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# The Decidability of the Existential Theory of the Poset of Recursively Enumerable Degrees with Jump Relations

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We show that the existential theory of the recursively enumerable degrees in the language $\mathcal{L}$ containing predicates for order and $n$-jump comparability for all $n$, and constant symbols for least and greatest elements, is decidable. The decidability follows from our main theorem, where we show that any finite $\mathcal{L}$-structure which is consistent with the order relation, the order-preserving property of the jump operator, and the property of the jump operator that the jump of an element is strictly greater than the element, can be embedded into the r.e. degrees. © 1996 Academic Press, Inc.

CONTENTS



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## 0. Introduction

Decidability and undecidability of (fragments of) elementary theories of recursion-theoretic structures have been central topics of research in recursion theory for more than two decades. Results of this nature have been obtained by Lachlan [La1, La2], Simpson [Si], Herrmann [He], Harrington (unpublished), Lerman and Soare [LrSo], Schmerl (cf. [Lr1]), Epstein [Ep], Shore [Sh1, Sh2], Lerman [Lr1], Lerman and Shore [LrSh], Sacks [Sa1], Harrington and Slaman [HaSl] (cf. [SlWo]), Jockusch and Soare (unpublished, cf. [Lr1]), Jockusch and Slaman [JoSl] and Hinman and Slaman [HiSl]. Sharp results have been obtained for the poset of degrees $\mathscr{D} = \langle \mathbf{D}, \leqslant \rangle$. In this case, Lachlan [La1] showed that $\mathrm{Th}(\mathscr{D})$ is undecidable, and Simpson [Si] showed that this theory is recursively isomorphic to second order arithmetic; Shore [Sh1] and Lerman [Lr1] showed that $\forall\exists \cap \mathrm{Th}(\mathscr{D})$ (the $\forall\exists$-fragment of the elementary theory of $\mathscr{D}$) is decidable, and this decidability result has been extended by Jockusch and Slaman [JoSl] to $\forall\exists \cap \mathrm{Th}(\langle \mathbf{D}, \leqslant, \cup \rangle)$. Schmerl (cf. [Lr1]) showed that $\exists\forall\exists \cap \mathrm{Th}(\mathscr{D})$ is undecidable. Sharp results have also been obtained for the elementary theory of the poset $\mathscr{D}[\mathbf{0}, \mathbf{0}'] = \langle [\mathbf{0}, \mathbf{0}'], \leqslant \rangle$ of the degrees below $\mathbf{0}'$. Epstein [Ep] and Lerman [Lr1] showed that $\mathrm{Th}(\mathscr{D}[\mathbf{0}, \mathbf{0}'])$ is undecidable and Shore [Sh2] showed that this theory has degree $\mathbf{0}^{(\omega)}$; Lerman and Shore [LrSh] showed that $\forall\exists \cap \mathrm{Th}(\mathscr{D}[\mathbf{0}, \mathbf{0}'])$ is decidable, while Schmerl (cf. [Lr1]) showed that $\exists\forall\exists \cap \mathrm{Th}(\mathscr{D}[\mathbf{0}, \mathbf{0}'])$ is undecidable. Gaps in our knowledge remain for other structures.

There are some natural operations on degree structures which motivate the study of decidability in languages other than the language of posets. Most degree structures are uppersemilattices and so support a *join operator* $\cup$. The join operator is definable from $\leqslant$ by an $\forall$-predicate. Hence $\exists \cap \mathrm{Th}(\langle \mathbf{D}[\mathbf{0}, \mathbf{0}'], \leqslant, \cup \rangle)$ is decidable. $\mathscr{D}$ also supports the jump operator, $'$, an order-preserving function of one variable on degrees such that $\mathbf{a}' > \mathbf{a}$ for all $\mathbf{a}$. Cooper [Co1, Co2] has shown that the jump operator is definable over $\mathscr{D}$, but Lerman and Shore [LrSh] have shown that the definition cannot be an $\forall\exists$-definition. Thus the study of the elementary theory of $\mathscr{D}' = \langle \mathbf{D}, \leqslant, ' \rangle$ is more complex than the study of $\mathrm{Th}(\mathscr{D})$. The results of Lachlan [La1] and Simpson [Si] cover $\mathrm{Th}(\mathscr{D}')$ as well; thus $\mathrm{Th}(\mathscr{D}')$ is undecidable, and is recursively isomorphic to second-order arithmetic. On the other hand, Jockusch and Soare (cf. [Lr1]) have shown that $\mathrm{Th}(\langle \mathbf{D}, ' \rangle)$ is decidable.

In this paper, we develop methods which enable us to build configurations of recursively enumerable degrees while simultaneously controlling the configurations of the $m$th jumps of these degrees. Our results will yield a decision procedure for the existential theory of the recursively enumerable degrees in an expanded language, and can be used to give a decision procedure for a fragment of $\exists \cap \mathrm{Th}(\langle \mathbf{D}, \leqslant, ', \mathbf{0}\rangle)$. We expect the methods introduced in this paper to be useful in providing a decision procedure for $\exists \cap \mathrm{Th}(\langle \mathbf{D}, \leqslant, ', \mathbf{0}\rangle)$. (Hinman and Slaman [HiSl] have recently proved that $\exists \cap \mathrm{Th}(\mathscr{D}')$ is decidable using a forcing argument.)

For every degree $\mathbf{a}$, we let $\mathscr{R}(\mathbf{a}) = \langle \mathbf{R}(\mathbf{a}), \leqslant\rangle$ be the poset of degrees r.e. in $\mathbf{a}$, and we set $\mathscr{R} = \mathscr{R}(\mathbf{0})$. For each $m \in \mathbf{N}$ ($\mathbf{N}$ is the set of natural numbers) and $\mathbf{a}, \mathbf{b} \in \mathbf{R}$, we define

$$\mathbf{a} \leqslant_m \mathbf{b} \Leftrightarrow \mathbf{a}^{(m)} \leqslant \mathbf{b}^{(m)}.$$

A *jump poset* is a 5-tuple $\langle P, \leqslant, P', \leqslant', f\rangle$, such that $\langle P, \leqslant\rangle$ and $\langle P', \leqslant'\rangle$ are posets of cardinality $\geqslant 2$ with least and greatest elements, and $f$ is an order-preserving map from $P$ onto $P'$. An *m-jump poset* is a structure

$$\mathscr{P} = \langle P_0, \leqslant_0, P_1, \leqslant_1, f_1, ..., P_m, \leqslant_m, f_m\rangle$$

such that for each $k < m$, $\langle P_k, \leqslant_k, P_{k+1}, \leqslant_{k+1}, f_{k+1}\rangle$ is a jump poset. We define a $<\omega$-jump poset analogously.

We now state our main theorem.

THEOREM 7.8. *Fix $n \in \mathbf{N}$, and let $\mathscr{P} = \langle P_0, \leqslant_0, P_1, \leqslant_1, f_1, ..., P_m, \leqslant_m, f_m\rangle$ be a finite m-jump poset such that $P_0$ has least element* 0 *and greatest element* 1. *Then there is a finite set $\mathbf{G}_0$ of r.e. degrees, and there are finite sets $\mathbf{G_k} = \{\mathbf{d}\colon \exists \mathbf{a} \in \mathbf{G}_0(\mathbf{a}^{(k)} = \mathbf{d})\}$ for each $k \in [1, m]$ such that the diagram of Fig.* 1 *commutes. Furthermore, the embedding maps $0 \in P_0$ to $\mathbf{0}$ and $1 \in P_0$ to $\mathbf{0}'$. (In fact, the proof of Theorem* 7.8 *can easily be extended to countable $<\omega$-jump posets.)*

$$\begin{tikzcd}
\langle P_0, \leq_0\rangle \arrow[r, "f_1"] \arrow[d, "\cong"'] & \langle P_1, \leq_1\rangle \arrow[r, "f_2"] \arrow[d, "\cong"'] & \cdots \arrow[r, "f_m"] & \langle P_m, \leq_m\rangle \arrow[d, "\cong"'] \\
\langle G_0, \leq\rangle \arrow[r] \arrow[d, "\subseteq"'] & \langle G_1, \leq\rangle \arrow[r] \arrow[d, "\subseteq"'] & \cdots \arrow[r] & \langle G_m, \leq\rangle \arrow[d, "\subseteq"'] \\
\mathscr{R} \arrow[r] & \mathscr{R}(\mathbf{0}') \arrow[r] & \cdots \arrow[r] & \mathscr{R}(\mathbf{0}^{(m)})
\end{tikzcd}$$

FIGURE 1.

We specify a finite set of axioms for $m$-jump posets. These axioms assert that $\langle P_i, \leqslant_i \rangle$ is a poset for each $i \leqslant m$, 0 is the least element of $P_0$, 1 is the greatest element of $P_0$, and $f_i$ is a surjective order-preserving map from $\langle P_{i-1}, \leqslant_{i-1} \rangle$ onto $\langle P_i, \leqslant_i \rangle$. Given any existential sentence in the language $\mathscr{L} = \langle 0, 1, \leqslant_0, \leqslant_1, ..., \leqslant_m, ... \rangle$, the sentence asserts that one of a finite number of diagrams is consistent with the axioms of $m$-jump posets, i.e., can be embedded into an $m$-jump poset. We can recursively determine whether or not one of these diagrams is consistent. If not, then the sentence is false; if so, then by Theorem 7.8, the sentence is true. Furthermore, this process is uniform in $m$, and mentions only finitely many relations $\leqslant_k$. Hence:

COROLLARY 7.9. *The existential theory of* $\mathscr{R}^{(<\omega)} = \langle \mathbf{R}, \mathbf{0}, \mathbf{0}', \leqslant, \leqslant_1, ..., \leqslant_m, ... \rangle$ *is decidable.*

This corollary extends the result in [LmLr2] where it is shown that the existential theory of $\langle \mathbf{R}, \mathbf{0}, \mathbf{0}', \leqslant, \leqslant_1 \rangle$ is decidable.

Fix $m$. The simplest sentences of this existential theory in the language $\mathscr{L}$ require us to construct a degree $\mathbf{a}$ such that $\mathbf{0}^{(k)} < \mathbf{a}^{(k)} < \mathbf{0}^{(k+1)}$ for all $k \leqslant m$. The Sacks Jump Inversion Theorem [Sa2] allows us to construct such degrees. One begins with a degree $\mathbf{d}_m$ such that $\mathbf{0}^{(m)} < \mathbf{d}_m < \mathbf{0}^{(m+1)}$ and $\mathbf{d}_m$ is r.e. in $\mathbf{0}^{(m)}$. The jump inversion theorem is now applied to obtain a degree $\mathbf{d}_{m-1}$ which is r.e. in $\mathbf{0}^{(m-1)}$ such that $(\mathbf{d}_{m-1})' = \mathbf{d}_m$ and $\mathbf{0}^{(m-1)} < \mathbf{d}_{m-1} < \mathbf{0}^{(m)}$. This procedure can be iterated, producing $\mathbf{d}_0 = \mathbf{a}$. Attempts were made to decide the full $\exists$-Theory in this way, but the Shore Non-Inversion Theorem [Sh3] showed that such attempts were doomed to failure. Our approach is to construct the r.e. degrees directly. To do this, we introduce a $\mathbf{0}^{(n)}$-priority argument for each $n \in \mathbf{N}$. Priority arguments of this sort were developed by Ash [A1, A2] and Knight [Kn] for recursive model theory, but their approach does not seem to be applicable here. Groszek and Slaman [GS] have been developing a different general framework for $\mathbf{0}^{(n)}$-priority arguments, and our proof has been influenced by the techniques of Ash, Groszek and Slaman. In particular, although our trees are different from those used by Ash, the properties used in the tree decomposition are, in many cases, based on ideas introduced in Ash [A1]. A framework is also being developed by Kontostathis [Ko1, Ko2, Ko3]. Other theorems proved using our framework can be found in [LmLr1, LmLr2, and LLW], and an overview of the framework is presented in [LmLr3]. Our treatment of individual requirements is modeled after the solution to the "deep degree" problem by Lempp and Slaman [LmSl].

We use the following notation. If $A \subseteq \mathbf{N}$, then we let $\overline{A}$ denote the complement of $A$. For $A, B \subseteq \mathbf{N}$, we let $A \backslash B$ denote the difference of $A$ and $B$. Given a set $P$, we let $|P|$ denote the cardinality of $P$. A $k$-dimensional space is a set $S=\{\bar{a}\}\times\mathbf{N}^k\times\{\bar{b}\}$ for some choice of finite sequences $\bar{a}$ and $\bar{b}$ of elements of $\mathbf{N}$; in this case, we write $\dim(S)=k$. If $A=\{\bar{a}\}\times\mathbf{N}^k\times\{\bar{b}\}$ and $i\in\mathbf{N}$, then we let $A^{[i]}$ denote $\{\langle\bar{a}, i, \bar{x}, \bar{b}\rangle : \bar{x}\in\mathbf{N}^{k-1}\}\cap A$ and call $A^{[i]}$ a *section* of $A$.

We depart a little from the standard classification of sentences, although our classification is equivalent to the standard classification. Thus a $\Sigma_0$- or $\Pi_0$-formula is one in which all quantifiers are bounded. A $\Sigma_m$-formula is one of the form $Q_1\bar{x}_1 \cdots Q_k\bar{x}_k\exists\bar{y}R(\bar{x}_1, ..., \bar{x}_k, \bar{y})$, where each $Q_i\bar{x}_i$ is a finite block of bounded universal quantifiers or a finite block of bounded existential quantifiers and $R$ is a $\Pi_{m-1}$-formula. Similarly, a $\Pi_m$-formula is one of the form $Q_1\bar{x}_1 \cdots Q_k\bar{x}_k\forall\bar{y}R(\bar{x}_1, ..., \bar{x}_k, \bar{y})$, where each $Q_i\bar{x}_i$ is a finite block of bounded universal quantifiers or a finite block of bounded existential quantifiers and $R$ is a $\Sigma_{m-1}$-formula.

Let $\gamma$ be a $\Sigma_m$- or $\Pi_m$-sentence. Then $\gamma$ can either be written as $\bar{Q}\bar{x}\,\exists\bar{y}\delta(\bar{x}, \bar{y})$ where $\delta$ is $\Pi_{m-1}$, or as $\bar{Q}\bar{x}\,\forall\bar{y}\delta(\bar{x}, \bar{y})$ where $\delta$ is $\Sigma_{m-1}$. The formula $\gamma^{[z]}$ is obtained from $\gamma$ by replacing the first block of unbounded quantifiers $\exists\bar{y}$ or $\forall\bar{y}$ with a similar block where all variables are restricted to numbers $\leqslant z$.

A *string* is a finite sequence of letters from an alphabet. If $S$ is an alphabet, we let $S^{<\omega}$ be the set of all strings from $S$. We write $\sigma\subset\tau$ if $\tau$ properly extends $\sigma$, and $\sigma\mid\tau$ if $\sigma$ and $\tau$ are incomparable. We say that $\sigma$ *lies along* $\tau$ if $\sigma\subseteq\tau$. For $\sigma, \tau\in S^{<\omega}$, we let $\mathrm{lh}(\sigma)$ denote the cardinality of the domain of $\sigma$. If $\sigma\neq\langle\ \rangle$ (the empty string), then $\sigma^-$ is the unique $\tau\subset\sigma$ such that $\mathrm{lh}(\tau)=\mathrm{lh}(\sigma)-1$. We define the string $\sigma\hat{\ }\tau$ by

$$\sigma\hat{\ }\tau(x)=\begin{cases}\sigma(x) & \text{if} \quad x<\mathrm{lh}(\sigma)\\ \tau(x-\mathrm{lh}(\sigma)) & \text{if} \quad \mathrm{lh}(\sigma)\leqslant x<\mathrm{lh}(\sigma)+\mathrm{lh}(\tau).\end{cases}$$

If $x\leqslant\mathrm{lh}(\sigma)$, then $\sigma\upharpoonright x$, *the restriction of* $\sigma$ *to* $x$, is the string $\tau$ of length $x$ such that $\tau(y)=\sigma(y)$ for all $y<x$. Restriction is defined similarly for infinite sequences from an alphabet. We also use interval notation for strings. Thus $[\sigma, \tau]=\{\rho : \sigma\subseteq\rho\subseteq\tau\}$. $\sigma\wedge\tau$ denotes the longest $\rho$ such that $\rho\subseteq\sigma, \tau$, and if $\sigma$ and $\tau$ are comparable, then $\sigma\vee\tau$ is the longer of $\sigma$ and $\tau$.

A *tree* is a set of strings which is closed under restriction. The *paths* through a tree $T$ are the infinite sequences $\Lambda$ such that $\Lambda\upharpoonright x\in T$ for all $x\in\mathbf{N}$. We let $[T]$ denote the set of paths through $T$.

The *high/low hierarchy* for $\mathscr{R}$ is defined as follows. For $n\geqslant 0$, we say that $\mathbf{a}$ is *low*$_n$ ($\mathbf{a}\in\mathbf{L}_n$) if $\mathbf{a}^{(n)}=\mathbf{0}^{(n)}$, and $\mathbf{a}$ is *high*$_n$ ($\mathbf{a}\in\mathbf{H}_n$) if $\mathbf{a}^{(n)}=\mathbf{0}^{(n+1)}$. If $\mathbf{0}^{(n)}<\mathbf{a}^{(n)}<\mathbf{0}^{(n+1)}$ for all $n$, then we say that $\mathbf{a}$ is *intermediate*.

$\langle\Phi_e^k : e\in\mathbf{N}\rangle$ will be the standard enumeration of all partial recursive functionals of $k$ variables. (We will frequently suppress the superscript, writing $\mathbf{\Phi}_e$ for $\mathbf{\Phi}_e^k$.) Thus $\Phi_e^k(A; x_1, ..., x_k)=y$ if the $e$th partial recursive functional of $k$ variables, computing from oracle $A$ and input $x_1, ..., x_k$, outputs the value $y$. For each $e, k \in \mathbf{N}$, we will have a recursive approximation $\langle \Phi_{e,s}^k : s \in \mathbf{N} \rangle$ to $\Phi_e^k$. We say that $\Phi_{e,s}^k(A; x_1, ..., x_k)\downarrow$ if we obtain an output from this computation in fewer than $s$ steps; otherwise, $\Phi_{e,s}^k(A; x_1, ..., x_k)\uparrow$. If $\Phi_{e,s}^k(A; x_1, ..., x_k)\downarrow$, then we let the *use* of this computation be the greatest element $u$ for which a question "$u \in A$?" is asked of the $A$ oracle during the computation. We will work under the convention that:

$$\text{If } u \text{ is the use of a computation at stage } s, \text{ then } u < s. \tag{0.1}$$

We will be constructing partial recursive functionals within a recursive construction by *declaring axioms* $\Delta(\sigma; \bar{x})=y$ to reflect the fact that the partial recursive functional $\Delta$ with input $\bar{x}$ produces output $y$ when computing from the oracle $\sigma$, (so from any oracle $A \supset \sigma$). If $\bar{x}=\langle x_1, ..., x_m\rangle$ and $\bar{z}=\langle z_{m+1}, ..., z_k\rangle$, then $\lim_{\bar{x}} \Phi_e^k(A; \bar{x}, \bar{z})$ denotes $\lim_{x_1} \cdots \lim_{x_m} \Phi_e^k(A; x_1, ..., x_m, z_{m+1}, ..., z_k)$. Other notation follows [So].

## 1. The Basic Modules

We will introduce the basic modules for requirements of dimensions 1 and 2 in this section. While the proof of the theorem will need requirements of higher dimensions, the descriptions of the basic modules for these higher dimension requirements is similar to the descriptions for requirements of dimensions 1 and 2, requiring only more iterations of the limit process.

Fix a finite $m$-jump poset $\mathscr{P}=\langle P_0, \leqslant_0, P_1, \leqslant_1, f_1, ..., P_m, \leqslant_m, f_m\rangle$, and assume that $P_0$ has least element 0 and greatest element 1. Let $g_0$ be the identity function, and for each $k \in [1, m]$, let $g_k = f_k \circ f_{k-1} \circ \cdots \circ f_1$. Assume, also, that there are $d \neq \tilde{d} \in P_0$ with the following properties:

(1.1) For all $k \leqslant m$, $g_k(0) <_k g_k(d) <_k g_k(1)$ and for all $c \in P_0 \backslash \{d\}$ such that $g_k(0) <_k g_k(c) <_k g_k(1)$, $g_k(d)$ and $g_k(c)$ are incomparable.

(1.2) For all $k \leqslant m$, $g_k(0) <_k g_k(\tilde{d}) <_k g_k(1)$ and for all $c \in P_0 \backslash \{\tilde{d}\}$ such that $g_k(0) <_k g_k(c) <_k g_k(1)$, $g_k(\tilde{d})$ and $g_k(c)$ are incomparable.

These conditions will reduce the number of different types of requirements needed for our construction. In particular, we will not have to treat, as special cases, requirements to make any of the sets which we are constructing non-low$_k$ or non-high$_k$ for any $k \leqslant m$.

We will construct an r.e. set $A_b$ for each $b \in P_0$. We specify that $A_0=\varnothing$ and $A_1$ is the complete r.e. set $K$. We will also be constructing partial recursive functionals $\Delta$ with subscripts and superscripts designating the requirement for which $\Delta$ is acting.

DEFINITION 1.1. Uniformly in $A \subseteq \mathbf{N}$ and $e, r \in \mathbf{N}$, we fix a sentence $\beta_r(A; e)$ which is $\Sigma_{r+1}$ and whose validity agrees with that of $e \in A^{(r)}$ if $r$ is odd, and which is $\Pi_{r+1}$ and whose validity agrees with that of $e \notin A^{(r)}$ if $r$ is even.

For all $b, c \in P_0$ and $k \leqslant m$, we will have to show that

$$g_k(b) \leqslant g_k(c) \Leftrightarrow A_b^{(k)} \leqslant_T A_c^{(k)}.$$

Each such equivalence will be satisfied if we satisfy the following conditions for sufficiently many $b$ and $c$: There is a partial recursive functional $\Delta$ (depending on the condition) such that for all $e \in \mathbf{N}$:

$$\begin{aligned}
(1.3)\quad & R_{e,b,c}^{0,k+1}\colon g_k(c) \not\leqslant g_k(b) \Rightarrow \Delta(A_c) \text{ is total } \& \\
& \quad \forall x(\lim_{u_1} \cdots \lim_{u_k} \Delta(A_c; u_1, ..., u_k, x)\downarrow) \ \& \\
& \quad \exists x(\lim_{u_1} \cdots \lim_{u_k} \Delta(A_c; u_1, ..., u_k, x)\downarrow \\
& \quad \neq \lim_{v_1} \cdots \lim_{v_k} \Phi_e^{k+1}(A_b; v_1, ..., v_k, x)). \\
(1.4)\quad & R_{e,b,c}^{1,k}\colon g_k(b) \leqslant g_k(c) \ \& \ b \neq 1 \Rightarrow \\
& \quad \forall e \exists q \leqslant 1((\lim_{u_1} \cdots \lim_{u_k} \Delta(A_c; u_1, ..., u_k, e)\downarrow = q) \ \& \\
& \quad (q = 1 \text{ iff } \beta_k(A_b; e) \text{ is true})). \\
(1.5)\quad & R_{e,b,c}^{2,k+1}\colon g_k(1) \leqslant g_k(c) \Rightarrow \\
& \quad \forall e\, \exists q \leqslant 1((\lim_{u_1} \cdots \lim_{u_k} \Delta(A_c; u_1, ..., u_k, e)\downarrow = q) \ \& \\
& \quad (q = 1 \text{ iff } \beta_{k+1}(\varnothing; e) \text{ is not true})).
\end{aligned}$$

(Conditions (1.1) and (1.2) and properties of the jump operator will allow us to be selective about those $b$ and $c$ for which we satisfy requirements. The second superscript in a requirement $R_{e,b,c}^{i,k}$ is the *dimension* of the requirement, and corresponds to the particular tree of strategies on which we begin to split the requirement up.) We refer to requirements listed in (1.3) as *incomparability* requirements, to those listed in (1.4) as *comparability* requirements, and to those listed in (1.5) as *highness* requirements.

*The Basic Module*: *Dimension* 0 *Comparability Requirements.* $R_{e,b,c}^{1,0}$ is satisfied by coding $A_b$ into $A_c$. The construction will have the following property:

(1.6) If $e \in A_b^{s+1} \backslash A_b^s$, and $b \leqslant c$, then $e \in A_c^{s+1} \backslash A_c^s$.

Thus to decide whether $e \in A_b$, we ask if $e \in A_c$. If the answer is no, then $e \notin A_b$. If the answer is yes, then we find the least $s$ such that $e \in A_c^s$, and note that $e \in A_b$ iff $e \in A_b^s$. We have thus proved:

LEMMA 1.1. *Let $b, c \in P_0$ be given such that $b \leqslant c$. Assume that the construction satisfies* (1.6). *Then $A_b \leqslant_T A_c$.* ∎

THE BASIC MODULE: *Dimension* 1 *Incomparability Requirements.* We satisfy $\{R_{e,b,c}^{0,1}\colon e \in \mathbf{N}\}$ through a modification of the Friedberg-Muchnik strategy. We construct a partial recursive functional $\Delta$ such that $\Delta(A_c)$ is total. We will appoint a witness $x$ and try to guarantee that $\Phi_e^1(A_b; x) \neq \Delta(A_c; x)\downarrow$. We begin by *activating* this requirement. To do so, we declare an axiom $\Delta(A_c^s; x) = 0$ with use $x$. If, at some later stage $t$, we find that $\Phi_{e,t}^1(A_b^t; x)\downarrow = 0$, with use $q$, then we will *restrain* $A_b \upharpoonright (q+1)$ from changing after stage $t$, and place $x$ into $A_c$. This will allow us to redefine $\Delta(A_c; x) = 1$ with use $x$. There are two possible types of outcomes. If, during the construction, we never see a computation $\Phi_{e,t}^1(A_b^t; x) = 0$, then $\Delta(A_c; x) = 0$ and either $\Phi_e^1(A_b; x)\uparrow$ or $\Phi_e^1(A_b; x)\downarrow \neq 0$. If we eventually place $x$ into $A_c$, then $\Delta(A_c; x) = 1 \neq 0 = \Phi_e^1(A_b; x)$.

THE BASIC MODULE: *Dimension* 1 *Comparability Requirements.* If $f_1(b) \leqslant f_1(c)$ and $f_1(1) \nleqslant f_1(c)$, then we will want

$$\lim_u \Delta(A_c; u, e) = \begin{cases} 1 & \text{if } \beta_1(A_b; e) \text{ is true (i.e., } e \in A_b') \\ 0 & \text{if } \beta_1(A_b; e) \text{ is not true (i.e., } e \notin A_b'). \end{cases}$$

We wait for a stage $s$ at which $\beta_1(A_b^s; e)$ is true with use $< s$, declaring axioms $\Delta(A_c^s; u, e) = 0$ for progressively larger $u$ until such an $s$ is found. If we ever find such a stage $s$, we restrain $A_b^s \upharpoonright s$ (so $A_b \upharpoonright s = A_b^s \upharpoonright s$), and declare axioms $\Delta(A_c^t; v, e) = 1$ for all $t \geqslant s$ and all sufficiently large $v$, having use $s$.

THE BASIC MODULE: *Dimension* 2 *Incomparability Requirements.* $R_{e,b,c}^{0,2}$ will be treated as a connected infinite set of dimension 1 incomparability requirements, producing the desired result in the limit. Fix $\Delta$. We will appoint a witness $x$, and try to guarantee that if $\{v\colon \Phi_e^2(A_b; v, x) = 0\}$ is infinite, then $\{u\colon \Delta(A_c; u, x) = 1\}$ is cofinite, and if $\{v\colon \Phi_e^2(A_b; v, x) = 0\}$ is finite, then $\{u\colon A_c; u, x) = 0\}$ is cofinite. We thus begin to declare axioms $\Delta(A_c; u, x) = 0$ for progressively larger $u$, with large use $p$. (At each stage, only finitely many axioms of this kind are declared, and several may share the same use.) If, at some later stage $t$, we find some $v_0$ such that $\Phi_{e,t}^2(A_b^t; v_0, x) = 0$, then we restrain $A_b^t \upharpoonright t$ (so $A_b \upharpoonright t = A_b^t \upharpoonright t$), and place $p$ into $A_c$ at stage $t$, allowing us to declare new axioms $\Delta(A_c^t; u, x) = 1$ with use $p$ for each $u$ for which we have previously declared an axiom with output 0. We now repeat this procedure for larger values $u$ for which no axiom has yet been declared, trying to find $v_1 > v_0$ for $\Phi_e^2$. Thus either $\{v\colon \Phi_e^2(A_b; v, x)\downarrow = 0\}$ is finite and $\{u\colon \Delta(A_c; u, x)\downarrow = 0\}$ is cofinite, or $\{v\colon \Phi_e^2(A_b; v, x)\downarrow = 0\}$ is infinite and $\{u\colon \Delta(A_c; u, x)\downarrow = 1\} = \mathbf{N}$. In either case, $R_{e,b,c}^{0,2}$ is satisfied.

THE BASIC MODULE. *Dimension* 2 *Highness Requirements.* If $f_1(1) \leqslant_1 f_1(c)$ and $K$ is the complete recursively enumerable set, then we will want to satisfy

$$\lim_u \Delta(A_c; u, e) = \begin{cases} 0 & \text{if } \; e \in K' (\text{i.e., } \beta_2(\emptyset, e) \text{ is not true or,} \\ & \qquad \text{equivalently, } \beta_1(K, e) \text{ is true}) \\ 1 & \text{if } \; e \notin K' (\text{i.e., } \beta_2(\emptyset, e) \text{ is true or,} \\ & \qquad \text{equivalently, } \beta_1(K, e) \text{ is not true}). \end{cases}$$

The strategy for satisfying this requirement is similar to that for the dimension 2 incomparability requirements. While $\Phi^1_{e,s}(K^s; e)\uparrow$, we declare axioms $\Delta(A^s_c; u, e) = 1$ for progressively larger $u$, with use 0. If we discover that $\Phi^1_{e,s}(K^s; e)\downarrow$ with use $v$, then we declare axioms $\Delta(A^s_c; u, e) = 0$ with large use $p$ for progressively larger $u$ for which axioms have not yet been defined. If $K^t \upharpoonright v \neq K^s \upharpoonright v$ at some later stage $t$, then we place $p$ into $A_c$ at stage $t$, and reset the axiom $\Delta(A^t_c; u, e) = 1$ with use $p$ for values of $u$ for which the axiom was previously set to 0, and begin again to declare axioms having output 1 and use $r \geqslant t$ for yet larger values of $u$. Thus either $\Phi^1_e(K; e)\downarrow$ and $\{u: \Delta(A_c; u, x)\downarrow = 0\}$ is cofinite, or $\Phi^1_e(K; e)\uparrow$ and $\{u: \Delta(A_c; u, x)\downarrow = 1\} = \mathbf{N}$. In either case, $R^{2,2}_{e,1,c}$ is satisfied.

Comparability requirements of dimension 2, and requirements of dimension 3 or greater are handled in a way similar to that in which their counterparts of lower dimension are handled, except that more iterations of the limit operation are required. As no new strategies are involved, we will not discuss basic modules for these requirements.

Conflicts between requirements are resolved by placing requirements on iterated trees of strategies, and using the trees to determine when requirements should act. We fix the maximum dimension $n$ of the requirements to be satisfied, and assign each requirement to all nodes of a given level of the tree $T^n$. Associated with a requirement is a sentence of the form $(\phi^n \to \psi^n)$ & $(\neg\phi^n \to \chi^n)$, where $\phi^n$ is a sentence which determines when to initiate action during the construction, and $\psi^n$ and $\chi^n$ are properties which must result from this action. For $k < n$, nodes of a tree $T^k$ will be derivatives of nodes of a tree $T^{k+1}$. Each derivative is to generate action based on the truth of a sentence $\varphi^k$ obtained from the sentence $\varphi^{k+1}$ assigned to the node from which it is derived, by appropriately bounding the outer block of quantifiers. And the action must result in satisfying some property derived from $\psi^n$ and $\chi^n$. Rather than bounding quantifiers for the sentence obtained from $\psi^n$ and $\chi^n$, we assign spaces to the nodes on which we ensure that a given functional has an iterated limit (sometimes requiring a specified value for this limit). When we reach $T^0$, we will be able to recursively specify when action should be taken and what that action should be. We then piece together the various sentences and actions on $T^0$ taken to show that the sentence assigned to $T^n$ is satisfied.

The processes of assigning derivatives and of determining which derivatives should act are delicate, and differ with the type of requirement.

This assignment must be done in such a way that the action taken by derivatives can be pieced together to show that the original sentence on $T^n$ is true. The priority argument is hidden in this decomposition; thus if the decomposition is done correctly, there are no conflicts between requirements on $T^0$ as we have determined when nodes can act in a way which avoids conflicts (which, however, are seen on $T^1$). The key is to determine the derivative responsible for the definition of a given axiom. For incomparability requirements, the nodes specifying axioms for a given functional and argument in the limit are all derived from a single node of $T^n$, and control must be expanded to ensure that limits exist when the node is not on the true path of the construction. In the case of comparability and highness requirements, nodes on each path through $T^n$ may specify axioms for a given functional and argument. In these cases, we must define control of axioms carefully, dividing control among many nodes. We also take advantage of the fact that if $k$ is the dimension of the requirement, axioms will frequently be corrected (since the oracle set will change below the use of the axiom) whenever the true path approximation changes its mind about the ultimate node of $T^k$ which is responsible for defining the axioms.

The notion of control will determine the axioms for which a node is responsible. Determining when a node should act will involve additional concepts, such as freeness, admissibility, implication chains, and links. We will define the assignment of requirements to trees in Section 2, and prove some lemmas about paths in Section 3. Links will be analyzed in Section 4. Implication chains and backtracking will be discussed in Section 5. Control will be discussed in Section 6. The construction and proof based on the machinery introduced in previous sections will be discussed in Section 7.

## 2. The Requirements and Systems of Trees

The framework for our priority argument uses systems of trees, and much of it can be presented independently of the set of requirements to be satisfied. Systems of trees are introduced in this section, and the mechanism for assigning requirements to the trees is described. (The reader is referred to [LmLr1], where systems of trees are used to prove some standard theorems of recursion theory. The framework there is a little different, as some of the subtleties needed here do not occur at the lower levels, but the many similarities in the approaches might be helpful.) Fix $n \in \mathbf{N}$ henceforth.

Definition 2.1 (Definition of Trees). We set $T^{-1} = \{0, \infty\}$ and $T^0 = \{0, \infty\}^{<\omega}$. If $0 < k \leqslant n$ and $T^{k-1}$ has been defined, let

$$T^k = \{\sigma \in (T^{k-1})^{<\omega} : \forall i < \mathrm{lh}(\sigma)\ \forall j < \mathrm{lh}(\sigma)(i < j \rightarrow \sigma(i) \subset \sigma(j))\}.$$

$T^k = \langle T^k, \subseteq \rangle$ is the $k$th *tree of strategies*, ordered by inclusion. We refer to the elements of $T^k$ as nodes of $T^k$, and view each node of $T^k$ as following its immediate predecessor by a designated node of $T^{k-1}$. If $\sigma \in T^k$, $\xi \in T^{k-1}$, and $\sigma = \sigma^- {}^\wedge \langle \xi \rangle$, then we say that $\sigma^-$ has *outcome* $\xi$ *along* $\sigma$, and define $\text{out}(\sigma) = \xi$. If $j \leqslant k$, then we define $\text{out}^j(\sigma)$ by reverse induction; $\text{out}^k(\sigma) = \sigma$, and $\text{out}^{j-1}(\sigma) = \text{out}(\text{out}^j(\sigma))$. Outcomes are of two types, activated or validated. If $k = 0$, $\tau \supseteq \sigma$, and $\text{lh}(\sigma) > 0$, then we say that $\sigma^-$ *is activated* (*validated*, resp.) *along* $\tau$ if $\text{out}(\sigma) = 0$ ($\text{out}(\sigma) = \infty$, resp.). If $k > 0$, then $\sigma^-$ *is activated* (*validated*, resp.) *along* $\tau$ if $\text{out}(\sigma)^-$ is activated (validated, resp.) along $\text{out}(\sigma)$. (Activation and validation represent different ways of satisfying a requirement depending on whether the sentence generating action is true or false. The steps taken when a requirement associated with the node $\sigma^-$ is first activated may be later extended when that requirement is validated.) If $\sigma \subseteq \tau \in T^k$ and $\text{lh}(\sigma) > 0$, then we say that $\sigma^-$ *has finite* (*infinite*, *resp.*) *outcome along* $\tau$ if either $k = 0$ and $\text{out}(\sigma) = 0$ ($\text{out}(\sigma) = \infty$, resp.), or $k > 0$ and $\text{out}(\sigma)^-$ has infinite (finite, resp.) outcome along $\text{out}(\sigma)$. (Note that $\sigma^-$ is activated (validated, resp.) along $\sigma$ if either $k$ is even and $\sigma^-$ has finite (infinite, resp.) outcome along $\sigma$, or $k$ is odd and $\sigma^-$ has infinite (finite, resp.) outcome along $\sigma$.)

