it must be the case that $(\sigma^u)^- = \bar{\rho}^u$. Furthermore, as (5.18)(i)(a) and (5.18)(ii)(a) fail to hold for σ , it follows from (5.18) that σ is nonswitching, so by Lemma 3.3 (λ -Behavior), $\rho^u = \bar{\rho}^u$. As $S \neq \langle \rangle$, it follows from (5.21) or (5.22) that ρ is not implication-free; hence by (5.1), $u < \dim((\sigma^u)^-)-1$. As $\sigma^u \in U \setminus S$ and σ^u requires extension, it now follows from (5.5) and

Lemma 3.3 (λ -Behavior) that $\rho^{u+1} = \bar{\rho}^{u+1}$ is a primary completion, and that ρ is the 0-completion corresponding to ρ^{u+1} . But then (5.18)(i)(a) holds, contradicting the hypotheses of (iii).

(5.27) follows from (i)-(iii); and (5.28) follows by induction as σ is not pseudotrue. Hence (iv) holds. \square

The next lemma keeps track of the relationship between various nodes, as we follow the step-by-step process of going from a node which requires extension to its primary completion. At a given step in the process, we will begin with a node $\rho \in T^0$ which is completion-consistent via a sequence $S = \langle \eta_i : i \leq m \rangle$ and extend ρ to σ such that $\sigma^- = \rho$ and σ is completion-consistent via a sequence U . (We will allow $m = -1$, but only if $U \neq \langle \rangle$.) U has been characterized by Lemma 5.9 (Completion-Consistency); let $w = |U| - 1$. For each $i \leq m$, fix $k(i)$ such that $\eta_i \in T^{k(i)}$, and if $\lambda^j(\sigma)$ requires extension for some j such that $\lambda^j(\sigma) \neq \lambda^j(\rho)$, let $k(m+1)$ be that j and let $\eta_{m+1} = \lambda^{k(m+1)}(\sigma)$. For each $i \leq w$, let $\delta_i = (\eta_i)^-$, and let η_i require extension for v_i .

Clauses (i) and (ii) of Lemma 5.10 specify that each element of $\{\text{up}(v_i) : i \leq w\}$ lies along the branch of $T^{k(i)+1}$ computed by σ , and that the inclusion ordering of elements of this set which lie on the same tree agrees with the ordering on the indices of these nodes, and so by (5.26), is the same as the ordering induced on the subset of U corresponding to the same indices. And clause (v) will be shown to imply that the immediate successors of the elements in $\{\text{up}(v_i) : i \leq w\}$ which lie along this branch of $T^{k(i)+1}$ require extension in the order specified by the indices which agrees with the order induced by inclusion, and none has a primary completion along the next node which requires extension. We cannot specify the ordering of $\{v_i : i \leq w\}$ lying on the same tree, but clauses (iii) and (vii) specify that each v_i is shorter than δ_{i+1} , causing a component for a PL set for δ_{i+1} to be formed. Clause (iv) is used to show that on this branch of $T^{k(i)+1}$, no elements of $T^{k(i)+1}$ except those in $\{\text{up}(v_i) : i \leq w\}$ can require extension without having a primary completion along the path. And clause (vi) relates nodes on trees of successive dimension, and implies the property induced by (5.25), namely, that higher dimension nodes find completions before we encounter any new node on a lower dimensional tree which requires extension.

Lemma 5.10 (Component Lemma): Fix $m \geq -1$ and $\rho, \sigma \in T^0$ such that σ is preadmissible and $\sigma^- = \rho$. Assume that ρ is completion-consistent via $S = \langle \eta_i : i \leq m \rangle$, that σ is completion-consistent via U , and that if $m = -1$, then $|U| \neq 0$. For each $i \leq m$, fix $k(i)$ such that $\eta_i \in T^{k(i)}$, and if $\lambda^j(\sigma)$ requires extension for some j such that $\lambda^j(\sigma) \neq \lambda^j(\rho)$, let $k(m+1)$ be that j (which is unique by Lemma 5.6 (Uniqueness of Requiring Extension)) and let $\eta_{m+1} = \lambda^{k(m+1)}(\sigma)$. Let $w = m - 1$ if $U \subset S$, let $w = m$ if $U = S$, and let $w = m + 1$ otherwise. For each $i \leq w$, let $\delta_i = (\eta_i)^-$, and let η_i require extension for v_i . For all $j \leq n$, let $\sigma^j = \lambda^j(\sigma)$, $\rho^j = \lambda^j(\rho)$, and $\bar{\rho}^j = \text{up}^j(\rho)$. Then for all $i \leq w$:

- (i) $\text{up}(v_i) \subseteq \rho^{k(i)+1} \wedge \sigma^{k(i)+1} = \bar{\rho}^{k(i)+1}$.
- (ii) If $0 < i$ and $k(i) = k(i-1)$, then $\text{up}(v_{i\pm 1}) \subseteq \text{up}(v_i)$.
- (iii) If $0 < i$ and $k(i) = k(i-1)$, then $v_i \subseteq \delta_{i\pm 1}$.
- (iv) Let $\mu^{k(i)+1} \subseteq \text{up}(v_i) \subseteq (\xi^{k(i)+1})^- \subseteq \xi^{k(i)+1} \subseteq \sigma^{k(i)+1}$ be given such that $\xi^{k(i)+1}$ requires extension for $\mu^{k(i)+1}$. Then one of the following holds:
 - (a) There is a primary completion $\kappa^{k(i)+1} \subseteq \sigma^{k(i)+1}$ of $\xi^{k(i)+1}$ such that $\kappa^{k(i)+1}$ has infinite outcome along $\sigma^{k(i)+1}$.
 - (b) Either $i < w$, $k(i+1) = k(i)$, and $\text{up}(v_{i+1}) \subseteq (\xi^{k(i)+1})^-$, or $w = 0$ and $\text{up}(v_0) = (\xi^{k(0)+1})^-$.
 - (c) $\sigma^{k(m)}$ is the primary completion of η_m and $(\xi^{k(i)+1})^- = \text{up}(v_m)$.
- (v) If $\dim(v_i) > k(i)+1$, then the immediate successor τ_i of $\text{up}(v_i)$ along $\sigma^{k(i)+1}$ requires extension for some $\mu^{k(i)+1}$; and if $0 < i$ and $k(i) = k(i-1)$, then $\mu^{k(i)+1} \subseteq \text{up}(v_{i\pm 1})$.
- (vi) If $j < i$ and $k(j) = k(i)-1$, then $\text{up}(v_j) \subseteq v_i$.
- (vii) If $0 < i$ and $k(i) = k(i-1)$, then $\text{PL}(\delta_i, \sigma^{k(i)})$ is a component of $\text{PL}(\delta_{i\pm 1}, \sigma^{k(i)})$.

Proof: We proceed by induction on $\text{lh}(\sigma)$.

Case 1: $m = -1$, so $w \neq -1$ by hypothesis. By Lemma 5.9 (Completion-Consistency), $w = 0$, $\sigma = \text{out}^0(\eta_0)$, and ρ is completion-consistent via $\langle \rangle$. (ii), (iii), (vi), and (vii) are vacuous in this case since $w = 0$. We verify (i), (iv), and (v).

(i): By (5.3) and Lemma 4.3(i)(a) (Link Analysis), $\text{up}(v_0) \subseteq \sigma^{k(0)+1}$. As $\eta_0 = \sigma^{k(0)}$, $\sigma = \text{out}^0(\sigma^{k(0)})$, so as $m = -1 \neq w$, $\sigma^{k(0)} \neq \rho^{k(0)}$. Furthermore, by (5.2), $\delta_0 = (\sigma^{k(0)})^-$ has infinite outcome along $\eta_0 = \sigma^{k(0)}$. By (2.4) and as $m = -1$, $\sigma^{k(0)+1} = \lambda(\sigma^{k(0)}) = \text{up}(\delta_0) \wedge \langle \sigma^{k(0)} \rangle$, so $\sigma^{k(0)} = \text{out}(\sigma^{k(0)+1})$, and $(\sigma^{k(0)+1})^- \subseteq \rho^{k(0)+1}$. Hence by Lemma 3.3 (λ -Behavior), $\text{up}(\delta_0) = \rho^{k(0)+1} \wedge \sigma^{k(0)+1} = (\sigma^{k(0)+1})^-$. By (5.2), $\text{up}(v_0) \neq (\sigma^{k(0)+1})^-$. By Definition 5.1, $v_0 \subseteq \eta_0 = \sigma^{k(0)} = \text{out}(\sigma^{k(0)+1})$, so by Lemma 3.1(i) (Limit Path), $\text{up}(v_0) \neq \sigma^{k(0)+1}$. Thus $\text{up}(v_0) \subseteq (\sigma^{k(0)+1})^- = \rho^{k(0)+1} \wedge \sigma^{k(0)+1}$, and (i) follows.

(iv): As $\eta_0 = \lambda^{k(0)}(\sigma) \in T^{k(0)}$, and η_0 requires extension, it follows from Lemma 5.6 (Uniqueness of Requiring Extension) that $\sigma^{k(0)+1} = \lambda^{k(0)+1}(\sigma)$ cannot require extension. Fix $\xi^{k(0)+1} \subseteq \sigma^{k(0)+1}$ satisfying the hypotheses of (iv), and note that $\xi^{k(0)+1} \subseteq \sigma^{k(0)+1}$. If $\rho = \langle \rangle$, then (iv) is vacuous. Thus we may assume that ρ^\pm exists.

First suppose that ρ^\pm is completion-consistent via $\langle \rangle$. As noted in the proof of (i), $(\sigma^{k(0)+1})^- \subseteq \rho^{k(0)+1}$, so $\xi^{k(0)+1} \subseteq \rho^{k(0)+1}$. Hence by Lemma 5.5(iii) (Completion-Respecting), $\xi^{k(0)+1}$ has a primary completion $\kappa^{k(0)+1} \subseteq \rho^{k(0)+1}$ which has infinite outcome along $\rho^{k(0)+1}$. By (2.10), $\kappa^{k(0)+1} \subseteq \sigma^{k(0)+1}$, and $\kappa^{k(0)+1}$ will have infinite outcome along

$\sigma^{k(0)+1}$ unless η switches $\kappa^{k(0)+1}$. Thus if η does not switch $\kappa^{k(0)+1}$, then (iv)(a) holds. And if η switches $\kappa^{k(0)+1}$, then $\bar{\rho}^{k(0)+1} = \kappa^{k(0)+1}$, and by (5.5)(ii), $\text{up}(v_0) = (\xi^{k(0)+1})^-$, so (iv)(b) holds.

Now suppose that ρ^\pm is not completion-consistent via $\langle \cdot \rangle$. By the admissibility of ρ , ρ must be a 0-completion corresponding to a primary completion ρ^k for some k . Again by Definition 5.6, ρ^i is an initial derivative of ρ^k for all $i < k$, so by (2.4), $\rho^k = \bar{\rho}^k \subset \sigma^k$.

First suppose that $\bar{\rho}^k$ has finite outcome along σ^k . Then by Lemma 5.3(ii) (Implication Chain), $\sigma^{k\pm 1}$ requires extension, so $k = k(0)+1$. Furthermore, $\bar{\rho}^{k(0)+1} = \rho^{k(0)+1} = (\sigma^{k(0)+1})^-$. Thus $\xi^{k(0)+1} \subseteq \rho^{k(0)+1}$, so (iv)(a) will follow from Lemma 5.5(ii) (Completion-Respecting) unless $\text{up}(v_0) = (\xi^{k(0)+1})^-$; but this is ruled out by the hypotheses of (iv).

Now suppose that $\bar{\rho}^k$ has infinite outcome along σ^k . We compare k and $k(0)$. First suppose that $k(0) < k-1$. Then by Definition 5.6, $\rho^{k(0)+1} = \bar{\rho}^{k(0)+1}$ is an initial derivative, so $\bar{\rho}^{k(0)+1}$ cannot be a primary completion. As $\bar{\rho}^k$ is a primary k -completion and $\text{up}^k(\bar{\rho}^{k(0)}) = \bar{\rho}^k$, $\dim(\bar{\rho}^{k(0)}) > k(0)+1$. Hence (5.5)(ii) must hold for $\sigma^{k(0)}$, contradicting the fact that $\bar{\rho}^{k(0)+1}$ is not a primary completion.

Next suppose that $k(0) = k-1+2q$ for some $q \geq 0$. As $\bar{\rho}^k$ has infinite outcome along σ^k , it follows from Lemma 3.3 (λ -Behavior) that $\bar{\rho}^{k(0)}$ has finite outcome along $\sigma^{k(0)}$. But by (5.2), as $\sigma^{k(0)}$ requires extension, $\bar{\rho}^{k(0)}$ must have infinite outcome along $\sigma^{k(0)}$, a contradiction.

Finally, suppose that $k(0) = k+2q$ for some $q \geq 0$. Then by (5.5)(ii), (5.9), and (5.12), $\bar{\rho}^{k(0)} = \text{up}^{k(0)}(\bar{\rho}^k)$ is the middle element of a triple in an implication chain, and by (5.15) or (5.16), $\bar{\rho}^{k(0)}$ must be a primary completion or an amenable pseudocompletion, contrary to Lemma 5.7(i).

(v): As $\dim(v_i) > k(i)+1$, (5.5)(ii) holds and, as $i = 0$, implies (v).

Case 2: $m \geq 0$. Then $\text{out}^0(\eta_0) \subseteq \rho \subset \sigma$. There are two subcases.

Subcase 2.1: $i \leq m$. (ii), (iii), and (vi) follow by induction.

(i): First suppose that σ is not v -switching for any $v \leq k(i)+1$. By Lemma 3.3 (λ -Behavior), $\rho^{k(i)+1} \subseteq \sigma^{k(i)+1}$. By (i) inductively and Lemma 3.3 (λ -Behavior),

$$\text{up}(v_i) \subset \rho^{k(i)+1} = \rho^{k(i)+1} \wedge \sigma^{k(i)+1} \subseteq \sigma^{k(i)+1}.$$

Otherwise, by (5.18) and (5.25), σ is $(k(m)+1)$ -switching and $k(i) = k(m)$. Thus by the preadmissibility of σ , (5.18)(i)(a) or (5.18)(ii)(a) must hold. Suppose that (5.18)(i)(a) holds. Then there is an $\bar{\eta}^{k(m)} \in T^{k(m)}$ such that ρ^\pm is completion-consistent via $S \wedge \langle \bar{\eta}^{k(m)} \rangle$.

Let $\bar{\eta}^{k(m)}$ require extension for $\bar{v}^{k(m)}$. Then by (5.18)(i)(a), $\bar{\rho}^{k(m)}$ is the primary completion of $\bar{\eta}^{k(m)}$, so by (5.19), $\bar{\rho}^{k(m)+1} = \text{up}(\bar{v}^{k(m)})$. By (ii) inductively and Lemma 3.3 (λ -Behavior),

$$\text{up}(v_i) \subset \text{up}(\bar{v}^{k(m)}) = \bar{\rho}^{k(m)+1} = \rho^{k(i)+1} \wedge \sigma^{k(i)+1} \subset \sigma^{k(i)+1}.$$

Now suppose that (5.18)(ii)(a) holds. Then $\bar{\rho}^{k(m)+1}$ is the end of a primary $\rho^{k(m)+1}$ -link which restrains $\text{up}(v_m)$. By the case assumption, $i \leq m$. Hence by (ii) inductively and Lemma 3.3 (λ -Behavior),

$$\text{up}(v_i) \subseteq \text{up}(v_m) \subset \bar{\rho}^{k(m)+1} = \rho^{k(i)+1} \wedge \sigma^{k(i)+1} \subset \sigma^{k(i)+1}.$$

(i) now follows.

(iv): Assume the hypothesis of (iv). By (ii), it suffices to verify (iv) under the assumption that i is the largest integer for which the hypotheses of (iv) hold for $\xi^{k(i)+1}$ and $\mu^{k(i)+1}$.