In order to provide the reader with some intuition about these trees, we relate them to the *tree of strategies* approach introduced by Harrington, and indicate the relationship between the way certain concrete requirements are treated by these approaches. First consider a typical Friedberg–Muchnik requirement $\Phi(A) \neq B$. The standard tree of strategies approach assigns such a requirement to a node $\sigma$ of $T^1 = \{0, 1\}^{<\omega}$, and proceeds by stages. When $\sigma$ first appears on the true path, a *follower* $x$ is assigned to the requirement. As long as $\sigma$ is on the path through $T^1$ computed at stage $s$ and $\Phi_s(A^s) \neq 0$, the path computation at $s$ follows $\sigma^\wedge\langle 0 \rangle$, and we set $B^s(x) = 0$. (We now say that $\sigma$ is *activated.*) If, at some later stage $t$, we find that $\Phi_t(A^t; x) = 0$, the path computation at $t$ follows $\sigma^\wedge\langle 1 \rangle$, we set $B^t(x) = 1$, and never again consider this requirement. (We now say that $\sigma$ is *validated.*)

Our approach replaces stages by the tree $T^0 = \{0, \infty\}^{<\omega}$. (We use $\infty$ in place of 1 because we want to talk about *finite* and *infinite* outcomes.) In place of stage $t$, we form a $\delta$-block on $T^0$ which consists of *derivatives* of nodes of $T^1$ which lie on the true path computation through $T^1$ generated by $\delta \in T^0$, and such that the requirement for these nodes of $T^1$ has not yet been validated. Each derivative along the path gets a chance to try to satisfy its requirement when it is reached, and the block ends either when we newly validate a node, or begin to deal with one new requirement. The outcome $\xi = \nu^\wedge\langle \beta \rangle$ of $\sigma$ along $\rho = \sigma^\wedge\langle \xi \rangle \in T^1$ is used to code whether or not $\Phi(A; x) = 0$. $\xi$ will tell us whether or not the requirement assigned to $\sigma$ has been activated or validated, and in addition, that the decision to activate or validate was made based on the outcome of $\nu$ along $\xi$. Thus if $\sigma$ is activated along $\rho$, then $\sigma$ will have infinitely many derivatives along the true path $\Lambda^0$ through $T^0$, all of which will be activated. The outcome $\xi = \nu^\frown\langle\beta\rangle$ of $\sigma$ will indicate that $\nu$ is the derivative of $\sigma$ at which we made the decision to determine the outcome of $\sigma$ along $\rho$, namely, the first derivative of $\sigma$ along $\Lambda^0$ (we will call $\nu$ both the *initial* and *principal* derivative of $\sigma$ along $\Lambda^0$), and the outcome $\beta = 0$ of $\nu$ along $\Lambda^0$ indicates that $\nu$ is activated along $\Lambda^0$. If $\sigma$ is validated along $\rho$, then the outcome $\xi$ of $\sigma$ will determine the node $\nu$ of $T^0$ at which we made the decision to determine the outcome of $\sigma$ along $\rho$ (we will call $\nu$ the *principal derivative* of $\sigma$ along $\rho$), and the outcome $\beta = \infty$ of $\nu$ along $\Lambda^0$ indicates that $\nu$ is validated along $\xi$. If $\nu$ is not the first (i.e., initial) derivative $\mu$ of $\sigma$ along $\rho$, then we create a link from $\mu$ to $\nu$. These links partially correspond, in standard priority arguments, to initializing all extensions of $\rho$. At higher levels, they also serve the purpose of not allowing nodes restrained by the link to act and cause a change in the approximation to the true path. This allows us to show that when the outcome of a node is switched by the approximation, it must be switched because of action taken for the requirement for which the node is responsible. (We note that this approach differs from that in [LmLr1], where the outcome of $\nu$ was not coded along the outcome of $\sigma$, and $\infty$ was used in place of a node of $T^0$ to represent activation on $T^1$, i.e., denoting that $\sigma$ has infinitely many derivatives along $\Lambda^0$. This approach works for $\mathbf{0}'''$-constructions, i.e., constructions which do not require a tree beyond $T^3$. Once $T^4$ is reached, initial derivatives of $\sigma$, i.e. derivatives of a node $\sigma$ on $T^4$ which do not properly contain another derivative of $\sigma$, are no longer unique, and our approximations need to code these initial derivatives, rather than use a catch-all symbol $\infty$ to denote an infinite outcome.)

Next, consider a typical *thickness* requirement on $T^2$. We are given an infinite recursive set $R$, and activation corresponds to putting only finitely many elements of $R$ into a set $A$, while validation corresponds to putting all elements of $R$ into $A$. Suppose that this thickness requirement is assigned to a node $\sigma$ of the true path $\Lambda^2$ through $T^2$. Then $\sigma$ will have derivatives along the true path $\Lambda^1$ through $T^1$, each of which will have the role of placing finitely many elements of $R$ into $A$ if a certain $\Sigma_1$-sentence is true. First suppose that one of these sentences is false, say the one corresponding to the derivative $\xi$ of $\sigma$. Then $\xi$ will be the last (and *principal*) derivative of $\sigma$ along $\Lambda^1$, and will have infinite outcome along $\Lambda^1$, designating that no derivative of $\xi$ is validated along the true path $\Lambda^0$ on $T^0$, i.e., that no derivative of $\xi$ finds a witness for its existential sentence at the stage specified by the framework. No elements $\geqslant$ the least element of $R$ for which $\xi$ has responsibility are placed into $A$ in this case.

Now suppose that all of the sentences are true. Then $\sigma$ will have infinite outcome along $\Lambda^2$, indicating that $\sigma$ has infinitely many derivatives along $\Lambda^1$, each of which is validated. Each such node will place the elements of $R$ for which it is responsible into $A$. As each element of $R$ will be assigned to such a node, all elements of $R$ will be placed into $A$. (The infinite outcome of $\sigma$ along $\Lambda^2$ is the *initial* derivative of $\sigma$ along $\Lambda^1$ (which also is the *principal* derivative of $\sigma$ along $\Lambda^1$), followed by its first validated derivative $v$ along $\Lambda^0$ and the outcome $\infty$ for $v$ indicating that $v$ is validated.)

We will need a one-to-one *weight* function on elements of $T = \bigcup \{T^k : 0 \leqslant k \leqslant n\}$ which will $\omega$-order $T$. (We take the disjoint union here, differentiating between the empty nodes of the various trees. A similar function was called par in [LmLr1].) The weight function will have various properties, which will be used to show that constructions in which action is determined by weight are able to protect certain computations.

Definition 2.2. It is routine to check that a one-to-one recursive *weight function* $\mathrm{wt}: T \to \mathbf{N}$ can be defined to satisfy the following properties for all $\sigma, \tau \in T^k$:

$$\text{If } \sigma \subset \tau \text{ then } \mathrm{wt}(\sigma) < \mathrm{wt}(\tau). \tag{2.1}$$

$$\text{If } k > 0, \text{ then } \mathrm{wt}(\mathrm{out}(\sigma)) < \mathrm{wt}(\sigma). \tag{2.2}$$

$$\text{If } k > 0 \text{ and } \mathrm{out}(\sigma) \subset \mathrm{out}(\tau), \text{ then } \mathrm{wt}(\sigma) < \mathrm{wt}(\tau). \tag{2.3}$$

Definition 2.3. The action taken at each stage of the construction will be associated with a node of $T^0$. This node will be derived from a node of $T^k$ where $k$ is the *dimension* of the requirement, i.e., we begin to split up the requirement into subrequirements on $T^{k-1}$. Nodes of $T^k$ will be of one of two types. Each node on $T^k$ working on a given incomparability requirement can pick a different witness for its functional, and we such requirements *locally distributed.* For any comparability or highness requirement and argument $x$, each path through $T^k$ must contain a node working to define a value for a functional on argument $x$; such requirements are called *densely distributed.*

We will take action to ensure the satisfaction of requirements. This action will consist in placing numbers into certain sets, and in trying to keep numbers out of other sets. Certain sets will be associated with a requirement $R$. $\mathrm{OS}(R)$, the *oracle set* of $R$, will contain a particular oracle from which the requirement wants to define axioms. We will want to prevent numbers from entering the oracles in $\mathrm{RS}(R)$, the *restraint set* of $R$. And $\mathrm{TS}(R)$ will be the *target set* of $R$, a set of oracles into which numbers should be placed in order to satisfy the requirement while preserving the ability to satisfy other requirements. If a requirement is assigned to a node $\sigma$, then the above definitions and notation are inherited by $\sigma$ from $R$, and inherited by all derivatives of $\sigma$ from $\sigma$.

Fix an $m$-jump poset, $\mathscr{P} = \langle P_0, \leqslant_0, P_1, \leqslant_1, f_1, ..., P_m, \leqslant_m, f_m \rangle$. There will be three types of requirements, which we now define.

DEFINITION 2.4. Incomparability requirements have *dimension* $k \geqslant 1$ and *type* 0. They are locally distributed requirements, each associated with an element of

$$Z_{0,k} = \{\langle b, c\rangle \in P^2 \colon g_{k-1}(c) \nleqslant g_{k-1}(b) \;\&\; (g_k(c) \leqslant g_k(b) \text{ or } k = m+1)$$
$$\& \; g_{k-1}(b) \neq g_{k-1}(0) \;\&\; g_{k-1}(c) \neq g_{k-1}(1)\}.$$

We establish a requirement $R = R^{0,k}_{e,b,c}$ for each $\langle b, c\rangle \in Z_{0,k}$ and $e \in \mathbf{N}$ as described in Section 1, whose goal is to make the condition

$$\exists x(\lim_{u_1} \cdots \lim_{u_{k-1}} \Delta(A_c; u_1, ..., u_{k-1}, x) \neq \lim_{v_1} \cdots \lim_{v_{k-1}} \Phi^k_e(A_b; v_1, ..., v_{k-1}, x))$$

true, if the latter limit exists. (The construction will automatically ensure that the first of the above limits exists for all $x$.) We set $\mathrm{RS}(R) = \{A_a \colon g_{k-1}(a) \leqslant g_{k-1}(b)\}$, $\mathrm{OS}(R) = \{A_c\}$, and $\mathrm{TS}(R) = \{A_a \colon g_{k-1}(a) \nleqslant g_{k-1}(b) \;\&\; a \neq 1\}$.

DEFINITION 2.5. Comparability requirements have *type* 1 and *dimension* $k \geqslant 1$. They are densely distributed requirements, each associated with an element of

$$Z_{1,k} = \{\langle b, c\rangle \in P^2 \colon g_k(b) \leqslant g_k(c) \;\&\; g_{k-1}(b) \nleqslant g_{k-1}(c) \;\&\; g_k(1) \nleqslant g_k(c)\}.$$

We establish a requirement $R = R^{1,k}_{e,b,c}$ for each $\langle b, c\rangle \in Z_{1,k}$ and $e \in \mathbf{N}$ as described in Section 1, whose goal is to ensure that

$$\lim_{u_1} \cdots \lim_{u_k} \Delta(A_c; u_1, ..., u_k, e) = \begin{cases} A_b^{(k)}(e) & \text{if } k \text{ is odd} \\ \overline{A_b^{(k)}}(e) & \text{if } k \text{ is even.} \end{cases}$$

We set $\mathrm{RS}(R) = \{A_a \colon g_k(a) \leqslant g_k(b)\}$, $\mathrm{TS}(R) = \{A_a \colon g_k(a) \nleqslant g_k(b) \;\&\; a \neq 1\}$, and $\mathrm{OS}(R) = \{A_c\}$.

DEFINITION 2.6. Highness requirements have *type* 2 and dimension $k \geqslant 2$. They are densely distributed requirements, each associated with an element of

$$Z_{2,k} = \{\langle 1, c\rangle \in P^2 \colon g_{k-1}(1) \leqslant g_{k-1}(c) \;\&\; g_{k-2}(1) \nleqslant g_{k-2}(c)\}.$$

We establish a requirement $R = R^{2,k}_{e,1,c}$ for each $\langle 1, c\rangle \in Z_{2,k}$ and $e \in \mathbf{N}$ as described in Section 1, whose goal is to ensure that

$$\lim_{u_1} \cdots \lim_{u_{k-1}} \Delta(A_c; u_1, ..., u_{k-1}, e) = \begin{cases} \varnothing^{(k)}(e) & \text{if} \quad k \text{ is odd} \\ \overline{\varnothing^{(k)}}(e) & \text{if} \quad k \text{ is even.} \end{cases}$$

We set $\mathrm{RS}(R) = \varnothing$, $\mathrm{TS}(R) = \{A_a\colon g_{k-1}(1) \leqslant g_{k-1}(a) \;\&\; a \neq 1\}$, and $\mathrm{OS}(R) = \{A_c\}$.

LEMMA 2.1. *Let* $\mathscr{P} = \langle P, \leqslant, P_1, \leqslant_1, f_1, ..., P_m, \leqslant_m, f_m\rangle$ *be an m-jump poset with least element* 0 *and greatest element* 1. *Suppose that we have a map h from P to the r.e. sets given by* $b \to A_b$ *which maps* 0 *to* $\varnothing$, 1 *to K, and satisfies the following conditions for all* $b, c \in P$:

(i) $\langle b, c\rangle \in Z_{0,k} \Rightarrow A_c^{(k-1)} \nleqslant_T A_b^{(k-1)}$.

(ii) $b \leqslant c \Rightarrow A_b \leqslant_T A_c$.

(iii) $\langle b, c\rangle \in Z_{1,k} \Rightarrow A_b^{(k)} \leqslant_T A_c^{(k)}$.

(iv) $\langle 1, c\rangle \in Z_{2,k} \Rightarrow \varnothing^{(k)} \leqslant_T A_c^{(k-1)}$.

*Then the m-jump poset generated by the image of h in the r.e. degrees is isomorphic to* $\mathscr{P}$.

*Proof.* Fix $b, c \in P$. We proceed by cases.

*Case* 1: $g_k(b) \leqslant_k g_k(c)$. As the jump operator is order-preserving, we can assume that $k$ is the least $r$ such that $g_r(b) \leqslant_r g_r(c)$.

*Subcase* 1.1: $k = 0$. Then $A_b^{(k)} \leqslant_T A_c^{(k)}$ by (ii).

*Subcase* 1.2: $k > 0$.

*Subcase* 1.2.1: $g_k(1) \nleqslant_k g_k(c)$. Then $A_b^{(k)} \leqslant_T A_c^{(k)}$ by (iii).

*Subcase* 1.2.2: $g_k(1) \leqslant_k g_k(c)$. As the jump operator is order-preserving, $A_b^{(k)} \leqslant_T A_1^{(k)}$. By (iv), $A_1^{(k)} \leqslant_T A_c^{(k)}$. But the degrees form a poset, so $A_b^{(k)} \leqslant_T A_c^{(k)}$.

*Case* 2: $g_k(c) \nleqslant_k g_k(b)$. As the jump operator is order-preserving, it suffices to show that $A_c^{(k)} \nleqslant_T A_b^{(k)}$ under the assumption that $k$ is the largest $r \leqslant m$ such that $g_r(c) \nleqslant_r g_r(b)$.

*Subcase* 2.1: $g_k(b) \neq g_k(0)$ and $g_k(c) \neq g_k(1)$. Then $A_c^{(k)} \nleqslant_T A_b^{(k)}$ by (i).

*Subcase* 2.2: $g_k(b) = g_k(0)$ and $g_k(c) = g_k(1)$. As the jump operator has the property that $\mathbf{0}^{(k)} < \mathbf{0}^{(k+1)}$ and $h(1) = K$ has degree $\mathbf{0}'$, $A_c^{(k)} \nleqslant_T A_b^{(k)}$.

*Subcase* 2.3: $g_k(b) = g_k(0)$, $g_k(c) \neq g_k(1)$, and $c \neq d$. Then by (1.1), $g_k(c) \nleqslant_k g_k(d)$, so as $g_k(0) <_k g_k(d) <_k g_k(1)$ by (1.1), we can apply (i) to conclude that $A_c^{(k)} \nleqslant_T A_d^{(k)}$. As the jump operator is order-preserving,

$A_0^{(k)} \leqslant_T A_d^{(k)}$, so as the degrees form a poset, $A_c^{(k)} \nleqslant_T A_0^{(k)}$. By Case 1, $A_b^{(k)} \equiv_T A_0^{(k)}$, so $A_c^{(k)} \nleqslant_T A_b^{(k)}$.

*Subcase* 2.4: $g_k(b) = g_k(0)$, $g_k(c) \neq g_k(1)$, and $c = d$. We proceed as in Case 2.3, replacing $d$ with $\tilde{d}$ and (1.1) with (1.2).

*Subcase* 2.5: $g_k(c) = g_k(1)$, $g_k(b) \neq g_k(0)$ and $b \neq d$. By (1.1), $g_k(d) \nleqslant_k g_k(b)$, so as $g_k(0) <_k g_k(d) <_k g_k(1)$ by (1.1), we can apply (i) to conclude that $A_d^{(k)} \nleqslant_T A_b^{(k)}$. As the jump operator is order-preserving, $A_d^{(k)} \leqslant_T A_1^{(k)}$, so as the degrees form a poset, $A_1^{(k)} \nleqslant_T A_b^{(k)}$. By Case 1, $A_c^{(k)} \equiv_T A_1^{(k)}$. Hence $A_c^{(k)} \nleqslant_T A_b^{(k)}$.

*Subcase* 2.6: $g_k(c) = g_k(1)$, $g_k(b) \neq g_k(0)$ and $b = d$. We proceed as in Case 2.5, replacing $d$ with $\tilde{d}$ and (1.1) with (1.2). ∎

The next lemma relates target sets, oracle sets and restraint sets for various requirements. It is used to show that once we satisfy requirements, we can preserve this satisfaction if the requirement lies on the true path for the construction. It is also used to show that action taken for requirements which do not lie along the true path for the construction is corrected, when necessary, in the process of returning to the true path. This lemma provides a crucial connection between the general framework and the particular set of requirements which we must satisfy.

LEMMA 2.2 (Interaction Lemma). *Fix requirements* $R = R_{e,b,c}^{j,k}$ *and* $\tilde{R} = R_{\tilde{e},\tilde{b},\tilde{c}}^{\tilde{j},\tilde{k}}$ *such that* $\tilde{k} \geqslant k$. *Then the following conditions hold*:

(i) $\mathrm{TS}(R) \cap \mathrm{RS}(R) = \emptyset$.

(ii) *If* $\mathrm{tp}(R) \in \{0, 2\}$, *then* $\mathrm{OS}(R) \subseteq \mathrm{TS}(R)$.

(iii) *Suppose that* $\mathrm{tp}(R) = 1$, *that* $A_b \in \mathrm{TS}(\tilde{R})$, *and that if* $\mathrm{tp}(\tilde{R}) \in \{0, 2\}$ *then* $k < \tilde{k}$. *Then* $A_c \in \mathrm{TS}(\tilde{R})$.

(iv) *If* $\mathrm{tp}(R) = 2$, *then* $A_c \in \mathrm{TS}(\tilde{R})$.

*Proof.* We note that if $\mathrm{tp}(R) = 0$, then $g_{k-1}(c) \nleqslant_{k-1} g_{k-1}(b)$. (i) and (ii) are now routine to verify.

(iii) Suppose that $\mathrm{tp}(R) = 1$ and $A_b \in \mathrm{TS}(\tilde{R})$. Then $g_k(b) \leqslant_k g_k(c)$, so as $k \leqslant \tilde{k}$, $g_{\tilde{k}}(b) \leqslant_{\tilde{k}} g_{\tilde{k}}(c)$. (iii) now follows if $\mathrm{tp}(\tilde{R}) = 1$.

Suppose that $\mathrm{tp}(\tilde{R}) = 2$. Since $A_b \in \mathrm{TS}(\tilde{R})$, $g_{\tilde{k}-1}(1) \leqslant_{\tilde{k}-1} g_{\tilde{k}-1}(b)$. As $g_k(b) \leqslant_k g_k(c)$ and $k < \tilde{k}$, $g_{\tilde{k}-1}(1) \leqslant_{\tilde{k}-1} g_{\tilde{k}-1}(c)$. Hence $A_c \in \mathrm{TS}(\tilde{R})$.

Finally, suppose that $\mathrm{tp}(\tilde{R}) = 0$. Since $A_b \in \mathrm{TS}(\tilde{R})$, $g_{\tilde{k}-1} \nleqslant_{\tilde{k}-1} g_{\tilde{k}-1}(\tilde{b})$. As $g_k(b) \leqslant_k g_k(c)$ and $k < \tilde{k}$, $g_{\tilde{k}-1}(b) \leqslant_{\tilde{k}-1} g_{\tilde{k}-1}(c)$. It now follows that $g_{\tilde{k}-1}(c) \nleqslant_{\tilde{k}-1} g_{\tilde{k}-1}(\tilde{b})$, else by transitivity, $g_{\tilde{k}-1}(b) \nleqslant_{\tilde{k}-1} g_{\tilde{k}-1}(\tilde{b})$, yielding a contradiction. Hence $A_c \in \mathrm{TS}(\tilde{R})$.

(iv) Suppose that $\mathrm{tp}(R) = 2$. Then $g_{k-1}(1) \leqslant_{k-1} g_{k-1}(c)$. As $k \leqslant \tilde{k}$, $g_{\tilde{k}-1} \leqslant_{\tilde{k}-1} g_{\tilde{k}-1}(c)$. It is now easily checked that $A_c \in \mathrm{TS}(\tilde{R})$. ∎

DEFINITION 2.7. Fix a recursive ordering $\{R_i : i \in \mathbf{N}\}$ of all requirements. We say that $R_i$ *has higher priority than* $R_j$ if $i < j$. If $R_i = R^{j,k}_{e,b,c}$ is assigned to node $\sigma \in T^k$, then we say that $\mathrm{tp}(\sigma) = j$ and $\dim(\sigma) = k$.

Requirements of dimension $k$ will be assigned to nodes of trees $T^r$ for $r \geqslant k$, and subrequirements of these requirements will be assigned to nodes of $T^j$ for $j < k$. Whenever $R_i$ is assigned to two nodes $\sigma$ and $\tau$ and $\mathrm{tp}(\sigma) \in \{1, 2\}$, then we say that $\sigma \equiv \tau$. (We will extend the definition of $\equiv$ to additional pairs of nodes later, and then take the reflexive, symmetric, transitive closure of the relation defined to make $\equiv$ an equivalence relation. Equivalent nodes work on the same functional, and sometimes on the same arguments for that functional. To satisfy requirements for all $e$, we will define a given functional as a disjoint union of many partial recursive functionals. The union over nodes in many equivalence classes will define the functional (for a specified oracle) on a recursive domain. We will take steps to ensure that functionals are total on specified oracles, whenever this is required.)

The assignment of requirements to nodes of trees will proceed by induction on $k = n - j$ for $j \leqslant n$. ($n$ will be the largest dimension of a requirement in our list.) The inductive step of the definition will proceed in four steps. In Step 1, we will define the path generating function $\lambda$ on nodes of trees which have already had requirements assigned to all of their predecessors. If $\sigma \in T^k$, then $\lambda(\sigma)$ will be a node on $T^{k+1}$. Given a path $\Lambda \in [T^k]$, $\{\lambda(\sigma) : \sigma \subset \Lambda\}$ gives an approximation to a path $\lambda(\Lambda) \in [T^{k+1}]$. When $\lambda(\sigma) \mid \lambda(\sigma^-)$, a link will be formed on $T^k$. These links, defined in Step 2, will prevent action by nodes of $T^k$ which do not seem to come from the true path approximation for trees of higher dimension. We will have to decide which nodes of $T^{k+1}$ are eligible to assign subrequirements to a given node of $T^k$. Conditions ensuring consistency between the different trees enter into this decision, and these conditions are delineated in Step 3. The requirement assignment process is described in Step 4.

DEFINITION 2.8. We proceed by induction on $k = n - j$, assigning requirements to nodes of $T^k$ and dividing $T^k$ into blocks of requirements. If $k = n$, then the requirement $R_i$ is assigned to every node $\sigma$ of $T^n$ such that $\mathrm{lh}(\sigma) = i$. Each node of $T^n$ is a *block*. Thus for $\sigma \in T^n$, we say that $\sigma$ *lies in the $\sigma$-block*, that we *begin the $\sigma$-block at $\sigma$*, and that a *path through the $\sigma$-block is completed at $\sigma$*.

Suppose that $k < n$. There are four steps.

*Step* 1: Definition of the path generating function $\lambda$. Given a node $\eta \in T^k$ such that requirements have been assigned to all predecessors of $\eta$, the function $\lambda$ will define a node $\lambda(\eta) \in T^{k+1}$. The process is meant to capture the following situation. For each $\xi \subset \eta$, $\xi$ will be derived from a node

$\sigma \in T^{k+1}$. A sentence $M_\sigma$ will be associated with $\sigma$, and a fragment $M_\xi$ of that sentence will be associated with $\xi$. Suppose that the first unbounded quantifier of $M_\sigma$ is a universal quantifier. If $\sigma$ has dimension $\geqslant k+1$, we bound the leading block of universal quantifiers by numbers which are strictly increasing with $\mathrm{lh}(\xi)$. As long as each $\xi$ succeeds in satisfying its sentence $M_\xi$, the approximation given by $\lambda$ predicts that $\sigma \hat{} \langle v \hat{} \langle \beta \rangle \rangle \subseteq \lambda(\eta)$, where $v$ will be the initial derivative of $\sigma$ along $\eta$ (defined formally below) and $\beta$ is the outcome of $v$ along $\eta$. If we find a first $\xi$ for which $M_\xi$ is false, then $\sigma \hat{} \langle \xi \hat{} \langle \beta \rangle \rangle \subseteq \lambda(\eta)$, where $\beta$ is the outcome of $\xi$ along $\eta$. If the first unbounded quantifier of $M_\sigma$ is an existential quantifier, then we proceed as above after replacing $M_\xi$ with $\neg M_\xi$. (If $\dim(\sigma) \leqslant k$, then outcomes on $T^k$ give rise to unique outcomes on $T^{k+1}$.)

If $\eta = \langle\ \rangle$ then $\lambda(\eta) = \langle\ \rangle$. Suppose that $\eta \neq \langle\ \rangle$. By (2.4), it will follow by induction that $\mathrm{up}(\eta^-) \subseteq \lambda(\eta^-)$, where $\mathrm{up}(\eta^-)$ is the node of $T^{k+1}$ from which $\eta^-$ is derived. ($\mathrm{up}(\eta^-)$ has been defined inductively in Step 4 for $\eta^-$.)

$$\text{If either } \mathrm{up}(\eta^-) = \lambda(\eta^-) \text{ or } \eta^- \text{ has infinite outcome along } \eta, \text{ then we set } \lambda(\eta) = \mathrm{up}(\eta^-) \hat{} \langle \eta \rangle. \text{ We set } \lambda(\eta) = \lambda(\eta^-) \text{ otherwise.} \tag{2.4}$$

(It follows from Definition 2.1 that $\lambda(\eta) \in T^{k+1}$.) It follows from (2.4) that:

$$\text{If } \sigma \subseteq \lambda(\eta) \text{ then } \mathrm{out}(\sigma) \subseteq \eta \text{ and } \lambda(\mathrm{out}(\sigma)) = \sigma; \tag{2.5}$$

and

$$\text{If } \lambda(\eta^-) \supseteq \sigma \text{ and } \lambda(\eta) \not\supseteq \sigma, \text{ then for all } \delta \supseteq \eta,\ \lambda(\delta) \not\supseteq \sigma. \tag{2.6}$$

We define $\lambda^r(\eta)$ for $r \in [k, n]$ by $\lambda^k(\eta) = \eta$ and $\lambda^r(\eta) = \lambda(\lambda^{r-1}(\eta))$ for $r > k$. Given $\xi \subseteq \eta$, we say that $\xi$ is the *principal derivative of* $\mathrm{up}(\xi)$ (defined in Step 4) *along* $\eta$ if either $\xi$ has infinite outcome along $\eta$, or $\xi$ is the shortest derivative of $\mathrm{up}(\xi)$ along $\eta$ and for all $\gamma \subset \eta$, if $\mathrm{up}(\gamma) = \mathrm{up}(\xi)$, then $\gamma$ has finite outcome along $\eta$. (We do not require that $\mathrm{up}(\xi) \subseteq \lambda(\eta)$.) And if $r \geqslant k$ and $\zeta \in T^r$, we call $\xi$ the *principal derivative of* $\zeta$ *along* $\eta$ if either $r = k$ and $\xi = \zeta \subset \eta$, or $r > k$, $\xi$ is the principal derivative of $\mathrm{up}(\xi)$ along $\eta$ and $\mathrm{up}(\xi)$ is the principal derivative of $\zeta$ along $\lambda(\eta)$.

*Step* 2: Links. We will place restrictions on the stages of the construction at which nodes are eligible to be *switched* by the approximation to the true path. One restriction requires a node to be *free* when it is switched by the true path approximation, i.e., that it not be contained in any *link*. Links are formed when a switch occurs, and can be broken when the outcome of a switched node is switched back. (Links correspond to initialization, after injury, in the standard approach to infinite injury priority arguments. Suppose that a node $\sigma \in T^2$ has initial derivative $v$

(defined inductively in Step 4) along a path $\Lambda^1$ through $T^1$, and principal derivative $\pi \supset \nu$ along $\eta \subset \Lambda^1$. Then we form a primary $\eta$-link $[\nu, \pi]$ from $\nu$ to $\pi$, thereby restraining any node $\xi \in [\nu, \pi)$ from acting and destroying computations declared by $\pi$. (Note that if $[\nu, \pi]$ is an $\eta$-link, then $\pi$ is not restrained by $[\nu, \pi]$. However, as we can have $[\nu, \pi) = [\nu, \delta)$ as intervals with $\pi \neq \delta$, we use closed interval notation $[\nu, \pi]$ for $\eta$-links to make sure that there is a one-to-one correspondence between intervals which determine links, and the links themselves.) Any such $\xi \in [\nu, \pi)$ will either be a derivative of a node which is no longer on the approximation to the true path, or a derivative of a node $\rho \subseteq \sigma$. In the former case, condition (2.7) of the definition of $\eta$-consistency which is presented in Step 3 will also prevent derivatives of $\xi$ from acting. The links are aimed at preventing derivatives of $\rho \subseteq \sigma$ from acting. Derivatives of such a node $\rho$ which extend $\pi$ will be able to act, and we will show that there is no harm in preventing derivatives of $\rho$ restrained by the link from acting. We will allow derivatives of $\pi$ to act, and so do not restrain $\pi$ in this link.)

A node $\eta \in T^k$ such that $\mathrm{lh}(\eta) > 0$ is said to be *switching* if there is an $r > k$ such that $\lambda^r(\eta^-) \mid \lambda^r(\eta)$. For the least such $r$, we say that $\eta$ is *r-switching*. If $j \in [r, n]$ and $\eta$ is $r$-switching, we say that $\eta$ *switches* $\mathrm{up}^j(\eta^-)$.

Fix $\eta \in T^k$. Each $\eta$-link will be derived from a *primary* $\lambda^j(\eta)$-link for some $j \geqslant k$, and will have either *finite* or *infinite outcome*. We define the *$\eta$-links* of $T^k$ by induction on $n-k$. If $k = n$, then there are no $\eta$-links. Suppose that $k < n$.

We first determine the *primary $\eta$-links*. Suppose that $\xi \subseteq \eta$ and $\xi^-$ is the principal derivative of $\gamma = \mathrm{up}(\xi^-)$ along $\eta$, but is not the initial derivative of $\gamma$ along $\eta$. Let $\mu$ be the initial derivative of $\gamma$ along $\eta$. Then $[\mu, \xi^-]$ is a *primary $\eta$-link* and has *infinite outcome*.

$\eta$-links can also be created by pulling down $\lambda(\eta)$-links. Suppose that $[\rho, \tau]$ is a $\lambda(\eta)$-link on $T^{k+1}$. Assume that the initial derivative $\mu$ of $\rho$ along $\eta$ and the principal derivative $\pi$ of $\tau$ along $\eta$ both exist. Then $[\mu, \pi]$ is an *$\eta$-link derived from* $[\rho, \tau]$. $[\mu, \pi]$ *has finite outcome* if $[\rho, \tau]$ has infinite outcome, and has infinite outcome otherwise.

If $[\rho, \tau]$ is derived from some link $[\zeta, \kappa]$, then every link derived from $[\rho, \tau]$ is *derived from* $[\zeta, \kappa]$. We say that $\xi$ *is* $\eta$-restrained if there is an $\eta$-link $[\mu, \pi]$ such that $\mu \subseteq \xi \subset \pi$. In this case, we say that $\xi$ is *$\eta$-restrained by* $[\mu, \pi]$. $\xi$ is *$\eta$-free* if $\xi$ is not $\eta$-restrained. $\xi$ *is free* if $\xi$ is $\xi$-free.

*Step* 3: $\eta$-consistency. We decide, in this step, whether a node $\sigma \in T^{k+1}$ is allowed to assign subrequirements at $\eta$. This will depend on four conditions. The first condition, (2.7), requires $\eta$ to predict that $\sigma$ is on the true path of $T^r$ for all $r \in [k+1, n]$. The second condition, (2.8), requires that if $\sigma \in T^{k+1}$, once a witness $\xi \subset \eta$ for an existential sentence associated with $\sigma$ is found, no derivatives of $\sigma$ can extend $\xi$. In this case, $\eta$ has all the information needed to correctly predict the outcome of $\sigma$. However, we do not search for such witnesses on $T^k$ if $k \geqslant \dim(\sigma)$, as we have not yet begun to decompose the sentence associated with $\sigma$, in this case. Rather, we will require an outcome of $\sigma$ to code the outcome of a unique derivative of $\sigma$ along a path of $T^k$, and so impose condition (2.9) requiring that there be a unique such derivative. Condition (2.10) requires that $\sigma$ be $\lambda(\eta)$-free. Lemma 4.4 will show that this condition implies condition (2.7), but for now, it is convenient to require both conditions.

For $\eta \in T^k$, we say that $\sigma \in T^{k+1}$ *is $\eta$-consistent* if the following conditions hold:

(2.7) $\mathrm{up}^r(\sigma) \subseteq \lambda^r(\eta)$ for all $r \in [k+1, n]$.

(2.8) If $\sigma \subset \lambda(\eta)$ and $\dim(\sigma) > k$, then for all $\nu \subset \eta$, if $\mathrm{up}(\nu) = \sigma$ then $\nu$ has finite outcome along $\eta$.

(2.9) For all $\nu \subset \eta$, if $\dim(\sigma) \leqslant k$ then $\mathrm{up}(\nu) \neq \sigma$.

(2.10) $\sigma$ is $\lambda(\eta)$-free.

(We note that our definition of $\eta$-consistency differs from that in [LmLr1] in that we impose (2.10) as an additional restriction. This restriction is needed to show that whenever a path approximation is switched precisely at $\sigma \in T^r$, then for all $k \geqslant r$, the path approximation on $T^k$ is switched precisely at the node from which $\sigma$ is derived. Many of the lemmas we prove rely on this fact. If $k \leqslant 3$, however, this property of switching is automatic, so (2.10) does not need to be imposed.)

*Step* 4: Assignment of Derivatives. Let $\eta \in T^k$ be given such that requirements have been assigned to all predecessors of $\eta$, but not to $\eta$. We want to assign a requirement to $\eta$. The requirement chosen will be one which has been assigned to some $\eta$-consistent node of $T^{k+1}$.