By (5.18) and (5.25), if σ is v -switching, then $v \geq k(m)+1 \geq k(i)+1$. And by the choice of the largest i in the preceding paragraph, $\xi^{k(i)+1} \neq \sigma^{k(i)+1}$ (else by Lemma 5.9 (Completion-Consistency), $w = m+1$ and we would choose $i = w$ for $\sigma^{k(i)+1}$). Now by Lemma 3.3 (λ -Behavior), $(\sigma^{k(i)+1})^- \subseteq \rho^{k(i)+1}$. We conclude that $\xi^{k(i)+1} \subseteq \rho^{k(i)+1}$. Thus by induction, one of (iv)(a)-(c) must hold at $\rho^{k(i)+1}$. We consider each possibility.

Assume that (iv)(a) holds at $\rho^{k(i)+1}$. If $\rho^{k(i)+1} \subseteq \sigma^{k(i)+1}$, then (iv)(a) will hold at $\sigma^{k(i)+1}$. If $\rho^{k(i)+1} \not\subseteq \sigma^{k(i)+1}$ and (iv)(a) holds at $\rho^{k(i)+1}$ but not at $\sigma^{k(i)+1}$, then by (5.18) and (5.25), $\sigma^{k(i)+1}$ is $(k(i)+1)$ -switching and, by (2.10), must switch the primary completion $\bar{\rho}^{k(i)+1} = \kappa^{k(i)+1}$ of $\xi^{k(i)+1}$. Thus $\kappa^{k(i)+1}$ will have finite outcome along $\sigma^{k(i)+1}$, so by Lemma 5.3(ii) (Implication Chain) and Lemma 5.2 (Requires Extension), $\sigma^{k(i)}$ requires extension. As $\sigma = \text{out}^0(\sigma^{k(i)})$, $w = m+1$ and $\eta_w = \sigma^{k(i)}$. But then by Lemma 5.2 (Requires Extension), $(\xi^{k(i)+1})^- = \text{up}(v_w)$, so (iv)(b) follows from (5.25) and (ii) inductively.

If (iv)(b) holds at $\rho^{k(i)+1}$, then by the maximality of i and as $k(m) = k(w)$, (iv)(b) will hold at $\sigma^{k(i)+1}$ unless $w = m-1 = i$; so assume that (iv)(b) fails and $w = m-1$. By Lemma 5.9 (Completion-Consistency), $\sigma^{k(m)}$ is the primary completion of η_m , and by Definition 5.6, σ is nonswitching; hence $\rho^{k(i)+1} \subseteq \sigma^{k(i)+1}$. Now either (iv)(a) will hold for $\xi^{k(i)+1} \supset \text{up}(v_m)$ at $\rho^{k(i)+1}$, and so at $\sigma^{k(i)+1}$, or $(\xi^{k(i)+1})^- = \text{up}(v_m)$. In the latter case, (iv)(c) holds at $\sigma^{k(i)+1}$.

Suppose that (iv)(c) holds at $\rho^{k(i)+1}$. By (5.19), if α^k is a primary completion with corresponding 0-completion α , then α is an initial derivative of α^k and all nodes $\beta \in (\text{out}^0(\alpha^k), \alpha]$ are nonswitching; so by Lemma 3.1(i) (Limit Path), $\lambda^k(\beta) = \lambda^k(\text{out}^0(\alpha^k))$ for all such β . Thus (iv)(c) will hold at σ unless ρ is a 0-completion. In the latter case, by

(5.18)(i), σ switches $\bar{\rho}^{k(m)+1}$. By (5.5)(ii) and the maximality of i , $k(i) = k(m)$ and $\bar{\rho}^{k(m)+1} = \text{up}(v_m) \subset \xi^{k(i)+1}$. But $\xi^{k(i)+1} \subseteq \rho^{k(m)+1}$ and by Lemma 3.3 (λ -Behavior), $\rho^{k(m)+1} \wedge \sigma^{k(m)+1} = \bar{\rho}^{k(m)+1}$; so $\xi^{k(i)+1} \not\subseteq \sigma^{k(m)+1}$, contrary to hypothesis.

(v): Assume the hypotheses of (v). By (i), $\text{up}(v_i) \subset \rho^{k(i)+1} \wedge \sigma^{k(i)+1}$. Fix $\tau_i \subseteq \rho^{k(i)+1} \wedge \sigma^{k(i)+1}$ such that $(\tau_i)^- = \text{up}(v_i)$. As $\tau_i \subseteq \rho^{k(i)+1}$, (v) follows by induction.

(vii): By (5.18) and (5.25), σ is not v -switching for any $v \leq k(i)$. Hence by Lemma 3.3(i) (λ -Behavior), $\rho^{k(i)} \subseteq \sigma^{k(i)}$. By induction, $\text{PL}(\delta_i, \rho^{k(i)})$ is a component of $\text{PL}(\delta_{i\pm 1}, \rho^{k(i)})$, and by hypothesis, δ_i has no primary completion along $\rho^{k(i)}$. It now follows from Definition 5.3 that $\text{PL}(\delta_i, \sigma^{k(i)})$ is a component of $\text{PL}(\delta_{i\pm 1}, \sigma^{k(i)})$.

Subcase 2.2: $i = w = m+1$. By Lemma 5.9 (Completion-Consistency), (5.18)(ii)(a) holds, and hence by (5.18)(ii) and (5.25), σ is $(k(w)+1)$ -switching, $k(w)+1 \geq k(m)+1$, and $\bar{\rho}^{k(w)+1}$ is a primary completion. By Lemma 5.3(ii) (Implication Chain) and Lemma 5.9(ii) (Completion-Consistency), $\langle \text{up}(v_w), \bar{\rho}^{k(w)+1}, \sigma^{k(w)+1} \rangle$ is the last triple of an r -implication chain for some r , and by the assumptions of Case 2, $m \geq 0$ so $\dim(v_w) > k(w)+1$. Hence by (5.5)(ii), if $\bar{\tau}^{k(w)+1}$ is the immediate successor of $\text{up}(v_w)$ along $\bar{\rho}^{k(w)+1}$, then $\bar{\tau}^{k(w)+1}$ requires extension for some $\bar{\mu}^{k(w)+1}$ which we fix, and $\bar{\tau}^{k(w)+1}$ has primary completion $\bar{\rho}^{k(w)+1}$. By (5.18)(ii)(a), $[\bar{\mu}^{k(w)+1}, \bar{\rho}^{k(w)+1}]$ is a primary $\rho^{k(w)+1}$ -link, and a $\rho^{k(m)+1}$ -link derived from this link restrains $\text{up}(v_m)$. By Lemma 5.9(ii) (Completion-Consistency), $\sigma^{k(w)} = \eta_w$ requires extension and by (5.1) and as $S \neq \langle \rangle$, (5.5)(ii) must hold; hence $\delta_w = \bar{\rho}^{k(w)}$. Furthermore, as σ is $(k(w)+1)$ -switching, $\rho^{k(w)} = \bar{\rho}^{k(w)}$. We verify (i)-(vii).

(i): As $\langle \text{up}(v_w), \bar{\rho}^{k(w)+1}, \sigma^{k(w)+1} \rangle$ is the last triple of a $(k(w)+1)$ -implication chain, it follows from (5.8)(i) that $\text{up}(v_w) \subset \bar{\rho}^{k(w)+1}$. By Lemma 3.3 (λ -Behavior), $\bar{\rho}^{k(w)+1} = \rho^{k(w)+1} \wedge \sigma^{k(w)+1}$. (i) now follows.

(ii): We assume that $k(w) = k(m)$, else there is nothing to verify. We assume that (ii) fails, and derive a contradiction. By (i), $\text{up}(v_w)$ and $\text{up}(v_m)$ are comparable. First assume that $\text{up}(v_w) = \text{up}(v_m)$ in order to obtain a contradiction. By (i), $\bar{\tau}^{k(w)+1} \subseteq \rho^{k(w)+1} \wedge \sigma^{k(w)+1}$, and we have noted that $\bar{\rho}^{k(w)+1}$ is the primary completion of $\bar{\tau}^{k(w)+1}$. Furthermore, as $(\bar{\tau}^{k(w)+1})^- = \text{up}(v_w) = \text{up}(v_m)$, $\bar{\tau}^{k(w)+1}$ requires extension and $S \neq \langle \rangle$, it follows from (5.5) that $\dim(\text{up}(v_m)) > k(w)+1$. Hence by (5.5)(ii) for both η_m and $\eta_w = \sigma^{k(w)}$, it must be the case that both δ_m and $\delta_w = \rho^{k(w)}$ have infinite outcome along $\sigma^{k(w)}$, and $\text{up}(\delta_m) = \text{up}(\delta_w) = \bar{\rho}^{k(w)+1}$. Hence by (2.8), $\eta_m = \eta_w = \sigma^{k(w)}$. But as $k(m) = k(w)$, $\eta_m \subseteq \rho^{k(w)} = \bar{\rho}^{k(w)} \subset \sigma^{k(w)}$, yielding a contradiction.

Next suppose that $\text{up}(v_m) \supset \text{up}(v_w)$ in order to obtain a contradiction. As $\bar{\tau}^{k(w)+1}$ requires extension and $\bar{\rho}^{k(w)+1}$ is the primary completion of $\bar{\tau}^{k(w)+1}$, all nodes in $[\bar{\tau}^{k(w)+1}, \bar{\rho}^{k(w)+1}]$ are implication-restrained. Hence by (5.1) and (i) for v_m , $\dim(v_m) > k(m)+2$. By (i), if τ_m is the immediate successor of $\text{up}(v_m)$ along $\sigma^{k(w)+1}$, then $\tau_m \subseteq$

$\rho^{k(w)+1}$, so by (v) inductively, τ_m requires extension. Hence by Lemma 5.5(i) (Completion Respecting), τ_m has a primary completion $\kappa_m \subset \bar{\rho}^{k(w)+1}$, and κ_m has infinite outcome along $\bar{\rho}^{k(w)+1} \subset \sigma^{k(w)+1}$. Thus by (2.4), all derivatives of κ_m along $\sigma^{k(w)}$ have finite outcome along $\sigma^{k(w)}$. But by (5.5)(ii) and (5.2) for η_m , $\text{up}(\delta_m) = \kappa_m$ and δ_m has infinite outcome along $\eta_m \subseteq \sigma^{k(w)}$, yielding a contradiction. (ii) now follows.

(iii): We assume that $k(m) = k(w)$, else there is nothing to show. (5.5)(ii) specifies the relationship between requires extension situations on $T^{k(w)}$ and $(k(w)+1)$ -implication chains, and specifies that v_w is the initial derivative of $\text{up}(v_w)$ along $\sigma^{k(w)}$. We have noted that $\bar{\tau}^{k(w)+1}$ requires extension; let $\bar{\tau}^{k(w)} = \text{out}(\bar{\tau}^{k(w)+1})$, and note that $\bar{\tau}^{k(w)} \subset \sigma^{k(w)}$ by (2.5). We compare the locations of $\bar{\tau}^{k(w)}$ and η_m , noting that they are comparable as both are $\subset \sigma^{k(w)}$. First assume that $\bar{\tau}^{k(w)} \subset \eta_m$. (We will show that this is the only case which can actually occur.) Now $\bar{\rho}^{k(w)+1}$ is the primary completion of $\bar{\tau}^{k(w)+1}$ and $\bar{\rho}^{k(w)+1} \subset \sigma^{k(w)+1}$; hence by Lemma 3.1(i) (Limit Path), $\bar{\rho}^{k(w)+1}$ has an initial derivative $\bar{\mu}^{k(w)} \subset \sigma^{k(w)}$ which is the $k(w)$ -completion of $\bar{\rho}^{k(w)+1}$. By (5.25) and (5.27), new nodes on lower dimension trees cannot require extension until all nodes on higher dimension trees which previously required extension have found their 0-completions; hence as $k(w) < k(w)+1$, no node in $[\bar{\tau}^{k(w)}, \bar{\mu}^{k(w)}]$ can require extension. Hence $\bar{\mu}^{k(w)} \subset \eta_m$. By Lemma 3.1(i) (Limit Path), and as $\text{up}(v_w) \subset \bar{\rho}^{k(w)+1}$ by (i), it must be the case that $v_w \subset \bar{\tau}^{k(w)} \subset \bar{\mu}^{k(w)} \subset \eta_m$, so (iii) holds in this case.

$\bar{\tau}^{k(w)} \not\subseteq \eta_m$, else we would contradict Lemma 5.6 (Uniqueness of Requiring Extension).

Finally, assume that $\bar{\tau}^{k(w)} \supset \eta_m$. By (vi) inductively for $\bar{\tau}^{k(w)+1}$, $\text{up}(v_m) \subset \bar{\mu}^{k(w)+1}$, contrary to (5.18)(ii)(a). Hence (iii) holds.

(iv): Assume that the hypotheses of (iv) hold for $\xi^{k(w)+1} \subseteq \sigma^{k(w)+1}$ which requires extension. By the hypotheses of (iv), $\text{up}(v_w) \subset (\xi^{k(w)+1})^-$, so $\bar{\tau}^{k(w)+1} \neq \xi^{k(w)+1}$. As $\bar{\tau}^{k(w)+1} \subset \bar{\rho}^{k(w)+1} \subset \sigma^{k(w)+1}$, it must be the case that $\bar{\tau}^{k(w)+1} \subset \xi^{k(w)+1}$. As σ is $(k(w)+1)$ -switching, it follows from Lemma 3.3(ii) (λ -Behavior) and (2.4) that $(\sigma^{k(w)+1})^- = \bar{\rho}^{k(w)+1}$ and $\bar{\rho}^{k(w)+1}$ has finite outcome along $\sigma^{k(w)+1}$. Thus by (5.2), $\xi^{k(w)+1} \subseteq \bar{\rho}^{k(w)+1} \subset \rho^{k(w)+1}$. As $\bar{\rho}^{k(w)+1}$ is the primary completion of $\bar{\tau}^{k(w)+1}$, (iv)(a) follows from Lemma 5.5(i) (Completion Respecting).

(v): We have already noted that the immediate successor, $\bar{\tau}^{k(w)+1}$, of $\text{up}(v_w)$ along $\bar{\rho}^{k(w)+1} \subset \sigma^{k(w)+1}$ requires extension for $\bar{\mu}^{k(w)+1}$. For the second clause of (v), we assume that $k(w) = k(m)$, else there is nothing to verify. As ρ is completion-consistent via $\langle \eta_i; i \leq m \rangle$ and (5.18)(ii)(a) holds at ρ with $\bar{\rho}^{k(w)+1}$ a primary completion, $[\bar{\mu}^{k(w)+1}, \bar{\rho}^{k(w)+1}]$ is a primary $\rho^{k(w)+1}$ -link which restrains $\text{up}(v_m)$, so $\bar{\mu}^{k(w)+1} \subseteq \text{up}(v_m)$.

We assume that $\bar{\mu}^{k(w)+1} = \text{up}(v_m)$ and derive a contradiction. As $[\bar{\mu}^{k(w)+1}, \bar{\rho}^{k(w)+1}]$ is a primary $\rho^{k(w)+1}$ -link and by (i), $\bar{\mu}^{k(w)+1} \neq \bar{\rho}^{k(w)+1}$, $\bar{\mu}^{k(w)+1}$ has finite outcome along $\bar{\rho}^{k(w)+1}$

$\subset \sigma^{k(w)+1}$, and $\text{up}(\bar{\mu}^{k(w)+1}) = \text{up}(\bar{\rho}^{k(w)+1})$, so $\dim(\bar{\mu}^{k(w)+1}) = \dim(v_m) > k(m)+1$. Thus by (v) for v_m , the immediate successor τ_m of $\text{up}(v_m)$ along $\sigma^{k(w)+1}$ requires extension, so by (5.2), $\text{up}(v_m) = \bar{\mu}^{k(w)+1}$ has infinite outcome along $\tau_m \subseteq \sigma^{k(w)+1}$, a contradiction. (v) now follows.