Requirements are assigned in *blocks*. (Blocks on $T^0$ are the counterpart of *stages* in the usual approach to priority constructions. Thus if a block is begun at $\delta \in T^0$ and a path through the block is completed at $\xi \in T^0$, then $[\delta, \xi]$ corresponds to a set of substages of a given stage.) We *begin* a block at $\delta \in T^k$ if either $\delta = \langle\ \rangle$ or a path through a block was completed at $\delta^-$. If we begin a block at $\delta$, then this block is called the *$\delta$-block*. A *path through the $\delta$-block is completed at* $\xi \supseteq \delta$ if $\mathrm{up}(\xi)$ completes a path through some block of $T^{k+1}$ and $\xi$ is an initial derivative of $\mathrm{up}(\xi)$. We say that $\gamma$ *lies in* the $\delta$-block if $\delta \subseteq \gamma$ and no path through the $\delta$-block has been completed at any $\beta \subset \gamma$.

Fix $\delta$ such that $\eta$ is in the $\delta$-block. If either $\eta = \langle\ \rangle$, $\eta = \delta$, or $\eta$ is switching, set $\rho = \langle\ \rangle$. Otherwise, fix $\rho \subseteq \lambda(\eta)$ such that $\rho^- = \mathrm{up}(\eta^-)$. (By induction using (2.7), $\mathrm{up}(\eta^-) \subseteq \lambda(\eta^-)$ and $\eta$ provides an outcome for a derivative of $\mathrm{up}(\eta^-)$; hence by (2.7), $\mathrm{up}(\eta^-) \subset \lambda(\eta)$ so such a $\rho$ must exist.)

Fix the shortest $\sigma$ such that $\rho \subseteq \sigma \subseteq \lambda(\eta)$ and $\sigma$ is $\eta$-consistent. (We note that for any $j \geqslant k$, any $\lambda^j(\eta)$-link $[\mu^j, \pi^j]$ satisfies $\pi^j \subset \lambda^j(\eta)$, so $\lambda(\eta)$ is $\lambda(\eta)$-free. Furthermore, (2.7) for $k$ will follow from (2.7) for $k+1$. It thus follows that $\lambda(\eta)$ is $\eta$-consistent, so $\sigma$ must exist.) Let $R_i$ be the requirement assigned to $\sigma$. We assign $R_i$ to $\eta$, designate $\eta$ as a *derivative* of $\sigma$, and say that $\mathrm{up}(\eta) = \sigma$. We assign a type, dimension, oracle set, target set, and restraint set to $\eta$ in the same way as these were assigned to $\sigma$.

The derivative operation can be iterated; thus for every $\zeta$ such that $\sigma$ is a derivative of $\zeta$, we call $\eta$ a *derivative* of $\zeta$. $\eta$ is also a derivative of $\eta$. If $r > k$, $\zeta \in T^r$, and $\eta$ is a derivative of $\zeta$ then we write $\mathrm{up}^r(\eta) = \zeta$. If there is no $\xi \subset \eta$ such that $\mathrm{up}(\xi) = \sigma$, then for all $\nu \supseteq \eta$, we call $\eta$ the *initial derivative of $\sigma$ along $\nu$*, and if $\sigma$ is the initial derivative of $\zeta$ along $\sigma$, then $\eta$ is also the *initial derivative of $\zeta$ along any $\nu \supseteq \eta$*. We specify that $\eta \equiv \sigma$. $\zeta$ is an *antiderivative* of $\xi$ if $\xi$ is a derivative of $\zeta$.

If $\Lambda^k$ is a path through $T^k$, then we let $\lambda(\Lambda^k) = \lim_s \{\lambda(\Lambda^k \upharpoonright s)\}$, and define $\Lambda^{k+1} = \lambda(\Lambda^k)$. (We will show in Lemma 3.2 that $\mathrm{lh}(\Lambda^{k+1})$ exists and is infinite.) For $\Lambda^k \in [T^k]$, the *$\Lambda^k$-links* are the $\eta$-links for those $\eta$ such that $\lambda^j(\eta) \subset \Lambda^j$ for all $j \in [k, n]$. We now define $\xi$ to be *$\Lambda^k$-restrained* or *$\Lambda^k$-free* as in Step 2, with $\Lambda^k$ in place of $\eta$.

The description of the assignment of requirements to nodes is now complete. We take the reflexive, symmetric, and transitive closure of $\equiv$ as defined in Step 4 and before Step 1 to generate an equivalence relation. ∎

We note an important relationship between the functions wt and $\lambda$. Suppose that $k < n$, $\sigma \subset \tau \in T^k$, and $\lambda(\sigma) \neq \lambda(\tau)$. By (2.5), $\mathrm{out}(\lambda(\sigma)) \subseteq \sigma$ and $\mathrm{out}(\lambda(\tau)) \subseteq \tau$, so by (2.4) and (2.5) and as $\sigma \subset \tau$, $\mathrm{out}(\lambda(\sigma)) \subseteq \sigma \subset \mathrm{out}(\lambda(\tau)) \subseteq \tau$. It now follows from (2.3) that:

(2.11) For all $k < n$ and $\sigma \subset \tau \in T^k$, if $\lambda(\sigma) \neq \lambda(\tau)$, then $\mathrm{wt}(\lambda(\sigma)) < \mathrm{wt}(\lambda(\tau))$.

We now indicate how to specify the sentence which generates the action for a requirement assigned to a given node. Our requirements will be of the form $(\varphi \to \psi) \mathbin{\&} (\neg\varphi \to \chi)$. We will show that for requirements, all of whose antiderivatives lie on the true paths determined by the construction, $\psi$ is true if $\varphi$ is true, and $\chi$ is true if $\varphi$ is false. To achieve this goal, we will have to correct action taken when it seemed that $\varphi$ was false if we later discover that $\varphi$ is true. The interplay between this correction feature and the determination of the node which controls the definition of a given axiom is the essence of priority arguments. Furthermore, as requirements will be introduced on $T^k$ for $k > 0$ and the construction takes place on $T^0$, we must work with fragments of $\varphi$ on $T^0$ rather than $\varphi$ itself. When introduced, $\varphi$ is assigned to a node $\sigma$ of $T^k$, and fragments of $\varphi$, obtained by bounding some of the quantifiers of $\varphi$, are assigned to derivatives of $\sigma$. We now define the sentences and describe the decomposition process. In order to avoid notational confusion later, we use $M$ in place of $\varphi$.

DEFINITION 2.9 (Sentences, Base Step). For each $\sigma \in T^k$ such that $\dim(\sigma) = k$, there is a requirement $R = R^{j,k}_{e,b,c}$ which is assigned to $\mathrm{up}^n(\sigma)$. We will assign a sentence $M_\sigma$ to $\sigma$ such that $M_\sigma$ is $\Pi_{k+1}$ if $k$ is even, and is $\Sigma_{k+1}$ if $k$ is odd. Thus we require $M_\sigma$ to have, as its final quantifier, a universal quantifier.

Suppose that $R^{j,k}_{e,b,c}$, is assigned to node $\sigma$ of $T^k$. For $k \geqslant 1$, let $\gamma^k(e, x, b)$ be the formula with free variable $x$

$$\exists x_0 \forall y_0 \geqslant x_0 \forall x_1 \exists y_1 \geqslant x_1 \cdots \forall x_{k-2} \exists y_{k-2} \geqslant x_{k-2} \exists s \forall t \geqslant s (\Phi^t_e(A^t_b; \bar{y}, x) = 0)$$

if $k$ is odd (there is no block $\exists x_0 \cdots \exists y_{k-2} \geqslant x_{k-2}$ when $k = 1$), and

$$\forall x_0 \exists y_0 \geqslant x_0 \exists x_1 \forall y_1 \geqslant x_1 \cdots \forall x_{k-2} \exists y_{k-2} \geqslant x_{k-2} \exists s \forall t \geqslant s (\Phi^t_e(A^t_b; \bar{y}, x) = 0)$$

if $k$ is even. If $j = 0$, we let $M_\sigma$ be the sentence $\gamma^k(e, \mathrm{wt}(\sigma), b)$. If $A$ is recursively enumerable, then by Definition 1.1, we can fix a sentence $\beta_r(A; e)$, such that, if $r$ is odd, then $\beta_r(A; e)$ is a $\Sigma_{r+1}$ sentence whose truth agrees with the truth of "$e \in A^{(r)}$", and if $r$ is even, then $\beta_r(A; e)$ is a $\Pi_{r+1}$ sentence whose truth agrees with the truth of "$e \notin A^{(r)}$". Furthermore, we can write $\beta_r(A; e)$ as $\bar{Q}\bar{x}\, \exists s\, \forall t \geqslant s \tilde{\beta}_r(A^t; e)$, where $\bar{Q}\bar{x}$ is a quantifier block and $\tilde{\beta}_r(A^t, e)$ is quantifier free. Suppose that the requirement $R^{j,k}_{e,b,c}$ is assigned to $\sigma$ for $j \in \{1, 2\}$. We let $M_\sigma$ be the sentence $\beta_k(A_b, e)$ if $j = 1$, and $\beta_k(\emptyset, e)$ if $j = 2$. Our construction will have the property that if $\mathrm{up}^r(\sigma) = \zeta^r$ lies on the true path of $T^r$ for all $r \in [k, n]$, then $M_{\zeta^r}$ is true iff $\Delta_{\zeta^r}(A_c; \bar{y})$ takes the value which ensures the satisfaction of the requirement assigned to $\zeta^r$.

DEFINITION 2.10 (Sentences, Inductive Step). Suppose that $\dim(\sigma) > k$. Define $M_\sigma = (M_{\mathrm{up}(\sigma)})^{[\mathrm{wt}(\sigma)]}$.

## 3. PATHS AND SWITCHING

In this section, we prove some technical lemmas about properties of the path generation process. The first lemma shows that the paths through the trees are infinite, and that initial and principal derivatives exist. This lemma is used many times to analyze the process of decomposing requirements.

LEMMA 3.1 (Limit Path Lemma). *Fix $k \in [0, n)$ and a path $\Lambda^k \in [T^k]$, and let $\Lambda^{k+1} = \lambda(\Lambda^k) = \lim\{\lambda(\eta) : \eta \subset \Lambda^k\}$. Then:*

(i) *If* $\sigma \subset \lambda(\Lambda^k)$, *then* $\sigma$ *has an initial derivative* $\nu$ *along* $\Lambda^k$ *and* $\lambda(\nu) = \sigma$.

(ii) *If* $\sigma \subset \lambda(\Lambda^k)$, *then there is a* $\pi \subseteq \Lambda^k$ *such that* $\pi^-$ *is the principal derivative of* $\sigma$ *along* $\Lambda^k$, $\lambda(\pi)^- = \sigma$, *and for all* $\eta \subseteq \Lambda^k$, $\lambda(\pi) \subseteq \lambda(\eta)$ *iff* $\pi \subseteq \eta$.

(iii) *For any* $\delta$*-block such that* $\delta \subset \Lambda^k$, *there is a* $\xi \subset \Lambda^k$ *such that* $\xi$ *completes a path through the* $\delta$*-block.*

(iv) $\mathrm{lh}(\Lambda^{k+1}) = \infty$.

*Proof.* We proceed by induction on $j = n - k$.

(i) By (2.4) and as $\sigma \subset \lambda(\Lambda^k)$, $\sigma$ must have a derivative along $\Lambda^k$. Hence if $\nu$ is the shortest derivative of $\sigma$ along $\Lambda^k$, then $\nu$ is the initial derivative of $\sigma$ along $\Lambda^k$. By (2.7), $\lambda(\nu) \supseteq \sigma$. By (2.4) and (2.7), no $\tau \supset \sigma$ can have a derivative $\mu \subset \nu$. Hence by (2.4), $\lambda(\nu) = \sigma$.

(ii) If $\dim(\sigma) \leqslant k$, then by (2.9), the initial derivative $\nu$ of $\sigma$ along $\Lambda^k$ is the principal derivative of $\sigma$ along $\Lambda^k$. (ii) follows in this case from (i), (2.5) and (2.6).

Suppose that $\dim(\sigma) > k$. By (i), let $\nu$ be the initial derivative of $\sigma$ along $\Lambda^k$. If there is no $\pi \subset \Lambda^k$ such that $\mathrm{up}(\pi^-) = \sigma$ and $\pi^-$ has infinite outcome along $\pi$, then it follows as in the case for $\dim(\sigma) \leqslant k$ that $\nu$ is the principal derivative of $\sigma$ along $\Lambda^k$. Otherwise, fix the shortest such $\pi$. We note that $\pi^-$ is the principal derivative of $\sigma$ along $\Lambda^k$. By (2.4), induction, (2.7) and (2.6), $\lambda(\pi) = \sigma\,\hat{}\,\langle\pi\rangle \subseteq \Lambda^{k+1}$, and if $\eta \subset \Lambda^k$ then $\lambda(\eta) \supseteq \sigma\,\hat{}\,\langle\pi\rangle$ iff $\eta \supseteq \pi$.

(iii, iv) It follows easily from (2.7) that $\Lambda^{k+1} = \lambda(\Lambda^k) = \lim\{\lambda(\eta) : \eta \subset \Lambda^k\}$ exists. First suppose that $\mathrm{lh}(\Lambda^{k+1}) = \infty$. By (iii) inductively, there are infinitely many blocks along $\Lambda^{k+1}$, so infinitely many $\tau \subset \Lambda^{k+1}$ such that $\tau$ completes a path through a block. By (i), each such $\tau$ has an initial derivative along $\Lambda^k$. Hence by Definition 2.8, Step 4, there are infinitely many $\xi \subset \Lambda^k$ which complete paths through blocks, and (iii) holds in this case.

Now suppose that $\mathrm{lh}(\Lambda^{k+1}) < \infty$ in order to obtain a contradiction. Then by (2.7), there is an $\eta \subset \Lambda^k$ such that for all $\xi$ satisfying $\eta \subseteq \xi \subset \Lambda^k$, $\lambda(\xi) = \Lambda^{k+1}$. If $\eta \subseteq \xi \subset \Lambda^k$ and $\xi$ completes a path through a block, then $\xi$ must be an initial derivative of some node $\subseteq \Lambda^{k+1}$. As this is possible only finitely often and $\mathrm{lh}(\Lambda^k) = \infty$, we can assume without loss of generality that there is no $\xi$ such that $\eta \subseteq \xi \subset \Lambda^k$ and $\xi$ completes a path through any block. By (2.6) and the choice of $\rho$ in Step 4 of Definition 2.8, if $\eta \subseteq \xi \subset \delta \subset \Lambda^k$ then $\xi$ is nonswitching, so $\mathrm{up}(\xi) \subset \mathrm{up}(\delta) \subseteq \Lambda^{k+1}$. But this is impossible if $\mathrm{lh}(\Lambda^{k+1}) < \infty$ so $\mathrm{lh}(\Lambda^k) = \infty$. ▮

From now on, whenever we write $\Lambda^k \in [T^k]$, we assume that there is a $\Lambda^0 \in [T^0]$ such that $\Lambda^k = \lambda^k(\Lambda^0)$. Furthermore, if we write $\eta \in T^k$, we assume that $\eta \subset \Lambda^k$ for some $\Lambda^k \in [T^k]$. If this is not the case, then $\eta$ and $\Lambda^k$ are irrelevant to our construction.

The next lemma describes some useful properties of the out function.

LEMMA 3.2 (Out Lemma). *Fix $k \leqslant n$ and $\rho^k \in T^k$. Then*:

(i) *If $k > 0$ then* $\lambda(\mathrm{out}(\rho^k)) = \rho^k$.

(ii) *If $k < n$ and* $\mathrm{lh}(\rho^k) > 0$, *then there is a unique $\rho^{k+1} \in T^{k+1}$ such that* $\mathrm{out}(\rho^{k+1}) = \rho^k$.

*Proof.* (i) $\rho^k = (\rho^k)^- {}^\wedge \langle \mathrm{out}(\rho^k) \rangle$, and $(\mathrm{out}((\rho^k))^-$ is the principal derivative of $(\rho^k)^-$ along $\mathrm{out}(\rho^k)$. Hence (i) follows from Lemma 3.1(ii) (Limit Path).

(ii) Let $\nu^k = (\rho^k)^-$, $\nu^{k+1} = \mathrm{up}(\nu^k)$, and $\rho^{k+1} = \nu^{k+1} {}^\wedge \langle \rho^k \rangle$. Then $\mathrm{out}(\rho^{k+1}) = \rho^k$. To see uniqueness, we note that if $\mathrm{out}(\tau^{k+1}) = \rho^k$, then $\tau^{k+1} = (\tau^{k+1})^- {}^\wedge \langle \rho^k \rangle$, and $\mathrm{up}((\rho^k)^-) = (\tau^{k+1})^-$. Hence $(\tau^{k+1})^- = \nu^{k+1}$ and $\tau^{k+1} = \rho^{k+1}$. ∎

Our next lemma analyzes the behavior of the function $\lambda$. Suppose that $\eta$ extends $\eta^-$ on $T^k$ in Step 4 of Definition 2.8. We discuss the relationship of the path computed by $\lambda^j(\eta^-)$ to the path computed by $\lambda^j(\eta)$ for all $j$ such that $k \leqslant j \leqslant n$. Three types of phenomena can occur, and one will occur for each $j$. These phenomena induce a partition of $[k, n]$ into three intervals.

There will be a largest $p \geqslant k$ such that for all $j \in [k, p]$, $\lambda^j(\eta)$ is an immediate successor of $\lambda^j(\eta^-) = \mathrm{up}^j(\eta^-)$. $\eta$ is not $j$-switching for any $j \in [k, p]$.

If $p \neq n$, then there will be two possibilities. The first is that $\mathrm{up}^p(\eta^-) = (\lambda^p(\eta))^-$ has infinite outcome along $\lambda^p(\eta)$. Then $\eta$ will be $(p+1)$-switching, and will switch $\mathrm{up}^j(\eta^-)$ for all $j \in [p+1, n]$. $\eta$ will switch the outcome of $\mathrm{up}^{p+1}(\eta^-)$ from infinite along $\lambda^{p+1}(\eta^-)$ to finite along $\lambda^{p+1}(\eta)$. It will follow from (2.10) that for all $j \geqslant p+1$, $\eta$ will switch the outcome of $\mathrm{up}^j(\eta^-)$ from infinite along $\lambda^j(\eta^-)$ to finite along $\lambda^j(\eta)$ if $j-(p+1)$ is even, and from finite along $\lambda^j(\eta^-)$ to infinite along $\lambda^j(\eta)$ if $j-(p+1)$ is odd. There will be a largest $s \in (p, n]$ such that for all $j \in [p, s)$, $\mathrm{up}^j(\eta^-)$ will be the principal derivative of $\mathrm{up}^{j+1}(\eta^-)$ along $\lambda^j(\eta)$, and $\lambda^j(\eta)$ will be an immediate successor of $\mathrm{up}^j(\eta^-)$. $[p+1, s]$ is the interval where the second type of phenomenon occurs.

If $s < n$, then the third type of phenomenon begins at $s+1$ (we set $s = p$ if $\mathrm{up}^p(\eta^-) = (\lambda^p(\eta))^-$ has finite outcome along $\lambda^p(\eta)$, which is the second possibility alluded to in the preceding paragraph, and if this is the case, then $\eta$ is not switching). Here $\mathrm{up}^s(\eta^-) = (\lambda^s(\eta))^-$ will have finite outcome along $\lambda^s(\eta)$ and will not be the principal derivative of $\mathrm{up}^{s+1}(\eta^-)$ along $\lambda^s(\eta)$. Thus by (2.4), $\lambda^{s+1}(\eta) = \lambda(\mathrm{up}^s(\eta^-))$, so $\lambda^t(\eta) = \lambda^t(\mathrm{up}^s(\eta^-))$ for all $t > s$.

The three types of phenomena mentioned above can be observed if we consider the usual way for satisfying a thickness requirement on $T^2$. $\eta$ decides, for $\eta^-$, whether to place an additional element $x$ into a set $S$ which is to be either a finite or a thick subset of a recursive set $R$. We assume that some elements have already been placed in $S$. If $\eta$ is the first stage at which we consider $x$, then we set $p = s = 1$ if we decide to place $x$ into $S$ and we set $p = 1$ and $s = 2$ otherwise. And if $\eta$ is not the first stage at which we consider $x$, and we decide to place $x$ into $S$ at $\eta$, then we set $p = 0$ and $s = 1$.

LEMMA 3.3 ($\lambda$-Behavior Lemma). *Fix $\eta \in T^k$ and assume that a requirement has been assigned to $\eta$. Then there are $p$ and $s$ such that $k \leqslant p \leqslant s \leqslant n$ and the following conditions hold*:

(i) *For all $i \in [k, p]$, $\lambda^i(\eta^-) = \mathrm{up}^i(\eta^-) = (\lambda^i(\eta))^-$, if $i < p$ then $\lambda^i(\eta^-)$ is the initial derivative of $\lambda^{i+1}(\eta^-)$ along $\lambda^i(\eta)$, and if $i > k$ then* $\mathrm{out}(\lambda^i(\eta)) = \lambda^{i-1}(\eta)$.

(ii) *For all $i \in (p, s]$, $\mathrm{up}^i(\eta^-) = \lambda^i(\eta)^-$, $\lambda^i(\eta^-) \mid \lambda^i(\eta) = \lambda^i(\eta)^- {}^\wedge \langle \lambda^{i-1}(\eta) \rangle$.*

(iii) *For all $i \in (s, n]$, $\lambda^i(\eta) = \lambda^i((\lambda^s(\eta))^-)$.*

*Proof.* We verify (i)–(iii) by induction on $\mathrm{lh}(\eta)$, analyzing what can happen when requirements are assigned in Step 4 of Definition 2.8 for $\eta^-$.

If $i = k$, then $\eta^- = \lambda^k(\eta^-) = (\lambda^k(\eta))^- = \mathrm{up}^k(\eta^-)$. Fix the largest $p \leqslant n$ such that for all $i \in [k, p]$, $\lambda^i(\eta^-) = (\lambda^i(\eta))^- = \mathrm{up}^i(\eta^-)$. (i) now follows from (2.4).

If $\eta$ is nonswitching, then we set $s = p$ and note that (ii) holds vacuously, and that (iii) holds vacuously if $s = n$. So suppose that $s < n$. As it is not the case that $\lambda^{s+1}(\eta^-) = (\lambda^{s+1}(\eta))^- = \mathrm{up}^{s+1}(\eta^-)$, it follows from (2.4) that $(\lambda^s(\eta))^-$ cannot be an initial derivative of $\mathrm{up}^{s+1}(\eta^-)$. As $\eta$ is nonswitching, it follows from (2.4) that $(\lambda^s(\eta))^- = \lambda^s(\eta^-)$ has finite outcome along $\lambda^s(\eta)$. Thus by (2.4), $\lambda^{s+1}(\eta) = \lambda^{s+1}(\eta^-)$, so for all $i > s$,

$$\lambda^i(\eta) = \lambda^i(\lambda^{s+1}(\eta)) = \lambda^i(\lambda^{s+1}(\eta^-)) = \lambda^i(\lambda^s(\eta^-)) = \lambda^i((\lambda^s(\eta))^-),$$

and (iii) must also hold.

Suppose that $\eta$ is switching. By (i), let $\zeta^p = \mathrm{up}^p(\eta^-) = (\lambda^p(\eta))^-$, and let $\zeta^{p+1} = \mathrm{up}^{p+1}(\eta^-)$. We first show that $\eta$ is $(p+1)$-switching, and that $\zeta^p$ has infinite outcome along $\lambda^p(\eta)$. For suppose that $\zeta^p$ has finite outcome along $\lambda^p(\eta)$ in order to obtain a contradiction. $\zeta^p$ cannot be an initial derivative of $\zeta^{p+1}$, else $p$ would have been chosen $\geqslant p+1$. But if $\zeta^p$ is not an initial derivative of $\zeta^{p+1}$, then as $\zeta^p$ has finite outcome along $\lambda^p(\eta)$, $\lambda^{p+1}(\eta)=\lambda(\zeta^p)=\lambda(\lambda^p(\eta^-))$ and so $\eta$ is nonswitching, contrary to assumption. We conclude that $\zeta^p$ has infinite outcome along $\lambda^p(\eta)$, and so by (2.4) and (i), that $\lambda^{p+1}(\eta)=\zeta^{p+1}{}^\wedge\langle\lambda^p(\eta)\rangle$. Thus (ii) holds for $i=p+1$. Fix the least $s\in(p,n)$, if any, such that the conditions of (ii) fail for $i=s+1$; otherwise, let $s=n$. (ii) now follows. (iii) holds vacuously for $s=n$, so assume that $s<n$.

As $s+1>p+1$, we note that $\mathrm{up}^s(\eta^-)=(\lambda^s(\eta))^-$ cannot have infinite outcome along $\lambda^s(\eta)$, else $\mathrm{up}^s(\eta^-)$ would be the principal derivative of $\mathrm{up}^{s+1}(\eta^-)$ along $\lambda^s(\eta)$, so by (2.4), the condition specified in (ii) would hold for $i=s+1$. For the same reason, $\mathrm{up}^s(\eta^-)=(\lambda^s(\eta))^-$ cannot be the initial derivative of $\mathrm{up}^{s+1}(\eta^-)$ along $\lambda^{s+1}(\eta)$. Thus by (2.4), $\lambda^{s+1}(\eta)=\lambda(\lambda^s(\eta)^-)$, so $\lambda^i(\eta)=\lambda^i((\lambda^s(\eta))^-)$ for all $i\in(s,n]$. ▮

Definition 3.1. Fix $k\leqslant r\leqslant n$, $\xi\subseteq\eta\in T^k$ and $\Lambda\in[T^k]$. We say that $\xi$ *is* $(\eta,r)$*-true* if $\lambda^j(\xi)\subseteq\lambda^j(\eta)$ for all $j\in[k,r]$, and that $\xi$ *is* $(\Lambda,r)$*-true* if $\lambda^j(\xi)\subset\lambda^j(\Lambda)$ for all $j\in[k,r]$. $\xi$ *is* $\eta$*-true* if $\xi$ is $(\eta,n)$-true, and $\xi$ *is* $\Lambda$*-true* if $\xi$ is $(\Lambda,n)$-true. $\xi$ *is true* if $\xi$ is $\xi$-true.

The next lemma, which is an easy corollary of Lemma 3.1 (Limit Path), proves the existence of many true nodes.

Lemma 3.4 (True Node Lemma). *Fix* $k\leqslant r\leqslant n$ *and* $\eta\subseteq\Lambda^k\in[T^k]$. *Then*:

(i) *Every* $\zeta\subseteq\lambda^r(\eta)$ *is* $(\lambda^r(\eta),r)$*-true.*

(ii) *If* $\sigma$ *is* $(\lambda(\eta),r)$*-true, then the initial derivative of* $\sigma$ *along* $\eta$ *is* $(\eta,r)$*-true.*

(iii) *If* $\sigma$ *is* $(\lambda(\eta),r)$*-true,* $\xi\subseteq\eta$, *and* $\xi^-$ *is the principal derivative of* $\sigma^-$ *along* $\eta$, *then* $\xi$ *is* $(\eta,r)$*-true.*

*Proof.* (i) follows by definition. (ii) and (iii) follow from clauses (i) and (ii), respectively, of Lemma 3.1 (Limit Path). ▮

We now turn our attention to an analysis of the possible ways of extending paths. We first show that we can always take nonswitching extensions.

Lemma 3.5 (Nonswitching Extension Lemma). *Fix* $\nu\in T^k$. *Then either* $\nu^\wedge\langle\beta\rangle$ *is nonswitching for every* $\beta\in T^{k-1}$ *such that* $\nu^\wedge\langle\beta\rangle\in T^k$ *and* $\beta^-$ *has infinite outcome along* $\beta$, *or* $\nu^\wedge\langle\beta\rangle$ *is nonswitching for every* $\beta\in T^{k-1}$ *such that* $\nu^\wedge\langle\beta\rangle\in T^k$ *and* $\beta^-$ *has finite outcome along* $\beta$. *Moreover, if* $\nu^\wedge\langle\beta\rangle$ *is a nonswitching extension of* $\nu$, *then for all* $j\geqslant k$, *the* $\lambda^j(\nu)$*-links and the* $\lambda^j(\nu^\wedge\langle\beta\rangle)$*-links coincide.*

*Proof.* We proceed by induction on $n-k$. We note that no node on $T^n$ is switching, and that there are no links on $T^n$.

Suppose that $k<n$ and $\nu$ is not the initial derivative of $\mathrm{up}(\nu)$ along $\nu$. Fix $\beta \in T^{k-1}$ such that $\beta^-$ has finite outcome along $\beta$. By (2.4), $\lambda(\nu \frown \langle \beta \rangle) = \lambda(\nu)$. Thus

$$\lambda^j(\nu \frown \langle \beta \rangle) = \lambda^j(\lambda(\nu \frown \langle \beta \rangle)) = \lambda^j(\lambda(\nu)) = \lambda^j(\nu)$$

for all $j \in (k, n]$, so $\nu \frown \langle \beta \rangle$ is not switching.

Suppose that $k<n$ and $\nu$ is the initial derivative of $\mathrm{up}(\nu)$ along $\nu$. By Lemma 3.1(i) (Limit Path), $\mathrm{up}(\nu) = \lambda(\nu)$. By induction, either $\mathrm{up}(\nu) \frown \langle \xi \rangle$ is nonswitching for all $\xi \in T^k$ such that $\mathrm{up}(\nu) \frown \langle \xi \rangle \in T^{k+1}$ and $\xi^-$ has infinite outcome along $\xi$, or $\mathrm{up}(\nu) \frown \langle \xi \rangle$ is nonswitching for all $\xi \in T^k$ such that $\mathrm{up}(\nu) \frown \langle \xi \rangle \in T^{k+1}$ and $\xi^-$ has finite outcome along $\xi$. If $\dim(\nu) \leqslant k$, then $\lambda(\nu \frown \langle \beta \rangle) = \mathrm{up}(\nu) \frown \langle \nu \frown \langle \beta \rangle \rangle$ by (2.9), (2.4), and Lemma 3.1(ii) (Limit Path), so by Lemma 3.1(i) (Limit Path),

$$\begin{aligned}\lambda^j(\nu \frown \langle \beta \rangle) &= \lambda^j(\lambda(\nu \frown \langle \beta \rangle)) = \lambda^j(\mathrm{up}(\nu) \frown \langle \nu \frown \langle \beta \rangle \rangle) \\ &\supseteq \lambda^j(\mathrm{up}(\nu)) = \lambda^j(\lambda(\nu)) = \lambda^j(\nu) \end{aligned} \tag{3.1}$$

for all $j \in (k, n]$, and $\nu \frown \langle \beta \rangle$ is nonswitching. Otherwise, $\dim(\nu) > k$. If $\mathrm{up}(\nu) \frown \langle \xi \rangle$ is nonswitching for every $\xi \in T^k$ such that $\mathrm{up}(\nu) \frown \langle \xi \rangle \in T^{k+1}$ and $\xi^-$ has infinite (finite, resp.) outcome along $\xi$, fix $\beta \in T^{k-1}$ such that $\nu \frown \langle \beta \rangle \in T^k$ and $\beta^-$ has finite (infinite, resp.) outcome along $\beta$. By (2.4) and Lemma 3.1(i) (Limit Path), $\lambda(\nu \frown \langle \beta \rangle) = \mathrm{up}(\nu) \frown \langle \nu \frown \langle \beta \rangle \rangle$. Hence (3.1) holds for all $j \in (k, n]$, and $\nu \frown \langle \beta \rangle$ is nonswitching.

Fix $j \geqslant k$. As $\lambda^j(\nu \frown \langle \beta \rangle) \supseteq \lambda^j(\nu)$, any primary $\lambda^j(\nu)$-link is a primary $\lambda^j(\nu \frown \langle \beta \rangle)$-link. Fix $p$ and $s$ as in the proof of Lemma 3.3 ($\lambda$-Behavior). As $\nu \frown \langle \beta \rangle$ is a nonswitching extension of $\nu$, it follows from Lemma 3.3(ii) ($\lambda$-Behavior) that $p = s$. If $j \leqslant p$, then by Lemma 3.3(i) ($\lambda$-Behavior), $(\lambda^j(\nu \frown \langle \beta \rangle))^- = \lambda^j(\nu)$ and $\lambda^j(\nu)$ is the initial derivative of $\mathrm{up}(\lambda^j(\nu))$ along $\lambda^j(\nu \frown \langle \beta \rangle)$; and if $j > p$, then by Lemma 3.3(iii) ($\lambda$-Behavior), $\lambda^j(\nu \frown \langle \beta \rangle) = \lambda^j(\nu)$. In either case, any primary $\lambda^j(\nu \frown \langle \beta \rangle)$-link is a primary $\lambda^j(\nu)$-link. The lemma now follows from Step 2 of Definition 2.8. ∎

The next lemma, together with the Nonswitching Extension Lemma, allows us to analyze the computation of $\lambda$.

LEMMA 3.6 (Switching Lemma). *Fix $k \leqslant n$ and $\xi \in T^k$. Then either:*

(i) *$\xi \frown \langle \beta \rangle$ is switching for all $\beta \in T^{k-1}$ such that $\beta^-$ has finite outcome along $\beta$; or*

(ii) *$\xi^\frown\langle\beta\rangle$ is switching for all $\beta\in T^{k-1}$ such that $\beta^-$ has infinite outcome along $\beta$; or*

(iii) *$\xi$ is the initial derivative of* $\mathrm{up}^j(\xi)$ *along $\lambda^j(\xi)$ for all $j\in[k,n]$.*

*Proof.* We proceed by induction on $n-k$. (iii) holds if $k=n$. Suppose that $k<n$. Let $\mu$ be the initial derivative of $\mathrm{up}(\xi)$ along $\xi$. If $\mu\subset\xi$, then by (2.7) and (2.8), $\mathrm{up}(\xi)$ must have infinite outcome along $\lambda(\xi)$, so if $\beta$ is an infinite outcome for $\xi$, then $\lambda(\xi)\,|\,\lambda(\xi^\frown\langle\beta\rangle)$, so (ii) holds. Suppose that $\mu=\xi$. By Lemma 3.1(i) (Limit Path), $\mathrm{up}(\xi)=\lambda(\xi)$. If $\mathrm{up}(\xi)^\frown\langle\nu\rangle$ is switching for all $\nu\in T^k$ such that $\nu^-$ has finite outcome along $\nu$, then (ii) holds. If $\mathrm{up}(\xi)^\frown\langle\nu\rangle$ is switching for all $\nu\in T^k$ such that $\nu^-$ has infinite outcome along $\nu$, then (i) holds. Otherwise, by induction, $\mathrm{up}(\xi)$ is the initial derivative of $\mathrm{up}^j(\mathrm{up}(\xi))=\mathrm{up}^j(\xi)$ along $\lambda^j(\lambda(\xi))=\lambda^j(\xi)$ for all $j\in[k+1,n]$, so (iii) holds. ∎

Our next lemma shows that if $\Lambda^k\in[T^k]$, then there is a nice approximation to $\lambda^{k+2}(\Lambda^k)$ from $\Lambda^k$. This lemma will enable us to show that, under certain circumstances, nodes not along $\lambda^{k+2}(\Lambda^k)$ will not declare many axioms. The lemma is a standard infinite injury lemma, stating that for any node $\tau$ which is not along the true path, i.e., not along $\lambda^{k+2}(\Lambda^k)$, there will only be finitely many nodes $\alpha$ along $\Lambda^k$ which think that $\tau$ is along the true path, i.e., such that $\lambda^{k+2}(\alpha)\supseteq\tau$. The machinery which we develop does not require us to look, locally, beyond the interaction of nodes from three consecutive levels. This is also the case with Harrington's approach to $\mathbf{0}^{(n)}$-priority arguments, which uses the Recursion Theorem.

LEMMA 3.7 (Infinite Injury Lemma). *Fix $k\leqslant n-2$ and $\Lambda^k\in[T^k]$. Let $\Lambda^{k+1}=\lambda(\Lambda^k)$ and $\Lambda^{k+2}=\lambda(\Lambda^{k+1})$. Fix $\tau\in T^{k+2}$ such that $\tau\not\subset\Lambda^{k+2}$. Then $\{\alpha\subset\Lambda^k\colon\lambda^{k+2}(\alpha)\supseteq\tau\}$ is finite.*

*Proof.* Let $\rho=\Lambda^{k+2}\wedge\tau$. As $\tau\not\subset\Lambda^{k+2}$, $\rho\subset\tau$. Fix $\xi\in T^{k+1}$ such that $\rho^\frown\langle\xi\rangle\subseteq\tau$. By (2.4), if $\alpha\subset\Lambda^k$ and $\lambda^{k+2}(\alpha)\supseteq\tau$, then $\lambda(\alpha)\supseteq\xi$. If $\xi\not\subset\Lambda^{k+1}$, then by Lemma 3.1 (Limit Path), $\{\alpha\subset\Lambda^k\colon\lambda(\alpha)\supseteq\xi\}$ is finite, and the lemma follows.

Suppose that $\xi\subset\Lambda^{k+1}$. Fix $\eta\neq\xi$ such that $\rho^\frown\langle\eta\rangle\subset\Lambda^{k+2}$. By (2.4), $\eta\subset\Lambda^{k+1}$, so $\xi$ and $\eta$ are comparable. $\dim(\rho)>k+1$, else $\xi=\eta$ by (2.9), a contradiction. $\xi^-$ cannot have infinite outcome along $\xi$, else by (2.8), $\eta\subseteq\xi$, so by (2.4) and as $\xi\subset\Lambda^{k+1}$, $\rho^\frown\langle\xi\rangle\subseteq\Lambda^{k+2}$. Hence by (2.4), $\xi\subset\eta$, $\xi^-$ has finite outcome along $\Lambda^{k+1}$ and $\eta^-$ has infinite outcome along $\Lambda^{k+1}$. By Lemma 3.1 (Limit Path), we can fix $\beta\subset\Lambda^k$ such that for all $\gamma$, if $\beta\subseteq\gamma\subset\Lambda^k$ then $\lambda(\gamma)\supseteq\eta$. But then for all such $\gamma$, if $\lambda^{k+2}(\gamma)\supseteq\rho$, then $\lambda^{k+2}(\gamma)\supseteq\rho^\frown\langle\eta\rangle$. ∎

## 4. Links

In this section, we analyze the effect of link formation on the path generation process. We show that links must be nested, so that the process of removing links by switching paths must be done in an orderly way, no matter how the path is extended. We relate the restraint of $\xi \in T^k$ by an $\eta$-link to the satisfaction of whether or not the antiderivatives of $\xi$ lie along the paths computed by $\eta$. If $\eta$ switches $\xi$, then by (2.10), $\xi$ will be $\eta^-$-free; we show that $\xi$ is also $\eta$-free. And we show that if $\mathrm{up}^j(\xi)$ is $\Lambda^j$-free for all $j \geqslant k$ and $k \leqslant \dim(\xi)$, then $\xi$ has sufficiently many $\Lambda^{k-1}$-free derivatives.

We will need to show that we can return a node to the true path by taking switching extensions which change the outcomes at the ends of links. We will need to determine which links must be switched in this way. This determination depends on the fact that $\eta$-links on $T^k$ are either nested or disjoint.

Lemma 4.1 (Nesting Lemma). *Fix $k \leqslant n$ and $\eta \in T^k$. Suppose that, for $i \leqslant 1$, $[\mu_i, \pi_i]$ is an $\eta$-link and that $\pi_0 \subset \pi_1$. Then $\mu_1 \subseteq \mu_0$ or $\pi_0 \subset \mu_1$.*

*Proof.* We proceed by induction on $n-k$. We note that there are no links on $T^n$, so the lemma holds trivially for $k=n$. Assume that the lemma is true for $k+1$ in place of $k$.

For each $i \leqslant 1$, fix the $\lambda(\eta)$-link $[\rho_i, \tau_i]$ from which $[\mu_i, \pi_i]$ is derived if such a link exists; otherwise, $[\mu_i, \pi_i]$ is a primary $\eta$-link, and we set $\rho_i = \tau_i = \mathrm{up}(\mu_i) = \mathrm{up}(\pi_i)$. It follows from the definition of links that for $i \leqslant 1$, $\mu_i$ is the initial derivative of $\rho_i$ along $\eta$ and $\pi_i$ is the principal derivative of $\tau_i$ along $\eta$. As $\pi_0 \subset \pi_1$, it follows from Lemma 3.1(ii) (Limit Path) that $\tau_0 \subset \tau_1$. If $\tau_0 \subset \rho_1$, then by Lemma 3.1(i, ii) (Limit Path), $\mu_0 \subset \pi_0 \subset \mu_1 \subset \pi_1$ and the lemma holds. Otherwise, $\rho_1 \subseteq \tau_0 \subset \tau_1$. If $\rho_0 = \tau_0$, then $\rho_1 \subseteq \rho_0$. And if $\rho_0 \neq \tau_0$, then by induction, $\rho_1 \subseteq \rho_0$. So in either case, it follows from Lemma 3.1(i) (Limit Path), that $\mu_1 \subseteq \mu_0$. ▮

Our next definition traces links back to higher trees. Again, a node is *free* if it is not restrained by a link. We will show that all free nodes have all their antiderivatives along the computed paths. The converse of this statement is not true.

Definition 4.1. Fix $k \leqslant r \leqslant n$ and $\xi \subset \eta \in T^k$. Let $[\mu, \pi]$ be an $\eta$-link. We say that $[\mu, \pi]$ is an $(\eta, r)$-*link* if $[\mu, \pi]$ is derived from a primary $\lambda^r(\eta)$-link. $[\mu, \pi]$ is an $(\eta, \leqslant_r)$-*link* if $[\mu, \pi]$ is an $(\eta, j)$-link for some $j \in [k, r]$. We say that $\xi$ *is* $(\eta, r)$-*restrained* if there is an $(\eta, r)$-link $[\mu, \pi]$ which $\eta$-restrains $\xi$. In this case, we say that $\xi$ *is* $(\eta, r)$-*restrained by* $[\mu, \pi]$. We say that $\xi$ is $(\eta, r)$-*free* if $\xi$ is not $(\eta, r)$-restrained. We say that $\xi$ is $r$-*free* if $\xi$ is $(\xi, r)$-free. If $\Lambda^k \in [T^k]$, then we say that $\xi$ *is* $(\Lambda^k, r)$-*free* if $\xi$

is $(\eta, r)$-free for all $(\Lambda^k, r)$-true $\eta \supseteq \xi$, and that $\xi$ *is* $\Lambda^k$*-free* if $\xi$ is $(\Lambda^k, n)$-free.

Recall that if $[\mu, \pi]$ is an $\eta$-link, then $\pi$ is *not* restrained by $[\mu, \pi]$. However, as we can have $[\mu, \pi] = [\mu, \delta)$ with $\pi \neq \delta$ for intervals, we used closed interval notation $[\mu, \pi]$ for $\eta$-links to make sure that there is a one-one correspondence between intervals which determine links, and the links themselves.

The next lemma identifies the outcome of a link with the actual outcome of the node ending the link.

LEMMA 4.2 (Faithful Outcome Lemma). *Fix $\mu \subset \pi \subset \eta \in T^k$ such that $[\mu, \pi]$ is an $\eta$-link. Then $[\mu, \pi]$ has finite outcome iff $\pi$ has finite outcome along $\eta$.*

*Proof.* We proceed by induction on $n - k$. As there are no links on $T^n$, the lemma holds for $k = n$. Assume that $k < n$.

If $[\mu, \pi]$ is a primary $\eta$-link, then $\pi$ has infinite outcome along $\eta$, and by Step 2 of Definition 2.8, $[\mu, \pi]$ has infinite outcome. Otherwise, $[\mu, \pi]$ is derived from some $\lambda(\eta)$-link $[\rho, \tau]$. By Definition 2.1, induction, and Step 2 of Definition 2.8,

> $\pi$ has finite outcome along $\eta$ iff $\mathrm{up}(\pi) = \tau$ has infinite outcome along $\lambda(\eta)$ iff $[\rho, \tau]$ has infinite outcome iff $[\mu, \pi]$ has finite outcome. ∎

The next lemma relates the presence of antiderivatives of a node $\xi$ on a path computed by $\eta$ to the restraint of $\xi$ by an $\eta$-link. It will follow from this lemma that (2.10) implies (2.7).

LEMMA 4.3 (Link Analysis Lemma). *Fix $k \leqslant r \leqslant n$ and $\xi \subseteq \eta \in T^k$. Then:*

(i) (a) *If $\mathrm{up}(\xi) \nsubseteq \lambda(\eta)$, then there is a primary $\eta$-link $[\mu, \pi]$ such that $\mathrm{up}(\mu) \neq \mathrm{up}(\xi)$ and $[\mu, \pi]$ $\eta$-restrains all $\delta \subseteq \eta$ such that $\mathrm{up}(\delta) = \mathrm{up}(\xi)$.*

(b) *If $\xi \subset \delta \subset \eta$, $\mathrm{up}(\xi) \subseteq \mathrm{up}(\delta)$ and $\mathrm{up}(\xi) \nsubseteq \lambda(\eta)$, then there is a primary $\eta$-link $[\mu, \pi]$ which $\eta$-restrains both $\xi$ and $\delta$.*

(c) *If $[\tilde{\mu}, \tilde{\pi}]$ is an $\eta$-link which $\eta$-restrains the principal derivative $\xi$ of $\mathrm{up}(\xi)$, then either there is a primary $\eta$-link $[\mu, \pi]$ which $\eta$-restrains $\xi$ and $\mathrm{up}(\xi) \nsubseteq \lambda(\eta)$, or $[\tilde{\mu}, \tilde{\pi}]$ is derived from a $\lambda(\eta)$-link which $\lambda(\eta)$-restrains $\mathrm{up}(\xi)$.*

(d) *If there is a primary $\eta$-link $[\mu, \pi]$ which $\eta$-restrains the initial derivative of $\mathrm{up}(\xi)$, then either $\mathrm{up}(\xi) \nsubseteq \lambda(\eta)$ or $\mu$ is the initial derivative of $\mathrm{up}(\xi)$ along $\eta$.*

(ii) *If* $\mathrm{up}(\xi) \subseteq \lambda(\eta)$ *and* $\mathrm{up}(\xi)$ *is* $(\lambda(\eta), r)$*-restrained by* $[\rho, \tau]$*, then* $\xi$ *is* $(\eta, r)$*-restrained by an* $\eta$*-link* $[\mu, \pi]$ *derived from* $[\rho, \tau]$.

(iii) *Suppose that* $j \leqslant r$ *and* $\mathrm{up}^j(\xi)$ *is* $(\lambda^j(\eta), r)$*-restrained by* $[\rho, \tau]$*. Then* $\xi$ *is* $(\eta, \leqslant_r)$*-restrained. Furthermore, if* $\mathrm{up}^i(\xi) \subseteq \lambda^i(\eta)$ *for all* $i \in [k, j]$*, then* $\xi$ *is* $(\eta, r)$*-restrained by an* $\eta$*-link derived from* $[\rho, \tau]$.

*Proof.* (ia) Let $\kappa = \mathrm{up}(\xi) \wedge \lambda(\eta)$ and let $\delta$ be any derivative of $\mathrm{up}(\xi)$ along $\eta$. As $\mathrm{up}(\xi) \nsubseteq \lambda(\eta)$, $\kappa \subset \mathrm{up}(\xi)$. Hence by Lemma 3.1(i, ii) (Limit Path), $\kappa$ has an initial derivative $\mu \subset \delta$ and a principal derivative $\pi \subset \eta$ such that $\mu \subset \pi$. By (2.4) and as $\kappa \subset \mathrm{up}(\xi)$, $\pi \nsubseteq \delta$. Hence $\delta$ is $\eta$-restrained by the primary $\eta$-link $[\mu, \pi]$. As $\mathrm{up}(\mu) = \kappa \subseteq \lambda(\eta)$, $\mathrm{up}(\mu) \neq \mathrm{up}(\xi)$.

(ib) As $\mathrm{up}(\xi) \nsubseteq \lambda(\eta)$, it follows from the proof of (ia) that there is a primary $\eta$-link $[\mu, \pi]$ which restrains $\xi$, and that $\mathrm{up}(\mu) = \mathrm{up}(\pi) \subset \mathrm{up}(\xi)$. Fix $\rho \subseteq \eta$ such that $\rho^- = \pi$. By (2.4), $\mathrm{up}(\mu)$ must have finite outcome along $\lambda(\rho)$, but infinite outcome along $\mathrm{up}(\xi) \subseteq \mathrm{up}(\delta)$. If $\pi \subset \delta$, then by (2.4), $\delta$ cannot lie along $\delta$, contradicting (2.7). $\pi \neq \delta$ as $\mathrm{up}(\pi) \neq \mathrm{up}(\delta)$. But $\pi, \delta \subset \eta$, so $\pi \supset \delta$.

(ic) Suppose that $[\mu, \pi]$ is a primary $\eta$-link which $\eta$-restrains the principal derivative of $\mathrm{up}(\xi)$ along $\eta$. We assume that $\mathrm{up}(\xi) \subseteq \lambda(\eta)$ and derive a contradiction. We compare the relative locations of $\mathrm{up}(\xi)$ and $\mathrm{up}(\mu) = \mathrm{up}(\pi)$ on $T^{k+1}$.

First suppose that $\mathrm{up}(\mu) \subset \mathrm{up}(\xi)$. Fix $\beta \in T^k$ such that $\mathrm{up}(\mu)^\wedge\langle\beta\rangle \subseteq \mathrm{up}(\xi) \subseteq \lambda(\eta)$. By (2.5), $\beta \subseteq \xi$. By (2.4), $\beta^-$ is the principal derivative of $\mathrm{up}(\mu)$ along $\eta$, so $\beta^- = \pi$. But then $\pi \subset \xi$ so $[\mu, \pi]$ cannot $\eta$-restrain $\xi$.

Suppose that $\mathrm{up}(\mu) = \mathrm{up}(\xi)$. As $\xi$ and $\pi$ are, respectively, the principal derivatives of $\mathrm{up}(\xi)$ and $\mathrm{up}(\pi)$ along $\eta$ and $\mathrm{up}(\mu) = \mathrm{up}(\pi)$, $\xi = \pi$. So $[\mu, \pi]$ cannot $\eta$-restrain $\xi$.

Suppose that $\mathrm{up}(\xi) \subset \mathrm{up}(\mu)$. Fix $\beta \in T^k$ such that $\mathrm{up}(\xi)^\wedge\langle\beta\rangle \subseteq \mathrm{up}(\mu)$. By (2.5), $\beta \subseteq \mu$. By (2.4), $\beta^-$ is the principal derivative of $\mathrm{up}(\xi)$ along $\pi \subseteq \eta$; as $\xi \subset \pi$ and $\xi$ is the principal derivative of $\mathrm{up}(\xi)$ along $\eta$, it follows from (2.4) that $\beta^- = \xi$. But then $\xi \subset \mu$ so $[\mu, \pi]$ cannot $\eta$-restrain $\xi$.

Suppose that $\mathrm{up}(\xi) | \mathrm{up}(\mu)$. Let $\tau = \mathrm{up}(\xi) \wedge \mathrm{up}(\mu)$, and fix $\alpha$, $\beta$ such that $\tau^\wedge\langle\alpha\rangle \subseteq \mathrm{up}(\xi)$ and $\tau^\wedge\langle\beta\rangle \subseteq \mathrm{up}(\mu)$. As $\xi$ is $\eta$-restrained by the primary $\eta$-link $[\mu, \pi]$ and $\mathrm{up}(\xi) \neq \mathrm{up}(\mu) = \mathrm{up}(\pi)$, $\mu \subset \xi \subset \pi$. By (2.7), $\tau^\wedge\langle\alpha\rangle \subseteq \lambda(\xi)$ and $\tau^\wedge\langle\beta\rangle \subseteq \lambda(\mu), \lambda(\pi)$, contradicting (2.6). Thus $[\mu, \pi]$ cannot $\eta$-restrain $\xi$.

Now suppose that $[\tilde{\mu}, \tilde{\pi}]$ is an $\eta$-link which $\eta$-restrains the principal derivative $\xi$ of $\mathrm{up}(\xi)$, but that there is no primary $\eta$-link which $\eta$-restrains $\xi$. By (ia), $\mathrm{up}(\xi) \subseteq \lambda(\eta)$, so $\mathrm{up}(\xi)$, $\mathrm{up}(\tilde{\mu})$, and $\mathrm{up}(\tilde{\pi})$ are all comparable, and $[\mathrm{up}(\tilde{\mu}), \mathrm{up}(\tilde{\pi})]$ is a $\lambda(\eta)$-link. As $[\tilde{\mu}, \tilde{\pi}]$ $\eta$-restrains $\xi$, $\tilde{\mu} \subseteq \xi \subset \tilde{\pi}$. As $\xi$ is the principal derivative of $\mathrm{up}(\xi)$ along $\eta$, it follows from Lemma 3.1(ii) (Limit Path) that $\mathrm{up}(\tilde{\mu}) \subseteq \mathrm{up}(\xi)$. And as $\tilde{\pi}$ is the principal derivative of $\mathrm{up}(\tilde{\pi})$

along $\eta$, it follows from Lemma 3.1(ii) (Limit Path) that $\mathrm{up}(\xi) \subset \mathrm{up}(\tilde{\pi})$. Hence $[\mathrm{up}(\tilde{\mu}), \mathrm{up}(\tilde{\pi})]$ $\lambda(\eta)$-restrains $\mathrm{up}(\xi)$.

(id) Suppose that $[\mu, \pi]$ is a primary $\eta$-link which $\eta$-restrains the initial derivative $\nu$ of $\mathrm{up}(\xi)$, and that $\mathrm{up}(\xi) \subseteq \lambda(\eta)$. We assume that $\mu \neq \nu$ and derive a contradiction. By Lemma 4.1 (Nesting), there can be no primary $\nu$-link which $\nu$-restrains $\mu$, so by (ia), $\mathrm{up}(\mu) \subseteq \lambda(\nu)$. By (2.7), $\mathrm{up}(\nu) \subseteq \lambda(\nu)$, so $\mathrm{up}(\mu)$ and $\mathrm{up}(\nu)$ are comparable. By Lemma 3.1(i) (Limit Path), $\mathrm{up}(\mu) \subseteq \mathrm{up}(\nu)$. As $[\mu, \pi]$ is a primary $\eta$-link and $\nu \subset \pi$, all derivatives of $\mathrm{up}(\mu)$ which are $\subset \nu$ must have finite outcome along $\nu$, but $\pi$ has infinite outcome along $\eta$. As $\mu \neq \nu$, it follows that $\mathrm{up}(\mu)$ has infinite outcome along $\lambda(\nu)$, but finite outcome along $\lambda(\eta)$. But $\mathrm{up}(\mu) \subset \mathrm{up}(\nu) = \mathrm{up}(\xi) \subseteq \lambda(\eta)$, so by (2.7), $\mathrm{up}(\nu) \subseteq \lambda(\nu), \lambda(\eta)$. Thus $\mathrm{up}(\mu)$ must have the same outcome along both $\lambda(\nu)$ and $\lambda(\eta)$, yielding a contradiction.

(ii) Suppose that $[\rho, \tau]$ is a $\lambda(\eta)$-link which restrains $\mathrm{up}(\xi)$ and is derived from the primary $\lambda^r(\eta)$-link $[\kappa, \zeta]$. By Lemma 3.1 (Limit Path), let $\mu$ ($\pi$, resp.) be the initial (principal, resp.) derivative of $\rho$ ($\tau$, resp.) along $\eta$. As $\mathrm{up}(\xi) \in [\rho, \tau)$, it follows from (2.10) and Lemma 3.1(i, ii) (Limit Path) that $\xi \in [\mu, \pi)$, so $\xi$ is $\eta$-restrained by $[\mu, \pi]$ which is derived from $[\kappa, \zeta]$.

(iii) Immediate by induction from (i) and (ii). ∎

We now show that free nodes lie along the computed path, so (2.10) implies (2.7).

Lemma 4.4 (Free Implies True Path Lemma). *Fix $k \leqslant r \leqslant n$ and $\xi \subset \eta \subseteq \Lambda^k \in [T^k]$ such that $\xi$ is $(\eta, r)$-free. Then for all $j \in [k, r]$, $\mathrm{up}^j(\xi) \subseteq \lambda^j(\eta)$.*

*Proof.* Immediate from Lemma 4.3(ia, iii) (Link Analysis). ∎

When a requirement is assigned to $\delta \in T^0$, then by (2.10) and the process of pulling links down from tree to tree, $\mathrm{up}^i(\delta)$ is $\lambda^i(\delta)$-free for all $i \leqslant n$. If $\eta^- = \delta$ and $i \leqslant n$, i.e., $\eta$ determines an outcome for $\delta$, then $\eta$ may or may not switch $\mathrm{up}^i(\delta)$. We show that in either case, no $\lambda^i(\eta)$-link restrains $\mathrm{up}^i(\delta)$ for any $i \leqslant n$. In fact, we show that this happens not only for $\delta \in T^0$, but also for all $k \leqslant n$, $\delta \in T^k$ and all $i \in [k, n]$.

Lemma 4.5 (Free Extension Lemma). *Fix $k \leqslant n$ and $\eta \in T^k$. Then for all $i \in [k, n]$, $\mathrm{up}^i(\eta^-)$ is $\lambda^i(\eta)$-free. Furthermore, if $r \in [k+1, n]$ and $\eta$ is $r$-switching, then for all $i \in [r, n]$, $\lambda^i(\eta^-) \mid \lambda^i(\eta)$, $\mathrm{up}^i(\eta^-) = \lambda^i(\eta^-) \wedge \lambda^i(\eta)$, and $\mathrm{up}^i(\eta^-)$ has finite outcome along $\lambda^i(\eta)$ iff $\mathrm{up}^i(\eta^-)$ has infinite outcome along $\lambda^i(\eta^-)$.*

*Proof.* Fix $p$ and $s$ as in Lemma 3.3 ($\lambda$-Behavior). First suppose that $\eta$ is nonswitching. Then by Lemma 3.3 ($\lambda$-Behavior), $p=s$, $\text{up}^i(\eta^-)=(\lambda^i(\eta))^-=\lambda^i(\eta^-)$ if $i\in[k,s]$, and $\lambda^i(\eta)=\lambda^i(\text{up}^s(\eta^-))=\lambda^i(\lambda^s(\eta^-))=\lambda^i(\eta^-)$ if $i\in[s+1,n]$. It follows from Lemma 3.5 (Nonswitching Extension) that for all $i\in[k,n]$, the primary $\lambda^i(\eta)$-links and the primary $\lambda^i(\eta^-)$-links coincide. By (2.10), for all $i\in[k,n]$, $\text{up}^i(\eta^-)$ is $\lambda^i(\eta^-)$-free. Hence for all $i\in[k,n]$, $\text{up}^i(\eta^-)$ must be $\lambda^i(\eta)$-free.

Now suppose that $\eta$ is switching. We first show that for all $i\in[k,n]$, $\text{up}^i(\eta^-)\subseteq\lambda^i(\eta)$. If $i\in[k,p]$, then by Lemma 3.3 ($\lambda$-Behavior), $\text{up}^i(\eta^-)=(\lambda^i(\eta))^-=\lambda^i(\eta^-)$; so as, by (2.7), $\text{up}^i(\eta^-)\subseteq\lambda^i(\eta^-)$, it must be the case that $\text{up}^i(\eta^-)\subseteq\lambda^i(\eta)$. If $i\in[p+1,s]$, then by Lemma 3.3 ($\lambda$-Behavior), $\lambda^i(\eta^-)\,|\,\lambda^i(\eta)$ and $\text{up}^i(\eta^-)=\lambda^i(\eta^-)\wedge\lambda^i(\eta)$; so $\text{up}^i(\eta^-)\subseteq\lambda^i(\eta)$. If $i\in[s+1,n]$, then by Lemma 3.3 ($\lambda$-Behavior), $\lambda^i(\eta)=\lambda^i(\text{up}^s(\eta^-))$. By (2.7), $\text{up}^i(\text{up}^s(\eta^-))\subseteq\lambda^i(\text{up}^s(\eta^-))$. Hence $\text{up}^i(\eta^-)=\text{up}^i(\text{up}^s(\eta^-))\subseteq\lambda^i(\eta)$.

We next show that for all $i\in[k,n]$, $\text{up}^i(\eta^-)$ is both $\lambda^i(\eta^-)$-free and $\lambda^i(\eta)$-free. By (2.10), $\text{up}^i(\eta^-)$ is $\lambda^i(\eta^-)$-free. Suppose that $\text{up}^i(\eta^-)$ is not $\lambda^i(\eta)$-free for some $i$, which we fix in order to obtain a contradiction. Then there is a $\lambda^i(\eta)$-link which restrains $\text{up}^i(\eta^-)$. We note that for any $m\in[k,n]$ and any $\lambda^m(\eta)$-link $[\mu^m,\pi^m]$, $\pi^m\subset\lambda^m(\eta)$ and $\pi^m$ is not $\lambda^m(\eta)$-restrained by $[\mu^m,\pi^m]$. By Lemma 3.3(i, ii) ($\lambda$-Behavior), if $i\leqslant s$ then $\text{up}^i(\eta^-)=(\lambda^i(\eta))^-$. Hence it must be the case that $i>s$. But then by Lemma 3.3 ($\lambda$-Behavior), $\lambda^i(\eta)=\lambda^i(\text{up}^s(\eta^-))$, and by (2.10), $\text{up}^i(\text{up}^s(\eta^-))$ is $\lambda^i(\text{up}^s(\eta^-))$-free. Hence $\text{up}^i(\eta^-)=\text{up}^i(\text{up}^s(\eta^-))$ is $\lambda^i(\eta)$-free, yielding the desired contradiction.

Finally, we must show that for all $i\in[p+1,n]$, $\lambda^i(\eta^-)\,|\,\lambda^i(\eta)$, $\text{up}^i(\eta^-)=\lambda^i(\eta^-)\wedge\lambda^i(\eta)$, and $\text{up}^i(\eta^-)$ has finite outcome along $\lambda^i(\eta)$ iff $\text{up}^i(\eta^-)$ has infinite outcome along $\lambda^i(\eta^-)$. This follows from Lemma 3.3(ii) ($\lambda$-Behavior) for $i\in[p+1,s]$. We now proceed by induction on $i\in[s+1,n]$. By (2.7) and as $\text{up}^i(\eta^-)\subseteq\lambda^i(\eta)$, $\text{up}^i(\eta^-)\subseteq\lambda^i(\eta^-)\wedge\lambda^i(\eta)$. Suppose that $\text{up}^i(\eta^-)\subset\lambda^i(\eta^-)\wedge\lambda^i(\eta)$ in order to obtain a contradiction. Fix $\rho^i\subseteq\lambda^i(\eta^-)\wedge\lambda^i(\eta)$ such that $(\rho^i)^-=\text{up}^i(\eta^-)$. Then by (2.4), $(\text{out}(\rho^i))^-$ is the principal derivative of $\text{up}^i(\eta^-)$ along $\text{out}(\rho^i)\subseteq\lambda^{i-1}(\eta^-)\wedge\lambda^{i-1}(\eta)$. Now $(\text{out}(\rho^i))^-\subset\text{up}^{i-1}(\eta^-)$, else either $\lambda^{i-1}(\eta^-)$ and $\lambda^{i-1}(\eta)$ are comparable, or $\text{up}^{i-1}(\eta^-)\neq\lambda^{i-1}(\eta^-)\wedge\lambda^{i-1}(\eta)$. Hence by (2.8), $(\text{out}(\rho^i))^-$ is the initial derivative of $\text{up}^i(\eta^-)$ along both $\lambda^{i-1}(\eta^-)$ and $\lambda^{i-1}(\eta)$, and $(\rho^i)^-$ has infinite outcome along both $\lambda^i(\eta^-)$ and $\lambda^i(\eta)$. But then all derivatives of $\text{up}^i(\eta^-)$ must have finite outcome along both $\lambda^{i-1}(\eta^-)$ and $\lambda^{i-1}(\eta)$. As $\text{up}^{i-1}(\eta^-)$ is such a derivative, we have contradicted our induction hypothesis.

Suppose that $\text{up}^i(\eta^-)$ has finite outcome along $\lambda^i(\eta)$ ($\lambda^i(\eta^-)$, resp.). Then there is a derivative $\nu^{i-1}$ of $\text{up}^i(\eta^-)$ which has infinite outcome along $\lambda^{i-1}(\eta)$ ($\lambda^{i-1}(\eta^-)$, resp.). Since $\text{up}^{i-1}(\eta^-)$ is $\lambda^{i-1}(\eta)$-free ($\lambda^{i-1}(\eta^-)$-free, resp.), $\nu^{i-1} = \mathrm{up}^{i-1}(\eta^-)$. By induction, $\mathrm{up}^{i-1}(\eta^-)$ has finite outcome along $\lambda^{i-1}(\eta^-)$ ($\lambda^{i-1}(\eta)$, resp.). As $\mathrm{up}^{i-1}(\eta^-)$ is $\lambda^{i-1}(\eta^-)$-free ($\lambda^{i-1}(\eta)$-free, resp.), all derivatives of $\mathrm{up}^i(\eta^-)$ along $\lambda^{i-1}(\eta^-)$ ($\lambda^{i-1}(\eta)$, resp.) must have finite outcome along $\lambda^{i-1}(\eta^-)$ ($\lambda^{i-1}(\eta)$, resp.). Hence by (2.4), $\mathrm{up}^i(\eta^-)$ has infinite outcome along $\lambda^i(\eta^-)$ ($\lambda^i(\eta)$, resp.). Now $\mathrm{up}^i(\eta^-)$ cannot have infinite outcome along both $\lambda^i(\eta^-)$ and $\lambda^i(\eta)$, else by (2.4), $\mathrm{up}^{i-1}(\eta^-)$ would have finite outcome along both $\lambda^{i-1}(\eta^-)$ and $\lambda^{i-1}(\eta)$ contrary to our induction assumption. Hence $\mathrm{up}^i(\eta^-)$ has finite outcome along $\lambda^i(\eta)$ iff $\mathrm{up}^i(\eta^-)$ has infinite outcome along $\lambda^i(\eta^-)$. ∎

The nodes which are $\Lambda^k$-free are the nodes which have the responsibility to determine the value of most of the axioms. However, we will be unable to recursively identify these nodes, and so, will be unable to prevent other nodes from defining a large number of axioms. We will have to show that the nodes to which we want to assign responsibility for defining most of the axioms can automatically transfer this responsibility to their derivatives. In order for this transfer to occur, we will need to show that principal derivatives of free nodes are free, and that if a free node has infinite outcome along the true path $\Lambda^{k+1}$ through $T^{k+1}$, then it has infinity many free derivatives along the true path $\Lambda^k$ through $T^k$. We show this in our next lemma. We will later show that these nodes also have the opportunity to correct many axioms defined by other nodes. We first note an important fact, whose proof we leave to the reader.

(4.1) Fix $k \leqslant r \leqslant n$, $\beta \subseteq \Lambda^k \in [T^k]$, and $\mu \subset \nu \subset \eta$ such that $\eta$ is $(\beta, r)$-true. Then $[\mu, \nu]$ is a $(\beta, r)$-link iff $[\mu, \nu]$ is an $(\eta, r)$-link.

LEMMA 4.6 (Free Derivative Lemma). *Fix $k < n$ and $\Lambda^k \in [T^k]$. For all $r \in [k, n]$, let $\Lambda^r = \lambda^r(\Lambda^k)$. Suppose that $\sigma \subset \Lambda^{k+1}$ is $\Lambda^{k+1}$-free. Then*:

(i) *If $\delta \subset \Lambda^k$ is the principal derivative of $\sigma$ along $\Lambda^k$, then $\delta$ is $\Lambda^k$-free.*

(ii) *If $\sigma$ has infinite outcome along $\Lambda^{k+1}$ and $\dim(\sigma) > k$, then there are infinitely many $\Lambda^k$-free derivatives of $\sigma$.*

*Proof.* (i) By Lemma 3.1(ii) (Limit Path), fix the principal derivative $\delta$ of $\sigma$ along $\Lambda^k$. Suppose that $\delta$ is $\Lambda^k$-restrained by a $\Lambda^k$-link $[\mu, \pi]$ in order to obtain a contradiction. By repeated applications of Lemma 3.1(i) (Limit Path) we may fix $\eta \subset \Lambda^k$ such that $\pi \subset \eta$ and $\eta$ is $\Lambda^k$-true. By (4.1), $[\mu, \pi]$ is an $\eta$-link. Without loss of generality, we may assume that $[\mu, \pi]$ is a primary $\eta$-link. (Else by Lemma 4.3(ic) (Link Analysis), $\sigma$ would be $\lambda(\eta)$-restrained by a $\lambda(\eta)$-link. But then $\lambda(\eta)$ is $\Lambda^{k+1}$-true, so by (4.1), $\sigma$

would not be $\Lambda^{k+1}$-free, a contradiction.) Let $\mathrm{up}(\mu) = \mathrm{up}(\pi) = \tau$. By (2.7) and assumption, $\sigma \subseteq \lambda(\delta), \lambda(\eta)$, so by (2.6), $\sigma \subseteq \lambda(\pi)$. By (2.7), $\tau \subseteq \lambda(\pi)$; hence $\sigma$ and $\tau$ are comparable. By Lemma 3.1(ii) (Limit Path) and as $\delta \subset \pi$, it must be the case that $\sigma \subset \tau$. $\delta$ cannot be the initial derivative of $\sigma$ along $\delta$, else by Lemma 3.1(i) (Limit Path) and as $\mu \subseteq \delta$, $\tau \subseteq \sigma$, yielding a contradiction. Let $\nu$ be the initial derivative of $\sigma$ along $\eta$. By Lemma 3.1(i) (Limit Path), $\nu \subset \mu \subset \delta$, and we have already shown that $\delta \subset \pi$. But $[\nu, \delta]$ and $[\mu, \pi]$ are $\eta$-links, contradicting Lemma 4.1 (Nesting).

(ii) We note by (i) that if $\zeta \subset \Lambda^n$, then for all $j$ such that $k \leqslant j \leqslant n$, the principal derivative $\zeta^j$ of $\zeta$ along $\Lambda^j$ is $\Lambda^j$-free. As $\mathrm{lh}(\Lambda^k) = \infty$, it follows inductively from Lemma 3.1(iv) (Limit Path) that $\mathrm{lh}(\Lambda^j) = \infty$ for all $j$ such that $k \leqslant j \leqslant n$. Hence there are infinitely many $\zeta \subset \Lambda^n$ such that $\zeta^j$ extends $\mathrm{up}^j(\sigma)$ for all $j$ such that $k \leqslant j \leqslant n$. Fix such a node $\xi$. It suffices to show that $\sigma$ has a free derivative along $\Lambda^k$ which extends $\zeta^k$.