(vi): We assume that $k(w) = k(m)+1$, else there is nothing to verify. By (5.18)(ii)(a) and as σ is $(k(w)+1)$ -switching, there is a primary $\rho^{k(w)+1}$ -link $[\bar{\mu}^{k(w)+1}, \bar{\rho}^{k(w)+1}]$ such that the $\rho^{k(w)}$ -link $[\mu^{k(w)}, \pi^{k(w)}]$ derived from $[\bar{\mu}^{k(w)+1}, \bar{\rho}^{k(w)+1}]$ restrains $\text{up}(v_m)$. Also, $\bar{\rho}^{k(w)+1}$ is the primary completion of the immediate successor $\bar{\tau}^{k(w)+1}$ of $\text{up}(v_w)$ along $\bar{\rho}^{k(w)+1}$, so $\pi^{k(w)}$ is the corresponding $k(w)$ -completion. By (5.2), $\text{up}(v_w)$ has infinite outcome along $\bar{\tau}^{k(w)+1}$, so by (2.4), the initial derivative of $\text{up}(v_w)$ along $\pi^{k(w)}$ is the principal derivative of $\text{up}(v_w)$ along $\pi^{k(w)}$ and has finite outcome along $\pi^{k(w)}$; hence again by (5.2), this initial derivative must be v_w . So by (2.4), if $\bar{\tau}^{k(w)} = \text{out}(\bar{\tau}^{k(w)+1})$, then $(\bar{\tau}^{k(w)})^- = v_w$. By (5.25) and (5.27), no node in $[\bar{\tau}^{k(w)}, \pi^{k(w)}]$ can require extension (else we would contradict the dimension ordering of (5.25)); so as, by (v), the immediate successor τ_m of $\text{up}(v_m)$ along $\pi^{k(w)} \subset \sigma^{k(w)}$ requires extension, it must be the case that $\tau_m \subset \bar{\tau}^{k(w)}$. Thus $\text{up}(v_m) = (\tau_m)^- \subset (\bar{\tau}^{k(w)})^- = v_w$, and (vi) holds.

(vii): By (iii) and hypothesis, if $k(m) = k(w)$, then η_m requires extension and $v_w \subset \delta_m \subset \eta_m \subseteq \delta_w \subset \eta_w = \sigma^{k(w)}$. (vii) now follows from Definition 5.3, as (5.14) holds. \square

The next lemma provides a step-by-step analysis of the effect of extending ρ to σ , as specified by (5.18), on the PL sets corresponding to each element in the sequence via which ρ is completion-consistent.

Lemma 5.11 (Amenable Backtracking Lemma): Fix hypotheses as in Lemma 5.10 (Component). Then for all $i \leq m$:

- (i) If either (5.18)(i)(a) holds for ρ with $k(m+1) = k(i)$, or (5.18)(ii)(a) holds for ρ with $j = k(i)+1$, then

$$\text{PL}(\delta_i, \sigma^{k(i)}) = \text{PL}(\delta_i, \rho^{k(i)}) \cup \{\rho^{k(i)}\}$$
 and the union is disjoint,

$$\bar{\rho}^{k(i)+1} \in \text{PL}(\text{up}(v_i), \rho^{k(i)+1}),$$
 and

$$\text{PL}(\text{up}(v_i), \sigma^{k(i)+1}) = \text{PL}(\text{up}(v_i), \rho^{k(i)+1}) \setminus \{\bar{\rho}^{k(i)+1}\}.$$
- (ii) If the hypotheses of (i) fail, then $\text{PL}(\delta_i, \sigma^{k(i)}) = \text{PL}(\delta_i, \rho^{k(i)})$ and

$$\text{PL}(\text{up}(v_i), \sigma^{k(i)+1}) = \text{PL}(\text{up}(v_i), \rho^{k(i)+1}).$$
- (iii) $\{\text{up}(\xi^{k(i)}): \xi^{k(i)} \in \text{PL}(\delta_i, \sigma^{k(i)})\} \cup \text{PL}(\text{up}(v_i), \sigma^{k(i)+1}) =$
 $\{\text{up}(\xi^{k(i)}): \xi^{k(i)} \in \text{PL}(\delta_i, \rho^{k(i)})\} \cup \text{PL}(\text{up}(v_i), \rho^{k(i)+1}),$
 and the unions are disjoint.
- (iv) If $w = m+1$, then $\text{PL}(\delta_w, \sigma^{k(w)}) = \emptyset$.
- (v) If σ is a 0-completion of η_m , then $\text{PL}(\text{up}(v_m), \sigma^{k(m)+1}) = \emptyset$.

(vi) If σ^\pm is not completion-consistent via $\langle \rangle$, then σ satisfies (5.29)(i).

Proof: As $\text{up}(\rho^{k(i)}) = \bar{\rho}^{k(i)+1}$ by (5.18), (5.25), and Lemma 3.3 (λ -Behavior), (iii) follows from (i) and (ii) by induction on $\text{lh}(\rho^{k(i)})$. We first prove (iv) and (v).

If $w = m+1$, then $(\sigma^{k(w)})^- = \delta_w$ by Lemma 5.9 (Completion-Consistency), so there can be no primary $\sigma^{k(w)}$ -link restraining δ_w or $\sigma^{k(w)}$. Furthermore, $\text{PL}(\delta_w, \sigma^{k(w)})$ can have a component only if $\text{lh}(\sigma^{k(w)}) - \text{lh}(\delta_w) \geq 2$. (iv) now follows from Definition 5.3. Suppose that σ is a 0-completion of η_m . By Lemma 5.9 (Component), $w = m-1$. By (5.19), $\text{up}(v_m) = \text{up}(\sigma^{k(m)})$, so by (2.10), $\text{up}(v_m)$ is $\sigma^{k(m)+1}$ -free. Suppose that elements are placed in $\text{PL}(\text{up}(v_m), \sigma^{k(m)+1})$ through (5.14) in order to obtain a contradiction. Then there are $\mu^{k(m)+1} \subset \text{up}(v_m) \subset (\xi^{k(m)+1})^- \subset \xi^{k(m)+1} \subseteq \sigma^{k(m)+1}$ such that $\xi^{k(m)+1}$ requires extension for $\mu^{k(m)+1}$. As $w = m-1$, the hypothesis of Lemma 5.10(iv) (Component) holds for ρ in place of σ , so one of conditions (iv)(a)-(iv)(c) must hold for $i = m$. (iv)(b) cannot hold, as the w corresponding to ρ is m , so v_{m+1} is undefined. (iv)(c) cannot hold, else $\rho^{k(m)}$ would be a primary completion, so by (5.18), σ would be a switching extension of ρ ; but σ is a 0-completion, and by Definition 5.6, 0-completions are nonswitching, yielding a contradiction. Suppose that (iv)(a) holds. Then $\xi^{k(m)+1}$ has a primary completion $\kappa^{k(m)+1}$ which has infinite outcome along $\rho^{k(m)+1}$. Thus by Definition 5.6, $[\mu^{k(m)+1}, \kappa^{k(m)+1}]$ is a primary $\rho^{k(m)+1}$ -link restraining $\text{up}(v_m)$, so $\text{up}(v_m)$ is not $\rho^{k(m)+1}$ -free, a contradiction. It now follows from Definition 5.3 that $\text{PL}(\text{up}(v_m), \sigma^{k(m)+1}) = \emptyset$, so (v) holds.

We now verify (i) and (ii). We proceed by induction on $\text{lh}(\sigma)$. There are several cases to consider, depending on the manner in which σ extends ρ . By (5.18) and (5.25), we see that if σ is v -switching, then $v > k(m)$ if $m \geq 0$. In the first two cases, $v > k(i)+2$ or σ is nonswitching, and $v = k(i)+2$, we show that the hypothesis and conclusion of (ii) hold. The final case is when $v = k(i)+1$, in which case (i) will be followed.

We first note the following:

Claim: If $\alpha^k \subset \beta^k \subset \gamma^k \in T^k$, $(\gamma^k)^- = \beta^k$, $\text{out}^0(\beta^k)$ is completion-consistent via a nonempty set, and γ^k is not u -switching for any $u \leq k+1$, then $\text{PL}(\alpha^k, \beta^k) = \text{PL}(\alpha^k, \gamma^k)$.

Proof: By Lemma 5.1(i) (PL Analysis), $\text{PL}(\alpha^k, \beta^k) \subseteq \text{PL}(\alpha^k, \gamma^k)$. If $\text{PL}(\alpha^k, \gamma^k) \setminus \text{PL}(\alpha^k, \beta^k) \neq \emptyset$, then by Lemma 5.1(iii) (PL Analysis), either β^k is the end of a primary γ^k -link and so γ^k is u -switching for some $u \leq k+1$, or γ^k requires extension; and in the latter case, it follows from Lemma 5.9 (Completion-Consistency) that (5.18)(ii) holds for σ with $j = k+1$, so γ^k is $(k+1)$ -switching. But this is contrary to our case assumption. The claim now follows. \square

Case 1: σ is v -switching for some $v > k(i)+2$ or is nonswitching. (i) is vacuous, and (ii) is immediate from the claim.

Case 2: σ is $(k(i)+2)$ -switching. By the claim, $PL(\delta_i, \sigma^{k(i)}) = PL(\delta_i, \rho^{k(i)})$. It follows from (5.18)(iii) that either (5.18)(i)(a) or (5.18)(ii)(a) holds. By (5.25), $k(i) \leq k(m)$, and by (5.18), if σ is v -switching, then $v > k(m)$. Hence there are two subcases to consider; $k(i) = k(m)$, and $k(i) = k(m)-1$. We note that in both cases, (i) is vacuous as $k(m+1) > k(i)$ if (5.18)(i) is followed, so it suffices to verify (ii).

Subcase 2.1: $k(i) = k(m)$. By Lemma 3.3 (λ -Behavior), $\rho^{k(m)+1} \subset \sigma^{k(m)+1}$.

Subcase 2.1.1: (5.18)(i)(a) holds and $i = m$, and so as σ is $(k(m)+2)$ -switching, $k(m+1) = k(m)+1$. Then there are $\bar{v}_{m+1}, \bar{\eta}_{m+1} \in T^{k(m)+1}$ such that $\rho^{k(m)+1}$ is a primary completion of $\bar{\eta}_{m+1}$ for \bar{v}_{m+1} and by Lemma 5.9 (Completion-Consistency), $(\rho^{k(m)+1})^-$ is completion-consistent via $S^\wedge \langle \bar{\eta}_{m+1} \rangle$. By Proposition 5.1(i)(iii), $PL(\text{up}(v_m), \rho^{k(m)+1}) \subseteq PL(\text{up}(v_m), \sigma^{k(m)+1})$ and $PL(\text{up}(v_m), \sigma^{k(m)+1}) \setminus PL(\text{up}(v_m), \rho^{k(m)+1}) \subseteq \{\rho^{k(m)+1}\}$. By Lemma 5.10(vi) (Component), $\text{up}(v_m) \subset \bar{v}_{m+1}$. Now $[\bar{v}_{m+1}, \rho^{k(m)+1}]$ is the only primary $\sigma^{k(m)+1}$ -link which is not a $\rho^{k(m)+1}$ -link, and by Lemma 5.7(i) (Primary Completion), $\sigma^{k(m)+1}$ does not require extension. Hence neither (5.13) nor (5.14) can place $\rho^{k(m)+1}$ into $PL(\text{up}(v_m), \sigma^{k(m)+1})$, so $PL(\text{up}(v_m), \sigma^{k(m)+1}) = PL(\text{up}(v_m), \rho^{k(m)+1})$.

Subcase 2.1.2: (5.18)(ii)(a) holds and $i = m$. Then there is a $\rho^{k(m)+1}$ -link $[\mu^{k(m)+1}, \pi^{k(m)+1}]$ which restrains $\text{up}(v_m)$ and is derived from a primary $\rho^{k(m)+2}$ -link $[\mu^{k(m)+2}, \pi^{k(m)+2}]$, and σ switches $\pi^{k(m)+2}$. Note that $[\pi^{k(m)+1}, \rho^{k(m)+1}]$ is the only primary $\sigma^{k(m)+1}$ -link which is not a $\rho^{k(m)+1}$ -link, and $\text{up}(v_m) \subset \pi^{k(m)+1}$; hence any node placed into $PL(\text{up}(v_m), \sigma^{k(m)+1})$ via (5.13) is already in $PL(\text{up}(v_m), \rho^{k(m)+1})$. By Lemma 5.10(vi) (Component), if $\sigma^{k(m)+1}$ requires extension for some \bar{v}_{m+1} which we fix, then $\text{up}(v_m) \subset \bar{v}_{m+1}$; hence any node placed into $PL(\text{up}(v_m), \sigma^{k(m)+1})$ via (5.14) is already in $PL(\text{up}(v_m), \rho^{k(m)+1})$. Thus by Definition 5.3 and Lemma 5.1(i) (PL Analysis), $PL(\text{up}(v_m), \sigma^{k(m)+1}) = PL(\text{up}(v_m), \rho^{k(m)+1})$.

Subcase 2.1.3: $i < m$. By Lemma 5.10(i) (Component), $PL(\text{up}(v_i), \rho^{k(m)+1}) \subseteq PL(\text{up}(v_i), \sigma^{k(m)+1})$. By Proposition 5.1(ii) (PL Analysis) and since $\rho^{k(m)+1} \subset \sigma^{k(m)+1}$, $PL(\text{up}(v_i), \sigma^{k(m)+1}) \setminus PL(\text{up}(v_i), \rho^{k(m)+1}) \subseteq \{\rho^{k(m)+1}\}$. For each $j \in [i+1, m]$, let $\tau_j^{k(m)+1}$ be the immediate successor of $\text{up}(v_j)$ along $\sigma^{k(m)+1}$ and note that, by Lemma 5.10(v) (Component), $\tau_j^{k(m)+1}$ requires extension for each such j . For each $j \in [i+1, m]$, let $\xi_j^{k(m)+1}$ be the primary completion of $\tau_j^{k(m)+1}$ along $\sigma^{k(m)+1}$ if it exists, and let $\xi_j^{k(m)+1} = \sigma^{k(m)+1}$

otherwise. As $i+1 > 0$, it follows from (5.1) that $\dim(\text{up}(v_{i+1})) > k(m)+1$. Hence by Lemma 5.10(v) (Component) and (5.14), we see that $\text{PL}(\text{up}(v_{i+1}), \xi_i^{k(m)+1})$ is a component of $\text{PL}(\text{up}(v_j), \sigma^{k(m)+1})$. Furthermore, by Lemma 5.10(iv) (Component), every component of $\text{PL}(\text{up}(v_j), \sigma^{k(m)+1})$ which has $\rho^{k(m)+1}$ as an element must be of the form $\text{PL}(\text{up}(v_j), \xi_j^{k(m)+1})$ for some $j \in [i+1, m]$. By Lemma 5.10(i),(ii) (Component), if $j \in [i, m]$, then $\text{up}(v_j) \subseteq \text{up}(v_m) \subset \rho^{k(m)+1}$. It now follows by induction on $m-i$ that if $\rho^{k(m)+1} \in \text{PL}(\text{up}(v_j), \sigma^{k(m)+1})$, then $\rho^{k(m)+1} \in \text{PL}(\text{up}(v_m), \sigma^{k(m)+1})$, and so by Subcase 2.1.2, that $\text{PL}(\text{up}(v_j), \sigma^{k(m)+1}) = \text{PL}(\text{up}(v_j), \rho^{k(m)+1})$.

Subcase 2.2: $k(i) = k(m)-1$. By Lemma 3.3 (λ -Behavior), $\rho^{k(m)} \subset \sigma^{k(m)}$.

Subcase 2.2.1: (5.18)(i)(a) holds. Then there are $\bar{v}_{m+1}, \bar{\eta}_{m+1} \in T^{k(m)}$ such that $\rho^{k(m)}$ is a primary completion of $\bar{\eta}_{m+1}$ for \bar{v}_{m+1} and by Lemma 5.9 (Completion-Consistency), $(\rho^{k(m)})^-$ is completion-consistent via $S^\wedge \langle \bar{\eta}_{m+1} \rangle$. By Lemma 5.10(vi) (Component), $\text{up}(v_i) \subset \bar{v}_{m+1}$. Now by (5.19) and (5.18)(i), $[\bar{v}_{m+1}, \rho^{k(m)}]$ is the only primary $\sigma^{k(m)}$ -link which is not a $\rho^{k(m)}$ -link, and by Lemma 5.7(i) (Primary Completion), $\sigma^{k(m)}$ does not require extension. Hence by Lemma 5.1(i),(iii) (PL Analysis), $\text{PL}(\text{up}(v_i), \sigma^{k(m)}) = \text{PL}(\text{up}(v_i), \rho^{k(m)})$.