By (2.4) and Lemma 3.1(ii) (Limit Path), if we fix $\gamma^k \subset \Lambda^k$ such that $(\gamma^k)^- = \zeta^k$, then $\gamma^k$ is $\Lambda^k$-true. Hence by (4.1), for all $j$ such that $k \leqslant j \leqslant n$ and all $\delta^j \subseteq \lambda^j(\gamma^k)$, $\delta^j$ is $\Lambda^j$-free iff $\delta^j$ is $\lambda^j(\gamma^k)$-free. In particular, $\sigma$ is $\lambda(\gamma^k)$-free. It thus follows easily from Lemma 4.4 (Free Implies True Path) and hypothesis that $\sigma$ is $\gamma^k$-consistent. Furthermore, as $\zeta^j$ is the principal derivative of $\zeta^{j+1}$ along $\lambda^j(\gamma^k)$ for all $j \in [k, n-1]$ and $(\gamma^k)^- = \zeta^k$, it follows from Lemma 3.3 ($\lambda$-Behavior) that either $\gamma^k$ is switching, or $(\gamma^k)^-$ is the initial derivative of $\zeta$ along $\Lambda^k$. In either case, we set $\rho = \langle\ \rangle$ in Definition 2.8, Step 4 when we are ready to assign a requirement to $\gamma^k$. As $\sigma \subset \lambda(\gamma^k) \subset \Lambda^{k+1}$, it follows from (2.4) and (2.6) that we will take non-switching extensions in Definition 2.8, Step 4, beginning at $\gamma^k$, and reach a node $\beta^k$ at which $\sigma$ is the shortest node eligible to determine a derivative along $\Lambda^k$. By Lemma 3.5 (Nonswitching Extension), no new links are formed when nonswitching extensions are taken. Since $\sigma \subset \lambda(\gamma^k)$, it will be the case that $\lambda(\beta^k) = \lambda(\gamma^k)$, and so, that $\sigma$ is $\beta^k$-free and $\beta^k$-consistent. By Definition 2.8, Step 4, we define $\mathrm{up}(\beta^k) = \sigma$, and $\beta^k$ will be $\Lambda^k$-true and $\Lambda^k$-free. Hence by (4.1), $\beta^k$ will be a $\Lambda^k$-free derivative of $\sigma$. ∎

## 5. Implication Chains

In order to coordinate the action of nodes working for the same (densely distributed) requirement of type 1 or 2 so that iterated limits will exist, we will have to force extensions of certain nodes to follow specified paths, so that we can form implication chains. This will allow us to show that the nodes work together to specify the same outcome for their axioms. Suppose that $\sigma$ and $\hat{\sigma}$ are two such nodes. If $\sigma$ and $\hat{\sigma}$ are incomparable, then the notion of *control* defined in Section 6 allows us to prevent the node which is off the true path from declaring too many axioms. So we restrict our attention in this section to the case where $\sigma$ and $\hat{\sigma}$ are comparable. We try to arrange that, whenever possible, either both $\sigma$ and $\hat{\sigma}$ are activated or both $\sigma$ and $\hat{\sigma}$ are validated. (Such attempts begin on $T^{\dim(\sigma)-1}$, as the notion of control is used to coordinate action taken by the construction for this requirement at nodes on trees $T^k$ for $k < \dim(\sigma) - 1$, allowing us to verify the existence of iterated limits, except for the outermost iteration.) Whenever faced with a path along which this is not the case, we try to force an extension of paths which causes one of these two nodes to switch before declaring any new axioms. As we also want these nodes to act in accordance with the validity of the sentences which generate their action, we try to construct implication chains between nodes which yield implications either from $M_\sigma$ to $M_{\hat{\sigma}}$ or from $M_{\hat{\sigma}}$ to $M_\sigma$ (see Definitions 2.9 and 2.10). These implication chains are carried down to $T^0$, where decisions on action can be made effectively, based on the truth of the sentences.

We now describe the construction of implication chains in more detail. Fix $\Lambda^0 \in [T^0]$ and assume that $\Lambda^0$ is the true path for the construction. For all $i \leqslant n$, let $\Lambda^i = \lambda^i(\Lambda^0)$. Suppose that we have $\sigma^r \subset \hat{\sigma}^r \subset \Lambda^r$ such that $r = \dim(\sigma^r) - 1$, $\mathrm{up}(\sigma^r) \neq \mathrm{up}(\hat{\sigma}^r)$, and $\sigma^r$ and $\hat{\sigma}^r$ are working for the same densely distributed requirement $R$. Then at most one of $\sigma^r$ and $\hat{\sigma}^r$ will have all of its antiderivatives on $T^i$ lying along $\Lambda^i$ for all $i \in [r, n]$, but we will not be able to recursively identify if either of these nodes has this property, and if so, which one has the property. We may then be forced to define infinitely many axioms for $R$ for derivatives of both $\sigma^r$ and $\hat{\sigma}^r$. Such axioms have value determined by the prediction of the truth of certain sentences derived from the sentence assigned to $R$. However, we can only show that these predictions are correct, and hence that the proper value is specified, when all antiderivatives of the node lie on the true path. Thus the values produced by derivatives of $\hat{\sigma}^r$ and derivatives of $\sigma^r$ may be different, preventing us from computing limits needed to satisfy $R$. We must therefore try to coordinate the actions taken for $\sigma^r$ and $\hat{\sigma}^r$.

The same sentence, $M_{\mathrm{up}(\sigma^r)}$, will be assigned to both $\mathrm{up}(\sigma^r)$ and $\mathrm{up}(\hat{\sigma}^r)$. The sentences $M_{\sigma^r}$ and $M_{\hat{\sigma}^r}$, assigned to $\sigma^r$ and $\hat{\sigma}^r$, respectively, will be obtained by bounding all quantifiers in the first quantifier block of $M_{\mathrm{up}(\sigma^r)}$ by numbers $\mathrm{wt}(\sigma^r)$ and $\mathrm{wt}(\hat{\sigma}^r)$, respectively, where $\mathrm{wt}(\sigma^r) < \mathrm{wt}(\hat{\sigma}^r)$. If the quantifier block is a block of universal quantifiers, then $M_{\hat{\sigma}^r}$ will formally imply $M_{\sigma^r}$, and if it is a block of existential quantifiers, then $M_{\sigma^r}$ will formally imply $M_{\hat{\sigma}^r}$. Assume the latter, and so, that $r$ is even, for concreteness.

The coordination problem arises when we reach $\tau^r$ such that $(\tau^r)^- = \hat{\sigma}^r$, and $\sigma^r$ has finite outcome along $\tau^r$ iff $\hat{\sigma}^r$ is has infinite outcome along $\tau^r$. We briefly describe the attempt to coordinate action. There are two cases to consider, depending on whether $\sigma^r$ has finite or infinite outcome along $\tau^r$.

*Case* 1. First suppose that $\sigma^r$ has finite outcome along $\tau^r$, and so, that $\hat{\sigma}^r$ has infinite outcome along $\tau^r$. Recall that $\sigma^r \subset \hat{\sigma}^r \subset \tau^r$. We will only need to follow this case if $\mathrm{up}(\sigma^r) \subset \mathrm{up}(\hat{\sigma}^r)$, so we assume that this latter condition holds. (If this is not the case, then we will be able to show that there are too few conflicting axioms to prevent the existence of iterated limits.) The sentences $M_{\sigma^r}$ and $M_{\hat{\sigma}^r}$ assigned to $\sigma^r$ and $\hat{\sigma}^r$ are obtained by bounding the leading unbounded quantifier block (a block of existential quantifiers) in $M_{\mathrm{up}(\sigma^r)} = M_{\mathrm{up}(\hat{\sigma}^r)}$ by numbers $\mathrm{wt}(\sigma^r) < \mathrm{wt}(\hat{\sigma}^r)$, respectively. As $\sigma^r$ has finite outcome along $\tau^r$, we are predicting that $M_{\sigma^r}$ is false, so do not have a formal implication from the truth of $M_{\sigma^r}$ to the truth of $M_{\hat{\sigma}^r}$. But if it were the case that $\mathrm{wt}(\sigma^r) \geqslant \mathrm{wt}(\hat{\sigma}^r)$ and as we are predicting that $M_{\hat{\sigma}^r}$ is true, $M_{\hat{\sigma}^r}$ would formally imply $M_{\sigma^r}$. We thus try to create an implication between sentences by replacing $\sigma^r$ with a derivative $\tilde{\sigma}^r$ of $\mathrm{up}(\sigma^r)$ which extends $\tau^r$. (The process of obtaining $\tilde{\sigma}^r$ will require us to switch certain nodes which are the ends of primary links, or which caused other implication chains to be created. We may need to iterate this process down to $T^0$, and nodes of $T^1$ which are switched will place elements into sets, so could injure the truth of the instance of $M_{\hat{\sigma}^r}$ on $T^1$. We will be able to check to see if this is the case, and will show that it will be unnecessary to pass from $\tau^r$ to $\tilde{\sigma}^r$ in this situation, as the construction will resolve conflicts between axioms declared by derivatives of $\sigma^r$ and axioms declared by derivatives of $\hat{\sigma}^r$ automatically. We will try to provide more intuition as to how this occurs later.) Suppose that we decide to extend $\tau^r$ to $\tilde{\sigma}^r$. (In this case, we say that $\tau^r$ *requires extension* for $\sigma^r$.) We now look at $\tilde{\tau}^r$ such that $(\tilde{\tau}^r)^- = \tilde{\sigma}^r$. If $\tilde{\sigma}^r$ has finite outcome along $\tilde{\tau}^r$, we proceed as in Case 2 below (with $\tilde{\sigma}^r$ in place of $\hat{\sigma}^r$ and $\hat{\sigma}^r$ in place of $\sigma^r$), where it is assumed that $\sigma^r$ has infinite outcome along $\tau^r$. If $\tilde{\sigma}^r$ has infinite outcome along $\tilde{\tau}^r$, then we will have switched the outcome of $\mathrm{up}(\sigma^r)$, thus forcing $\mathrm{up}(\hat{\sigma}^r)$ off the true path, and will have prevented derivatives of $\hat{\sigma}^r$ from defining any axioms which might prevent the computation of an iterated limit, as we have delayed the declaration of axioms by derivatives of $\hat{\sigma}^r$.

*Case* 2. Suppose that $\sigma^r$ has infinite outcome along $\tau^r$. We now have a formal implication from $M_{\sigma^r}$ which seems to be true, to $M_{\hat{\sigma}^r}$ which seems to be false. (We will not allow this to happen for $r = 0$.) If the immediate successor of $\sigma^r$ along $\tau^r$ does not require extension, then we call $\hat{\sigma}^r$ a *pseudocompletion* of $\sigma^r$. We form an *r-implication chain* $\langle\langle \sigma^r, \hat{\sigma}^r, \tau^r \rangle\rangle$, to try to resolve this discrepancy on $T^{r-1}$. This discrepancy is first observed at $\tilde{\tau}^{r-1} = \mathrm{out}(\tau^r)$ along $\Lambda^{r-1}$. We will then have $\tilde{\sigma}^{r-1} \subset \hat{\sigma}^{r-1} \subset \tilde{\tau}^{r-1}$ such that $\tilde{\sigma}^{r-1}$ and $\hat{\sigma}^{r-1}$ are, respectively, the principal derivatives of $\sigma^r$ and $\hat{\sigma}^r$ along $\tilde{\tau}^{r-1}$, and $(\tilde{\tau}^{r-1})^- = \hat{\sigma}^{r-1}$. Furthermore, $\tilde{\sigma}^{r-1}$ has finite outcome along $\tilde{\tau}^{r-1}$ and $\hat{\sigma}^{r-1}$ has infinite outcome $\tilde{\tau}^{r-1}$. We now have the situation for $r-1$ which we discussed in Case 1 for $r$. If $\tilde{\tau}^{r-1}$ requires extension for $\tilde{\sigma}^{r-1}$, then the $r$-implication chain $\langle\langle\sigma^r, \hat{\sigma}^r, \tau^r\rangle\rangle$ will be called *amenable*, and we will either be able to extend our implications between sentences to level $r-1$ and build an amenable $(r-1)$-implication chain, or will switch paths as described above.

Once we have an $(r-1)$-implication chain, we repeat this process. There are three possibilities. Either we eventually switch $\sigma^r$, thus removing $\hat{\sigma}^r$ from the current path. Or we switch $\hat{\sigma}^r$ (this can occur when we try to build a $j$-implication chain for $j$ even), thus resolving the conflict by forcing derivatives of $\sigma^r$ and $\hat{\sigma}^r$ to define axioms with identical outputs (no axioms are defined by $\hat{\sigma}^r$ while we are resolving the conflict), or we reach $T^0$ and do not allow the construction of a 0-implication chain. We show that the action of the construction on $T^0$ is still in accordance with the potential truth of the sentences described.

The process of defining implication chains requires us to define several notions by simultaneous induction on $\mathrm{lh}(\eta)$ for $\eta \in T^0$. We begin by defining $\eta^k$ *requires extension for* $v^k$, where $\eta^k = \lambda^k(\eta)$. (In Case 1 of our intuitive remarks, $\eta^k$ corresponds to $\tau^r$ and $v^k$ to $\sigma^r$.) When $\eta^k$ *requires extension for* $v^k$, then either $k = \dim(v^k) - 1$ and we will be beginning an attempt to construct a $k$-implication chain, or $k < \dim(v^k) - 1$ and we will be attempting to extend a $(k+1)$-implication chain which has been defined by the time $\eta^k$ is reached, to a $k$-implication chain. If $\eta^k$ *requires extension for* $v^k$, then we will begin a process of defining the $k$*-completion* of $\eta^k$ for $v^k$. (The $k$-completion will correspond to the node $\tilde{\sigma}^r$ when $k = r$ in Case 1). We may need to switch nodes while constructing a $k$-completion, and may thereby discover a new node which requires extension, and so wants to find a $j$-completion. In order to resolve potential conflicts about which completion to pursue, we stipulate that we obtain the $j$-completion of the new node before continuing with the process of finding the $k$-completion of the original node. (We will show that this process is finitary.) Nodes which are first encountered during the process of finding a $k$-completion will not be *implication-free*, and so will not be allowed to control the declaration of axioms. The decision as to whether $\eta^k$ requires extension for $v^k$ will depend on the elements in $\mathrm{PL}(\mathrm{up}(v^k), \lambda(\eta^k))$, a set of ends of primary links along $\lambda(\eta^k)$ which restrain $\mathrm{up}(v^k)$, and nodes extending $\mathrm{up}(v^k)$ which caused implication chains to be created. These are nodes which will have to be switched in order to obtain the $k$-completion of $\eta^k$, and in the iterative process of finding a 0-implication chain, could place elements into sets which might destroy the truth of the instance $M_{\hat{\sigma}^1}$ of the sentence whose truth at a given stage caused us to try to construct the implication chain. (We note that this can only occur for requirements of type 1.) Should such a destruction occur, then $\eta^k$ will not require extension for $v^k$; we will show that if $v^k$ really is on the true path for the construction, then any way of returning

$\nu^k$ to the true path will cause such a destruction, and that such a destruction will also allow us to correct axioms. The *amenable* implication chains are those which give rise to sets $\mathrm{PL}(\mathrm{up}(\nu^k), \lambda(\eta^k))$ for which no such destruction will occur.

There are five conditions which must be satisfied in order for a node to *require extension*. Fix nodes $\nu^k \subset \delta^k \subset \eta^k$ (the nodes corresponding to $\sigma^r$, $\hat{\sigma}^r$ and $\tau^r$, respectively, in Case 1 of our intuitive remarks), and let $\xi^k$ be the immediate successor of $\nu^k$ along $\eta^k$. Condition (5.1) requires that, if $k = r$, then for all $i \leqslant k$, the principal derivatives of $\nu^k$ along $\mathrm{out}^i(\xi^k)$ and $\delta^k$ along $\mathrm{out}^i(\eta^k)$ are *implication-free* (see Definition 5.7). This will correspond to assuming that all action to find $j$-completions for $j \geqslant k$ which was started before $\mathrm{out}^0(\eta^k)$ has been completed, so we are free to try to resolve the current conflict between sentences. (If $k < r$, then we must try to build completions even when a node is not implication-free as part of the process for finding completions for other nodes.) We also require that $\mathrm{out}^0(\xi^k)$ is pseudotrue; should this condition fail, then $\xi^k$ will not be allowed to define axioms. Condition (5.2) implies that two nodes disagree about the value to be assigned to a newly declared axiom, but there is no implication between the sentences. This corresponds to Case 1 of our intuitive remarks, and $\nu^k$ in (5.2) corresponds to $\sigma^r$ in Case 1. By (5.2) and Lemma 4.3(ia) (Link Analysis), condition (5.3) will imply that $\mathrm{up}(\nu^k) \subset \mathrm{up}(\delta^k)$; and the failure of (5.3) will imply that $\mathrm{up}(\nu^k)$ and $\mathrm{up}(\delta^k)$ are incomparable, so by (2.6), no derivatives of $\mathrm{up}(\nu^k)$ can extend $\delta^k$. In the latter case, it is impossible to carry out the extension process needed to find a completion. (5.4) is the condition which determines if any node which must be switched during the iteration process for finding completions will place elements into the restraint set for the sentence whose apparent truth caused us to want to act; the condition requires that such nodes do not exist. This will always be the case for requirements of type 2, so (5.4) only applies to requirements of type 1. Condition (5.5)(i) is the condition required to start building an $r$-implication chain, and condition (5.5)(ii) describes the situation which arises in extending a $(k+1)$-implication chain to a $k$-implication chain.

Definition 5.1. Suppose that $k \leqslant r < n$ and $\nu^k \subset \xi^k \subseteq \delta^k \subset \eta^k \in T^k$ are given such that $(\eta^k)^- = \delta^k$, $(\xi^k)^- = \nu^k$, and $r = \dim(\nu^k) - 1$. We say that $\eta^k$ *requires extension for* $\nu^k$ if $\nu^k$ is the shortest node for which the following conditions hold:

(5.1) If $k = r$, then for all $i \leqslant r$, the principal derivatives of $\nu^r$ along $\mathrm{out}^i(\xi^r)$ and $\delta^r$ along $\mathrm{out}^i(\eta^r)$ are implication-free (see Definition 5.7), and $\mathrm{out}^0(\xi^r)$ is pseudotrue (see Definition 5.9).

(5.2) $\mathrm{tp}(\nu^k) \in \{1, 2\}$, $\nu^k \equiv \delta^k$, $\mathrm{up}(\delta^k) \neq \mathrm{up}(\nu^k)$, $\delta^k$ has infinite outcome along $\eta^k$, $\nu^k$ is the principal derivative of $\mathrm{up}(\nu^k)$ along $\eta^k$, and $\nu^k$ has finite outcome along $\eta^k$ (so $\nu^k$ is the initial derivative of $\mathrm{up}(\nu^k)$ along $\eta^k$).

(5.3) There is no primary $\eta^k$-link which restrains $\nu^k$.

(5.4) If $k = r$ and $\mathrm{tp}(\delta^k) = 1$, then for every $\pi^{k+1} \in \mathrm{PL}(\mathrm{up}(\nu^k), \lambda(\eta^k))$, $\mathrm{TS}(\pi^{k+1}) \cap \mathrm{RS}(\delta^k) = \emptyset$ (see Definition 5.3 for the definition of PL sets).

(5.5) One of the following conditions holds:

(i) $r = k$.

(ii) There is an amenable $(k+1)$-implication chain $\langle \langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle : r \geqslant j \geqslant k+1 \rangle$ along $\lambda(\eta^k)$ such that $\eta^k = \mathrm{out}(\tau^{k+1})$, and $\delta^k$ ($\nu^k$, resp.) is the principal derivative of $\hat{\sigma}^{k+1}$ ($\sigma^{k+1}$, resp.) along $\eta^k$. (See Definitions 5.4 and 5.2 for the definitions of *amenable* and *implication chain.*)

We say that $\eta^k$ *requires extension* if $\eta^k$ requires extension for some $\nu^k$. ▮

Implication chains keep track of the implications between sentences for a requirement. The first and second coordinates of the triple at a given level of the implication chain determine the nodes which are potentially responsible for defining axioms for the requirement. The third coordinate keeps track of the conflicting outcomes of the first and second coordinates. The $k$-implication chain follows the implications of sentences from the starting level, $T^r$, down to $T^k$. The conditions mentioned in Definition 5.2 are described in the motivation at the beginning of the section. In addition, we require the principal derivatives of $\sigma^r$ along $\mathrm{out}^i(\bar{\tau}^r)$ and $\hat{\sigma}^r$ along $\mathrm{out}^i(\tau^r)$ to be implication-free for all $i \leqslant r$ (Condition (5.10)). This will correspond to assuming that all action to find $j$-completions for $j \geqslant k$ which was started before $\mathrm{out}^i(\bar{\tau}^r)$ or $\mathrm{out}^i(\tau^r)$ was completed before that node is reached, so we are free to try to resolve the current conflict between sentences. If $k < r$, then we have already begun building the implication chain, and must continue to extend it within other implication chains; thus the principal derivatives of $\sigma^k$ and $\hat{\sigma}^k$ along $\mathrm{out}^i(\tau^k)$ need not be implication-free. (We note that Condition (5.6) below allows $\mathrm{up}(\sigma^r) \mid \mathrm{up}(\hat{\sigma}^r)$.)

We will also need to describe the situation when the first triple of an implication chain can be formed by taking an immediate extension of a node $\hat{\sigma}^r$ in the absence of a requires extension configuration; such a $\hat{\sigma}^r$ will be called a *pseudocompletion.*

Definition 5.2. Fix $k \leqslant r \leqslant n$. A *k-implication chain* is a sequence $\langle \langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle : r \geqslant j \geqslant k \rangle$ such that:

(5.6) $\sigma^r \equiv \hat{\sigma}^r$ and $\mathrm{up}(\sigma^r) \neq \mathrm{up}(\hat{\sigma}^r)$.

(5.7) $\mathrm{tp}(\sigma^r) \in \{1, 2\}$, $\dim(\sigma^r) = r + 1$.

(5.8) (i) $\sigma^k \subset \hat{\sigma}^k$.

(ii) $\hat{\sigma}^k = (\tau^k)^- \subset \tau^k$.

(5.9) If $k < r$, then $\mathrm{up}(\sigma^k) = \hat{\sigma}^{k+1}$ and $\mathrm{up}(\hat{\sigma}^k) = \sigma^{k+1}$.

(5.10) (i) Fix $\bar{\tau}^r \subseteq \hat{\sigma}^r$ such that $(\bar{\tau}^r)^- = \sigma^r$. Then for all $i \leqslant r$, the principal derivative of $\sigma^r$ along $\mathrm{out}^i(\bar{\tau}^r)$ is implication-free (see Definition 5.7), and $\hat{\sigma}^r$ is implication-free.

(ii) For all $i \leqslant r$, the principal derivative of $\hat{\sigma}^r$ along $\mathrm{out}^i(\tau^r)$ is implication-free.

(5.11) (i) $\sigma^k$ has infinite outcome along $\hat{\sigma}^k$.

(ii) $\hat{\sigma}^k$ has finite outcome along $\tau^k$.

(5.12) If $k < r$, then $\langle \langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle : r \geqslant j \geqslant k+1 \rangle$ is a $(k+1)$-implication chain along $\tau^{k+1}$ and $\mathrm{out}(\tau^{k+1}) \subset \tau^k$.

We say that this implication chain is *along* $\rho^k \in T^k$ ($\Lambda^k \in [T^k]$, resp.) if $\tau^k \subseteq \rho^k$ ($\tau^k \subset \Lambda^k$, resp.).

Suppose that $k = r$, Conditions (5.6), (5.7), (5.8)(i), (5.9), (5.10)(i), and (5.11)(i) hold, and $\hat{\sigma}^r$ is an initial derivative. In this case, we call $\hat{\sigma}^r$ a *pseudocompletion* of $\sigma^r$. $\hat{\sigma}^r$ is a *pseudocompletion* if it is a pseudocompletion of some node. ∎

The process of building a new $r$-implication chain, or of extending a $(k+1)$-implication chain to a $k$-implication chain, will require us to build completions. We will define PL sets, which keep track of the antiderivatives of those nodes of $T^1$ which will eventually have to be switched (and might thereby injure restraint sets), should we need to pull the implication chains down to $T^0$ during the process of building completions. Consider the situation wherein a node of $T^k$ requires extension. Thus assume that we have $\nu^k \subset \delta^k \subset \eta^k \in T^k$ such that $(\eta^k)^- = \delta^k$ and $\eta^k$ requires extension for $\nu^k$. We wish to construct a $\kappa^k \supset \eta^k$ such that $\mathrm{up}(\kappa^k) = \mathrm{up}(\nu^k) = \nu^{k+1}$. By (2.10), this requires taking extensions of $\eta^k$ with the goal of making $\nu^{k+1}$ a $\lambda(\kappa^k)$-free node. Thus we must eliminate the links which restrain $\nu^{k+1}$.

Let $\eta^u = \lambda^u(\eta^k)$ for all $u \in [k, n]$. We will show later that, in this situation, there is an $\eta^{k+1}$-link which restrains $\nu^{k+1}$ and $\nu^{k+1} \subset \delta^{k+1} = \mathrm{up}(\delta^k)$. By Lemma 4.1 (Nesting), there will be an $\eta^{k+1}$-link $[\mu^{k+1}, \pi^{k+1}]$ which restrains $\nu^{k+1}$ and contains all $\eta^{k+1}$-links which restrain $\nu^{k+1}$, and $\pi^{k+1}$ will be $\eta^{k+1}$-free. By (2.10), we must eliminate this link in order to make $\nu^{k+1}$ free; this is done as follows. Let $[\mu^{k+1}, \pi^{k+1}]$ be derived from the primary $\eta^j$-link $[\mu^j, \pi^j]$ (we allow $j = k+1$). By Lemma 3.5 (Nonswitching Extension) and since all blocks defined in Section 2 are finite, we will be able to find a nonswitching extension $\tilde{\eta}^k$ of $\eta^k$ such that $\mathrm{up}^j(\tilde{\eta}^k) = \pi^j$ and $\tilde{\eta}^k$ is an initial derivative of $\mathrm{up}^{j-1}(\tilde{\eta}^k) \supset \eta^{j-1}$. By Lemma 3.6 (Switching), we can find $\hat{\eta}^k$ such that $(\hat{\eta}^k)^- = \tilde{\eta}^k$, $\lambda^i(\hat{\eta}^k) \supset \lambda^i(\tilde{\eta}^k)$ for all $i < j$, and $\hat{\eta}^k$ switches $\pi^j$. $[\mu^{k+1}, \pi^{k+1}]$ will not be a $\lambda^{k+1}(\hat{\eta}^k)$-link, and every $\lambda^{k+1}(\hat{\eta}^k)$-link which restrains $\nu^{k+1}$ will be properly contained in the interval $[\mu^{k+1}, \pi^{k+1}]$. Hence barring other considerations, we can repeat this process for the longest $\lambda^{k+1}(\hat{\eta}^k)$-link which restrains $\nu^{k+1}$, and eventually find a new derivative $\kappa^k$ of $\nu^{k+1}$ of $T^k$. (There may be additional considerations, but for this paragraph, assume that there are none.) This procedure will be induced by taking extensions of nodes on $T^0$ which will be nonswitching except when needed to switch one of the above nodes ending a link. $\mathrm{out}^0(\kappa^k)$ will act according to the validity of its sentence unless $k = 0$, in which case we force $\kappa^0$ to have infinite outcome, and show that this action is in accordance with the validity of the sentence assigned to $\kappa^0$. If $k > 0$ and the action of $\mathrm{out}^0(\kappa^k)$ produces an immediate successor $\bar{\eta}$ of $\mathrm{out}^0(\kappa^k)$ such that $\kappa^k$ has infinite outcome along $\lambda^k(\bar{\eta})$, then the process halts since we will then have switched $\nu^{k+1}$, so will have forced $\delta^{k+1}$ not to lie along $\Lambda^{k+1}$. Otherwise, we will have constructed a $k$-implication chain, and $\lambda^{k-1}(\bar{\eta})$ will require extension, so we can repeat this process. $\pi^j$ is placed in $\mathrm{PL}(\nu^{k+1}, \eta^{k+1})$ via (5.13) whenever $j = k+1$, i.e., whenever $[\mu^{k+1}, \pi^{k+1}]$ is a primary $\eta^{k+1}$-link. Each such $\pi^j$ will be the last node of a primary $\eta^{k+1}$-link which restrains $\nu^{k+1}$. The nodes in $\mathrm{PL}(\nu^{k+1}, \eta^{k+1})$ are those which cause a small element to be placed in a set when we carry out the backtracking process for $k = 0$, and may thereby injure the oracle of the computation which has generated the implication chain. We will check to see, for all nodes in $\mathrm{PL}(\nu^{k+1}, \eta^{k+1})$, whether this action causes an element to be placed into this oracle. If not, then the implication chains constructed during this process are called *amenable* (see Definition 5.4). The derivative operation will provide a one-one correspondence between $\mathrm{PL}(\nu^{k+1}, \eta^{k+1})$ and $\mathrm{PL}(\delta^k, \lambda^k(\bar{\eta}))$, so it will suffice to consider only the nodes in PL sets.

There are additional considerations which we need to take into account. Our proof requires that we follow the backtracking process for a node whenever that node requires extension. In the preceding paragraph, $\lambda^{j-1}(\tilde{\eta}^k)$ will have infinite outcome along $\lambda^{j-1}(\hat{\eta}^k)$. It is thus possible that $\lambda^{j-1}(\hat{\eta}^k)$ will require extension for some $\gamma^{j-1}$. Furthermore, it is possible that for such a $\gamma^{j-1}$, if $\gamma^j = \mathrm{up}(\gamma^{j-1})$, then there is a $\lambda^j(\hat{\eta}^k)$-link which restrains $\gamma^j$, but no $\lambda^{k+1}(\hat{\eta}^k)$-link derived from this $\lambda^j(\hat{\eta}^k)$-link restrains $\nu^{k+1}$, so this situation is not covered by (5.13).

Suppose that $\lambda^{j-1}(\hat{\eta}^k)$ requires extension for $\gamma^{j-1}$. By (5.1) and since $\lambda^{j-1}(\hat{\eta}^k)$ is implication-restrained, $\dim(\gamma^j) > j$. In order to make our construction cohere, we must perform the backtracking process for $\lambda^{j-1}(\hat{\eta}^k)$ (which entails removing *all* links around $\gamma^j$) before proceeding as in the preceding paragraph for the next link which restrains $\nu^{k+1}$. This may require us to switch additional primary links, say $[\rho^t, \hat{\tau}^t]$ on $T^t$ for $t \geqslant j$, with $\nu^{k+1} \subset \mathrm{out}^{k+1}(\rho^t) \subset \delta^{k+1}$. Also, once we have found a new derivative $\tau^{j-1}$ of $\gamma^j$, we must force it to have infinite outcome along its immediate extension in order to preclude the existence of a $(j-1)$-implication chain along the true path. In either case, we have to switch nodes on $T^{k+1}$ if $t = k+1$ or $j = k+1$, respectively, until we complete the backtracking process for $\lambda^{j-1}(\hat{\eta}^k)$, i.e., until we reach the primary completion of $\lambda^{j-1}(\hat{\eta}^k)$. For the first case, we put all nodes $\tau^{k+1} \in \mathrm{PL}(\gamma^{k+1}, \eta^{k+1})$ into $\mathrm{PL}(\nu^{k+1}, \eta^{k+1})$ via (5.14)(ii) as these nodes have to be switched in order to backtrack $\lambda(\hat{\eta}^k)$, and call $\mathrm{PL}(\gamma^{k+1}, \eta^{k+1})$ a *component* of $\mathrm{PL}(\nu^{k+1}, \eta^{k+1})$. For the second case, we put $\gamma^{k+1}$ into $\mathrm{PL}(\nu^{k+1}, \eta^{k+1})$ via (5.14)(i).

In the preceding paragraphs, we have tried to motivate the definition of $\mathrm{PL}(\nu^{k+1}, \eta^{k+1})$ by looking ahead to some $\hat{\eta}^k \supset \mathrm{out}(\eta^{k+1})$, and seeing which nodes $\subseteq \eta^{k+1}$ need to be switched in order to carry out the backtracking process beginning at $\hat{\eta}^k$. However, in the definition of PL sets below, we will want to inductively describe this set in advance, as we pass from $\nu^{k+1}$ to $\eta^{k+1}$, in anticipation of later finding $\hat{\eta}^k$ and having to carry out the corresponding backtracking process. When we wanted to place an element $\tau^{k+1}$ into $\mathrm{PL}(\nu^{k+1}, \eta^{k+1})$ through (5.13), it was the case that $\tau^{k+1}$ was the end of a primary $\eta^{k+1}$-link restraining $\nu^{k+1}$, so these nodes are readily identified in advance. We will show that the other case, described in the preceding paragraph and specified in (5.14), corresponds precisely to a reversal of a backtracking process beginning at a node $\delta^{k+1}$ which requires extension for some $\mu^{k+1} \subset \nu^{k+1}$ with $\nu^{k+1} \subset (\delta^{k+1})^-$, and so we can again identify these nodes in advance. ($(\delta^{k+1})^-$ will be the $\gamma^{k+1}$ of the preceding paragraph.) Once we complete the backtracking process for $\delta^{k+1}$, i.e., once we find a primary completion $\kappa^{k+1}$ of $\delta^{k+1}$, the component corresponding to action for $\delta^{k+1}$ does not place elements $\supset \kappa^{k+1}$ into $\mathrm{PL}(\nu^{k+1}, \eta^{k+1})$. Thus the node $\xi^{k+1}$ in (5.14) (for $j = k+1$) must satisfy $\xi^{k+1} \subseteq \kappa^{k+1}$.

The *backtracking process* is induced by the process described above, starting at $\mathrm{out}^0(\eta^k)$ and ending at $\mathrm{out}^0(\kappa^k)$. Thus we begin at $\mathrm{out}^0(\eta^k)$, and proceed as described above by taking extensions on $T^0$ which are never $j$-switching for any $j \leqslant k$, until we reach a node $\kappa^0 \supset \mathrm{out}^0(\eta^k)$ which has the properties of $\mathrm{out}^0(\kappa^k)$. As activated and validated outcomes are unique on $T^0$, there will be a unique way to carry out the backtracking process, as long as we decide to follow activated outcomes of nodes when not otherwise specified. Assume that $\kappa^k$ has been defined in this way. For all $i \leqslant k$, $\mathrm{out}^i(\kappa^k)$ will be called the *i-completion* of $\eta^k$ for $\nu^k$, and will be defined in Definition 5.6. In the definition of the PL sets, which we now present, it would be helpful for the reader to think of $j$ as the $k+1$ of the preceding remarks. The definition is an inductive definition, proceeding by induction on $n - j$ and then by induction on $\mathrm{lh}(\eta^j) - \mathrm{lh}(\nu^j)$.

DEFINITION 5.3. Fix $j<n$ and $\nu^j \subset \eta^j \in T^j$. We place $\tau^j \subset \eta^j$ into $\mathrm{PL}(\nu^j, \eta^j)$ if one of the following conditions holds:

(5.13) There is a $\mu^j$ such that $\mu^j \subseteq \nu^j \subset \tau^j$ and $[\mu^j, \tau^j]$ is a primary $\eta^j$-link.

(5.14) There are $\mu^j$, $\delta^j$, and $\xi^j$ such that $\mu^j \subset \nu^j \subset (\delta^j)^- \subset \delta^j \subseteq \xi^j \subseteq \eta^j$, $\delta^j$ requires extension for $\mu^j$ and has no $j$-completion with infinite outcome along $\xi^j$, and either:

(i) $\tau^j = (\delta^j)^-$; or

(ii) $\tau^j \in \mathrm{PL}((\delta^j)^-, \xi^j)$.

If nodes satisfying the hypotheses of (5.14) exist, then we call $\mathrm{PL}((\delta^j)^-, \xi^j)$ a *component* of $\mathrm{PL}(\nu^j, \eta^j)$.

LEMMA 5.1 (PL Analysis Lemma). *Fix $j \leqslant n$ and $\nu^j \subseteq \bar{\rho}^j \subset \sigma^j \subseteq \eta^j \in T^j$ such that $(\sigma^j)^- = \bar{\rho}^j$. Then:*

(i) $\mathrm{PL}(\nu^j, \bar{\rho}^j) \subseteq \mathrm{PL}(\nu^j, \sigma^j)$.

(ii) $\mathrm{PL}(\nu^j, \sigma^j) \backslash \mathrm{PL}(\nu^j, \bar{\rho}^j) \subseteq \{\bar{\rho}^j\}$.

(iii) *If* $\mathrm{PL}(\nu^j, \sigma^j) \backslash \mathrm{PL}(\nu^j, \bar{\rho}^j) \neq \emptyset$, *then either* $\mathrm{PL}(\nu^j, \sigma^j) \backslash \mathrm{PL}(\nu^j, \bar{\rho}^j) = \{\bar{\rho}^j\}$ *and $\bar{\rho}^j$ is the last node of a primary $\sigma^j$-link, or $\sigma^j$ requires extension.*

(iv) *If $\bar{\rho}^j$ has finite outcome along $\sigma^j$ then* $\mathrm{PL}(\nu^j, \sigma^j) = \mathrm{PL}(\nu^j, \bar{\rho}^j)$.

(v) *If $\bar{\rho}^j$ is $\eta^j$-free and for every $\delta^j$ and $\mu^j$ such that $\delta^j$ requires extension for $\mu^j$ and $\mu^j \subset \nu^j \subset (\delta^j)^- \subset \delta^j \subseteq \eta^j$, it is the case that there is a $\kappa^j \subset \eta^j$ such that $\kappa^j$ is the j-completion of $\delta^j$ and $\kappa^j$ has infinite outcome along $\eta^j$, then* $\mathrm{PL}(\nu^j, \sigma^j) = \mathrm{PL}(\nu^j, \eta^j)$.

(vi) *If $\xi^j \subseteq \eta^j$ and $\mathrm{PL}(\bar{\rho}^j, \xi^j)$ is a component of $\mathrm{PL}(\nu^j, \eta^j)$, then* $\mathrm{PL}(\bar{\rho}^j, \xi^j) \cup \{\bar{\rho}^j\} \subseteq \mathrm{PL}(\nu^j, \eta^j)$.

(vii) *If $\mathrm{PL}(\bar{\rho}^j, \eta^j)$ is a component of $\mathrm{PL}(\nu^j, \eta^j)$ and $(\eta^j)^- \in \mathrm{PL}(\nu^j, \eta^j)$, then $(\eta^j)^- \in \mathrm{PL}(\bar{\rho}^j, \eta^j)$ or $(\eta^j)^- = \bar{\rho}^j$.*

(viii) *If every $\delta^j \subseteq \bar{\rho}^j$ which requires extension has a j-completion $\subseteq \bar{\rho}^j$ with infinite outcome along $\sigma^j$, then* $\mathrm{PL}(\nu^j, \eta^j) \subseteq \mathrm{PL}(\nu^j, \bar{\rho}^j) \cup \mathrm{PL}(\bar{\rho}^j, \eta^j) \cup \{\bar{\rho}^j\}$.

(ix) *Given $\tilde{\rho}^j$ such that $\mathrm{PL}(\tilde{\rho}^j, \eta^j)$ is a component of $\mathrm{PL}(\bar{\rho}^j, \eta^j)$ and $\mathrm{PL}(\bar{\rho}^j, \eta^j)$ is a component of $\mathrm{PL}(\nu^j, \eta^j)$, then* $\mathrm{PL}(\tilde{\rho}^j, \eta^j) \cup \{\tilde{\rho}^j\} \subseteq \mathrm{PL}(\nu^j, \eta^j)$.

*Proof.* (i) By definition.

(ii, iii) Any primary $\sigma^j$-link which is not a primary $\bar{\rho}^j$-link has $\bar{\rho}^j$ as its last element. And new components can first appear at $\sigma^j$ only if $\sigma^j$

requires extension. Hence (iii) holds. (ii) now follows from (5.13), (5.14)(i), and induction on $\mathrm{lh}(\eta^j)-\mathrm{lh}(\nu^j)$ for (5.14)(ii).

(iv) If $\bar{\rho}^j$ has finite outcome along $\sigma^j$, then $\bar{\rho}^j$ is not the last element of a primary $\sigma^j$-link. By (5.2), if $\sigma^j$ requires extension, then $\bar{\rho}^j$ has infinite outcome along $\sigma^j$. (iv) now follows from (i) and (iii).

(v) As $\bar{\rho}^j$ is $\eta^j$-free, it follows from (4.1) that the primary $\eta^j$-links which restrain $\nu^j$ coincide with the primary $\sigma^j$-links restraining $\nu^j$. Hence all nodes placed in $\mathrm{PL}(\nu^j, \eta^j)$ via (5.13) are already in $\mathrm{PL}(\nu^j, \sigma^j)$. Suppose that $\nu^j \subset (\delta^j)^- \subset \delta^j \subseteq \eta^j$ and $\delta^j$ requires extension for $\mu^j \subset \nu^j$. By the hypothesis of (v), there is a $\kappa^j$ such that $[\mu^j, \kappa^j]$ is a primary $\eta^j$-link which restrains $\nu^j$. As $\bar{\rho}^j$ is $\eta^j$-free and $\nu^j \subseteq \bar{\rho}^j$, $\kappa^j \subseteq \bar{\rho}^j = (\sigma^j)^-$. Hence by (5.14), all elements placed in $\mathrm{PL}(\nu^j, \eta^j)$ via (5.14) are already in $\mathrm{PL}(\nu^j, \sigma^j)$, so (v) follows.

(vi) Immediate from (5.14).

(vii) We proceed by induction on $\mathrm{lh}(\bar{\rho}^j)-\mathrm{lh}(\nu^j)$. By definition, if $\mathrm{PL}(\bar{\rho}^j, \eta^j)$ is a component of $\mathrm{PL}(\nu^j, \eta^j)$, then $\nu^j \subset \bar{\rho}^j$ and there is a $\bar{\mu}^j \subset \nu^j$ such that $\sigma^j$ requires extension for $\bar{\mu}^j$. Hence if $(\eta^j)^-$ enters $\mathrm{PL}(\nu^j, \eta^j)$ via (5.13), then the corresponding primary link $[\mu^j, (\eta^j)^-]$ also restrains $\bar{\rho}^j$. Thus $(\eta^j)^- \in \mathrm{PL}(\bar{\rho}^j, \eta^j)$ as desired.

Suppose that $(\eta^j)^-$ enters $\mathrm{PL}(\nu^j, \eta^j)$ via (5.14). Then there are $\delta^j \subset \tau^j \subseteq \eta^j$ such that $\nu^j \subset \delta^j = (\tau^j)^-$, $\tau^j$ requires extension for some $\mu^j \subset \nu^j$, $\mathrm{PL}(\delta^j, \eta^j)$ is a component of $\mathrm{PL}(\nu^j, \eta^j)$, and either $(\eta^j)^- = \delta^j$ or $(\eta^j)^- \in \mathrm{PL}(\delta^j, \eta^j)$. If $\bar{\rho}^j \subset \delta^j$, then as $\mu^j \subset \nu^j \subset \bar{\rho}^j$, $\mathrm{PL}(\delta^j, \eta^j)$ is a component of $\mathrm{PL}(\bar{\rho}^j, \eta^j)$, so (vii) follows from (5.14). If $\bar{\rho}^j = \delta^j$, then (vii) is immediate. Otherwise, as $\delta^j, \bar{\rho}^j \subset \eta^j$, it follows that $\delta^j \subset \bar{\rho}^j \subset \eta^j$; hence as $\bar{\mu}^j \subset \nu^j \subset \delta^j$, $\mathrm{PL}(\bar{\rho}^j, \eta^j)$ is a component of $\mathrm{PL}(\delta^j, \eta^j)$. Now $\mathrm{lh}(\bar{\rho}^j)-\mathrm{lh}(\delta^j) < \mathrm{lh}(\bar{\rho}^j)-\mathrm{lh}(\nu^j)$ and $(\eta^j)^- \neq \delta^j$, so by induction, $(\eta^j)^- \in \mathrm{PL}(\delta^j, \eta^j)$. But $\mathrm{PL}(\delta^j, \eta^j)$ is a component of $\mathrm{PL}(\nu^j, \eta^j)$ and $\mathrm{lh}(\delta^j)-\mathrm{lh}(\nu^j) < \mathrm{lh}(\bar{\rho}^j)-\mathrm{lh}(\nu^j)$, so by induction, either $(\eta^j)^- = \bar{\rho}^j$ or $(\eta^j)^- \in \mathrm{PL}(\bar{\rho}^j, \eta^j)$.

(viii) Suppose that $\tau^j \in \mathrm{PL}(\nu^j, \eta^j)$. First assume that (5.13) holds for $\tau^j$. Then there is a $\mu^j \subset \tau^j$ such that $[\mu^j, \tau^j]$ is a primary $\eta^j$-link restraining $\nu^j$. If $\tau^j \subset \bar{\rho}^j$, then $\tau^j$ is placed into $\mathrm{PL}(\nu^j, \bar{\rho}^j)$ by (5.13). And if $\tau^j \supset \bar{\rho}^j$, then $\tau^j$ is placed into $\mathrm{PL}(\bar{\rho}^j, \eta^j)$ by (5.13).

Next assume that $\tau^j$ is placed into $\mathrm{PL}(\nu^j, \eta^j)$ by (5.14), because of the component $\mathrm{PL}(\delta^j, \bar{\xi}^j)$ associated with some $\delta^j \supset \nu^j$ which requires extension for some $\mu^j \subset \nu^j$. If $\delta^j \subseteq \bar{\rho}^j$, then by hypothesis, $\delta^j$ has a $j$-completion $\kappa^j \subseteq \bar{\rho}^j$ which has infinite outcome along $\bar{\rho}^j$, so by the properties of $\xi^j$ in (5.14), $\tau^j \subset \bar{\xi}^j \subseteq \bar{\rho}^j$. Hence either $\tau^j = \bar{\rho}^j$, or $\tau^j$ is placed into $\mathrm{PL}(\nu^j, \bar{\rho}^j)$ by (5.14). Otherwise, $\bar{\rho}^j \subset \delta^j$. As $\mu^j \subset \nu^j \subseteq \bar{\rho}^j$, $\mathrm{PL}(\delta^j, \bar{\xi}^j)$ is a component of $\mathrm{PL}(\bar{\rho}^j, \eta^j)$, so $\tau^j$ is placed into $\mathrm{PL}(\bar{\rho}^j, \eta^j)$ by (5.14).

(ix) Immediate from (vi). ∎

As mentioned earlier, the process of extending a $k$-implication chain to a 0-implication chain may injure the validity of a sentence whose truth we are trying to preserve. When this happens, we will not act to extend the $k$-implication chain. Our next definition allows us to differentiate between the $k$-implication chains which we want to extend (the *amenable* implication chains), and those which we do not want to extend (the *nonamenable* implication chains). Condition (5.15) applies when $\mathrm{up}(\hat{\sigma}^r)$ has an initial derivative $\subset \sigma^r$, specifying that in this case, when we first observe the $(k+1)$-implication chain along a path of $T^{k+1}$ generated by a node on $T^k$, then we have a configuration of nodes on $T^k$ which gives rise to a *requires extension* situation, so Condition (5.4) will be applicable. Condition (5.16) imposes a restriction similar to that imposed by (5.4) when $\mathrm{up}(\hat{\sigma}^r)$ does not have an initial derivative $\subset \sigma^r$. This restriction requires the ability to preserve certain computations while the backtracking process is carried out. (Note that at the beginning level $r$ for an implication chain, it is possible to have an implication chain which arises without a requires extension situation, if, for example, $\hat{\sigma}^r$ is an initial derivative.) We will show later that similar restrictions are automatically carried down to lower levels. A similar restriction needs to apply to separate the pseudocompletions which potentially give rise to amenable implication chains from those which do not. Thus we also define amenable pseudocompletions.

DEFINITION 5.4. Suppose that $k=r$ and that $\hat{\sigma}^r$ is a pseudocompletion of $\sigma^r$. We say that $\hat{\sigma}^r$ is an *amenable pseudocompletion* of $\sigma^r$ if either $\mathrm{tp}(\sigma^r) \neq 1$, or for every $\pi^r \in \mathrm{PL}(\sigma^r, \hat{\sigma}^r)$, $\mathrm{TS}(\pi^r) \cap \mathrm{RS}(\sigma^r) = \emptyset$.

Now suppose that $\langle\langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle : r \geqslant j \geqslant k\rangle$ is a $k$-implication chain along $\rho^k$, and for each $j \in [k, r]$, fix $\bar{\tau}^j \subset \tau^j$ such that $(\bar{\tau}^j)^- = \sigma^j$. Let $v^k$ be the principal derivative of $\mathrm{up}(\hat{\sigma}^k)$ along $\tau^k$. We say that $\langle\langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle$: $r \geqslant j \geqslant k\rangle$ is *amenable* if one of conditions (5.15) and (5.16) below holds, and if $k=r$, then $\sigma^k$ is the shortest string satisfying (5.6)–(5.11), and (5.15) or (5.16) for $\hat{\sigma}^k$ and $\tau^k$.

(5.15) $\bar{\tau}^k$ requires extension for $v^k$ and $\hat{\sigma}^k$ is the primary $k$-completion of $\bar{\tau}^k$. (See Definition 5.6 for the definition of $k$-completion.)

(5.16) $k=r$ and $\hat{\sigma}^r$ is an amenable pseudocompletion of $\sigma^r$.

A *nonamenable* implication chain is an implication chain which is not amenable. ∎

The backtracking process requires us to keep track, along each path, of the nodes which require extension but have no 0-completion along the path, and to find 0-completions for these nodes in reverse order of the order in which we discover that they require extension. This ordering is defined in Definition 5.8, and depends on the definition of completions (Definition 5.6). In order to show that backtracking can be carried out, we require that the final paths through $T^0$ be admissible (see Definition 5.9). There is a potential circularity here, which we avoid by requiring these nodes to be *preadmissible*. Preadmissibility ensures that nodes will only be switched when they do not interfere with the backtracking process; and in the course of finding completions, nodes will be switched only as required by the backtracking process. Completions are then defined as the nodes reached when the backtracking process has been completed.

Because of the interdependence of the next five definitions, we will explain some of the terminology used in the next definition. A node $\rho$ will be *completion-respecting* if for all $j \leqslant n$, any node along $\lambda^j(\rho)$ which requires extension has a completion along $\lambda^j(\rho)$. $\rho$ is *completion-consistent via* the sequence $S$ if the paths determined by $\rho$ are compatible with primary completions of all nodes of $S$, where the nodes in $S$ are those which require extension but have not yet found primary completions, and the order in which the completions are to be found is the reverse of the ordering of $S$. $\rho$ is *implication-free* if $\rho$ is not a derivative of any node which is captured in the backtracking process, and is *implication-restrained* otherwise. Implication-restrained nodes will not define too many axioms during the construction, so there is no harm in forcing their outcomes.

The clauses of Definition 5.5 spell out the extensions which allow us to maintain compatibility with the backtracking process. Condition (5.17)(i) requires that we take switching extensions of primary 0-completions and pseudocompletions on $T^0$, and Condition (5.17)(ii) requires that non-switching extensions be taken for all nodes which are not captured during a backtracking process but are derivatives of captured nodes. Condition (5.18) covers extensions taken during the backtracking process. Clause (i) requires that we take $(k+1)$-switching extensions of nodes of $T^k$ which are primary completions. This condition is needed to maintain compatibility with all completions which are forced to be taken during the construction. Clause (ii) requires us to switch outcomes of primary links in a minimal way, in order to return a designated node to the true path. Clause (iii) specifies that no other nodes captured by the backtracking process have switching extensions. (The reader may want to refer back to the remarks following Definition 5.2 for intuition.)

Definition 5.5. Fix $\sigma \in T^0$. If $\mathrm{lh}(\sigma) > 0$, let $\rho = \sigma^-$, and assume that $\rho$ is completion-consistent via some sequence $S = \langle \eta_i : i < m \rangle$ for some $m \geqslant 0$ (see Definition 5.8), and for each $i < m$, fix $k(i)$ such that $\eta_i \in T^{k(i)}$ and $\nu_i$ such that $\eta_i$ requires extension for $\nu_i$. We say that $\sigma$ is *preadmissible* if either $\sigma = \langle\ \rangle$, or $\sigma \neq \langle\ \rangle$, $\rho$ is admissible (see Definition 5.9), and the following conditions hold:

(5.17) (i) If either $\rho$ is a primary 0-completion or an amenable pseudocompletion, then $\rho$ has infinite outcome along $\sigma$.

(ii) If the hypotheses of (i) fail, $S = \langle\ \rangle$, and $\rho$ is implication-restrained (see Definition 5.7), then $\sigma$ is a nonswitching extension of $\rho$.

(5.18) If $S \neq \langle\ \rangle$, then one of the following conditions holds:

(i) (a) There are $k(m)$ and $\eta_m \in T^{k(m)}$ such that $\rho^-$ is completion-consistent via $\langle \eta_i : i \leqslant m \rangle$, $\rho$ is a 0-completion of $\eta_m$, and

(b) $\sigma$ is a $(k(m)+1)$-switching extension of $\rho$.

(ii) (a) (i.a) fails, and there is a $j > k(m-1)$ and a $\lambda^{k(m-1)+1}(\rho)$-link $[\mu^{k(m-1)+1}, \pi^{k(m-1)+1}]$ restraining $\mathrm{up}(\nu_{m-1})$ of shortest length which is derived from a primary $\lambda^j(\rho)$-link $[\mu^j, \pi^j]$ such that $\rho$ is the initial derivative of $\mathrm{up}^{j-1}(\rho)$ along $\rho$ and $\mathrm{up}^{j-1}(\rho)$ is a derivative of $\pi^j$; and

(b) $\sigma$ is a $j$-switching extension of $\rho$.

(iii) (i.a) and (ii.a) fail, $\sigma$ is a nonswitching extension of $\rho$, and if there are two nonswitching immediate extensions of $\rho$, then $\rho$ is activated along $\sigma$.

(We note that the extensions specified by (5.18) are unique.) ∎

We described role of completions earlier. We will need to show later that completions never require extension. This will follow from our requirement that completions be nonswitching extensions.

Definition 5.6. Fix $k \leqslant n$ and $\kappa^k \in T^k$. We say that $\kappa^k$ is the *k-completion* if $\mathrm{out}^0(\kappa^k)$ is nonswitching and either:

(5.19) There are $m \geqslant 0$, $\gamma^k \subset \rho^k \subset \kappa^k$, and a sequence $S = \langle \eta_i : i \leqslant m \rangle$ such that $\eta_m = \rho^k$ requires extension for $\gamma^k$, $\mathrm{up}(\gamma^k) = \mathrm{up}(\kappa^k)$, both $\mathrm{out}^0(\rho^k)$ and $\mathrm{out}^0((\kappa^k)^-)$ are completion-consistent via $S$ (see Definition 5.8), and there is no $k$-completion $\tilde{\kappa}^k$ of $\rho^k$ such that $\tilde{\kappa}^k \subset \kappa^k$ (in this case, we say that $\kappa^k$ is the *primary k-completion of $\rho^k$ (for $\gamma^k$)*); or

(5.20) There is a $j > k$ and a $\kappa^j \in T^j$ such that $\kappa^j$ is a primary $j$-completion of some $\rho^j$ and $\kappa^k$ is an initial derivative of $\kappa^j$. (In this case, we say that $\kappa^k$ is the *k-completion of $\rho^j$*.)

We say that $\kappa^k$ is a *completion* if $\kappa^k$ is a $k$-completion of some $\rho^j$. (We note that if $\kappa$ is a 0-completion, then it must be a 0-completion of the last element, $\rho^k$, of the sequence via which $\kappa^-$ is completion-consistent, and cannot be the 0-completion of any other node. It also follows from (5.18) that for all $j \leqslant k$, there is at most one $j$-completion of $\rho^k$.) ∎

The process of finding a 0-completion of $\eta^k$ may force paths to follow nodes on $T^j$ for all $j \leqslant n$ which were not previously followed. For $j \leqslant k$, the new nodes will be those in the interval $[\text{out}^j(\eta^k), \kappa^j]$, where $\kappa^j$ is the $j$-completion of $\eta^k$. We will not want to switch any of these nodes except for $\kappa^j$, unless we are forced to do so during the backtracking process (it is here that we need to add Condition (5.14) to the definition of PL). Thus we call nodes in this interval *primarily implication-restrained* (Condition (5.21)) if $j=k$ and *hereditarily implication-restrained* (Condition (5.22)) if $j<k$. In addition, we do not want derivatives of implication-restrained nodes to be switched, unless we are forced to switch these derivatives during the backtracking process; so we specify that all derivatives of implication-restrained nodes are also *implication-restrained* (Condition (5.23)).

DEFINITION 5.7. A node $\xi^k \in T^k$ is *primarily implication-restrained* if:

(5.21) There is an $\eta^k \subseteq \xi^k$ which requires extension, but there is no $k$-completion $\kappa^k \subseteq \xi^k$ of $\eta^k$.

$\xi^k$ is *hereditarily implication-restrained* if:

(5.22) There are $j>k$ and $\eta^j$ such that $\text{out}^k(\eta^j) \subseteq \xi^k$, $\eta^j$ requires extension, and there is no $k$-completion $\kappa^k \subseteq \xi^k$ of $\eta^j$.

$\xi^k$ is *inductively implication-restrained* if the following condition holds:

(5.23) $\text{up}^j(\xi^k)$ is implication-restrained for some $j \in (k, n]$.

$\xi^k$ is *implication-restrained* if $\xi^k$ is either primarily, hereditarily, or inductively implication-restrained. $\xi^k$ is *implication-free* if $\xi^k$ is not implication-restrained. (By Definition 2.1, the implication-restrained nodes can be recursively recognized.) ▮

Suppose that $\xi^k \in T^k$. $\xi^k$ is *completion-respecting* if for all $j \in [k, n]$ and all $\rho^j \subseteq \lambda^j(\xi^k)$, if $\rho^j$ requires extension, then $\rho^j$ has a $j$-completion along $\lambda^j(\xi^k)$. It is possible for such a node $\rho^j \subseteq \lambda^j(\xi^k)$ to have a $k$-completion along $\xi^k$ but not to have a $j$-completion along $\lambda^j(\xi^k)$. This will happen only during an iteration of the backtracking process, and in this case, $\rho^j$ will have an $i$-completion along $\lambda^i(\delta^k)$ for all $i \in [k, j)$. Such a $\rho^j$ has already found a $j$-completion, and does not need to find another one; in fact, an attempt to maintain compatibility with its $j$-completion may conflict with being able to carry out a finitary backtracking process. Thus we will need to determine the nodes $\rho^j \subseteq \lambda^j(\xi^k)$ which require extension but do not have $k$-completions along $\xi^k$. These are the nodes for which we need to find $k$-completions, and are placed in the *completion-deficient* set at $\xi^k$. These nodes are ordered into a sequence by the order of the appearance of their images under $\text{out}^k$ on the path of $T^k$ under construction. This ordering is *completion-consistent* if it respects the dimension ordering of the trees on which the nodes appear, refined by the length of nodes on trees of the same dimension. We will show that the backtracking process produces completions in the reverse order to the completion-consistent ordering, if paths through trees are *admissible*, as defined in Definition 5.9.

DEFINITION 5.8. Fix $k \leqslant n$, $\xi^k \in T^k$ and a set $S$ of nodes of $\bigcup \{\rho^j \subseteq \lambda^i(\xi^k): k \leqslant j \leqslant n\}$. We say that $\xi^k$ is *completion-deficient for S* if the following condition holds:

(5.24) For all $j \in [k, n]$ and $\rho^j \subseteq \lambda^j(\xi^k)$, $\rho^j \in S$ iff $\rho^j$ requires extension and has no $k$-completion $\subseteq \xi^k$.

$\xi^k$ is *completion-respecting* if for all $j \in [k, n]$ and $\rho^j \subseteq \lambda^j(\xi^k)$, if $\rho^j$ requires extension, then there is a $j$-completion $\kappa^j \subseteq \lambda^j(\xi^k)$ of $\rho^j$.

Given $S$ such that $\xi^k$ is completion-deficient for $S$, let $\bar{S} = \langle \eta_i : i < m \rangle$ be the linear ordering of $S$ induced by the inclusion ordering on $\mathrm{out}^k(v)$ for $v \in S$. By (2.5) and Lemma 5.6 (Uniqueness of Requiring Extension), this ordering will be well-defined. For all $i < m$, fix $k(i)$ such that $\eta_i \in T^{k(i)}$. (Note that, by Lemma 3.2(ii) (Out) and Lemma 3.1(ii) (Limit Path), this ordering will be independent of $k$ as long as $k \leqslant k(i)$ for all $i < m$.) We say that $\xi^k$ is *completion-consistent* via $\bar{S}$ if the following conditions hold:

(5.25) If $i < j < m$, then $k(i) \leqslant k(j)$.

(5.26) If $i < j < m$ and $k(i) = k(j)$, then $\eta_i \subset \eta_j$.

$\xi^k$ is *hereditarily completion-consistent* if every $\rho^k \subseteq \xi^k$ is completion-consistent. ▮

*Admissible* nodes, as defined below, are nodes which are preadmissible, hereditarily completion-consistent in a uniform manner as specified by Condition (5.27), act in a way to preclude the existence of amenable implication chains along the final paths through the trees as specified in (5.28), and preserve a correspondence between PL sets on consecutive trees, as specified in (5.29)(i–iii). Condition (5.29)(i) specifies that when the extension of a path on $T^k$ causes the path on $T^{k+1}$ to switch and a node to leave a viable PL set on $T^{k+1}$, then a derivative of that node enters a corresponding PL set on $T^k$. If the above happens during the backtracking process for a node, then (5.29)(ii) specifies that immediately at the end of that process, the PL set on $T^k$ for the predecessor of the node requiring extension consists exactly of derivatives of all nodes in a corresponding PL set on $T^{k+1}$ at the beginning of the backtracking process. Furthermore, if no additional nodes need to go through the backtracking process at this point, then (5.29)(iii) specifies that the node completing the backtracking process is implication-free. *Pseudotrue* nodes are nodes which are not involved in the backtracking process, so action of the construction at these nodes is according to the truth of the sentences generating action.

DEFINITION 5.9. Fix $k \leqslant n$ and $\sigma^k \in T^k$, and let $\sigma = \mathrm{out}^0(\sigma^k)$. We say that $\sigma^k$ is *k-completion-free* if for every $j \in [k, n]$, $\lambda^j(\sigma)$ is not a primary completion, and if $k = 0$, we say that $\sigma = \sigma^k$ is *completion-free* if $\sigma$ is 0-completion-free. We say that $\sigma$ is *pseudotrue* if $\sigma$ is preadmissible, completion-consistent via $\langle\ \rangle$, and completion-free. We say that $\sigma$ is *admissible* if $\sigma$ is preadmissible, hereditarily completion-consistent, completion-consistent via a sequence $S$, and the following conditions hold:

(5.27) If $\xi \subset \sigma$ is completion-consistent via $\tilde{S}$ and $\eta \in \tilde{S}$, then either $\eta$ has a 0-completion $\kappa \subseteq \sigma$, or $\eta \in S$.

(5.28) If $\eta \subseteq \sigma$ is pseudotrue, then there is no amenable $j$-implication chain along $\lambda^j(\eta)$ for any $j \leqslant n$.

(5.29) (i) For all $k < n$ and $\mu^k \subset \nu^k \subset \eta^k \subseteq \lambda^k(\sigma) \in T^k$, if $\mathrm{up}(\mu^k) \subset \mathrm{up}(\nu^k)$, $\lambda(\eta^k)$ and $\nu^k$ is implication-free, then

$$\mathrm{PL}(\mathrm{up}(\mu^k), \mathrm{up}(\nu^k)) \subseteq \{\mathrm{up}(\xi^k) : \xi^k \in \mathrm{PL}(\nu^k, \eta^k)\} \cup \mathrm{PL}(\mathrm{up}(\mu^k), \lambda(\eta^k)).$$

(ii) For all $k < n$ and $\mu^k \subset \nu^k = (\eta^k)^- \subset \eta^k \subset \kappa^k \subseteq \lambda^k(\sigma) \in T^k$, if $\eta^k$ requires extension for $\mu^k$ and $\kappa^k$ is the primary completion of $\eta^k$, then

$$\mathrm{PL}(\mathrm{up}(\mu^k), \lambda(\eta^k)) = \{\mathrm{up}(\xi^k) : \xi^k \in \mathrm{PL}(\nu^k; \kappa^k)\}.$$

(iii) If $\eta \subseteq \sigma$ is completion-consistent via $\langle\ \rangle$ and $\eta$ is a 0-completion, then $\eta$ is implication-free.

$\Lambda^0 \in [T^0]$ is *admissible* if every $\sigma \subset \Lambda^0$ is admissible. $\Lambda^k \in [T^k]$ is *admissible* if $\Lambda^k = \lambda^k(\Lambda^0)$ for some admissible $\Lambda^0 \in [T^0]$. ∎

We now show that an amenable $k$-implication chain gives rise to a node on $T^{k-1}$ which requires extension.

LEMMA 5.2 (Requires Extension Lemma). *Fix $k$ such that $0 < k < n$ and fix $\sigma^k \in T^k$. Let $r = \dim(\sigma^k) - 1$, and assume that $k \leqslant r$. Suppose that $\langle\langle \sigma^j, \hat{\sigma}^j, \tau^j\rangle : r \geqslant j \geqslant k\rangle$ is an amenable $k$-implication chain. Let $\eta^{k-1} = \mathrm{out}(\tau^k)$ and let $\nu^{k-1}$ be the principal derivative of $\sigma^k$ along $\eta^{k-1}$. Assume that $\eta = \mathrm{out}^0(\eta^{k-1})$ is preadmissible. Then $\eta^{k-1}$ requires extension for $\nu^{k-1}$.*

*Proof.* Let $\delta^{k-1} = (\eta^{k-1})^-$. As $\eta^{k-1} = \mathrm{out}(\tau^k)$ and as, by (5.8)(ii), $\hat{\sigma}^k = (\tau^k)^-$, $\delta^{k-1}$ is the principal derivative of $\hat{\sigma}^k$ along $\eta^{k-1}$. We verify (5.1)–(5.5).

Condition (5.1) is vacuous. By (5.6) and (5.9), $\sigma^k \equiv \hat{\sigma}^k \equiv \sigma^r$, so by (5.7), $\mathrm{tp}(\nu^{k-1}) \in \{1, 2\}$. Furthermore, $\nu^{k-1}$ and $\delta^{k-1}$ are, respectively, the principal derivatives of $\sigma^k$ and $\hat{\sigma}^k$ along $\eta^{k-1}$, so $\nu^{k-1} \equiv \delta^{k-1}$. By (5.6) and (5.9), $\mathrm{up}^{r+1}(\nu^{k-1}) \neq \mathrm{up}^{r+1}(\delta^{k-1})$, so $\mathrm{up}(\nu^{k-1}) \neq \mathrm{up}(\delta^{k-1})$. By (5.11), $\hat{\sigma}^k$ has finite outcome along $\tau^k$ and $\sigma^k$ has infinite outcome along $\tau^k$, so by

(2.4) and as $\nu^{k-1}$ and $\delta^{k-1}$ are, respectively, the principal derivatives of $\sigma^k$ and $\hat{\sigma}^k$ along $\eta^{k-1}$, $\nu^{k-1}$ has finite outcome along $\eta^{k-1}$ and $\delta^{k-1}$ has infinite outcome along $\eta^{k-1}$. Hence (5.2) holds.

By Lemma 3.2(i) (Out) and hypothesis, $\lambda(\eta^{k-1}) = \tau^k \supset \sigma^k = \mathrm{up}(\nu^{k-1})$. By (5.2), $\nu^{k-1}$ must be both the initial and principal derivative of $\mathrm{up}(\nu^{k-1})$ along $\lambda(\eta^{k-1})$, so cannot be the first node in a primary $\lambda(\eta^{k-1})$-link. Condition (5.3) now follows from Lemma 4.3(i)(d) (Link Analysis). Condition (5.4) is vacuous as $k-1<r$. Condition (5.5) follows from the hypothesis. The minimality of $\mathrm{lh}(\nu^{k-1})$ follows from the uniqueness of $\sigma^r$ for $\hat{\sigma}^r$ given by Definition 5.4, if $k=r$. And if $k<r$, then the minimality of $\mathrm{lh}(\nu^{k-1})$ follows from (5.15) and the fact that, by Definition 5.6, a primary completion along a preadmissible path is the primary completion of exactly one node. ▌

Suppose that $\eta^k$ requires extension for $\nu^k$, $\kappa^k$ is the $k$-completion of $\eta^k$, and $(\xi^k)^- = \kappa^k$. If $\kappa^k$ has finite outcome along $\xi^k$, then a $k$-implication chain will have been formed along $\xi^k$. Otherwise, we show that $[\nu^k, \kappa^k]$ is a primary $\xi^k$-link.

Lemma 5.3 (Implication Chain Lemma). *Fix $k \leqslant r < n$ and $\nu^k \subset \delta^k \subset \eta^k \subset \kappa^k \subset \xi^k \in T^k$ such that $k < \dim(\nu^k) = r+1$, $(\eta^k)^- = \delta^k$, $(\xi^k)^- = \kappa^k$, and $\mathrm{out}^0(\xi^k)$ is preadmissible. Assume that $\eta^k$ requires extension for $\nu^k$, and that $\kappa^k$ is the $k$-completion of $\eta^k$ for $\nu^k$. Then:*

(i) *If $\kappa^k$ has infinite outcome along $\xi^k$, then $[\nu^k, \kappa^k]$ is a primary $\xi^k$-link.*

(ii) *If $\kappa^k$ has finite outcome along $\xi^k$, then there is an amenable $k$-implication chain $\langle\langle \sigma^j, \hat{\sigma}^j, \tau^j\rangle : r \geqslant j \geqslant k\rangle$ such that $\tau^k = \xi^k$, $\hat{\sigma}^k = \kappa^k$, and $\sigma^k = \delta^k$.*

*Now fix $\bar{\delta}^k \subset \bar{\kappa}^k \subset \bar{\xi}^k \in T^k$ such that $(\bar{\xi}^k)^- = \bar{\kappa}^k$, $\bar{\kappa}^k$ has finite outcome along $\bar{\xi}^k$, $\bar{\kappa}^k$ is an amenable pseudocompletion of $\bar{\delta}^k$, and for all $i \leqslant k$, the principal derivative of $\bar{\kappa}^k$ along $\mathrm{out}^i(\bar{\xi}^k)$ is implication-free. Then:*

(iii) *$\langle\langle \bar{\delta}^k, \bar{\kappa}^k, \bar{\xi}^k\rangle\rangle$ is an amenable $k$-implication chain.*

*Proof.* We proceed by induction on $n-k$, and then by induction on $\mathrm{lh}(\kappa^k)$.

(i) By (5.19), $\mathrm{up}(\nu^k) = \mathrm{up}(\kappa^k)$, and by (5.2), $\nu^k$ is the initial derivative of $\mathrm{up}(\nu^k)$ along $\xi^k$. Since $\kappa^k$ has infinite outcome along $\xi^k$, $[\nu^k, \kappa^k]$ is a primary $\xi^k$-link.

(ii) We first show that (5.6)–(5.12) hold. By (5.19), $\mathrm{up}(\nu^k) = \mathrm{up}(\kappa^k)$. Hence (5.6) follows from (5.2) if $k=r$, and from (5.5)(ii) and (5.6) inductively if $k<r$. Condition (5.7) follows from (5.2), the definition of $r$, and

(5.5)(i) if $k=r$, and (5.5)(ii) and (5.7) inductively if $k<r$. Condition (5.8) follows from the definitions of $\sigma^k$, $\hat{\sigma}^k$, and $\tau^k$ in (ii). Conditions (5.9) and (5.12) follow from (5.5)(ii) and (5.19). Condition (5.10) follows from the definitions of $\sigma^k$, $\hat{\sigma}^k$, and $\tau^k$ in (ii), (5.1), and hypothesis if $k=r$, and by (5.5)(ii) and (5.10) inductively if $k<r$. Condition (5.11) follows from the definitions of $\sigma^k$, $\hat{\sigma}^k$, and $\tau^k$ in (ii), (5.2), and hypothesis. Hence $\langle\langle\sigma^j, \hat{\sigma}^j, \tau^j\rangle$: $r\geqslant j\geqslant k\rangle$ is a $k$-implication chain.

Condition (5.15) follows from hypothesis, so this $k$-implication chain is amenable. We complete the proof of (ii) by verifying the minimality condition for the case $k=r$. By the minimality of the choice of $v^r$ in Definition 5.1 and by (5.15), the minimality condition can only fail if there is a $\mu^r\subset v^r$ such that $\langle\langle\mu^r, \kappa^r, \xi^r\rangle\rangle$ is an amenable $r$-implication chain, which we assume in order to obtain a contradiction. Now by (5.2), $v^r$ is an initial derivative, and by Definition 5.6, $\mathrm{up}(v^r)=\mathrm{up}(\kappa^r)$ and $\kappa^r$ is not an initial derivative. Hence by Definition 5.2, $\kappa^r$ cannot be a pseudocompletion. Thus (5.16) must fail for $\langle\langle\mu^r, \kappa^r, \xi^r\rangle\rangle$. By Definition 5.6, $\kappa^r$ is the primary completion of only one node, and by hypothesis, that node must be $\eta^r$. Hence by (5.15) for $\langle\langle\mu^r, \kappa^r, \xi^r\rangle\rangle$, $\eta^r$ must require extension for $\mu^r$. By Definition 5.1, $\eta^r$ requires extension for at most one node. But $\eta^r$ requires extension for $v^r$, so $v^r=\mu^r$, yielding a contradiction.

(iii) Immediate from hypothesis and the definition of amenable pseudocompletions (Definitions 5.2 and 5.4). ▌

We will need to know that admissible paths are always compatible with completions, except when we are iterating the backtracking process to try to eliminate an amenable implication chain. In the latter case, by (5.18) and (5.24), the only completions which may be incompatible with the path under construction are the primary completions.