Subcase 2.2.2: (5.18)(ii)(a) holds. Then there is a primary $\rho^{k(m)+1}$ -link $[\mu^{k(m)+1}, \pi^{k(m)+1}]$ which restrains $\text{up}(v_m)$, so $\text{up}(v_m) \subset \pi^{k(m)+1}$. Let $\pi^{k(m)}$ be the initial derivative of $\pi^{k(m)+1}$ along $\sigma^{k(m)}$. Then $[\pi^{k(m)}, \rho^{k(m)}]$ is the only primary $\sigma^{k(m)}$ -link which is not a $\rho^{k(m)}$ -link, and by (5.2), v_m is the initial derivative of $\text{up}(v_m)$ along $\sigma^{k(m)}$; hence it follows from Lemma 3.1(i) (Limit Path) that $v_m \subset \pi^{k(m)}$. If $\sigma^{k(m)}$ requires extension for some \bar{v}_{m+1} , then by Lemma 5.10(vi) (Component), $\text{up}(v_i) \subset \bar{v}_{m+1}$. It now follows from Lemma 5.1(i) (PL Analysis) and Definition 5.3 that $\text{PL}(\text{up}(v_i), \sigma^{k(m)+1}) = \text{PL}(\text{up}(v_i), \rho^{k(m)+1})$.

Case 3: σ is $(k(i)+1)$ -switching. By (5.25), $k(i) \leq k(m)$, so σ must be $(k(m)+1)$ -switching, i.e., $k(i) = k(m)$. By Lemma 3.3 (λ -Behavior), $(\sigma^{k(m)+1})^- = \bar{\rho}^{k(m)+1}$. Now σ is preadmissible and $m \geq 0$, so ρ is not completion-consistent via $\langle \rangle$; hence (5.18)(i)(a) or (5.18)(ii)(a) must hold. We note that (ii) is vacuous in this case, and verify (i). We consider three subcases.

Subcase 3.1: (5.18)(i)(a) holds and $i = m$. By Lemma 5.9 (Completion-Consistency), there is a node $\bar{\eta}^{k(m)} \in T^{k(m)}$ such that $(\text{out}^0(\rho^{k(m)}))^-$ is completion-consistent via $S^\wedge \langle \bar{\eta}^{k(m)} \rangle$, $\rho^{k(m)}$ is the $k(m)$ -completion of $\bar{\eta}^{k(m)}$ for some $\bar{v}^{k(m)}$, and $\sigma^{k(m)}$ switches $\bar{\rho}^{k(m)+1}$. Let $\bar{\delta}^{k(m)} = (\bar{\eta}^{k(m)})^-$.

By Lemma 5.10(iii) (Component), $\bar{v}^{k(m)} \subset \delta_m$; and as $(\text{out}^0(\rho^{k(m)}))^-$ is completion-consistent via $S^\wedge\langle\bar{\eta}^{k(m)}\rangle$ and $\eta_m \in S$, $(\eta_m)^- = \delta_m \subset \rho^{k(m)}$. Thus $[\bar{v}^{k(m)}, \rho^{k(m)}]$ is a primary $\sigma^{k(m)}$ -link which restrains δ_m . It now follows from (5.13) that $\rho^{k(m)} \in \text{PL}(\delta_m, \sigma^{k(m)})$, so by Lemma 5.1(i),(ii) (PL Analysis), $\text{PL}(\delta_m, \sigma^{k(m)}) = \text{PL}(\delta_m, \rho^{k(m)}) \cup \{\rho^{k(m)}\}$.

As $\rho^{k(m)}$ is a primary completion of $\bar{\eta}^{k(m)}$ and $\bar{\eta}^{k(m)}$ requires extension for $\bar{v}^{k(m)}$, it follows from (5.19) that $\text{up}(\bar{v}^{k(m)}) = \text{up}(\rho^{k(m)}) = \bar{\rho}^{k(m)+1}$. Let $\tau^{k(m)+1}$ be the immediate successor of $\bar{\rho}^{k(m)+1}$ along $\rho^{k(m)+1}$. By Definition 5.6, $\bar{\eta}^{k(m)}$ and $(\rho^{k(m)})^-$ are completion-consistent via the same sequence, so by Definition 5.6 and Lemma 5.9 (Completion-Consistency), $\rho^{k(m)}$ and $\bar{\delta}^{k(m)}$ are completion-consistent via the same sequence, which is non-empty as $m \geq 0$. Hence as $\bar{\eta}^{k(m)}$ requires extension, it follows from (5.1) that $\dim(\bar{\delta}^{k(m)}) > k(m)+1$. By (5.3) and Lemma 4.3(i)(a) (Link Analysis), $\lambda(\bar{\eta}^{k(m)}) \supset \bar{\rho}^{k(m)+1} = \text{up}(\bar{v}^{k(m)})$; so as $\sigma^{k(m)}$ switches $\bar{\rho}^{k(m)+1}$ and $\bar{\eta}^{k(m)} \subset \rho^{k(m)}$, it follows that $\tau^{k(m)+1} \subseteq \lambda(\bar{\eta}^{k(m)})$; hence by (5.5)(ii) and (5.15), $\tau^{k(m)+1}$ requires extension for $\text{up}(\bar{\delta}^{k(m)})$. By Lemma 5.10(v) (Component) at ρ^- , $\text{up}(\bar{\delta}^{k(m)}) \subset \text{up}(v_m)$, and as $\text{up}(\bar{v}^{k(m)}) = \bar{\rho}^{k(m)+1}$, it follows from Lemma 5.10(ii) (Component) that $\text{up}(v_m) \subset \bar{\rho}^{k(m)+1}$. Thus by (5.14)(i), $\bar{\rho}^{k(m)+1} \in \text{PL}(\text{up}(v_m), \tau^{k(m)+1})$. As σ switches $\bar{\rho}^{k(m)+1}$ (and so, by Definition 5.6, cannot be a primary completion), it follows from Lemma 4.5 (Free Extension) that $\bar{\rho}^{k(m)+1}$ is $\sigma^{k(m)+1}$ -free; hence by Lemma 5.10(iv) (Component), we may apply Lemma 5.1(v) (PL Analysis) to conclude that $\text{PL}(\text{up}(v_m), \rho^{k(m)+1}) = \text{PL}(\text{up}(v_m), \tau^{k(m)+1})$; so by Lemma 5.1(i),(ii) (PL Analysis) and Definition 5.3,

$$\text{PL}(\text{up}(v_m), \rho^{k(m)+1}) = \text{PL}(\text{up}(v_m), \bar{\rho}^{k(m)+1}) \cup \{\bar{\rho}^{k(m)+1}\} \ \& \ \bar{\rho}^{k(m)+1} \notin \text{PL}(\text{up}(v_m), \bar{\rho}^{k(m)+1}).$$

As σ switches $\bar{\rho}^{k(m)+1}$ and is $(k(m)+1)$ -switching, it follows from Lemma 3.3 (λ -Behavior) that $(\sigma^{k(m)+1})^- = \bar{\rho}^{k(m)+1}$ and $\bar{\rho}^{k(m)+1}$ has finite outcome along $\sigma^{k(m)+1}$. Thus by Lemma 5.1(iv) (PL Analysis), $\text{PL}(\text{up}(v_m), \sigma^{k(m)+1}) = \text{PL}(\text{up}(v_m), \bar{\rho}^{k(m)+1})$. Thus

$$\text{PL}(\text{up}(v_m), \rho^{k(m)+1}) \setminus \text{PL}(\text{up}(v_m), \sigma^{k(m)+1}) = \{\bar{\rho}^{k(m)+1}\},$$

and (i) follows in this case.

Subcase 3.2: (5.18)(ii)(a) holds and $i = m$. Then there is a primary $\rho^{k(m)+1}$ -link $[\mu^{k(m)+1}, \bar{\rho}^{k(m)+1}]$ which restrains $\text{up}(v_m)$ such that σ switches $\bar{\rho}^{k(m)+1}$. By (5.13), $\bar{\rho}^{k(m)+1} \in \text{PL}(\text{up}(v_m), \rho^{k(m)+1})$. By induction (on T^0) using (i) and (ii), if $\eta_m \subset \xi^{k(m)} \subseteq \rho^{k(m)}$ then $\text{PL}(\text{up}(v_m), \lambda(\xi^{k(m)})) \subseteq \text{PL}(\text{up}(v_m), \lambda((\xi^{k(m)})^-))$. Thus $\text{PL}(\text{up}(v_m), \rho^{k(m)+1}) \subseteq \text{PL}(\text{up}(v_m), \lambda(\eta_m))$, so $\bar{\rho}^{k(m)+1} \in \text{PL}(\text{up}(v_m), \lambda(\eta_m))$, from which it follows that $\bar{\rho}^{k(m)+1} \subset \lambda(\eta_m)$. As $(\eta_m)^- = \delta_m$ and by (5.2), δ_m has infinite outcome along η_m , it follows from (2.4) that $(\lambda(\eta_m))^- = \text{up}(\delta_m)$, and so $\bar{\rho}^{k(m)+1} \subseteq \text{up}(\delta_m)$. By Lemma 3.1(i) (Limit Path),

$\bar{\rho}^{k(m)+1}$ has an initial derivative $\bar{\mu}^{k(m)} \subseteq \delta_m$. Now $(\sigma^{k(m)})^- = \rho^{k(m)}$ and σ is $(k(m)+1)$ -switching, so by (2.4), $\rho^{k(m)}$ has infinite outcome along $\sigma^{k(m)}$; so as $\text{up}(\rho^{k(m)}) = \bar{\rho}^{k(m)+1}$, $[\bar{\mu}^{k(m)}, \rho^{k(m)}]$ must be a primary $\sigma^{k(m)}$ -link restraining δ_m . By Definition 5.3, $\rho^{k(m)} \notin \text{PL}(\delta_m, \rho^{k(m)})$. It now follows from (5.13) and Lemma 5.1(i),(ii) (PL Analysis) that

$$\text{PL}(\delta_m, \sigma^{k(m)}) \setminus \text{PL}(\delta_m, \rho^{k(m)}) = \{\rho^{k(m)}\}.$$

By (5.18)(ii), $\bar{\rho}^{k(m)+1}$ is the last node of a primary $\rho^{k(m)+1}$ -link which restrains $\text{up}(v_m)$. Hence by (5.13), $\bar{\rho}^{k(m)+1} \in \text{PL}(\text{up}(v_m), \rho^{k(m)+1})$. As σ switches $\bar{\rho}^{k(m)+1}$ (and so, by Definition 5.6, cannot be a primary completion), it follows from (2.10) that $\bar{\rho}^{k(m)+1}$ is $\rho^{k(m)+1}$ -free; hence by Lemma 5.10(iv) (Component), we may apply Lemma 5.1(v) (PL Analysis) to conclude that if $\tau^{k(m)+1}$ is the immediate successor of $\bar{\rho}^{k(m)+1}$ along $\rho^{k(m)+1}$, then $\text{PL}(\text{up}(v_m), \rho^{k(m)+1}) = \text{PL}(\text{up}(v_m), \tau^{k(m)+1})$. Thus by Lemma 5.1(i),(ii) (PL Analysis) and Definition 5.3,

$$\text{PL}(\text{up}(v_m), \rho^{k(m)+1}) = \text{PL}(\text{up}(v_m), \bar{\rho}^{k(m)+1}) \cup \{\bar{\rho}^{k(m)+1}\} \ \& \ \bar{\rho}^{k(m)+1} \notin \text{PL}(\text{up}(v_m), \bar{\rho}^{k(m)+1}).$$

As σ switches $\bar{\rho}^{k(m)+1}$ and is $(k(m)+1)$ -switching, it follows from Lemma 3.3 (λ -Behavior) that $(\sigma^{k(m)+1})^- = \bar{\rho}^{k(m)+1}$ and $\bar{\rho}^{k(m)+1}$ has finite outcome along $\sigma^{k(m)+1}$. Thus by Lemma 5.1(iv) (PL Analysis), $\text{PL}(\text{up}(v_m), \sigma^{k(m)+1}) = \text{PL}(\text{up}(v_m), \bar{\rho}^{k(m)+1})$. Thus

$$\text{PL}(\text{up}(v_m), \rho^{k(m)+1}) \setminus \text{PL}(\text{up}(v_m), \sigma^{k(m)+1}) = \{\bar{\rho}^{k(m)+1}\} \ \& \ \bar{\rho}^{k(m)+1} \notin \text{PL}(\text{up}(v_m), \sigma^{k(m)+1}),$$

and (i) follows in this case.

Subcase 3.3: $i < m$. Recall that $k(i) = k(m)$ and $\rho^{k(m)} \subset \sigma^{k(m)}$. By Lemma 5.10(vii) (Component) and induction using Lemma 5.1(ix) (PL Analysis),

$$\{\rho^{k(m)}\} \cup \text{PL}(\delta_m, \sigma^{k(i)}) \subseteq \text{PL}(\delta_i, \sigma^{k(i)}).$$

By Definition 5.3, $\rho^{k(m)} \notin \text{PL}(\delta_i, \rho^{k(i)})$. Hence by Lemma 5.1(ii) (PL Analysis), $\text{PL}(\delta_i, \sigma^{k(i)}) = \text{PL}(\delta_i, \rho^{k(i)}) \cup \{\rho^{k(m)}\}$.

By Subcases 3.1 and 3.2, $\bar{\rho}^{k(m)+1} \in \text{PL}(\text{up}(v_m), \rho^{k(m)+1})$. By Definition 5.3 and Lemma 5.10(v) (Component), for all q such that $i \leq q < m$, $\text{PL}(\text{up}(v_{q+1}), \rho^{k(m)+1})$ is a component of $\text{PL}(\text{up}(v_q), \rho^{k(m)+1})$. Hence by induction on $m-i$, $\bar{\rho}^{k(m)+1} \in \text{PL}(\text{up}(v_i), \rho^{k(m)+1})$. Let $\tau^{k(m)+1}$ be the immediate successor of $\bar{\rho}^{k(m)+1}$ along $\rho^{k(m)+1}$. As σ switches $\bar{\rho}^{k(m)+1}$ (and so, by Definition 5.6, cannot be a primary completion), it follows from (2.10) that $\bar{\rho}^{k(m)+1}$ is $\rho^{k(m)+1}$ -free. Hence by Lemma 5.10(iv) (Component), we may

apply Lemma 5.1(v) (PL Analysis) to conclude that $PL(\text{up}(v_i), \rho^{k(m)+1}) = PL(\text{up}(v_i), \tau^{k(m)+1})$. By Lemma 5.1(i),(ii) (PL Analysis) and Definition 5.3, and as $\bar{\rho}^{k(m)+1} \in PL(\text{up}(v_i), \rho^{k(m)+1})$,

$$PL(\text{up}(v_i), \tau^{k(m)+1}) = PL(\text{up}(v_i), \bar{\rho}^{k(m)+1}) \cup \{\bar{\rho}^{k(m)+1}\} \ \& \ \bar{\rho}^{k(m)+1} \notin PL(\text{up}(v_i), \rho^{k(m)+1}).$$

Now $(\sigma^{k(m)+1})^- = \bar{\rho}^{k(m)+1}$ and $\bar{\rho}^{k(m)+1}$ has finite outcome along $\sigma^{k(m)+1}$, so by Lemma 5.1(iv) (PL Analysis), $PL(\text{up}(v_i), \sigma^{k(m)+1}) = PL(\text{up}(v_i), \bar{\rho}^{k(m)+1})$. Thus

$$PL(\text{up}(v_i), \sigma^{k(m)+1}) = PL(\text{up}(v_i), \rho^{k(m)+1}) \setminus \{\bar{\rho}^{k(m)+1}\}.$$

(i) now follows.

We complete the proof of the lemma by verifying (vi). Fix $k < n$ and $\mu^k \subset v^k \subset \sigma^k$ such that v^k is implication-free and $\text{up}(\mu^k) \subset \text{up}(v^k), \sigma^{k+1}$ in order to verify (5.29)(i). (Note that we may assume, by induction, that $\eta^k = \sigma^k$ in (5.29)(i).) Fix p and s for σ as in Lemma 3.3 (λ -Behavior). We proceed by cases.

Case 1: σ is not j -switching for any $j \leq k+1$. Then $(\sigma^k)^- = \rho^k$ and $(\sigma^{k+1})^- = \rho^{k+1}$. If $v^k \subset \rho^k$, then $PL(v^k, \rho^k) \subseteq PL(v^k, \sigma^k)$ and $PL(\text{up}(\mu^k), \rho^{k+1}) \subseteq PL(\text{up}(\mu^k), \sigma^{k+1})$, so (5.29)(i) follows by induction. Otherwise, $v^k = \rho^k$. But as v^k is implication-free and ρ is not completion-consistent via $\langle \rangle$, this is impossible.

Case 2: σ is j -switching for some $j \leq k$. By (5.18) and Definition 5.6, σ switches a principal derivative which is not an initial derivative on T^j , so by Lemma 3.3 (λ -Behavior), $p+1 = j = s$. If $j < k$, then there is a $\delta \subset \sigma$ such that $\lambda^k(\delta) = \sigma^k$, so (5.29)(i) follows by induction. Suppose that $j = k$. If $v^k \subset \bar{\rho}^k$, then (5.29)(i) holds at $\text{out}^0(\bar{\rho}^k)$, and as $j = k = s$, $\sigma^{k+1} = \lambda(\bar{\rho}^k)$. Hence (5.29)(i) follows by induction and Lemma 5.1(i) (PL Analysis). Otherwise, it must be the case that $v^k = \bar{\rho}^k$. By Lemma 4.5 (Free Extension), $\text{up}(v^k) \subseteq \sigma^{k+1}$, so by Lemma 5.1(i) (PL Analysis), $PL(\text{up}(\mu^k), \text{up}(v^k)) \subseteq PL(\text{up}(\mu^k), \sigma^{k+1})$, and (5.29)(i) holds.

Case 3: σ is $(k+1)$ -switching. Then by Lemma 5.1(i) (PL Analysis), $PL(v^k, \rho^k) \subseteq PL(v^k, \sigma^k)$. Suppose that $\pi^{k+1} \in (PL(\text{up}(\mu^k), \rho^{k+1}) \cap PL(\text{up}(\mu^k), \text{up}(v^k))) \setminus PL(\text{up}(\mu^k), \sigma^{k+1})$. (vi) will follow by induction once we show that a derivative of π^{k+1} lies in $PL(v^k, \sigma^k)$. There are several subcases.

Subcase 3.1: π^{k+1} is the end of a primary ρ^{k+1} -link restraining $\text{up}(\mu^k)$. By (2.10) and as $\text{up}(\mu^k) \subset \lambda(\sigma^k)$, σ must switch π^{k+1} . If $\pi^{k+1} \supseteq \text{up}(\nu^k)$, then $\text{PL}(\text{up}(\mu^k), \text{up}(\nu^k)) \subseteq \text{PL}(\text{up}(\mu^k), \sigma^{k+1})$, so (vi) holds. If $\pi^{k+1} \not\supseteq \text{up}(\nu^k)$, then $\pi^{k+1} \notin \text{PL}(\text{up}(\mu^k), \text{up}(\nu^k))$, contrary to hypothesis. Hence $\pi^{k+1} \subset \text{up}(\nu^k)$. By Lemma 3.1(i) (Limit Path), π^{k+1} has an initial derivative $\pi^k \subset \nu^k$. But $\text{up}(\rho^k) = \pi^{k+1}$, and as σ is $(k+1)$ -switching, ρ^k has infinite outcome along σ^k . Hence by (5.13), $\rho^k \in \text{PL}(\nu^k, \sigma^k)$.

Subcase 3.2: (5.14) places π^{k+1} into $\text{PL}(\text{up}(\mu^k), \rho^{k+1}) \setminus \text{PL}(\text{up}(\mu^k), \sigma^{k+1})$ through the component induced by some β^{k+1} requiring extension, and the conditions of (5.13) fail. Then β^{k+1} cannot have a primary completion with infinite outcome along ρ^{k+1} , else by (2.10) and Lemma 5.3(i), any component of $\text{PL}(\text{up}(\mu^k), \rho^{k+1})$ induced by β^{k+1} would be a component of $\text{PL}(\text{up}(\mu^k), \sigma^{k+1})$. Thus by Lemma 5.10(iv) (Component), $(\beta^{k+1})^- = \text{up}(\nu_i)$ for some $i \leq m$, which we fix. Now by (iii), there is a derivative π^k of π^{k+1} such that $\pi^k \in \text{PL}(\delta_i, \sigma^k) \setminus \text{PL}(\delta_i, \rho^k)$, so by Lemma 5.1(ii) (PL Analysis), $\pi^k = \rho^k$. By Definition 5.3, ρ^k has infinite outcome along σ^k . Now $\pi^{k+1} \in \text{PL}(\text{up}(\mu^k), \text{up}(\nu^k))$, so again by Definition 5.3, $\pi^{k+1} \subset \text{up}(\nu^k)$; thus $\pi^k \neq \nu^k$, and by Lemma 3.1(i) (Limit Path), π^{k+1} has an initial derivative $\tilde{\pi}^k \subset \nu^k$. We now see that $[\tilde{\pi}^k, \rho^k]$ is a primary σ^k -link restraining ν^k , so by (5.13), $\rho^k = \pi^k \in \text{PL}(\nu^k, \sigma^k)$. \square

Our next lemma specifies the correspondence between a PL set which is encountered when a node η^k requires extension, and another PL set which is defined at the k -completion κ^k of η^k .

Lemma 5.12 (PL Lemma): Fix $k \leq r \leq n$, and $\nu^k \subset \delta^k \subset \eta^k \subseteq \kappa^k \in T^k$ such that $(\eta^k)^- = \delta^k$ and $k < \dim(\nu^k)$, and let $\nu^{k+1} = \text{up}(\nu^k)$. Assume that η^k requires extension for ν^k , and that κ^k is the k -completion of η^k for ν^k , and is preadmissible. Then:

- (i) $\{\tau^{k+1} \subseteq \lambda(\eta^k): \exists \rho^k(\eta^k \subset \rho^k \subseteq \kappa^k \text{ \& } \rho^k \text{ switches } \tau^{k+1})\} = \text{PL}(\nu^{k+1}, \lambda(\eta^k))$.
- (ii) $\{\text{up}(\xi^k): \xi^k \in \text{PL}(\delta^k, \kappa^k)\} = \text{PL}(\nu^{k+1}, \lambda(\eta^k))$.
- (iii) If $\langle\langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle\rangle_{r \geq j \geq k+1}$ is an amenable $(k+1)$ -implication chain, $\text{up}(\nu^k) = \sigma^{k+1}$, and $\text{up}(\delta^k) = \hat{\sigma}^{k+1}$, then $\{\text{up}^r(\xi^k): \xi^k \in \text{PL}(\delta^k, \kappa^k)\} = \text{PL}(\sigma^r, \tau^r)$.
- (iv) (5.29)(ii) holds at κ^k .

Proof: (ii): By Lemma 5.11(iv),(v) (Amenable Backtracking), $\{\text{up}(\xi^k): \xi^k \in \text{PL}(\delta^k, \eta^k)\} = \text{PL}(\nu^{k+1}, \lambda(\kappa^k)) = \emptyset$, so (ii) follows by repeated applications of Lemma 5.11(iii) (Amenable Backtracking) to those ρ^k such that $\eta^k \subset \rho^k \subseteq \kappa^k$.

(i): By (5.18), $\{\tau^{k+1} \subseteq \lambda(\eta^k): \exists \rho^k(\eta^k \subset \rho^k \subseteq \kappa^k \text{ \& } \rho^k \text{ switches } \tau^{k+1})\}$ is identical with $V = \{\tau^{k+1} \subseteq \lambda(\eta^k): \exists \rho^k(\eta^k \subset \rho^k \subseteq \kappa^k \text{ \& } (\text{out}^0(\rho^k))^- \text{ satisfies (5.18)(i)(a) with } k(m) =$

k or (5.18)(ii)(a) with $j = k+1$). By Lemma 5.11(i),(ii),(iv) (Amenable Backtracking), $V = \{\text{up}(\xi^k): \xi^k \in \text{PL}(\delta^k, \kappa^k)\}$. Hence (i) follows from (ii).

(iii): Let $\hat{\sigma}^k = \kappa^k$ and $\sigma^k = \delta^k$. By induction and Lemma 5.1(iv) (PL Analysis), it suffices to show that for all $j \in [k, r)$, $\{\text{up}(\xi^j): \xi^j \in \text{PL}(\sigma^j, \hat{\sigma}^j)\} = \text{PL}(\sigma^{j+1}, \tau^{j+1}) = \text{PL}(\sigma^{j+1}, \hat{\sigma}^{j+1})$. Fix $j \in [k, r)$, and let $\bar{\tau}^j = \text{out}(\tau^{j+1})$. Note that $\bar{\tau}^k = \eta^k$. By (5.5)(ii) and Lemma 5.2 (Requires Extension), $\bar{\tau}^j$ requires extension for σ^j , and $\hat{\sigma}^j$ is the j -completion of $\bar{\tau}^j$. By (5.5)(ii) and (ii), $\{\text{up}(\xi^j): \xi^j \in \text{PL}(\sigma^j, \hat{\sigma}^j)\} = \text{PL}(\sigma^{j+1}, \tau^{j+1})$. By (5.11), $\hat{\sigma}^{j+1}$ has finite outcome along τ^{j+1} , so by Lemma 5.1(iv) (PL Analysis), $\text{PL}(\sigma^{j+1}, \tau^{j+1}) = \text{PL}(\sigma^{j+1}, \hat{\sigma}^{j+1})$.

(iv): Immediate from (ii). \square

Let $\Lambda^0 \in [T^0]$ be admissible, and for all $k \leq n$, let $\Lambda^k = \lambda^k(\Lambda^0)$. In order to show that all requirements are satisfied, we will need to show that if a node is Λ^k -free, then it is also implication-free, and so can act according to the truth of the sentence generating its action. We will be able to show this under the assumption that Λ^0 is admissible. (5.17)(i) may prevent a node which is a potential component of a 0-implication chain from acting according to the truth of the sentence generating its action, as it forces a specified outcome for certain implication-free nodes. However, we want to show that the implication-chain mechanism ensures that the action this node takes is in accordance with the truth of that sentence. The proof of this fact relies on the next lemma, which relates the implication-freeness of one of the first two nodes on T^k of a k -implication chain to the implication-freeness of the other node.

Lemma 5.13 (Free Amenable Implication Chain Lemma): Suppose that $k < r \leq n$, $\langle \langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle: r \geq j \geq k+1 \rangle$ is an amenable $(k+1)$ -implication-chain, that $\hat{\sigma}^k$ is the primary completion of $\bar{\tau}^k = \text{out}(\tau^{k+1})$, and that $\xi \in T^0$ is preadmissible and the 0-completion corresponding to $\hat{\sigma}^k$. Let $\sigma^k = (\bar{\tau}^k)$. Then σ^k is implication-free iff $\hat{\sigma}^k$ is implication-free. Furthermore, if ξ is completion-consistent via $\langle \rangle$, then ξ is implication-free.

Proof: We proceed by induction on $r-k$. First suppose that $\hat{\sigma}^k$ is implication-free. By Lemma 5.2 (Requires Extension), $\bar{\tau}^k$ requires extension, so by (5.5)(ii) and (5.19), $\text{up}(\hat{\sigma}^k) = \sigma^{k+1}$ and $\text{up}(\sigma^k) = \hat{\sigma}^{k+1}$. As (5.23) fails to hold for $\hat{\sigma}^k$, σ^{k+1} is implication-free. Hence by (5.1) if $r = k+1$ and by induction otherwise, $\hat{\sigma}^{k+1}$ is implication-free. As (5.21) and (5.22) fail for $\hat{\sigma}^k$, $\hat{\sigma}^k$ is completion-consistent via $\langle \rangle$. (We note that the completion-consistency of the admissible $\eta \in T^0$ via S implies the completion-consistency of $\lambda^k(\eta)$ via the subsequence of S consisting of those nodes which are on T^j for some $j \geq k$.) By Lemma 5.9(i) (Completion-Consistency) applied to T^k , $(\hat{\sigma}^k)$ must be completion-consistent via $\langle \bar{\tau}^k \rangle$, and by (5.19), $\bar{\tau}^k$ is completion-consistent via $\langle \bar{\tau}^k \rangle$. By Lemma 5.9

(Completion-Consistency) applied on T^k , σ^k is completion-consistent via $\langle \rangle$, so (5.21) and (5.22) fail to hold for σ^k . Thus we conclude that σ^k is implication-free.

Now suppose that σ^k is implication-free. By Lemma 5.2 (Requires Extension), $\bar{\tau}^k$ requires extension, so by (5.5)(ii) and (5.19), $\text{up}(\hat{\sigma}^k) = \sigma^{k+1}$ and $\text{up}(\sigma^k) = \hat{\sigma}^{k+1}$. As (5.23) fails to hold for σ^k , $\hat{\sigma}^{k+1}$ is implication-free. Hence by (5.1) if $r = k+1$ and by induction otherwise, σ^{k+1} is implication-free. As (5.21) and (5.22) fail for σ^k , σ^k is completion-consistent via $\langle \rangle$. (We again note that the completion-consistency of the admissible $\eta \in T^0$ via S implies the completion-consistency of $\lambda^k(\eta)$ via the subsequence of S consisting of those nodes which are on T^j for some $j \geq k$.) By Lemma 5.9(ii) (Completion-Consistency) applied on T^k , $\bar{\tau}^k$ must be completion-consistent via $\langle \bar{\tau}^k \rangle$, and by (5.19), $(\hat{\sigma}^k)^-$ is completion-consistent via $\bar{\tau}^k$. By Lemma 5.9 (Completion-Consistency) applied to T^k , $\hat{\sigma}^k$ is completion-consistent via $\langle \rangle$, so (5.21) and (5.22) fail to hold for $\hat{\sigma}^k$. We conclude that $\hat{\sigma}^k$ is implication-free.

Suppose that ξ is completion-consistent via $\langle \rangle$ but not implication-free, in order to obtain a contradiction. By (5.10), (5.5), and Definition 5.6, for all j such that $k \leq j \leq r = \dim(\hat{\sigma}^k) - 1$, $\text{out}^0(\sigma^j)$ and $\text{out}^0(\hat{\sigma}^j)$ are completion-consistent via the same sequence; so σ^j is primarily implication-restrained iff $\hat{\sigma}^j$ is primarily implication-restrained. Furthermore, by (5.10), neither σ^r nor $\hat{\sigma}^r$ is implication-restrained, and by (5.9), $\text{up}^j(\hat{\sigma}^k) \in \{\sigma^j, \hat{\sigma}^j\}$ for all $j \in [k, r]$. Hence we fix the largest j such that $\text{up}^j(\hat{\sigma}^k)$ is implication-restrained, and note that $j < r$, and that $\text{up}^j(\hat{\sigma}^k)$ is either primarily or hereditarily implication-restrained.

First suppose that $\text{up}^j(\hat{\sigma}^k)$ is primarily implication-restrained. By (5.6) and as $j < r$, $\text{up}^j(\hat{\sigma}^k) \in \{\sigma^j, \hat{\sigma}^j\}$, so by the preceding paragraph, both σ^j and $\hat{\sigma}^j$ are primarily implication-restrained and completion-consistent via the same sequence. Thus by (5.8)(i), there is an $\eta^j \subseteq \sigma^j$ which requires extension but has no j -completion $\subseteq \hat{\sigma}^j$. Fix δ^j such that $(\eta^j)^- = \delta^j$ and ν^j such that η^j requires extension for ν^j . Now $\delta^j \subseteq \sigma^j$, so by Lemma 5.5(ii) (Completion-Respecting) and as ξ is completion-consistent via $\langle \rangle$ and is not a 0-completion, η^j has a j -completion $\kappa^j \subseteq \lambda^j(\xi)$ which has infinite outcome along $\lambda^j(\xi)$. By Definition 5.6 and (2.7), $\text{up}^j(\hat{\sigma}^k) \subseteq \lambda^j(\xi)$, so as $\text{up}^j(\hat{\sigma}^k) \in \{\sigma^j, \hat{\sigma}^j\}$ and η^j has no j -completion $\subseteq \hat{\sigma}^j$, $\text{up}^j(\hat{\sigma}^k)$ is $\lambda^j(\xi)$ -restrained by the primary link $[\nu^j, \kappa^j]$, contradicting (2.10).