LEMMA 5.4 (Compatibility Lemma). *Fix $\rho\in T^0$ such that $\rho$ is preadmissible. Fix $i\leqslant n$, $\beta\subseteq\rho$, and $\eta^i\subseteq\lambda^i(\beta)$ such that $\eta^i$ requires extension, and suppose that $\kappa\subseteq\rho$ is the 0-completion of $\eta^i$. Fix $v<i$, let $\eta^v=\mathrm{out}^v(\eta^i)$, and suppose that $\eta^v\subseteq\lambda^v(\rho)=\rho^v$. Then for all $j\leqslant v$, $\rho^j=\lambda^j(\rho)\supseteq\lambda^j(\kappa)=\kappa^j$.*

*Proof.* We proceed by induction on $j\leqslant v$, noting that, by hypothesis, the lemma holds for $j=0$. Assume that $j>0$. As $\eta^v\subseteq\rho^v$, it follows from (2.5) that $\eta^q=\mathrm{out}^q(\eta^i)\subseteq\rho^q$ for all $q\leqslant j$. By (5.22), every $\xi^j$ such that $\eta^j\subseteq\xi^j\subset\kappa^j$ is implication-restrained. We note, by (5.18) and (5.25), that if $u<n$, $\sigma^u\in T^u$ requires extension and has $u$-completion $\tau^u$, $\sigma=\mathrm{out}^0(\sigma^u)$, $\tau$ is the 0-completion of $\sigma^u$ and is preadmissible, and $\sigma\subset\delta\subseteq\tau$, then $\delta$ cannot be $t$-switching for any $t\leqslant u$.

As $\rho$ is preadmissible, $\rho^-$ is admissible and thus hereditarily completion-consistent. Fix $\xi^j$ such that $\eta^j\subseteq\xi^j\subset\kappa^j$. As $j<i$, it follows from the above

paragraph that $\xi^j$ is not $(j+1)$-switching, so if $\xi^j$ is the principal derivative of up$(\xi^j)$ along $\kappa^j$, then $\xi^j$ must be the initial derivative of up$(\xi^j)$ along $\kappa^j$; thus there is no $\mu^j$ such $[\mu^j, \xi^j]$ is a primary $\kappa^j$-link. By (5.19) and (5.25) and as $j<i$, $\xi^j$ is not a primary completion or an amenable pseudocompletion. Fix $\delta$ such that $\kappa \subset \delta \subseteq \rho$ and $\mathrm{up}^j(\delta^-) = \xi^j$. If $\delta^-$ is primarily or hereditarily implication-restrained, then by (5.18), $\delta$ will not switch $\xi^j$. Otherwise, $\delta^-$ will be inductively implication-restrained. We will show that $\delta^-$ is neither a primary 0-completion nor an amenable pseudocompletion. It will then follow from (5.17)(ii) that $\delta$ does not switch $\xi^j$. Thus as $\kappa^{j-1} \subseteq \rho^{j-1}$ by induction, it follows from (2.4) that $\rho^j \supseteq \kappa^j$.

We complete the proof of the lemma by assuming that $\delta^-$ is either a primary 0-completion or an amenable pseudocompletion, and obtaining a contradiction. First assume that $\dim(\xi^j) > j$. If $j$ is even, then by repeated applications of (5.5)(ii), (5.9) and (5.15) ((5.16) cannot apply at any $t<j$), it follows that $\xi^j$ is a primary completion or an amenable pseudocompletion, contrary to the preceding paragraph. Suppose that $j$ is odd. By repeated applications of (5.5)(ii), (5.9) and (5.15) ((5.16) cannot apply at any $t<j$), it follows that $\mathrm{up}^{j-1}(\delta^-)$ is a primary completion, and that the immediate successor of $\xi^j$ along $\kappa^j$ requires extension, contrary to (5.25) which would require $j \geqslant i$. Thus in either case, we have a contradiction.

Now suppose that $\dim(\xi^j) \leqslant j$. By Lemma 3.1(i) (Limit Path), $\xi^j$ has an initial derivative $\xi^{j-1} \subset \kappa^{j-1}$, and as $\eta^{j-1} = \mathrm{out}(\eta^j)$ and $\eta^j \subseteq \xi^j$, it follows from (2.5) and Lemma 3.1(i) (Limit Path) that $\eta^{j-1} \subseteq \xi^{j-1}$. If $\dim(\xi^j) < j$, then by (2.9), $\xi^{j-1}$ is the only derivative of $\xi^j$ along $\kappa^{j-1}$, so it follows by induction that $\delta^-$ is neither a primary 0-completion nor an amenable pseudocompletion. Suppose that $\dim(\xi^j) = j$. By (5.9), (5.1), and (5.10), $\xi^{j-1}$ would have to be implication-free. But by Lemma 3.1(i) (Limit Path), $\xi^{j-1} \in [\eta^{j-1}, \kappa^{j-1}]$, so is hereditarily implication-restrained, yielding the desired contradiction. ∎

One consequence of the next lemma is that if $\eta$ is admissible and pseudotrue, then for all $j \leqslant n$, if $\rho^j \subseteq \lambda^j(\eta)$ requires extension, then $\rho^j$ has a primary completion along $\lambda^j(\eta)$. Hence for pseudotrue nodes, completion-respecting and completion-consistent via $\langle\ \rangle$ coincide. We will need a somewhat more general statement.

Lemma 5.5 (Completion-Respecting Lemma). (i) *Fix $k<n$ and $\delta^k \subset \xi^k \subseteq \kappa^k \in T^k$ such that $\kappa = \mathrm{out}^0(\kappa^k)$ is admissible, $\delta^k$ and $\xi^k$ both require extension, and $\kappa^k$ is the primary completion of $\delta^k$. Then $\xi^k$ has a primary completion $\tau^k \subset \kappa^k$, and $\tau^k$ has infinite outcome along $\kappa^k$.*

(ii) *Fix $\eta \in T^0$ such that $\eta$ is preadmissible and completion-consistent via $\langle\ \rangle$. Suppose that $\rho^j \subseteq \lambda^j(\eta)$ requires extension for $v^j$ and $\gamma^j = (\rho^j)^-$. If $\eta$ is the 0-completion corresponding to a primary k-completion $\hat{\sigma}^k$ and $\hat{\sigma}^k$*

*is the primary completion of the immediate successor of* $\sigma^k$ *along* $\hat{\sigma}^k$, *then assume further that it is neither the case that* $j \geqslant k$, $j-k$ *is odd and* $\mathrm{up}^j(\hat{\sigma}^k) = \gamma^j$, *nor the case that* $j \geqslant k$, $j-k$ *is even and* $\mathrm{up}^j(\sigma^k) = \gamma^j$. *Then* $\rho^j$ *has a primary completion* $\kappa^j \subset \lambda^j(\eta)$ *which has infinite outcome along* $\lambda^j(\eta)$.

(iii) *Fix* $\xi, \eta \in T^0$ *such that* $\eta$ *and* $\xi$ *are preadmissible and completion-consistent via* $S = \langle\ \rangle$, *and* $\xi^- = \eta$. *Suppose that* $\rho^j \subseteq \lambda^j(\eta)$ *requires extension. Then* $\rho^j$ *has a primary completion* $\kappa^j \subset \lambda^j(\xi)$ *which has infinite outcome along* $\lambda^j(\xi)$.

*Proof.* (i) By (5.26) and Definition 5.6, $\xi^k$ has a primary completion $\tau^k \subset \kappa^k$. By (5.18)(i) and as, by (5.18), (5.24), and (5.25), if $\kappa$ is the 0-completion corresponding to $\kappa^k$, then no node in $(\mathrm{out}^0(\delta^k), \kappa]$ can be $v$-switching for any $v \leqslant k$, $\tau^k$ has infinite outcome along $\kappa^k$.

(ii, iii) We prove (ii), and indicate the modifications needed for (iii) in parentheses. We assume that $\rho^j$ satisfies the hypotheses of (ii) or (iii), and either $\rho^j$ has no primary completion $\kappa^j \subset \lambda^j(\eta)$ ($\lambda^j(\xi)$, resp.), or that $\kappa^j$ exists and has finite outcome along $\lambda^j(\eta)$ ($\lambda^j(\xi)$, resp.), and derive a contradiction under the assumption, in the proof of (ii), that the exclusionary conditions in (ii) fail. (For (iii), fix $\nu^j$ and $\gamma^j$ such that $\rho^j$ requires extension for $\nu^j$ and $\gamma^j = (\rho^j)^-$.) Without loss of generality, we may assume that $j$ is the smallest number for which the conclusion fails for some $\rho^j \subseteq \lambda^j(\eta)$ ($\lambda^j(\xi)$, resp.) satisfying the hypothesis of (ii) ((iii), resp.). As $\eta$ is completion-consistent via $\langle\ \rangle$, $\rho^j$ has a 0-completion $\kappa \subseteq \eta$. Now for (ii), $\kappa \neq \eta$, else $\eta$ would be the 0-completion corresponding to the primary $j$-completion of $\rho^j$ and so $k = j$, $\hat{\sigma}^k = \kappa^j = \mathrm{up}^j(\eta)$, and $\sigma^k = \gamma^j$, contrary to hypothesis. Hence $\kappa \subset \eta$ ($\kappa \subset \xi$, resp.). If $j = 0$, then by (5.17)(i) or (5.18)(i), the immediate successor of $\kappa$ along $\eta$ ($\xi$, resp.) switches $\mathrm{up}^{j+1}(\kappa)$; so $\kappa$ has infinite outcome along $\eta$. Hence $j > 0$.

By Lemma 5.4 (Compatibility), $\rho^j$ has a $(j-1)$-completion $\kappa^{j-1} \subseteq \eta^{j-1} = \lambda^{j-1}(\eta)$, and by Definition 5.6, $\kappa^{j-1}$ is an initial derivative of the primary completion $\kappa^j$ of $\rho^j$, and $\kappa$ is an initial derivative of $\kappa^{j-1}$. As $\eta \supset \kappa$ ($\xi \supset \kappa$, resp.), it follows from (2.4) that $\kappa^{j-1} \subset \eta^{j-1}$ ($\kappa^{j-1} \subset \xi^{j-1} = \lambda^{j-1}(\xi)$, resp.). Fix $\tau^{j-1} \subseteq \eta^{j-1}$ ($\xi^{j-1}$, resp.) such that $(\tau^{j-1})^- = \kappa^{j-1}$. By Lemma 3.1(ii) (Limit Path), $(\lambda(\tau^{j-1}))^- = \kappa^j$.

We assume that all derivatives of $\kappa^j$ along $\eta^{j-1}$ ($\xi^{j-1}$, resp.) have finite outcome along $\eta^{j-1}$ ($\xi^{j-1}$, resp.), and derive a contradiction. Under this assumption and by (5.19), there is a primary $\lambda(\tau^{j-1})$-link $[\mu^j, \kappa^j]$ which restrains $\rho^j$ with $\mu^j \subset \rho^j \subset \kappa^j$. As $\rho^j \subseteq \lambda^j(\eta)$ ($\lambda^j(\xi)$, resp.), it follows from (2.6) and since $\tau^{j-1} \subseteq \eta^{j-1}$ ($\xi^{j-1}$, resp.) that $[\mu^j, \kappa^j]$ is a $\lambda(\beta^{j-1})$-link for all $\beta^{j-1}$ such that $\tau^{j-1} \subseteq \beta^{j-1} \subseteq \eta^{j-1}$ ($\xi^{j-1}$, resp.), so $[\mu^j, \kappa^j]$ is a $\lambda^j(\eta)$-link ($\lambda^j(\xi)$-link, resp.). But then $\kappa^j \subset \lambda^j(\eta)$ ($\lambda^j(\xi)$, resp.), and by (2.4), $\kappa^j$ has infinite outcome along $\lambda^j(\eta)$ ($\lambda^j(\xi)$, resp.), contrary to the choice of $j$.

We conclude that there is a derivative $\bar{\kappa}^{j-1} \subset \eta^{j-1}$ ($\xi^{j-1}$, resp.) of $\kappa^j$ which has infinite outcome along $\eta^{j-1}$ ($\xi^{j-1}$, resp.). Fix $\bar{\tau}^{j-1} \subseteq \eta^{j-1}$ ($\xi^{j-1}$, resp.) such that $(\bar{\tau}^{j-1})^- = \bar{\kappa}^{j-1}$. By Lemma 3.1(ii) (Limit Path), $(\lambda(\bar{\tau}^{j-1}))^- = \kappa^j$ and $\kappa^j$ has finite outcome along $\lambda(\bar{\tau}^{j-1})$. Hence by Lemma 5.3(ii) (Implication Chain) and Lemma 5.2 (Requires Extension), $\bar{\tau}^{j-1}$ requires extension for some derivative $\gamma^{j-1}$ of $\gamma^j$. We assume that one of the exclusionary conditions of (ii) holds for $\bar{\tau}^{j-1}$, and derive a contradiction. By (5.9), $\dim(\eta) > j$, so (5.5)(ii) must hold for the immediate successor of $\sigma^k$ along $\hat{\sigma}^k$; fix the corresponding $(k+1)$-implication chain $\langle\langle \sigma^i, \hat{\sigma}^i, \alpha^i \rangle$: $r \geqslant i \geqslant k+1\rangle$. First suppose that $k \leqslant j-1$, $(j-1)-k$ is odd, and $\mathrm{up}^{j-1}(\hat{\sigma}^k) = \bar{\kappa}^{j-1}$. By (5.9), $\mathrm{up}^{j-1}(\hat{\sigma}^k) = \sigma^{j-1}$ and so $\mathrm{up}^j(\hat{\sigma}^k) = \hat{\sigma}^j = \kappa^j$. Thus by (5.5), the failure of (5.16), and our assumptions, $\hat{\sigma}^j = \kappa^j$ is the primary completion both of the immediate successor of $\gamma^j$ along $\hat{\sigma}^j$, and of the immediate successor of $\sigma^j$ along $\hat{\sigma}^j$, so $\gamma^j = \sigma^j = \mathrm{up}^j(\sigma^k)$, $k \leqslant j$, and $k-j$ is even, and the exclusionary conditions of (ii) hold for $\rho^j$, contrary to our assumption. Finally, suppose that $k \leqslant j-1$, $k-(j-1)$ is even, and $\mathrm{up}^{j-1}(\sigma^k) = \bar{\kappa}^{j-1}$. By (5.9), $\mathrm{up}^{j-1}(\sigma^k) = \sigma^{j-1}$ and so $\mathrm{up}^j(\sigma^k) = \hat{\sigma}^j = \kappa^j$. Thus by (5.5), the failure of (5.16), and our assumptions, $\hat{\sigma}^j = \kappa^j$ is the primary completion both of the immediate successor of $\gamma^j$ along $\hat{\sigma}^j$, and of the immediate successor of $\sigma^j$ along $\hat{\sigma}^j$, so $\gamma^j = \sigma^j = \mathrm{up}^j(\hat{\sigma}^k)$, $k \leqslant j$, and $k-j$ is odd, and the exclusionary conditions of (ii) hold for $\rho^j$, contrary to our assumption.

By the minimality of the choice of $j$ and as $\bar{\tau}^{j-1}$ requires extension for some derivative $\gamma^{j-1}$ of $\gamma^j$, it follows that $\bar{\tau}^{j-1}$ has a primary completion $\bar{\gamma}^{j-1}$ which has infinite outcome along $\eta^{j-1}$ ($\xi^{j-1}$, resp.). By (5.19), $\mathrm{up}(\bar{\gamma}^{j-1}) = \gamma^j$, so by (2.4) and as $\gamma^j \subseteq \lambda^j(\eta)$, $\gamma^j$ has finite outcome along $\lambda^j(\eta)$ ($\lambda^j(\xi)$, resp.). But by (5.2), $\gamma^j$ has infinite outcome along $\rho^j \subseteq \lambda^j\eta)$ ($\lambda^j(\xi)$, resp.), a contradiction. ∎

We now show that at most one new string requires extension at any admissible $\eta$.

Lemma 5.6 (Uniqueness of Requiring Extension Lemma). *Fix $\eta \in T^0$ such that $\eta$ is preadmissible and* $\mathrm{lh}(\eta) > 0$. *Let $\eta$ and $\eta^-$ be completion-consistent via $S$ and $\tilde{S}$, respectively. Suppose that $i \leqslant j$ and $\lambda^i(\eta), \lambda^j(\eta) \in S \backslash \tilde{S}$. Then $i = j$.*

*Proof.* For all $u \leqslant n$, let $\eta^u = \lambda^u(\eta)$. Fix $p$ and $s$ as in Lemma 3.3 ($\lambda$-Behavior) for $\eta$. First assume that $j > s$, in order to obtain a contradiction. Then by Lemma 3.3 ($\lambda$-Behavior), there will be a $\xi \subset \eta$ such that $\lambda^j(\xi) = \eta^j$. By (5.27) for the admissible node $\eta^-$ and as $\eta^j \notin \tilde{S}$, there will be a 0-completion $\kappa$ of $\eta^j$ such that $\xi \subset \kappa \subseteq \eta^-$. Hence by Definition 5.8, $\eta^j \notin S$, contrary to hypothesis.

We conclude that $j \leqslant s$, and hence, by Lemma 3.3 ($\lambda$-Behavior), that $(\eta^j)^- = \mathrm{up}^j((\eta^i)^-)$. We assume that $j > i$ and derive a contradiction. Let $\tau^i = \eta^i$. By (5.5)(ii), there is an $r > i$ and an amenable $(i+1)$-implication chain $\langle\langle \sigma^t, \hat{\sigma}^t, \tau^t \rangle : r \geqslant t \geqslant i+1 \rangle$ such that $\tau^i = \mathrm{out}(\tau^{i+1})$. By Lemma 3.2(i) (Out), $\tau^{i+1} = \lambda(\mathrm{out}(\tau^{i+1})) = \lambda(\tau^i) = \eta^{i+1}$.

We now claim that if $i < t \leqslant j$ and $t - i$ is odd then:

$$\tau^t \subseteq \eta^t. \tag{5.30}$$

By the preceding paragraph, (5.30) is true for $t = i+1$. We proceed by induction, assuming that (5.30) holds, and verifying (5.30) with $t+2$ in place of $t$, under the assumption that $t+2 \leqslant j$. Let $\rho^t = (\tau^t)^-$, $\rho^{t+1} = \mathrm{up}(\rho^t)$ and $\rho^i = (\tau^i)^-$. By (5.8)(ii), $\rho^t = \hat{\sigma}^t$, and by (5.9), $\rho^t = \mathrm{up}^t(\rho^i)$ and $\rho^{t+1} = \mathrm{up}^{t+1}(\rho^i) = \sigma^{t+1}$. By Lemma 4.5 (Free Extension), $\rho^i$ is $\tau^i$-free. Hence $\rho^{t+1}$ is $\eta^{t+1}$-free. By (5.30) and (2.4), $\rho^{t+1} \subset \lambda^{t+1}(\eta)$. Hence we can fix $\bar{\tau}^{t+1} \subseteq \eta^{t+1}$ such that $(\bar{\tau}^{t+1})^- = \rho^{t+1}$.

By (5.2), $\rho^i$ has infinite outcome along $\tau^i$, so as $\tau^i = \eta^i$ and $\rho^i$ is $\tau^i$-free, it follows from (2.4) that $\rho^{i+1} = \mathrm{up}(\rho^i)$ has finite outcome along $\eta^{i+1}$; thus by (5.11) and (5.30), $\rho^t$ has finite outcome along $\eta^t$. As $\rho^i$ is $\tau^i$-free, $\rho^t = \mathrm{up}^t(\rho^i)$ must be $\eta^t$-free. So as $\tau^t \subseteq \eta^t$, it follows from the definition of links that all derivatives of $\rho^{t+1}$ along $\eta^t$ have finite outcome along $\eta^t$. Hence by (2.4), $\bar{\tau}^{t+1} = \rho^{t+1}{}^\wedge\langle \gamma^t \rangle \subseteq \eta^{t+1}$, where $\gamma^t \subseteq \eta^t$ and $(\gamma^t)^-$ is the initial derivative of $\rho^{t+1}$ along $\eta^t$. Thus $(\gamma^t)^- \subseteq \rho^t$, and so by (5.30), $\gamma^t \subseteq \tau^t$. By (5.12) and (5.30), $\mathrm{out}(\tau^{t+1}) \subseteq \tau^t \subseteq \eta^t$ and $\rho^{t+1} \subset \tau^{t+1}$. Now all derivatives of $\rho^{t+1}$ have finite outcome along $\mathrm{out}(\tau^{t+1}) \subseteq \eta^t$. Hence by (2.4), $\bar{\tau}^{t+1} \subseteq \tau^{t+1}$. Now by (5.12), $\mathrm{out}(\tau^{t+2}) \subseteq \tau^{t+1}$, and by (5.5)(ii), $(\mathrm{out}(\tau^{t+2}))^- = \rho^{t+1}$. So as $(\bar{\tau}^{t+1})^- = \rho^{t+1}$ and $\bar{\tau}^{t+1}$, $\mathrm{out}(\tau^{t+2}) \subseteq \tau^{t+1}$, we have $\bar{\tau}^{t+1} = \mathrm{out}(\tau^{t+2})$. As $\rho^{t+2} = \mathrm{up}(\rho^{t+1})$ is $\eta^{t+2}$-free and $\rho^{t+1}$ has infinite outcome along $\bar{\tau}^{t+1} \subseteq \tau^{t+1} \wedge \eta^{t+1}$, it follows from (2.4) that $\tau^{t+2} = \rho^{t+2}{}^\wedge\langle \bar{\tau}^{t+1} \rangle \subseteq \eta^{t+2}$, verifying (5.30) with $t+2$ in place of $t$. Furthermore, we note that $\rho^{t+2}$ has finite outcome along $\tau^{t+2}$, and by (5.8) and (5.9), $\rho^{t+2} = \mathrm{up}(\rho^{t+1}) = \mathrm{up}^{t+2}(\rho^i)$. Hence since $\mathrm{up}^j((\eta^i)^-) = (\eta^j)^-$ and by (5.2) and (5.30), $j > t+2$.

We conclude that $j - i$ is even, and that (5.30) holds for $t = j-1$. By (5.12) and (5.30), $\mathrm{out}(\tau^j) \subset \tau^{j-1} \subseteq \eta^{j-1}$. Iterating (5.5)(ii) and recalling that $(\eta^j)^- = \mathrm{up}^j((\eta^i)^-)$ and that $j - i$ is even, we see that $(\eta^j)^- = \sigma^j \subset \hat{\sigma}^j \subset \tau^j$; hence $\eta^j \not\supseteq \tau^j$ and so by (2.4) and as $\mathrm{out}(\tau^j) \subset \eta^{j-1} \subseteq \mathrm{out}(\eta^j)$, it must be the case that $\eta^j \not\subset \tau^j$ and so that $\eta^j \wedge \tau^j = \sigma^j$. By (5.2), $\sigma^j$ has infinite outcome $\beta^{j-1}$ along $\eta^j$, so by (2.4), all derivatives of $\sigma^j$ which are $\subset \eta^{j-1}$ must have finite outcome along $\eta^{j-1}$, and $(\beta^{j-1})^-$ is the initial derivative of $\sigma^j$ along $\eta^{j-1}$. As $\tau^{j-1} \subseteq \eta^{j-1}$, all derivatives of $\sigma^j$ which are $\subset \tau^{j-1}$ must have finite outcome along $\tau^{j-1}$. By Lemma 3.1(ii) (Limit Path), $\beta^{j-1} \subseteq \tau^{j-1}$ and by (2.4), $(\beta^{j-1})^-$ is the principal derivative of $\sigma^j$

along $\tau^{j-1}$. Hence since $\text{out}(\tau^j) \subset \tau^{j-1}$ and by (2.4), $\sigma^j \wedge \langle \beta^{j-1} \rangle \subseteq \tau^j$, so $\sigma^j \wedge \langle \beta^{j-1} \rangle \subseteq \tau^j \wedge \eta^j$, contradicting the fact that $\eta^j \wedge \tau^j = \sigma^j$. ∎

In order to show that the backtracking process is finitary, we will need to know that if a node requires extension, then its immediate predecessor is not a primary $j$-completion.

LEMMA 5.7 (Primary Completion Lemma). *Fix $j \leqslant n$ and $\eta^j \in T^j$ such that $\eta^j$ is preadmissible and requires extension, and let $\eta = \text{out}^0(\eta^j)$ and $\delta^j = (\eta^j)^-$. Then:*

(i) *$\delta^j$ is not a primary j-completion or an amenable pseudocompletion.*

(ii) *If $\eta^j \neq \lambda^j(\eta^-)$, then either $\eta$ is switching or $\eta^-$ is not primarily or hereditarily implication-restrained; hence $\eta$ is not a* 0*-completion.*

*Proof.* We prove (i) and (ii) simultaneously by induction on $r - j$. For all $i \leqslant n$, let $\eta^i = \lambda^i(\eta)$.

(i) Let $r = \dim(\delta^j) - 1$. Let $\eta^j$ require extension for $\mu^j$. To see that $\delta^j$ is not a primary $j$-completion or a pseudocompletion, we proceed by induction on $r - j$ and then by induction on $\text{lh}(\eta^j)$, assuming to the contrary and deriving a contradiction. Let $\eta^j$ require extension for $\mu^j$. There are several cases.

*Case* 1: $j = r$. There are two subcases, depending on whether we assume that $\delta^r$ is a primary completion or a pseudocompletion.

*Subcase* 1.1: $\delta^r$ is a primary completion of some $\rho^r$ which requires extension for some $\nu^r$. By Definition 5.6, $\text{up}(\nu^r) = \text{up}(\delta^r)$; and by (5.2) and the hypothesis of the lemma, $\delta^r$ has infinite outcome along $\eta^r$. Hence $[\nu^r, \delta^r]$ is a primary $\eta^r$-link. By (5.2), $\text{up}(\mu^r) \neq \text{up}(\delta^r)$, so $\mu^r \neq \nu^r$. By (5.3), it now follows that $\mu^r \subset \nu^r$. We show that (5.1)–(5.5) hold for $\mu^r \subset \gamma^r = (\rho^r)^- \subset \rho^r$, and thus contradict the minimality of $\text{lh}(\nu^r)$ for $\rho^r$ in Definition 5.1.

By Definition 5.1 and the preceding paragraph, $\mu^r \subset \nu^r \subset \rho^r \subset \delta^r$, and by (5.2), $\gamma^r = (\rho^r)^-$ has infinite outcome along $\eta^r$. Hence $\text{up}(\gamma^r) \neq \text{up}(\mu^r)$, else by (5.2) and the preceding paragraph, $[\nu^r, \delta^r]$ and $[\mu^r, \gamma^r]$ would be primary $\eta^r$-links, contradicting Lemma 4.1 (Nesting). (5.1)–(5.3) and (5.5) can now be routinely verified, using those same conditions and the assumptions that $\text{up}(\nu^r) = \text{up}(\delta^r)$, $\eta^r$ requires extension for $\mu^r$, and $\rho^r$ requires extension for $\nu^r$. By (2.7), (5.3), and Lemma 4.3(i)(a) (Link Analysis), $\text{up}(\mu^r) \subseteq \lambda(\mu^r), \lambda(\eta^r)$, so by (2.6), $\text{up}(\mu^r) \subseteq \lambda(\rho^r)$; and by (5.3) and Lemma 4.3(i)(a) (Link Analysis), $\text{up}(\nu^r) \subset \lambda(\rho^r)$. Thus $\text{up}(\mu^r)$ and $\text{up}(\nu^r)$ are comparable. By (5.2), $\mu^r$ and $\nu^r$ are both initial derivatives, so by Lemma 5.1(i) (Limit Path) and as $\mu^r \subset \nu^r$, it follows that $\text{up}(\mu^r) \subset \text{up}(\nu^r)$. As $(\eta^r)^- = \delta^r$ and $\text{up}(\nu^r) = \text{up}(\delta^r)$, it follows from Lemma 4.5 (Free

Extension) that $\mathrm{up}(\nu^r) \subseteq \lambda(\eta^r)$. By (5.2), $\nu^r$ has finite outcome along $\rho^r$ and is the principal derivative of $\mathrm{up}(\nu^r)$ along $\rho^r$; so by (2.4), $\mathrm{up}(\nu^r)$ has infinite outcome along $\lambda(\rho^r)$. Let $\beta^r$ be the immediate successor of $\nu^r$ along $\rho^r$. By (2.4), $\lambda(\beta^r)$ is the immediate successor of $\mathrm{up}(\nu^r)$ along $\lambda(\rho^r)$, and by (5.1), $\mathrm{out}^0(\beta^r) = \mathrm{out}^0(\lambda(\beta^r))$ is pseudotrue. By (5.1), $\nu^r$ is implication-free, so by (5.23), $\mathrm{up}(\nu^r)$ is implication-free. Hence by Lemma 5.5(iii) (Completion-Respecting) and Lemma 5.1(viii),(i) (PL Analysis),

$$\begin{aligned}\mathrm{PL}(\mathrm{up}(\mu^r), \lambda(\rho^r)) &\subseteq \mathrm{PL}(\mathrm{up}(\mu^r), \mathrm{up}(\nu^r)) \cup \{\mathrm{up}(\nu^r)\} \cup \mathrm{PL}(\mathrm{up}(\nu^r), \lambda(\rho^r))\\ &\subseteq \mathrm{PL}(\mathrm{up}(\mu^r), \lambda(\eta^r)) \cup \{\mathrm{up}(\nu^r)\} \cup \mathrm{PL}(\mathrm{up}(\nu^r), \lambda(\rho^r)).\end{aligned}$$

Thus (5.4) for $\mu^r \subset \gamma^r \subset \rho^r$ follows from Lemma 2.2(i) (Interaction) and (5.4) for $\mu^r \subset \delta^r \subset \eta^r$ and for $\nu^r \subset \gamma^r \subset \rho^r$, contradicting the minimality of $\mathrm{lh}(\nu^r)$ for $\rho^r$ in Definition 5.1.

*Subcase* 1.2: $\delta^r$ is an amenable pseudocompletion. Let $\delta^r$ be a pseudocompletion of $\nu^r$. By (5.2) and (5.11)(i), $\mu^r$ has finite outcome along $\eta^r$ and $\nu^r$ has infinite outcome along $\eta^r$, so $\nu^r \neq \mu^r$. We compare the locations $\mu^r$ and $\nu^r$.

*Subcase* 1.2.1: $\nu^r \subset \mu^r$. Let $\tau^r$ be the immediate successor of $\mu^r$ along $\eta^r$. By (5.2), $\mu^r$ is an initial derivative, so $\mathrm{up}(\mu^r) \neq \mathrm{up}(\nu^r)$. (5.6)–(5.12) are routinely verified for $\langle\langle \nu^r, \mu^r, \tau^r \rangle\rangle$, using the conditions obtained from the assumptions that $\eta^r$ requires extension for $\mu^r$ and that $\delta^r$ is a pseudocompletion of $\nu^r$. As $\mu^r \subset \delta^r$, it follows from Lemma 5.1(i) (PL Analysis) that $\mathrm{PL}(\nu^r, \mu^r) \subseteq \mathrm{PL}(\nu^r, \delta^r)$, so (5.16) follows from the amenability condition for pseudocompletions. Thus $\langle\langle \nu^r, \mu^r, \tau^r \rangle\rangle$ is an amenable implication chain along $\tau^r$. But by (5.1), $\mathrm{out}^0(\tau^r)$ is pseudotrue, so we have contradicted (5.28).

*Subcase* 1.2.2: $\mu^r \subset \nu^r$. Let $\xi^r$ be the immediate successor of $\nu^r$ along $\eta^r$. Recall that $\nu^r$ has infinite outcome along $\eta^r$, and by (5.2), $\mu^r$ is the principal derivative of $\mathrm{up}(\mu^r)$ along $\eta^r$, so $\mathrm{up}(\mu^r) \neq \mathrm{up}(\nu^r)$. Conditions (5.1)–(5.3) and (5.5) are now routinely verified for $\mu^r \subset \nu^r \subset \xi^r$, using the conditions obtained from the assumptions that $\eta^r$ requires extension for $\mu^r$ and that $\delta^r$ is a pseudocompletion of $\nu^r$. Recall that $\nu^r$ has infinite outcome along $\eta^r$, hence along $\xi^r$, so by Lemma 3.3 ($\lambda$-Behavior), $\mathrm{up}(\nu^r) = (\lambda(\xi^r))^-$ and $\mathrm{up}(\nu^r)$ has finite outcome along $\lambda(\xi^r)$. By (5.2), $\mu^r$ is an initial derivative, so by Lemma 3.1(i) (Limit Path), $\mathrm{up}(\mu^r) \subset \mathrm{up}(\nu^r)$. Hence by Lemma 5.1(iv) (PL Analysis),

$$\mathrm{PL}(\mathrm{up}(\mu^r), \lambda(\xi^r)) = \mathrm{PL}(\mathrm{up}(\mu^r), \mathrm{up}(\nu^r)).$$

Hence by (5.29)(i),

$$\mathrm{PL}(\mathrm{up}(\mu^r), \mathrm{up}(\nu^r)) \subseteq \{\mathrm{up}(\gamma^r): \gamma^r \in \mathrm{PL}(\nu^r, \eta^r)\} \cup \mathrm{PL}(\mathrm{up}(\mu^r), \lambda(\eta^r)).$$

Now by Lemma 5.1(ii) (PL Analysis),

$$\mathrm{PL}(\nu^r, \eta^r) \subseteq \mathrm{PL}(\nu^r, \delta^r) \cup \{\delta^r\}.$$

Now $\delta^r \equiv \mu^r$ by (5.2), so by (5.4) for $\mu^r$ and $\eta^r$, Definition 5.4 for $\nu^r$ and $\delta^r$, and Lemma 2.2(i) (Interaction), for all $\pi \in \mathrm{PL}(\mathrm{up}(\mu^r), \lambda(\eta^r)) \cup \mathrm{PL}(\nu^r, \delta^r) \cup \{\delta^r\}$, $\mathrm{TS}(\pi) \cap \mathrm{RS}(\mu^r) = \emptyset$. Thus for all $\pi \in \mathrm{PL}(\mathrm{up}(\mu^r), \lambda(\xi^r))$, $\mathrm{TS}(\pi) \cap \mathrm{RS}(\mu^r) = \emptyset$, so (5.4) holds for $\mu^r \subset \nu^r \subset \xi^r$. Thus $\xi^r$ requires extension for some $\alpha^r \subseteq \mu^r$. As $\nu^r \subset \delta^r$, it follows that $\xi^r \subseteq \delta^r$. Now $\delta = \eta^-$ is the principal derivative of $\delta^r$ along $\eta$, and by (5.1), $\delta$ is implication-free. So as $\delta$ is admissible, it follows from (5.27) that $\delta$ is completion-consistent via $\langle\ \rangle$. Furthermore, $\mathrm{up}^r(\delta) = \delta^r \supset \nu^r$. Hence by Lemma 5.5(ii) (Completion-Respecting), $\xi^r$ has a primary completion $\kappa^r \subset \delta^r$ which has infinite outcome along $\delta^r \subset \eta^r$. Thus $[\alpha^r, \kappa^r]$ is a primary $\eta^r$-link restraining $\mu^r$, contradicting (5.3) for $\mu^r \subset \delta^r \subset \eta^r$.

*Case* 2: $j = r-1$. By case assumption, $\delta^{r-1}$ must be a primary completion; fix $\rho^{r-1}$ such that $\delta^{r-1}$ is a primary completion of $\rho^{r-1}$. By (5.5)(ii), there is an amenable $r$-implication-chain $\langle\langle \sigma^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ such that $\rho^{r-1} = \mathrm{out}(\tau^r)$. As $\eta^{r-1}$ requires extension, it follows from (5.5)(ii) that there is an amenable $r$-implication-chain $\langle\langle \bar{\sigma}^r, \tilde{\sigma}^r, \bar{\tau}^r \rangle\rangle$ such that $\eta^{r-1} = \mathrm{out}(\bar{\tau}^r)$. By Definition 5.6 and (5.5)(ii), $\sigma^r = \mathrm{up}(\delta^{r-1}) = \tilde{\sigma}^r$, so by (5.8)(i), $\bar{\sigma}^r \subset \tilde{\sigma}^r = \sigma^r \subset \hat{\sigma}^r$. We show that $\langle\langle \bar{\sigma}^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ satisfies (5.6)–(5.12) and (5.15) or (5.16), contradicting the minimality of $\mathrm{lh}((\sigma^r)$ for $\langle\langle \sigma^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ in Definition 5.4.