Now suppose that $\text{up}^j(\hat{\sigma}^k)$ is hereditarily implication-restrained but not primarily implication-restrained. By (5.6) and as $j < r$, $\text{up}^j(\hat{\sigma}^k) \in \{\sigma^j, \hat{\sigma}^j\}$, so by the preceding paragraph, both σ^j and $\hat{\sigma}^j$ are hereditarily but not primarily implication-restrained and are completion-consistent via the same sequence. Thus there are $i > j$ and $\eta^i \in T^i$ such that η^i requires extension, has no j -completion $\subseteq \text{up}^j(\hat{\sigma}^k)$, and $\text{out}^j(\eta^i) \subseteq \sigma^j$. By Lemma 5.4 (Compatibility) and as ξ is completion-consistent via $\langle \rangle$, η^i has a j -completion κ^j such that $\hat{\sigma}^j \subseteq \kappa^j \subseteq \lambda^j(\beta)$ for some $\beta \subseteq \xi$. Let $\bar{\tau}^k \subseteq \hat{\sigma}^k$ be defined by $(\bar{\tau}^k)^- = \sigma^k$. By Definition 5.6,

$\bar{\tau}^k$ requires extension; but by Lemma 3.2 (Out), $\text{out}^0(\eta^i) \subset \text{out}^0(\bar{\tau}^k) \subset \text{out}^0(\kappa^j)$, contradicting (5.26). \square

We are now ready to show that completions exist. We proceed as described in the example preceding Definition 5.3. The definition proceeds by induction on $n-k$, and then by induction on the cardinality of PL sets. The process used, within the proof, to construct completions is called *backtracking*.

Lemma 5.14 (Completion Lemma): Fix $\eta \in T^0$ such that η is admissible, $\eta^k = \lambda^k(\eta)$ requires extension, and $\eta = \text{out}^0(\eta^k)$. Then there is an effectively obtainable admissible 0-completion $\kappa \supset \eta$ of η^k .

Proof: For all $j \leq n$, let $\eta^j = \lambda^j(\eta)$. Let η^k require extension for v^k , set $v^{k+1} = \text{up}(v^k)$, and note, by (5.3) and Lemma 4.3(i)(a) (Link Analysis), that $v^{k+1} \subset \eta^{k+1}$. Fix $\delta^k \subset \eta^k \in T^k$ such that $(\eta^k)^- = \delta^k$. Fix $u \geq 0$ and $S = \langle \alpha_i : i \leq u \rangle$ such that η is completion-consistent via S . We proceed by induction on $n-k$, and then by induction on the cardinality of $\text{PL}(v^{k+1}, \eta^{k+1})$. We carry out a *backtracking process*, constructing increasing sequences $\langle \xi_i^k \in T^k : i \leq m \rangle$ and $\langle \bar{\xi}_i \in T^0 : i \leq m \rangle$ of strings for some $m \geq u$ such that each ξ_i is an admissible extension of η , and $\bar{\xi}_i = \text{out}^0(\xi_i^k)$. m will be bounded by the length of the longest η^{k+1} -link restraining v^{k+1} plus 1. We also define a map $\xi_i^k \rightarrow \bar{\xi}_i^{k+1} \in T^{k+1}$ for $i \leq m$, yielding a decreasing sequence of strings on T^{k+1} .

We begin by setting $\xi_0 = \eta$, $\xi_0^k = \eta^k$ and $\bar{\xi}_0^{k+1} = \lambda(\eta^k)$. Suppose that ξ_i , ξ_i^k , and $\bar{\xi}_i^{k+1}$ have been defined for some fixed $i < m$. We assume by induction that:

- (5.31) (i) ξ_i is admissible and completion-consistent via S .
(ii) If $i > 0$, then ξ_i is switching.

- (5.32) (i) For all $j < i$, $\xi_j \subset \xi_i$, $\xi_j^k \subset \xi_i^k$, and $\bar{\xi}_j^{k+1} \subset \bar{\xi}_i^{k+1}$.
(ii) $\bar{\xi}_i^{k+1} \subseteq \lambda(\xi_i^k)$.

- (5.33) (i) $\bar{\xi}_i^{k+1}$ is $\lambda(\xi_i^k)$ -free.
(ii) (5.18)(i)(a), with ξ_i in place of ρ , is not satisfied.

At the end, we will ensure that (5.32) and (5.33)(i) also hold for $i = m$, and in addition:

- (5.34) $\bar{\xi}_m^{k+1} = v^{k+1}$ and ξ_m is admissible and the 0-completion of η^k .

First suppose that $i = 0$. (5.32) is vacuous. (5.31) follows by hypothesis.

(5.33)(i) follows from (2.10). By Definition 5.6, any node which is a 0-completion is nonswitching, and its immediate predecessor is implication-restrained. It follows from (5.1) that $\dim((\eta^k)^-) > k+1$, so (5.5)(ii) must be true when η^k requires extension; and by (5.5)(ii) and (5.18)(i) for $(\xi_i)^-$ and ξ_i , if $(\xi_i)^-$ is implication-restrained and $\lambda^j(\xi_i)$ requires extension, then ξ_i is switching. Hence η cannot be a 0-completion, and (5.33)(ii) holds.

We now assume that $i \geq 0$ and verify (5.31)-(5.34) for $i+1$. There are two cases:

Case 1: There is a $\lambda(\xi_i^k)$ -link $[\mu^{k+1}, \pi^{k+1}]$ which restrains v^{k+1} . By Lemma 4.1 (Nesting), we can assume that $[\mu^{k+1}, \pi^{k+1}]$ is the longest such link. Now there is a $t \geq k+1$ and a primary $\lambda^t(\xi_i^k)$ -link $[\mu^t, \pi^t]$ such that $[\mu^{k+1}, \pi^{k+1}]$ is derived from $[\mu^t, \pi^t]$. π^t is $\lambda^t(\xi_i^k)$ -free by Lemma 4.3(iii) (Link Analysis), and by (4.1), will be $\lambda^t(\xi_i^k)$ -free for any nonswitching extension $\bar{\xi}_i^k$ of ξ_i^k . And as $[\mu^t, \pi^t]$ is a primary $\lambda^t(\xi_i^k)$ -link, π^t has infinite outcome along $\lambda^t(\xi_i^k)$. Hence by Lemma 4.4 (Free Implies True Path), π^t is $\lambda^{t+1}(\xi_i^k)$ -consistent for all nonswitching extensions $\bar{\xi}_i^k$ of ξ_i^k . By Lemma 3.1(iii) (Limit Path), all blocks defined in Step 4 of Definition 2.8 are finite, so by repeated applications of Lemma 3.4 (Nonswitching Extension), we can keep taking nonswitching extensions of ξ_i , and will eventually reach the shortest such nonswitching $\beta \supset \xi_i$ such that $\text{up}^t(\beta^k) = \pi^t$, $\text{up}^{t+1}(\beta^k) \supset \lambda^{t+1}(\xi_i^k)$ and β is an initial derivative of $\text{up}^{t+1}(\beta^k)$. (When we apply the Nonswitching Extension Lemma to extend a string, and it is possible to take both activated and validated extensions and still be nonswitching, (5.18)(iii) requires that we take the activated extension, in order to uniquely define the process of taking nonswitching extensions in this induction.) As ξ_i is admissible, it follows from Lemma 5.9 (Completion-Consistency) that β will be admissible and completion-consistent via S unless there is a ρ such that $\xi_i \subseteq \rho \subset \beta$ and either (5.18)(i)(a) or (5.18)(ii)(a) holds for ρ , and that such a ρ will be admissible and completion-consistent via S . (The clauses of (5.29) not covered by Lemma 5.9 are covered by Lemmas 5.11-5.13. In particular, (5.29)(i) is covered by Lemma 5.11(vi), (5.29)(ii) by Lemma 5.12(iv), and (5.29)(iii) by Lemma 5.13.) So assume that such a ρ exists in order to obtain a contradiction.

By (4.1), for all $\gamma \in T^0$ such that $\text{lh}(\gamma) > 0$, if γ is a nonswitching extension of γ^- , then for all $q \leq n$, the $\lambda^q(\gamma)$ -links coincide with the $\lambda^q(\gamma^-)$ -links. Hence (5.18)(ii)(a) cannot hold for ρ , else $\rho = \beta$. As $\text{up}(\nu^k)$ is not ξ_i^k -free, $\text{up}(\nu^k)$ is not $\lambda^k(\rho)$ -free. So ρ cannot be the 0-completion of η^k , else by Definition 2.6, $\text{up}^{k+1}(\rho) = \text{up}(\nu^k)$, so by (2.10), $\text{up}(\nu^k)$ would have to be $\lambda^k(\rho)$ -free, which is impossible. It now follows that ρ is not a 0-completion, else by Definition 5.6, ρ would have to be the 0-completion of $\eta^k = \alpha_u$; hence (5.18)(i)(a) does not hold for ρ . The same proof shows that (5.18)(i)(a) does not hold for $\rho = \beta$.

We conclude that β is admissible and completion-consistent via S , and that (5.18)(i)(a) does not hold for β . By Lemma 3.6 (Switching), we can choose an extension

$\tilde{\beta}$ of β such that $\tilde{\beta}^- = \beta$ and $\tilde{\beta}$ induces an infinite outcome for $\text{up}^{\pm 1}(\beta)$ along $\lambda^{\pm 1}(\tilde{\beta})$, thus switching the outcome of π^t to finite along $\lambda^t(\tilde{\beta})$. Note that (5.18)(ii)(a) holds for β , so by (5.18)(ii), $\tilde{\beta}$ is preadmissible. Now $t-1$ is the p in Lemma 3.3 (λ -Behavior), so $\lambda^j(\beta) \subseteq \lambda^j(\tilde{\beta})$ iff $j \square t$.

Subcase 1.1: π^t is not a primary t -completion. Then by (5.18), ξ_i is not a 0-completion. Set $\xi_{i+1} = \tilde{\beta}$, $\xi_{i+1}^k = \lambda^k(\tilde{\beta})$ and $\bar{\xi}_{i+1}^{k+1} = (\lambda^{k+1}(\tilde{\beta}))^-$. We note that neither the hypothesis of Lemma 5.9(i) or of Lemma 5.9(ii) (Completion-Consistency) is satisfied, so by Lemma 5.9(iii) (Completion-Consistency), ξ_{i+1} is completion-consistent via S . Furthermore, by Lemma 5.9(iv) (Completion-Consistency) and again by Lemmas 5.11(vi), 5.12(iv), and 5.13, ξ_{i+1} is admissible. (5.31) now follows for ξ_{i+1} , and (5.32) and (5.33) follow from the properties of $\tilde{\beta}$, (2.10), and Lemma 4.5 (Free Extension).

Subcase 1.2: π^t is a primary t -completion. First suppose that $t > k+1$. Then by (2.4), $(\lambda^t(\tilde{\beta}))^- = \pi^t$ and π^t has finite outcome along $\lambda^t(\tilde{\beta})$. Hence by Lemma 5.3(ii) (Implication Chain) and Lemma 5.2 (Requires Extension), $\lambda^{\pm 1}(\tilde{\beta})$ requires extension, so by induction on $n-k$, we can find a 0-completion κ of $\lambda^{\pm 1}(\tilde{\beta})$, and, by Lemma 5.9(i),(iv), (Completion-Consistency) and again by Lemmas 5.11(vi), 5.12(iv), and 5.13, find an admissible $\tilde{\kappa}$ such that $\tilde{\kappa}^- = \kappa$ and $\tilde{\kappa}$ induces an infinite outcome for $(\lambda^{\pm 1}(\tilde{\kappa}))^-$ along $\lambda^{\pm 1}(\tilde{\kappa})$. By Lemma 5.7(i) (Primary Completion), $\lambda^{\pm 1}(\tilde{\kappa})$ does not require extension. We now set $\xi_{i+1} = \tilde{\kappa}$, $\xi_{i+1}^k = \lambda^k(\tilde{\kappa})$ and $\bar{\xi}_{i+1}^{k+1} = (\lambda^{k+1}(\tilde{\kappa}))^-$, and note that (5.31)-(5.33) follow from the properties of $\tilde{\kappa}$, (2.4), (2.10), and Lemma 4.5 (Free Extension) and Lemma 5.9 (Completion-Consistency). ((5.33)(ii) follows since $\tilde{\kappa}$ is switching, and by Definition 5.6, completions are nonswitching.) The induction step is now complete for this case.

Suppose that $t = k+1$. Then by (2.4), $(\lambda^{k+1}(\tilde{\beta}))^- = \pi^{k+1}$ and π^{k+1} has finite outcome along $\lambda^{k+1}(\tilde{\beta})$. By Lemma 5.3(ii) (Implication Chain), there is an $r < n$ and an amenable $(k+1)$ -implication chain $\langle\langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle\rangle: r \geq j \geq k+1$ such that $\hat{\sigma}^{k+1} = \pi^{k+1}$ and the immediate successor of σ^{k+1} along $\hat{\sigma}^{k+1}$ requires extension for μ^{k+1} . Note that as $[\mu^{k+1}, \pi^{k+1}]$ restrains ν^{k+1} , $\mu^{k+1} \subseteq \nu^{k+1}$. And by (5.2) and Lemma 5.10(v) (Component), ν^{k+1} has infinite outcome along π^{k+1} , so as, by (2.8), μ^{k+1} has finite outcome along π^{k+1} , $\mu^{k+1} \subset \nu^{k+1}$. By Lemma 5.2 (Requires Extension), $\tilde{\beta}^k = \lambda^k(\tilde{\beta})$ requires extension for some $\tilde{\gamma}^k$, and $\tilde{\beta}^k$ is not implication-free. Furthermore, by (5.5)(ii) and (5.8), $\text{up}(\tilde{\gamma}^k) = \sigma^{k+1} \subset \hat{\sigma}^{k+1} = \text{up}((\tilde{\beta}^k)^-) = \pi^{k+1} \subset \eta^{k+1}$. By Lemma 5.9(ii) (Completion-Consistency), $\tilde{\beta}$ is completion-consistent via $S \wedge \langle \tilde{\beta}^k \rangle$, so by Lemma 5.10(ii) (Component), $\nu^{k+1} \subset \text{up}(\tilde{\gamma}^k)$. By Lemma 3.3

(λ -Behavior), $(\lambda(\tilde{\beta}^k))^- = \pi^{k+1}$ and π^{k+1} has finite outcome along $\lambda(\tilde{\beta}^k)$, so by Lemma 5.1(iv) (PL Analysis), $\text{PL}(\text{up}(\tilde{\gamma}^k), \lambda(\tilde{\beta}^k)) = \text{PL}(\text{up}(\tilde{\gamma}^k), \pi^{k+1})$. Hence by (5.14), $\text{PL}(\text{up}(\tilde{\gamma}^k), \lambda(\tilde{\beta}^k))$ is a component of $\text{PL}(\mathbf{v}^{k+1}, \lambda(\tilde{\beta}^k))$, so by (5.14)(i) and Lemma 5.1(vi) (PL Analysis), $\text{up}(\tilde{\gamma}^k) \in \text{PL}(\mathbf{v}^{k+1}, \lambda(\tilde{\beta}^k)) \setminus \text{PL}(\text{up}(\tilde{\gamma}^k), \lambda(\tilde{\beta}^k))$. Thus $\text{PL}(\mathbf{v}^{k+1}, \lambda(\tilde{\beta}^k)) \supset \text{PL}(\text{up}(\tilde{\gamma}^k), \lambda(\tilde{\beta}^k)) = \text{PL}(\text{up}(\tilde{\gamma}^k), \pi^{k+1})$. We now proceed as in the preceding paragraph to find an admissible switching extension of a 0-completion of $\tilde{\beta}^k$, and justifying the existence of κ by induction on the cardinality of the PL sets.

Case 2: Otherwise. By the case assumption, there are no $\lambda(\xi_i^k)$ -links restraining \mathbf{v}^{k+1} . Hence \mathbf{v}^{k+1} is $\lambda(\xi_i^k)$ -free, so as in Case 1, we can keep taking nonswitching extensions of ξ_i , taking the activated extension when both the activated and validated extensions are nonswitching, and will eventually reach the shortest such nonswitching $\xi_{i+1} \supset \xi_i$ such that $\text{up}^{k+1}(\xi_{i+1}) = \mathbf{v}^{k+1}$ and ξ_{i+1} is admissible and completion-consistent via S. By (5.31)(ii) and Definition 5.6, ξ_i cannot be a 0-completion. Thus by Lemma 5.9 (Completion-Consistency) and Lemmas 5.10-5.12, ξ_{i+1} is the shortest nonswitching extension of ξ_i satisfying (5.18)(i)(a), and is admissible. We set $m = i+1$, $\xi_m^m = \lambda^k(\xi_{i+1})$, and $\bar{\xi}_m^{k+1} = \mathbf{v}^{k+1}$. (5.32), (5.33)(i), and (5.34) now follow. \blacksquare

We now show that admissible paths have nice properties; they are completion-respecting and do not extend amenable implication chains.

Lemma 5.15 (Admissibility Lemma): Let an admissible path $\Lambda^0 \in [T^0]$ be given, and for all $k \leq n$, let $\Lambda^k = \lambda^k(\Lambda^0)$. Then for all $k \leq n$:

- (i) Λ^k does not extend an amenable k -implication chain.
- (ii) Every $\eta^k \subset \Lambda^k$ which requires extension has a primary completion along Λ^k .

Proof: We proceed by induction on k . First let $k = 0$. (i) follows from (5.11)(ii) and (5.18)(i) for implication-restrained nodes, and from (5.11)(ii) and (5.17)(i) for implication-free nodes. And (ii) follows from Lemma 5.14 (Completion), the uniqueness of primary completions, and (5.18).

Suppose that $k > 0$. First suppose that $\langle\langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle\rangle: r \geq j \geq k$ is an amenable k -implication chain for some r , with $\tau^k \subset \Lambda^k$, in order to obtain a contradiction. By (2.5), $\bar{\tau}^{k+1} = \text{out}(\tau^k) \subset \Lambda^{k+1}$, and by Lemma 5.2 (Requires Extension), $\bar{\tau}^{k+1}$ requires extension. By (ii) inductively, $\bar{\tau}^{k+1}$ has a $(k-1)$ -completion $\kappa^{k+1} \subset \Lambda^{k+1}$, and by Lemma 5.3(ii) (Implication Chain) and (i) inductively, κ^{k+1} has infinite outcome along Λ^{k+1} . Fix $\xi^{k+1} \subset \Lambda^{k+1}$ such that $(\xi^{k+1})^- = \kappa^{k+1}$. By (5.5)(ii) and (5.19), $\text{up}(\kappa^{k+1}) = \sigma^k$ and so by (2.4), σ^k has finite outcome along Λ^k . By (5.11)(i), σ^k has infinite outcome along τ^k , so $\tau^k \not\subset \Lambda^k$, yielding the desired contradiction. Hence (i) holds for k .

Now suppose that $\eta^k \subset \Lambda^k$ requires extension for v^k . By (5.18), the uniqueness of completions, and Lemma 5.14 (Completion), η^k has a 0-completion $\subset \Lambda^0$, so by Lemma 5.4 (Compatibility), η^k has a $(k-1)$ -completion $\kappa^{k+1} \subset \Lambda^{k+1}$, so has a principal derivative $\bar{\kappa}^{k+1} \subset \Lambda^{k+1}$. First assume that $\bar{\kappa}^{k+1}$ has finite outcome along Λ^{k+1} . Fix $\bar{\tau}^{k+1} \subset \Lambda^{k+1}$ such that $(\bar{\tau}^{k+1})^- = \bar{\kappa}^{k+1}$, and let $\kappa^k = \text{up}(\bar{\kappa}^{k+1})$. Then $[v^k, \kappa^k]$ is a primary $\lambda(\bar{\tau}^{k+1})$ -link which restrains η^k , so by (2.6), (2.10) and as $\eta^k \subset \Lambda^k$, no β^{k+1} such that $\bar{\tau}^{k+1} \subseteq \beta^{k+1} \subset \Lambda^{k+1}$ can switch any node $\subset \kappa^k$, and so $\kappa^k \subset \Lambda^k$. (ii) now follows in this case.

Suppose that $\bar{\kappa}^{k+1}$ has infinite outcome along Λ^{k+1} , fix $\bar{\tau}^{k+1} \subset \Lambda^{k+1}$ such that $(\bar{\tau}^{k+1})^- = \bar{\kappa}^{k+1}$, and let $\kappa^k = \text{up}(\bar{\kappa}^{k+1})$. Then by (2.4), κ^k has finite outcome along $\lambda(\bar{\tau}^{k+1})$, and so by Lemma 5.3(ii) (Implication Chain), $\lambda(\bar{\tau}^{k+1})$ is the last node of the last triple of an amenable k -implication chain. By Lemma 5.2 (Requires Extension), $\bar{\tau}^{k+1}$ requires extension. By (ii) inductively, $\bar{\tau}^{k+1}$ has a $(k-1)$ -completion $\alpha^{k+1} \subset \Lambda^{k+1}$, and by Lemma 5.3(ii) (Implication Chain) and (i) inductively, α^{k+1} has infinite outcome along Λ^{k+1} . Fix $\xi^{k+1} \subset \Lambda^{k+1}$ such that $(\xi^{k+1})^- = \alpha^{k+1}$. By (5.5)(ii) and (5.19), $\text{up}(\alpha^{k+1}) = (\eta^k)^-$ and so by (2.4), $(\eta^k)^-$ has finite outcome along Λ^k . By (5.2), $(\eta^k)^-$ has infinite outcome along η^k , so $\eta^k \not\subset \Lambda^k$, yielding the desired contradiction. Hence (ii) holds for k . \square

In order to show that all requirements are satisfied, we will need to show that if a node is Λ^k -free, then it is also implication-free, so can act in accordance with the truth of the sentence trying to generate its action. In fact, we will need to apply this lemma to $\beta^k \subseteq \Lambda^k$ such that $\text{out}^0(\beta^k)$ is pseudotrue.

Lemma 5.16 (Implication-Freeness Lemma): Fix $k \leq n$. Suppose that $\beta^k \in T^k \cup [T^k]$ is admissible, and if $\text{lh}(\beta^k) < \infty$, then $\beta = \text{out}^0(\beta^k)$ is pseudotrue. For all $i \leq n$, let $\beta^i = \lambda^i(\beta)$. Let $\eta^{k+1} \subset \lambda(\beta^k)$ be $\lambda(\beta^k)$ -free and implication-free. Then:

- (i) For all $j \leq k$, the initial derivative v^j of η^{k+1} along β^j is implication-free.
- (ii) If $\eta^k \subset \beta^k$, $\text{up}(\eta^k) = \eta^{k+1}$, and η^k is β^k -free, then η^k is implication-free.
- (iii) If η^{k+1} is $(k+1)$ -completion-free, then the initial derivative v^k of η^{k+1} along β^k is k -completion-free.
- (iv) If $\dim(\eta^{k+1}) = k+1$, v^k is the initial derivative of η^{k+1} along β^k and has finite outcome along β^k , then $v = \text{out}^0(v^k)$ is completion-consistent via $\langle \rangle$.
If, in addition, δ^k is the immediate successor of v^k along β^k , then $\delta = \text{out}^0(\delta^k)$ is pseudotrue.

Proof: Recall that, if β exists, then β is pseudotrue, so β^k is completion-consistent via $\langle \rangle$ and β^j is j -completion-free for all $j \leq n$.

- (i): Fix $j \leq k$. By Lemma 3.1 (Limit Path), $v^j \subset \beta^j$. Suppose that v^j is not

implication-free, in order to obtain a contradiction. Then one of clauses (5.21)-(5.23) must cause v^j to be implication-restrained. If (5.21) holds, then there is a shortest $\xi^j \subseteq v^j$ such that ξ^j requires extension but there is no j -completion of ξ^j along v^j . Let ξ^j require extension for μ^j . By Lemma 5.5(ii) (Completion-Respecting), ξ^j has a j -completion $\kappa^j \subseteq \beta^j$ and κ^j has infinite outcome along β^j . Hence $\mu^j \subseteq v^j \subseteq \kappa^j \subseteq \beta^j$, and $[\mu^j, \kappa^j]$ is a primary β^j -link. By Lemma 4.3(i)(c) (Link Analysis), $\eta^{k+1} = \text{up}^{k+1}(v^j)$ cannot be $\lambda(\beta^j)$ -free, contrary to hypothesis.

Suppose that (5.22) holds in order to obtain a contradiction, and fix the largest i for which (i) fails because (5.22) holds for i . By Lemma 5.15(ii) (Admissibility) for $\text{lh}(\beta^k) = \infty$, and Lemma 5.4 (Compatibility) and since β is completion-free if $\text{lh}(\beta^k) < \infty$, there is a $q > i$ and a $\delta^q \in T^q$ such that δ^q requires extension, δ^q has an i -completion $\kappa^i \subseteq \beta^i$, and $\delta^i = \text{out}^i(\delta^q) \subseteq v^i \subseteq \kappa^i$. As $q > i$, it follows from (5.18) and (5.26) that no node in $(\delta^i, \kappa^i]$ is $(i+1)$ -switching. Let $\delta^{i+1} = \text{out}^{i+1}(\delta^q)$, and by Lemma 5.4 (Compatibility) let κ^{i+1} be the $(i+1)$ -completion of δ^q along $\lambda(\beta^i)$, and note, by Definition 5.6, that $\text{up}(\kappa^i) = \lambda(\kappa^i) = \kappa^{i+1}$ and κ^i is an initial derivative of κ^{i+1} . As v^i is an initial derivative of η^{i+1} , it follows from Lemma 3.1(i) (Limit Path) that $\delta^{i+1} \subseteq \text{up}(v^i) = v^{i+1} \subseteq \kappa^{i+1}$, so η^{i+1} is implication-restrained, contrary to the inductive hypothesis.

(5.23) cannot hold, by our induction.

(ii): Suppose that η^k is not implication-free, in order to obtain a contradiction. Then one of clauses (5.21)-(5.23) must cause η^k to be implication-restrained. (5.23) cannot hold by hypothesis. We assume that η^k is primarily or hereditarily implication-restrained, and obtain a contradiction. Fix the shortest $\delta^k \subseteq \eta^k$ such that for some $j \geq k$ and some $\mu^j \subseteq \delta^j = \lambda^j(\delta^k) \subseteq \beta^j$, δ^j requires extension for μ^j , but there is no k -completion of δ^j along η^k . By Lemma 5.5(ii) (Completion-Respecting), δ^j has a primary j -completion $\kappa^j \subseteq \beta^j$ which has infinite outcome along β^j . If $j = k$, then by Lemma 5.2(i) (Implication Chain), $[\mu^j, \kappa^j]$ is a primary β^j -link. And if $j > k$, then as we have assumed that $\kappa^k \not\subseteq \eta^k$, it follows that $\delta^k \subseteq \eta^k \subseteq \kappa^k$. Now if $[\mu^k, \kappa^k]$ is the β^k -link derived from the primary $\lambda^j(\beta^k)$ -link $[\mu^j, \kappa^j]$, then $\mu^k \subseteq \delta^k \subseteq \eta^k \subseteq \kappa^k$, so η^k is not β^k -free, contrary to hypothesis.

(iii): If η^{k+1} is $(k+1)$ -completion-free, then by Lemma 3.1(i) (Limit Path), for all $j \geq k+1$, $\lambda^j(\eta^{k+1}) = \lambda^j(v^k)$ is not a primary completion. By Definition 5.6, no primary completion is an initial derivative. (iii) now follows.

(iv): For all $i \leq k$, let $\delta^i = \text{out}^i(\delta^k)$, let $v^i = (\delta^i)^-$, and note that v^i is the principal derivative of v^k along δ^i and that $\delta^i \subseteq \beta^i$ by (2.5). We first show that v is completion-consistent via $\langle \rangle$. Suppose not in order to obtain a contradiction. Then we may fix the largest i such that v^i is implication-restrained. As v^k is implication-free, $i < k$; note that, by choice of i , v^i is either primarily or hereditarily implication-restrained. First suppose that v^i is hereditarily implication-restrained. By Lemma 5.4 (Compatibility), δ^i lies along the i -completion of the node witnessing that v^i is hereditarily implication-restrained, and by

(5.18) and (5.25), δ^i is not $(i+1)$ -switching. Hence v^i is the initial derivative of v^{i+1} along β^i . By Lemma 3.1 (Limit Path), an initial derivative can be hereditarily implication-restrained only if its immediate antiderivative is primarily or hereditarily implication-restrained; hence v^{i+1} is primarily or hereditarily implication-restrained, contrary to the choice of i .

Now suppose that v^i is primarily implication-restrained. Then there is an $\eta^i \subseteq v^i$ which requires extension but has no primary completion $\subseteq v^i$. Fix $\mu^i \subset \eta^i$ such that η^i requires extension for μ^i . By Lemma 5.5(ii) (Completion-Respecting), η^i has a primary completion $\kappa^i \subset \beta^i$ which has infinite outcome along β^i . Thus $[\mu^i, \kappa^i]$ is a primary β^i -link restraining v^i . But then by Lemma 4.3(i)(a) (Link Analysis), $v^{i+1} \not\subseteq \beta^{i+1}$, contradicting (2.5). This completes the proof of the first part of (iv).

We now note that for all i , $\delta^i = \lambda^i(\delta)$ is not a primary completion. For as v is completion-consistent via $\langle \rangle$ and $\delta^\pm = v$, it follows from (5.27) that if δ^i is a primary completion, then it is a primary completion of δ^i , contrary to Definition 5.6.

Finally, we show by contradiction that for all $i \leq n$, δ^i does not require extension. Fix the largest i such that δ^i requires extension in order to obtain a contradiction, and let δ^i require extension for μ^i . If $i > k$, then by (5.2), v^k is the principal derivative of η^{k+1} along δ^k , and $up^n(v^k)$ has a unique derivative along δ^j for all $j > k$; hence by (2.4), $(\delta^j)^- = up^j(v^k)$ for all $j > k$, contrary to the dimension requirements of Definition 5.1. Hence $i \leq k$. As v^k has finite outcome along δ^k and $\dim(\eta^{k+1}) = k+1$, it follows from (5.2) that $i < k$. As β is pseudotrue or $lh(\beta) = \infty$, it follows either from Lemma 5.15(ii) (Admissibility) or Lemma 5.5(ii) (Completion-Respecting) that δ^i has a primary completion $\kappa^i \subset \beta^i$ which has infinite outcome along β^i . By Definition 5.1, $\mu^i \subset v^i \subset \kappa^i$, and $[\mu^i, \kappa^i]$ is a primary β^i -link. By (5.2), v^i has infinite outcome along δ^i , so is the principal derivative of $up(v^i)$ along β^i . By Lemma 4.3(i)(c) (Link Analysis), $up(v^i) \not\subseteq \beta^{i+1}$. But as $i < k$, $up(v^i) = v^{i+1} \subset \delta^{i+1} \subseteq \beta^{i+1}$, a contradiction. Thus δ is pseudotrue. \square

Our next lemma shows that, under the assumption that Λ^0 is admissible, every requirement R is assigned to a free and implication-free node along Λ^n . Furthermore, if R has dimension k , then we will show that R is assigned to a unique free and implication-free node ζ^k along Λ^k , and that the principal derivative of ζ^k along Λ^{k+1} is free and implication-free. We will show later that, as a result of this lemma, the construction will act to satisfy R in accordance with the truth or falsity of the sentence which tries to determine the action for R . The implication-freeness of the nodes involved will enable us to show that sufficiently many derivatives of ζ^k will also be able to act consistently with their assigned sentences. Again we will need to apply the lemma not only to Λ^0 , but to pseudotrue $\beta \subset \Lambda^0$.

Lemma 5.17 (Assignment Lemma): Suppose that $\beta \in T^0 \cup [T^0]$ is admissible, and if $\text{lh}(\beta) < \infty$, then β is pseudotrue. Let R be a requirement of dimension k . For all $i \leq n$, let $\beta^i = \lambda^i(\beta)$. Then:

- (i) If $\text{lh}(\beta) = \infty$, then there is a $\zeta^n \subset \beta^n$ such that ζ^n is β^n -free, implication-free, and n -completion-free, and R is assigned to ζ^n .
- (ii) If R is assigned to $\zeta^n \subset \beta^n$, then there is a unique $\zeta^k \subset \beta^k$ such that $\text{up}^n(\zeta^k) = \zeta^n$, and ζ^k is β^k -free, implication-free, and k -completion-free.
- (iii) If ζ^k exists as in (ii), then the principal derivative $\zeta^{k\pm 1}$ of ζ^k along $\beta^{k\pm 1}$ is $\beta^{k\pm 1}$ -free and implication-free.
- (iv) If $j \leq n$, $\xi^j \subset \beta^j$ is β^j -free and implication-free, $\delta^j \subseteq \beta^j$, and $(\delta^j)^- = \xi^j$, then $\delta = \text{out}^0(\delta^j)$ is pseudotrue.
- (v) If $\zeta^n \subset \beta^n$ and $\text{lh}(\zeta^n) > 0$, then $\text{out}^0(\zeta^j)$ is pseudotrue, and the initial derivative of ζ^n along β is pseudotrue.

Proof: (i): Assume that $\text{lh}(\beta) = \infty$. By (5.28) or Lemma 5.15(i) (Admissibility), there are no amenable j -implication chains along β^j for any $j \leq n$. Fix i such that $R = R_i$. By Lemma 3.1(iii),(iv) all blocks along β^n are completed, so there are infinitely many blocks along β^n . Hence there is a $\zeta^n \subset \beta^n$ which completes the $(i+1)$ st block. By Lemma 3.1(i) (Limit Path), ζ^n has an initial derivative along $\beta^{n\pm 1}$, so a requirement must be assigned to ζ^n . Such a requirement can only be assigned when Step 4 of Definition 2.8 is followed, and the requirement assigned is R_i .

As there are no β^n -links, ζ^n is β^n -free. As all requirements have dimension $\leq n$, it follows from (5.2) and Definition 5.7 that ζ^n is implication-free. As no nodes on T^n require extension, ζ^n is n -completion-free.

(ii),(iii): By Lemma 3.1(ii) (Limit Path) inductively, for all $i \leq n$, ζ^n has a principal derivative $\zeta^i \subset \beta^i$, and by (2.9) for all i such that $k \leq i \leq n$, ζ^i is the unique derivative of ζ^n along β^i . For all $i \leq n$, it follows from Lemma 4.6(i) (Free Derivative), (i), and induction that ζ^i is β^i -free. Now by (i) and iterating Lemma 5.16(ii),(iii) (Implication-Freeness) inductively, we see that ζ^k is implication-free and k -completion-free. Again by Lemma 5.16(ii) (Implication-Freeness), $\zeta^{k\pm 1}$ is implication-free.

(iv): For all $i \leq j$, let $\delta^i = \text{out}^i(\delta^j)$, and for all $i > j$, let $\delta^i = \lambda^i(\delta^j)$. We note that by definition, for all $i \leq j$, $\xi^i = (\delta^i)^-$ is the principal derivative of ξ^j along β^j . Fix $i \leq n$. By Lemma 4.6(i) (Free Derivative), ξ^i is β^i -free, so by Lemma 5.16(ii) (Implication-Freeness), ξ^i is implication-free; thus $\xi = \xi^0 = \text{out}^0(\xi^i)$ is completion-consistent via $\langle \rangle$. Hence by Lemma 5.5(iii) (Completion-Respecting) applied to δ , every $\eta^i \subset \delta^i$ which requires extension has a primary completion $\subset \delta^i$. As no node can be its own primary completion, δ^i cannot be a primary completion.

We complete the proof of (iv) by assuming that δ^i requires extension, and obtaining

a contradiction. As β^i is admissible and if $\text{lh}(\beta^i) < \infty$ then β^i is pseudotrue and so $\text{out}^0(\beta^i)$ is completion-consistent via $\langle \rangle$ and is not a 0-completion, it follows from Lemma 5.5(ii) (Completion-Respecting) or Lemma 5.15(ii) (Admissibility) that $\delta^i \subseteq \beta^i$ has a primary completion $\kappa^i \subseteq \beta^i$. But β^i is pseudotrue so is not a primary completion; hence $\kappa^i \subset \beta^i$. Fix $\gamma^i \subseteq \beta^i$ such that $(\gamma^i)^- = \kappa^i$. If κ^i has infinite outcome along γ^i , then by Lemma 5.3(i) (Implication-Chain), there is a primary γ^i -link restraining ξ^i ; this link is then a primary β^i -link, contradicting the fact that ξ^i is β^i -free. Thus κ^i has finite outcome along γ^i , so by Lemma 5.3(ii) (Implication Chain), there is an amenable implication chain along β^i , contradicting (5.28) or Lemma 5.15(i) (Admissibility).

(v): By the proof of (i), every $\zeta^n \subset \beta^n$ is β^n -free and implication-free. The first conclusion of (v) now follows from (iv). Let ζ be the initial derivative of ζ^n along β . Then ζ is admissible, and by Lemma 3.1(i) (Limit Path) and as initial derivatives are not primary completions, for all $i \leq n$, $\text{up}^i(\zeta) = \lambda^i(\zeta)$ is not a primary completion. By Lemma 5.16(i) (Implication-Freeness), ζ is implication-free, hence completion-consistent via $\langle \rangle$. The second conclusion of (v) now follows. \square

In Lemma 5.12 (PL), we showed that the backtracking process yielded a one-to-one correspondence between the PL sets defined for any two triples of an amenable implication chain, and that this correspondence was provided by the up function. In order to successfully correct axioms, we will need to show that if ξ is pseudotrue, $\delta^1, \rho^1, \eta^1 \in T^1$, $\xi \supset \text{out}(\eta^1) = \eta$, and $\rho^1 \in \text{PL}(\delta^1, \eta^1) \setminus \text{PL}(\delta^1, \lambda(\xi))$, then some element of $(\eta, \xi]$ switches ρ^1 . The next lemma will allow us to draw such a conclusion when the need to correct is due to the existence of a nonamenable implication chain (the relationship of the nonamenable implication chain on $T^{\pm 1}$ to the situation on T^1 is not readily apparent, as it is absorbed in the control machinery of Section 6). We will also need an inclusion relation between PL sets at higher levels, in order to analyze the formation of implication chains.

Lemma 5.18 (Nonamenable Backtracking Lemma): Fix $k < n$, $\delta^{k+1}, \rho^{k+1}, \eta^{k+1} \in T^{k+1}$ and $\xi^k \supset \text{out}(\eta^{k+1}) = \eta^k$, such that $\delta^{k+1} \subset \eta^{k+1}, \lambda(\xi^k)$, δ^{k+1} is $\lambda(\xi^k)$ -free, $\xi = \text{out}^0(\xi^k)$ and $\text{out}^0(\eta^{k+1})$ are admissible and pseudotrue, and $\rho^{k+1} \in \text{PL}(\delta^{k+1}, \eta^{k+1})$. Then:

- (i) Some element of $(\eta^k, \xi^k]$ switches ρ^{k+1} .
- (ii) $\{\text{up}(\gamma^k): \gamma^k \in \text{PL}((\eta^k)^-, \xi^k)\} \supseteq \text{PL}(\delta^{k+1}, \eta^{k+1})$.

Proof: We first note that $\text{PL}(\delta^{k+1}, \lambda(\xi^k)) = \emptyset$. As δ^{k+1} is $\lambda(\xi^k)$ -free, (5.13) cannot place any elements into δ^{k+1} is $\lambda(\xi^k)$ -free. Suppose that $\tau^{k+1} \supset \delta^{k+1}$ requires extension for some $\mu^{k+1} \subset \delta^{k+1}$. As $\text{out}^0(\xi^k) = \text{out}^0(\lambda(\xi^k))$ is pseudotrue, $\text{out}^0(\lambda(\xi^k))$ is completion consistent via $\langle \rangle$ and is not a 0-completion. Hence by Lemma 5.5(ii) (Completion-Respecting), τ^{k+1} has a primary completion κ^{k+1} which has infinite outcome

along $\lambda(\xi^k)$. But then $[\mu^{k+1}, \kappa^{k+1}]$ is a primary $\lambda(\xi^k)$ -link restraining δ^{k+1} , so δ^{k+1} is not $\lambda(\xi^k)$ -free, contrary to hypothesis.

By (2.6), no element of $(\eta^k, \xi^k]$ can switch any $\gamma^{k+1} \subset \delta^{k+1}$. We first consider the case in which ρ^{k+1} is placed in $\text{PL}(\delta^{k+1}, \eta^{k+1})$ through (5.13), and show that (i) and (ii) are satisfied. As $\delta^{k+1} \subset \eta^{k+1}$, there is a $\mu^{k+1} \subset \eta^{k+1}$ and a primary η^{k+1} -link $[\mu^{k+1}, \rho^{k+1}]$ which restrains δ^{k+1} . As $\delta^{k+1} \subset \lambda(\xi^k)$ and $\text{PL}(\delta^{k+1}, \lambda(\xi^k)) = \emptyset$, it follows that $\rho^{k+1} \notin \text{PL}(\delta^{k+1}, \lambda(\xi^k))$; thus $[\mu^{k+1}, \rho^{k+1}]$ is not a $\lambda(\xi^k)$ -link. Hence by (2.6) and (2.10), some element τ^k of $(\eta^k, \xi^k]$ must switch ρ^{k+1} so (i) holds, and by (2.4), $\rho^k = (\tau^k)^-$ has infinite outcome along τ^k . Now $\rho^{k+1} \subset \eta^{k+1}$, so by Lemma 3.1(i), ρ^{k+1} has an initial derivative $\bar{\rho}^k \subset \eta^k$. Hence $[\bar{\rho}^k, \rho^k]$ is a primary ξ^k -link which restrains $(\eta^k)^-$, so by (5.13), $\rho^k \in \text{PL}((\eta^k)^-, \xi^k)$ and (ii) holds.

We complete the proof of (i) by showing that if ρ^{k+1} is placed into $\text{PL}(\delta^{k+1}, \eta^{k+1})$ by (5.14) as an element of the component $\text{PL}(\sigma^{k+1}, \zeta^{k+1})$ for some $\zeta^{k+1} \subseteq \eta^{k+1}$, or if $\rho^{k+1} = \sigma^{k+1}$ for this component, then some element of $(\eta^k, \xi^k]$ switches ρ^{k+1} . Let τ^{k+1} be the immediate successor of σ^{k+1} along η^{k+1} , and note that τ^{k+1} requires extension for some μ^{k+1} which we fix. By (5.14), $\mu^{k+1} \subset \delta^{k+1} \subset \sigma^{k+1}$. As $\text{out}^0(\eta^{k+1})$ is pseudotrue, $\text{out}^0(\eta^{k+1})$ is completion consistent via $\langle \rangle$ and is not a 0-completion. Hence by Lemma 5.5(ii) (Completion-Respecting), τ^{k+1} has a primary completion κ^{k+1} which has infinite outcome along η^{k+1} . By Lemma 5.4 (Compatibility), τ^{k+1} must have a k -completion $\bar{\kappa}^k \subseteq \eta^k \subset \xi^k$, so κ^{k+1} must have a principal derivative κ^k along ξ^k . Let α^k be the immediate successor of κ^k along ξ^k . Suppose first that κ^k has finite outcome along ξ^k , for the sake of obtaining a contradiction. Then by (2.4), $\kappa^k = \bar{\kappa}^k$ and κ^{k+1} has infinite outcome along $\lambda(\alpha^k)$, so $[\mu^{k+1}, \kappa^{k+1}]$ would be a primary $\lambda(\alpha^k)$ -link restraining ρ^{k+1} . By (2.6) and (2.10) and as κ^k is the principal derivative of κ^{k+1} along ξ^k , $[\mu^{k+1}, \kappa^{k+1}]$ must be a primary $\lambda(\xi^k)$ -link restraining δ^{k+1} , contrary to the hypothesis that δ^{k+1} is $\lambda(\xi^k)$ -free.

We conclude that κ^k has infinite outcome along ξ^k . As κ^{k+1} has infinite outcome along η^{k+1} , it follows from (2.4) and (2.8) that $\eta^k \subseteq \kappa^k$. By Lemma 5.3(ii) (Implication Chain) and Lemma 5.2 (Requires Extension), α^k will require extension for some σ^k such that $\text{up}(\sigma^k) = \sigma^{k+1}$. As ξ is pseudotrue, $\lambda^j(\xi)$ is not a primary completion for any $j \leq n$. Hence by Lemma 5.9 (Completion-Consistency), ξ^\pm is completion-consistent via $\langle \rangle$. Hence α^k must have a k -completion $\tilde{\kappa}^k \subseteq (\xi^k)^- \subset \xi^k$. As ξ is admissible, it follows from (5.28) that there are no amenable k -implication chains along ξ^k , so by Lemma 5.3(ii) (Implication Chain), $\tilde{\kappa}^k$ must have infinite outcome along ξ^k . Fix $\bar{\alpha}^k \subseteq \xi^k$ such that $(\bar{\alpha}^k)^- = \tilde{\kappa}^k$. By Definition 5.3 and since κ^{k+1} has infinite outcome along η^{k+1} , all elements of $\text{PL}(\delta^{k+1}, \eta^{k+1})$ coming from a component $\text{PL}(\sigma^{k+1}, \zeta^{k+1})$ for some $\zeta^{k+1} \subseteq \eta^{k+1}$ are elements of $\text{PL}(\sigma^{k+1}, \kappa^{k+1})$; and as κ^{k+1} has finite outcome along $\lambda(\alpha^k)$, it follows from Lemma

5.1(iv) (PL Analysis) that $\text{PL}(\sigma^{k+1}, \kappa^{k+1}) = \text{PL}(\sigma^{k+1}, \lambda(\alpha^k))$. By Lemma 5.12(i),(ii) (PL) and Lemma 5.11(v) (Amenable Backtracking), $\text{PL}(\sigma^{k+1}, \lambda(\tilde{\kappa}^k)) = \emptyset$, every node in $\text{PL}(\sigma^{k+1}, \lambda(\alpha^k))$ is switched by some element of $(\alpha^k, \tilde{\kappa}^k]$, and $\{\text{up}(\gamma^k): \gamma^k \in \text{PL}(\kappa^k, \tilde{\kappa}^k)\} = \text{PL}(\sigma^{k+1}, \kappa^{k+1})$. Furthermore, $\bar{\alpha}^k$ switches $\sigma^{k+1} = \text{up}(\tilde{\kappa}^k)$, so (i) holds.

By (5.2), σ^k is the initial derivative of σ^{k+1} along ξ^k . As $\sigma^{k+1} \subset (\eta^{k+1})^-$, it follows from Lemma 3.1(i) that $\sigma^k \subset \text{out}((\eta^{k+1})^-) \subseteq (\eta^k)^-$. We have shown that $\eta^k \subseteq \kappa^k$. Hence by (5.14), $\text{PL}(\kappa^k, \tilde{\kappa}^k)$ is a component of $\text{PL}((\eta^k)^-, \xi^k)$. Furthermore, as $\sigma^{k+1} \subset (\eta^{k+1})^-$, it follows from Lemma 3.1(i) (Limit Path) that $\sigma^k \subset (\eta^k)^-$. By Definition 5.6, $\text{up}(\tilde{\kappa}^k) = \sigma^{k+1}$. Hence $[\sigma^k, \tilde{\kappa}^k]$ is a primary ξ^k -link restraining $(\eta^k)^-$. (ii) now follows holds. \square