Conditions (5.6)–(5.12) for $\langle\langle \bar{\sigma}^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ follow routinely from (5.6)–(5.12) for $\langle\langle \bar{\sigma}^r, \tilde{\sigma}^r, \bar{\tau}^r \rangle\rangle$ and $\langle\langle \sigma^r, \hat{\sigma}^r, \tau^r \rangle\rangle$, once we recall that $\bar{\sigma}^r \subset \tilde{\sigma}^r \subset \hat{\sigma}^r$, and note that $\bar{\sigma}^r$ has infinite outcome along $\hat{\sigma}^r$ by (5.11)(i), so $\mathrm{up}(\bar{\sigma}^r) \neq \mathrm{up}(\hat{\sigma}^r)$ by (2.8). Let $\mu^r$ be the initial derivative of $\mathrm{up}(\hat{\sigma}^r)$ along $\hat{\sigma}^r$, and let $\tilde{\mu}^r$ be the initial derivative of $\mathrm{up}(\tilde{\sigma}^r)$ along $\tilde{\sigma}^r$. By (5.6), $\mu^r \neq \sigma^r$ and $\tilde{\mu}^r \neq \bar{\sigma}^r$.

*Subcase* 2.1. $\mu^r \subset \sigma^r$. Then (5.15) must hold for $\langle\langle \sigma^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ and so if we fix $\tilde{\tau}^r \subseteq \hat{\sigma}^r$ such that $(\tilde{\tau}^r)^- = \sigma^r$, then $\tilde{\tau}^r$ requires extension for $\mu^r$. But $\sigma^r = \tilde{\sigma}^r$, so by (5.15) or (5.16) for $\langle\langle \bar{\sigma}^r, \tilde{\sigma}^r, \bar{\tau}^r \rangle\rangle$, $\sigma^r$ is either a primary completion or an amenable pseudocompletion. As $r > j$, we have contradicted (i) by induction.

*Subcase* 2.2: $\sigma^r \subset \mu^r$. We show that $\langle\langle \bar{\sigma}^r, \hat{\sigma}^r, \tau^r \rangle\rangle$ is an amenable implication chain, contradicting the minimality condition in Definition 5.4 as $\bar{\sigma} \subset \sigma^r$.

It suffices to show that $\hat{\sigma}^r$ is an amenable pseudocompletion of $\bar{\sigma}^r$. The relevant conditions from (5.6)–(5.11) follow easily from our assumptions that $\langle\langle\bar{\sigma}^r, \tilde{\sigma}^r, \bar{\tau}^r\rangle\rangle$ and $\langle\langle\sigma^r, \hat{\sigma}^r, \tau^r\rangle\rangle$ are amenable implication chains. Fix $\tilde{\tau}^r \subseteq \hat{\sigma}^r$ such that $(\tilde{\tau}^r)^- = \sigma^r$. Then $\hat{\sigma}^r$ is a pseudocompletion of $\bar{\sigma}^r$, and by (5.10)(i), $\sigma^r$ and $(\text{out}^0(\tilde{\tau}^r))^-$ are implication-free. As any implication-free node on $T^0$ must be completion-consistent via $\langle\ \rangle$ (else (5.21) or (5.22) would cause it to be implication-restrained), $(\text{out}^0(\tilde{\tau}^r))^-$ is completion-consistent via $\langle\ \rangle$. Suppose that $\xi^r \subseteq \sigma^r$ requires extension. If $\xi^r \subset \sigma^r$, then as $(\text{out}^0(\tilde{\tau}))^-$ is a derivative of $\sigma^r$, the exclusionary conditions of Lemma 5.5(ii) (Completion-Respecting) cannot hold unless $\sigma^r$ is the primary completion of $\xi^r$, so $\xi^r$ has a primary completion with infinite outcome along $\sigma^r$. And if $\sigma^r$ is the primary completion of $\xi^r$, then $\sigma^r$ has infinite outcome along $\tilde{\tau}^r$. Hence Lemma 5.1(viii) (PL Analysis) can be applied (for $\tilde{\tau}^r$ as the $\sigma^j$ of the lemma).

*Subcase* 2.2.1: $\tilde{\mu}^r \subset \bar{\sigma}^r$. Then (5.15) must hold for $\langle\langle\bar{\sigma}^r, \tilde{\sigma}^r, \bar{\tau}^r\rangle\rangle$ and so if we fix $\hat{\tau}^r \subseteq \tilde{\sigma}^r$ such that $(\hat{\tau}^r)^- = \bar{\sigma}^r$, then $\hat{\tau}^r$ requires extension for $\tilde{\mu}^r$. But then by (5.15) for $\langle\langle\bar{\sigma}^r, \tilde{\sigma}^r, \bar{\tau}^r\rangle\rangle$, $\tilde{\sigma}^r = \sigma^r$ is the primary completion of $\hat{\tau}^r$. By Lemma 5.1(viii) (PL Analysis),

$$\text{PL}(\bar{\sigma}^r, \hat{\sigma}^r) \subseteq \text{PL}(\bar{\sigma}^r, \sigma^r) \cup \{\sigma^r\} \cup \text{PL}(\sigma^r, \hat{\sigma}^r).$$

By (5.29)(ii),

$$\{\text{up}(\xi^r) : \xi^r \in \text{PL}(\bar{\sigma}^r, \sigma^r)\} = \text{PL}(\text{up}(\tilde{\mu}^r), \lambda(\hat{\tau}^r)).$$

Hence by (5.4) for $\tilde{\mu}^r \subset \bar{\sigma}^r \subset \hat{\tau}^r$, Lemma 2.2(i) (Interaction) and Definition 5.4 for $\sigma^r \subset \hat{\sigma}^r$, for all $\pi \in \text{PL}(\bar{\sigma}^r, \sigma^r) \cup \{\sigma^r\} \cup \text{PL}(\sigma^r, \hat{\sigma}^r)$, $\text{TS}(\pi) \cap \text{RS}(\bar{\sigma}^r) = \emptyset$, so $\hat{\sigma}^r$ is an amenable pseudocompletion of $\bar{\sigma}^r$. As $r > j$, we have contradicted (i) by induction.

*Subcase* 2.2.2: $\tilde{\mu}^r \supset \bar{\sigma}^r$. Then (5.16) must hold for both $\langle\langle\bar{\sigma}^r, \tilde{\sigma}^r, \bar{\tau}^r\rangle\rangle$ and $\langle\langle\sigma^r, \hat{\sigma}^r, \tau^r\rangle\rangle$. By Lemma 5.1(viii) (PL Analysis), $\text{PL}(\bar{\sigma}^r, \tau^r) \subseteq \text{PL}(\bar{\sigma}^r, \sigma^r) \cup \text{PL}(\sigma^r, \tau^r) \cup \{\sigma^r\}$. Condition (5.16) for $\langle\langle\bar{\sigma}^r, \hat{\sigma}^r, \tau^r\rangle\rangle$ now follows from (5.16) for $\langle\langle\bar{\sigma}^r, \tilde{\sigma}^r, \bar{\tau}^r\rangle\rangle$ and $\langle\langle\sigma^r, \hat{\sigma}^r, \tau^r\rangle\rangle$ and Lemma 2.2(i) (Interaction). Thus $\langle\langle\bar{\sigma}^r, \hat{\sigma}^r, \tau^r\rangle\rangle$ satisfies (5.6)–(5.12) and (5.16), contradicting the minimality of $\text{lh}(\sigma^r)$ for $\langle\langle\sigma^r, \hat{\sigma}^r, \tau^r\rangle\rangle$ in Definition 5.4.

*Case* 3: $j < r-1$. Let $\delta^j$ be the $j$-completion of $\nu^j$ and let $\nu^j$ require extension for $\rho^j$. By (5.5)(ii), there is an amenable $r$-implication-chain $\langle\langle\sigma^i, \hat{\sigma}^i, \tau^i\rangle : r \geqslant i \geqslant j+1\rangle$ such that $\nu^j = \text{out}(\tau^{j+1})$. As $\eta^j$ requires extension, it follows from (5.5)(ii) that there is an amenable $r$-implication-chain $\langle\langle\bar{\sigma}^i, \tilde{\sigma}^i, \bar{\tau}^i\rangle : r \geqslant i \geqslant j+1\rangle$ such that $\eta^j = \text{out}(\bar{\tau}^{j+1})$. By (5.5)(ii) and (5.9), $\tilde{\sigma}^{j+1} = \text{up}(\delta^j) = \sigma^{j+1}$. As $j+1 < r$, the conditions of (5.16) at $j+1$ are not satisfied by either amenable implication chain, so (5.15) must hold at $j+1$

for both implication chains. By (5.5)(ii) for $\langle\langle\bar{\sigma}^i, \tilde{\sigma}^i, \tau^i\rangle : r \geqslant i \geqslant j+1\rangle$, $\tilde{\sigma}^{j+1}$ is a primary completion. By (5.5)(ii) for $\langle\langle\sigma^i, \hat{\sigma}^i, \tau^i\rangle : r \geqslant i \geqslant j+1\rangle$, if $\tilde{\tau}^{j+1}$ is the immediate successor of $\sigma^{j+1}$ along $\hat{\sigma}^{j+1}$, then $\tilde{\tau}^{j+1}$ requires extension. But $\tilde{\sigma}^{j+1} = \sigma^{j+1}$, so we have contradicted (i) inductively.

(ii) Let $r = \dim(\eta^-) - 1$. We assume that $\eta$ is nonswitching and $\eta^-$ is primarily or hereditarily implication-restrained, and derive a contradiction. By hypothesis, $\eta^j$ requires extension. As $\eta^j \neq \lambda^j(\eta^-)$ and $\eta$ is nonswitching, it follows from Lemma 3.3 ($\lambda$-Behavior) that $(\eta^j)^- = \mathrm{up}^j(\eta^-) = \lambda^j(\eta^-)$, so $\eta^-$ is the principal derivative of $(\eta^j)^-$ along $\eta = \mathrm{out}^0(\eta^j)$; and by (5.2), $\lambda^j(\eta^-)$ has infinite outcome along $\eta^j$. Now by Definition 5.1, $j \leqslant r$. By (2.4), $\lambda^j(\eta^-)$ is the principal derivative of $\mathrm{up}(\lambda^j(\eta^-))$ along $\eta^j$, so as $\eta$ is nonswitching, $(\eta^{j+1})^- = \mathrm{up}^{j+1}(\eta^-) = \lambda^{j+1}(\eta^-)$. By (5.1), (5.10)(ii), and as $\eta^-$ is implication-restrained, $j+1 < r$, so by (5.5)(ii), $\mathrm{up}^{j+1}(\eta^-)$ is a primary completion. But then as $\eta^-$ is implication-restrained, it follows from (5.18)(i) that $\eta$ is switching, contrary to hypothesis.

If $\eta$ is a 0-completion, then by Definition 5.6, $\eta$ is nonswitching and $\eta^-$ is implication-restrained. (ii) now follows. ∎

In order to show that $k$-completions exist, it will be necessary for the paths constructed to be admissible. We thus need to analyze the process of constructing paths, and to show that we can construct admissible paths. The proof will proceed by induction on $n-k$, and then by induction on $\mathrm{lh}(\eta^k)$ for $\eta^k \in T^k$. There are some induction hypotheses that will also need to be verified. We will need to know that admissible nodes are completion-consistent for some set. And we will need to show a relationship between certain PL sets on $T^k$ at $\eta^k$ and corresponding PL sets on $T^{k+1}$ at $\lambda(\eta^k)$ whenever $\eta^k$ is not completion-respecting. We prove several lemmas which will give us the desired information. The first lemma treats the case where the node to be extended is completion-consistent via $\langle\ \rangle$. We treat the case where extensions are taken during the backtracking process in Lemmas 5.9–5.14.

Lemma 5.8 (Completion-Respecting Admissible Extension Lemma). *Fix $\eta, \xi \in T^0$ such that $\xi^- = \eta$, $\xi$ is preadmissible, and $\eta$ is completion-consistent via $\langle\ \rangle$. Then $\xi$ is admissible and either $\xi$ is completion-consistent via $S = \langle\ \rangle$, or $\xi$ is completion-consistent via $S = \langle\lambda^j(\xi)\rangle$ for some $j \leqslant n$.*

*Proof.* As $\eta$ is admissible and completion-consistent via $S = \langle\ \rangle$, (5.27) is vacuous. We now verify (5.28). As $\eta$ is admissible, (5.28) follows by induction if $\xi$ is not pseudotrue. Hence we may suppose that $\xi$ is pseudotrue for the sake of proving (5.28). It then follows that $\lambda^j(\xi)$ is not a primary completion for any $j \leqslant n$.

Suppose that $\langle\langle \sigma^j, \hat{\sigma}^j, \tau^j\rangle : r \geqslant j \geqslant k\rangle$ is an amenable $k$-implication chain with $\tau^k \subseteq \lambda^j(\xi)$ in order to obtain a contradiction. Then by (5.15) or (5.16), $\hat{\sigma}^k$ is a primary completion or an amenable pseudocompletion, and by (5.11)(ii), $\hat{\sigma}^k$ has finite outcome along $\tau^k$. Thus $k > 0$, else by (5.17)(i) or (5.18)(i), $\hat{\sigma}^k$ would have infinite outcome along $\tau^k$. By Lemma 5.2 (Requires Extension), out$(\tau^k)$ requires extension; so as $\eta$ and $\xi$ are completion-consistent via $\langle\ \rangle$, it follows from Lemma 5.5(iii) (Completion-Respecting) that out$(\tau^k)$ has a primary completion $\kappa^{k-1}$ which has infinite outcome along its immediate successor $\beta^{k-1} \subseteq \lambda^{k-1}(\xi)$. But now by (5.19) and (5.5)(ii), $\beta^{k-1}$ switches $\sigma^k$, so $\tau^k \nsubseteq \lambda^k(\xi)$, a contradiction. Hence (5.28) holds.

We now verify (5.29)(i–iii). For all $i \leqslant n$, let $\xi^i = \lambda^i(\xi)$, $\eta^i = \lambda^i(\eta)$, and $\bar{\eta}^i = \mathrm{up}^i(\eta)$. By (2.5) and Lemma 3.1 (Limit Path), for any $\gamma^k \in T^k$, if $\gamma \in T^0$ is an initial derivative of $\gamma^k$, then for all $\beta$ such that out$^0(\gamma^k) \subseteq \beta \subseteq \gamma$, $\lambda^k(\beta) = \gamma^k$; and by (2.4) and Lemma 5.4 (Compatibility), if $\alpha \supset \gamma$, $\gamma^k \subseteq \lambda^k(\alpha)$, and $\gamma^k$ is a $k$-completion, then $\gamma^k \subset \lambda^k(\alpha)$. Hence if $\xi^k$ were a $k$-completion, then out$^0(\xi^k) \subseteq \xi$ and either $\xi$ would be a 0-completion or there would be no 0-completion corresponding to $\xi^k$ along $\xi$; in either case, it follows from Definition 5.6, that $\eta = \xi^-$ must be primarily or hereditarily implication-restrained so cannot be completion-consistent via $\langle\ \rangle$, contrary to our hypothesis. Conditions (5.29)(ii) and (5.29)(iii) now follow from (5.27) by induction.

We now note that if $\eta$ is implication-restrained, then $\xi$ does not switch $\eta$. For by (5.17), if $\xi$ were to switch $\eta$ and $\eta$ were implication-restrained, then $\eta$ would have to be either a primary completion or an amenable pseudocompletion. If $\eta$ is an amenable pseudocompletion, then $\dim(\eta) = 1$ and this is impossible by (5.10); and if $\eta$ is a primary completion, then $\dim(\eta) > 1$ and $\eta$ must be a 0-completion contrary to (5.29)(iii). Hence by (5.23), if $\xi$ is switching, then $\bar{\eta}^i$ is implication-free for all $i \leqslant n$.

We now verify (5.29)(i). Fix $k < n$ and $\mu^k \subset \nu^k \subset \xi^k$ such that $\nu^k$ is implication-free and $\mathrm{up}(\mu^k) \subset \mathrm{up}(\nu^k), \xi^{k+1}$, and fix $p$ and $s$ for $\xi$ as in Lemma 3.3 ($\lambda$-Behavior). Note that $\eta$ is admissible, so (5.29)(i) holds for all $\gamma \subseteq \eta$.

*Case* 1: $k+1 \leqslant p$. Then by Lemma 3.3 ($\lambda$-Behavior), $\nu^k \subseteq \eta^k = (\xi^k)^-$ and $\mathrm{up}(\nu^k) \subseteq \eta^{k+1}$, so as $\mathrm{up}(\mu^k) \subset \mathrm{up}(\nu^k)$, it follows that $\mathrm{up}(\mu^k) \subset \eta^{k+1}$. Hence by (5.29)(i) for $\eta$ if $\nu^k \subset \eta^k$, and by (2.7) and Lemma 5.1(i) (PL Analysis) if $\nu^k = \eta^k$,

$$\mathrm{PL}(\mathrm{up}(\mu^k), \mathrm{up}(\nu^k)) \subseteq \{\mathrm{up}(\alpha^k) : \alpha^k \in \mathrm{PL}(\nu^k, \eta^k)\} \cup \mathrm{PL}(\mathrm{up}(\mu^k), \eta^{k+1}).$$

Again by Lemma 3.3 ($\lambda$-Behavior), $\eta^{k+1} \subset \xi^{k+1}$, so by Lemma 5.1(i) (PL Analysis), $\mathrm{PL}(\nu^k, \eta^k) \subseteq \mathrm{PL}(\nu^k, \xi^k)$ and $\mathrm{PL}(\mathrm{up}(\mu^k), \eta^{k+1}) \subseteq \mathrm{PL}(\mathrm{up}(\mu^k), \xi^{k+1})$, so (5.29)(i) holds for $k$.

*Case* 2: $k+1>s$. By Lemma 3.3 ($\lambda$-Behavior), there are $i<k+1$ and $\sigma^i=(\xi^i)^-$ such that for all $q\in[i+1,n]$, $\xi^q=\lambda^q(\sigma^i)=\sigma^q$. As $\nu^k\subset\xi^k$, it follows that $\nu^k\subseteq\sigma^k$. Let $\sigma=\text{out}^0(\sigma^i)$, and note that by (2.5), $\sigma\subset\xi$. Thus by (2.5) and (5.29)(i) for $\sigma$ if $\nu^k\subset\sigma^k$, and by (2.7) and Lemma 5.1(i) (PL Analysis) if $\nu^k=\sigma^k$,

$$\text{PL}(\text{up}(\mu^k),\text{up}(\nu^k))\subseteq\{\text{up}(\alpha^k):\alpha^k\in\text{PL}(\nu^k,\sigma^k)\}\cup\text{PL}(\text{up}(\mu^k),\sigma^{k+1}).$$

We have noted that $\sigma^{k+1}=\xi^{k+1}$, and that $\sigma^k\subseteq\xi^k$; hence by Lemma 5.1(i) (PL Analysis), $\text{PL}(\nu^k,\sigma^k)\subseteq\text{PL}(\nu^k,\xi^k)$, so (5.29)(i) holds for $k$.

*Case* 3: $p<k+1\leqslant s$. By Lemma 3.3 ($\lambda$-Behavior), $\bar{\eta}^k=(\xi^k)^-$ and $\xi$ switches $\eta^{k+1}$. As $\nu^k\subset\xi^k$, it follows that $\nu^k\subseteq\bar{\eta}^k$. Let $\beta$ be the initial derivative of $\bar{\eta}^k$ along $\xi$. By (2.7), $\text{up}(\nu^k)\subseteq\lambda(\nu^k)$. Hence by (2.4), (2.6), and as $\text{up}(\mu^k)\subset\text{up}(\nu^k),\xi^{k+1}$, it follows that $\text{up}(\mu^k)\subset\lambda(\bar{\eta}^k)$. Hence by (5.29)(i) for $\beta$ if $\nu^k\subset\bar{\eta}^k$, and by Lemma 5.1(i) (PL Analysis) if $\nu^k=\bar{\eta}^k$,

$$\text{PL}(\text{up}(\mu^k),\text{up}(\nu^k))\subseteq\{\text{up}(\alpha^k):\alpha^k\in\text{PL}(\nu^k,\bar{\eta}^k)\}\cup\text{PL}(\text{up}(\mu^k),\lambda(\bar{\eta}^k)).$$

Suppose that $\rho^{k+1}\in(\text{PL}(\text{up}(\mu^k),\text{up}(\nu^k))\cap\text{PL}(\text{up}(\mu^k),\lambda(\bar{\eta}^k)))\backslash\text{PL}(\text{up}(\mu^k),\xi^{k+1})$. As $\xi$ is $q$-switching for some $q\leqslant k+1$, it follows from an earlier observation that $\bar{\eta}^k$ is implication-free. Furthermore, by Lemma 3.3 ($\lambda$-Behavior), $(\xi^k)^-=\bar{\eta}^k$ and $(\xi^{k+1})^-=\bar{\eta}^{k+1}=\text{up}(\bar{\eta}^k)$.

First suppose that (5.13) places $\rho^{k+1}$ into $\text{PL}(\text{up}(\mu^k),\text{up}(\nu^k))$. Then there is a $\gamma^{k+1}$ such that $[\gamma^{k+1},\rho^{k+1}]$ is a primary $\text{up}(\nu^k)$-link restraining $\text{up}(\mu^k)$, so $\rho^{k+1}\subset\text{up}(\nu^k)$. By Lemma 3.1(i) (Limit Path), $\rho^{k+1}$ has an initial derivative $\rho^k\subset\nu^k$. By (2.10) and as $\rho^{k+1}\in\text{PL}(\text{up}(\mu^k),\lambda(\bar{\eta}^k))\backslash\text{PL}(\text{up}(\mu^k),\xi^{k+1})$ and $\xi$ switches $\bar{\eta}^{k+1}$, $\bar{\eta}^{k+1}=\rho^{k+1}$, and $\rho^{k+1}$ has infinite outcome along $\lambda(\bar{\eta}^k)$ but finite outcome along $\xi^{k+1}$; hence by Lemma 3.3 ($\lambda$-Behavior), $\bar{\eta}^k$ has infinite outcome along $\xi^k$ and $[\rho^k,\bar{\eta}^k]$ is a primary $\xi^k$-link. Now $\nu^k\neq\bar{\eta}^k$, else $\text{up}(\nu^k)=\bar{\eta}^{k+1}=\rho^{k+1}$, so $\rho^{k+1}\notin\text{PL}(\text{up}(\mu^k),\text{up}(\nu^k))$, contrary to our assumption. Hence $[\rho^k,\bar{\eta}^k]$ is a primary $\xi^k$-link restraining $\nu^k$. But then (5.13) places $\bar{\eta}^k$ into $\text{PL}(\nu^k,\xi^k)$, as required by (5.29)(i).

Suppose that (5.14) places $\rho^{k+1}$ into $\text{PL}(\text{up}(\mu^k),\text{up}(\nu^k))$, but (5.13) does not. Now $\bar{\eta}^{k+1}=\text{up}^{k+1}(\eta)$, and we have noted that $\bar{\eta}^{k+1}$ is implication-free. Let $\text{PL}(\gamma^{k+1},\pi^{k+1})$ be a component of $\text{PL}(\text{up}(\mu^k),\text{up}(\nu^k))$ which causes $\rho^{k+1}$ to be placed into $\text{PL}(\text{up}(\mu^k),\text{up}(\nu^k))$, with $\pi^{k+1}$ as long as possible. It follows from Definition 5.3 that if $\pi^{k+1}\subset\text{up}(\nu^k)$, then $\pi^{k+1}$ has infinite outcome along $\text{up}(\nu^k)$. As $\nu^k$ is implication-free, it follows from (5.23) that $\text{up}(\nu^k)$ is implication-free; so by (5.21) and Definition 5.3, $\pi^{k+1}$ must be the primary completion of the immediate successor $\delta^{k+1}$ of $\gamma^{k+1}$ along $\pi^{k+1}$ for some $\bar{\mu}^{k+1}\subset\text{up}(\mu^k)$. By Definition 5.6, $\delta^{k+1}\subset\rho^{k+1}$, so as

$\rho^{k+1} \in \mathrm{PL}(\mathrm{up}(\mu^k), \lambda(\bar{\eta}^k))$, $\delta^{k+1} \subset \rho^{k+1} \subseteq \lambda(\bar{\eta}^k)$. By Lemma 5.5(ii) (Completion-Respecting), either $[\bar{\mu}^{k+1}, \pi^{k+1}]$ is a primary $\lambda(\bar{\eta}^k)$-link restraining $\mathrm{up}(\mu^k)$, or $\bar{\eta}^{k+1} = \pi^{k+1}$ or $\bar{\eta}^{k+1} = \gamma^{k+1}$.

*Subcase* 3.1: $[\bar{\mu}^{k+1}, \pi^{k+1}]$ is a primary $\lambda(\bar{\eta}^k)$-link restraining $\mathrm{up}(\mu^k)$. Now $\pi^{k+1} \subset \lambda(\bar{\eta}^k)$, and by (2.7), $\bar{\eta}^{k+1} \subseteq \lambda(\bar{\eta}^k)$; hence $\pi^{k+1}$ and $\bar{\eta}^{k+1}$ are comparable. By (2.10), $\bar{\eta}^{k+1} \notin [\bar{\mu}^{k+1}, \pi^{k+1})$. Also, $\bar{\eta}^{k+1} \not\subset \bar{\mu}^{k+1}$, else as $\mu^{k+1} \supset \bar{\mu}^{k+1}$ and $\xi$ switches $\bar{\eta}^{k+1}$, we would not have $\mu^{k+1} \subset \xi^{k+1}$. Hence $\pi^{k+1} \subseteq \bar{\eta}^{k+1} \subset \xi^{k+1}$, and so $\mathrm{PL}(\gamma^{k+1}, \pi^{k+1})$ is a component of $\mathrm{PL}(\mathrm{up}(\mu^k), \xi^{k+1})$. But then $\rho^{k+1} \in \mathrm{PL}(\mathrm{up}(\mu^k), \xi^{k+1})$, a contradiction.

*Subcase* 3.2: $\bar{\eta}^{k+1} = \pi^{k+1}$. Proceed as in the last two sentences of Subcase 3.1.

*Subcase* 3.3: $\bar{\eta}^{k+1} = \gamma^{k+1}$. Recall that $\delta^{k+1} \subseteq \mathrm{up}(\nu^k) \subseteq \eta^{k+1}$ requires extension. As $\eta$ is completion-consistent via $\langle\ \rangle$, $\delta^{k+1}$ has a 0-completion $\pi^0 \subseteq \eta$. And as $\delta^{k+1} \subseteq \eta^{k+1}$, it follows from Lemma 5.4 (Compatibility) that for all $i \leqslant k$, $\delta^{k+1}$ has an $i$-completion $\pi^i \subseteq \eta^i$; and by Definition 5.6, $\mathrm{up}(\pi^i) = \pi^{i+1}$ for all $i \leqslant k$. Now $\mathrm{up}^{k+1}(\pi^0) = \pi^{k+1} \neq \gamma^{k+1} = \mathrm{up}^{k+1}(\eta)$, so $\pi^0 \neq \eta$. Hence $\pi^0 \subset \eta$. Thus by induction using (2.4), $\pi^i \subset \eta^i$ for all $i \leqslant k$, so $\pi^k \subset \eta^k$.

First suppose that all derivatives of $\pi^{k+1}$ along $\eta^k$ have finite outcome along $\eta^k$. Then by (2.4), $[\bar{\mu}^{k+1}, \pi^{k+1}]$ is a primary $\eta^{k+1}$-link restraining $\gamma^{k+1} \supset \mathrm{up}(\mu^k)$. But then by (2.10), $\xi$ could not switch $\bar{\eta}^{k+1} = \gamma^{k+1}$, a contradiction.

We conclude that there is a derivative $\bar{\pi}^k$ of $\pi^{k+1}$ which has infinite outcome along $\eta^k$. Let $\sigma^k$ be the immediate successor of $\bar{\pi}^k$ along $\eta^k$. By (2.4), $\pi^{k+1}$ has finite outcome along $\lambda(\sigma^k)$, so by Lemma 5.3(ii) (Implication Chain) and Lemma 5.2 (Requires Extension), $\sigma^k$ requires extension for the initial derivative $\gamma^k$ of $\gamma^{k+1}$ along $\eta^k$. As $\eta$ is completion-consistent via $\langle\ \rangle$, $\sigma^k$ must have a 0-completion $\kappa \subseteq \eta$. First suppose that $\eta$ is a 0-completion corresponding to a primary $i$-completion. As $\mathrm{up}^{k+1}(\eta) = \bar{\eta}^{k+1} = \gamma^{k+1}$ and, by Lemma 5.7(i) (Primary Completion), $\gamma^{k+1}$ is not a primary completion, it follows from (5.5)(ii), (5.9) and (5.12) that $k+1-i$ is odd. Hence $k-i$ is even and by (5.5)(ii), (5.9) and (5.12), $\bar{\eta}^k$ is the primary completion of $\sigma^k$. Otherwise, by Lemma 5.5(ii) (Completion-Respecting), $\sigma^k$ must have a primary completion $\kappa^k \subseteq \eta^k$. Now $\gamma^{k+1} \subset \mathrm{up}(\nu^k)$, so by Lemma 3.1(i) (Limit Path) $\gamma^k \subset \nu^k$. Thus $\mathrm{PL}(\bar{\pi}^k, \kappa^k)$ is a component of $\mathrm{PL}(\nu^k, \xi^k)$ which places $\bar{\pi}^k$ into $\mathrm{PL}(\nu^k, \xi^k)$ via (5.14)(i), completing the proof for the case in which $\rho^{k+1} = \gamma^{k+1}$. Furthermore, by (5.29)(ii) and (5.14)(ii), $\mathrm{PL}(\gamma^{k+1}, \pi^{k+1}) \subseteq \{\mathrm{up}(\alpha^k) \colon \alpha^k \in \mathrm{PL}(\bar{\pi}^k, \kappa^k)\} \subseteq \{\mathrm{up}(\alpha^k) \colon \alpha^k \in \mathrm{PL}(\nu^k, \xi^k)\}$, so (5.29)(i) holds in this case.

As $\eta$ is completion-consistent via $S = \langle\ \rangle$, it follows from Lemma 3.1(i) (Limit Path) that any $\rho^j \subset \lambda^j(\xi)$ which requires extension satisfies

$\rho^j = \lambda^j(\gamma)$ for some $\gamma \subseteq \eta$, so has a 0-completion $\subseteq \eta$ by (5.27). Hence if $\xi$ is completion-deficient via some $Z \neq \langle\ \rangle$, then all elements of $Z$ are of the form $\lambda^j(\xi)$ for some $j$. The last conclusion of the lemma now follows from Lemma 5.6 (Uniqueness of Requiring Extension), so $\xi$ is admissible. ∎

When an admissible $\eta \in T^0$ requires extension, we will need to find an admissible 0-completion $\kappa$ of $\eta$. The process of constructing $\kappa$ is called *backtracking*. The next lemma indicates the manner in which backtracking preserves completion-consistency.

LEMMA 5.9 (Completion-Consistency Lemma). *Fix $m \geqslant 0$ and $\rho, \sigma \in T^0$ such that $\sigma$ is preadmissible, and $\sigma^- = \rho$. Assume that $\rho$ is completion-consistent via $S = \langle \eta_i : i \leqslant m \rangle$ and that $\sigma$ is completion-deficient for $V$. Let $U$ be the sequence obtained by ordering $V$ according to the inclusion relation induced by* $\mathrm{out}^0$ *on the elements of $V$. (Note that by Lemma* 5.6 *(Uniqueness of Requiring Extension), a linear ordering is obtained in this way.) For all $j \leqslant n$, let $\sigma^j = \lambda^j(\sigma)$, $\rho^j = \lambda^j(\rho)$, and $\bar{\rho}^j = \mathrm{up}^j(\rho)$. Then $\sigma$ is completion-consistent via $U$ and*:

(i) *If $\sigma$ is a* 0*-completion, then $U = \langle \eta_i : i < m \rangle$.*

(ii) *If* (5.18)(ii)(a) *holds for $\rho$, $j$ is defined as in* (5.18)(ii)(a), *$s$ is defined as in Lemma* 3.3 *($\lambda$-Behavior), $\bar{\rho}^{t+1}$ is a primary completion for some $t$ such that $j-1 \leqslant t \leqslant s$, and $\bar{\rho}^t$ has infinite outcome along $\sigma^t$, then $t = j-1$, $U = S^\wedge\langle \sigma^t \rangle$ and $\sigma^t$ requires extension.*

(iii) *If the hypotheses of* (i) *or* (ii) *are not satisfied, then $U = S$.*

(iv) *$\sigma$ satisfies* (5.27) *and* (5.28).

*Proof.* For each $i \leqslant m$, fix $k(i)$ such that $\eta_i \in T^{k(i)}$. If $\sigma^j$ requires extension for some $j$ such that $\sigma^j \neq \rho^j$, then by Lemma 5.6 (Uniqueness of Requiring Extension), we let $k(m+1)$ be the unique such $j$ and let $\eta_{m+1} = \sigma^{k(m+1)}$.

Fix $u \leqslant n$, and $\eta^u \subset \sigma^u$ such that $\eta^u$ requires extension and $\eta^u \notin S$. By (2.5), $\eta = \mathrm{out}^0(\eta^u) \subset \sigma$, so $\eta \subseteq \rho$. Furthermore, if $\bar{S}$ is the set via which $\eta$ is completion-consistent, then $\eta^u \in \bar{S}$. As $\eta^u \notin S$, it follows from (5.27) and the admissibility of $\rho$ that $\eta^u$ must have a 0-completion along $\rho \subset \sigma$. Hence $\eta^u \notin U$. We conclude that $U \backslash S \subseteq \{\sigma^u : u \leqslant n \;\&\; \mathrm{out}^0(\sigma^u) \nsubseteq \rho\}$. By the preceding paragraph, $U \backslash S$ has at most one element.

Suppose that $\sigma$ is a 0-completion. By Definition 5.6, $\sigma$ is nonswitching, so $\rho^j \subseteq \sigma^j$ for all $j \leqslant n$. By Lemma 5.7(ii) (Primary Completion), $\sigma^j$ cannot require extension for any $j \leqslant n$. Thus $U \subseteq S$ (as sets) by the preceding paragraph. By Definition 5.6, $\sigma$ must be a 0-completion of $\eta_m$ and, as noted in Definition 5.6, cannot be a 0-completion of $\eta_i$ for any $i < m$. Thus (i) follows.

Assume that $\sigma$ is not a 0-completion. (We note that this will be the ease if $\rho$ satisfies (5.18)(ii)(a), as then by (5.18)(ii)(b), $\sigma$ would be switching, and 0-completions are nonswitching.) We first show that $S \subseteq U$ as sets. Suppose that $\eta_i \in S$. By Definition 5.8, $\eta_i$ has no 0-completion along $\rho$. As $\sigma$ is switching, it follows from Definition 5.6 that $\sigma$ is not a 0-completion of $\eta_i$. Hence $\eta_i$ has no 0-completion along $\sigma$. By (5.18) and (5.25), $\sigma$ is not $u$-switching for any $u \leqslant k(i)$, so $\eta_i \subseteq \rho^{k(i)} \subseteq \sigma^{k(i)}$. Hence $\eta_i \in U$.

Suppose that the hypotheses of (ii) hold. Then by (5.18)(ii)(b), Lemma 5.2 (Requires Extension), and Lemma 5.3(ii) (Implication Chain), $\sigma^t \in U$. Now by (2.5), $\mathrm{out}^0(\sigma^t) = \sigma$, so $\sigma^t$ is the last element of $U$. By (5.18)(ii)(b), $t \geqslant k(m)$; so it follows by induction, our characterization of $U \backslash S$, and as $\sigma^t = \lambda^t(\sigma)$ that (5.25) and (5.26) hold.

In order to complete the proof of (ii), we must show that if $j$ is defined as in (5.18)(ii)(a), then $t = j-1$. By Lemma 3.3 ($\lambda$-Behavior), it must be the case that $(\sigma^t)^- = \bar{\rho}^t$. As $S \notin \langle\ \rangle$, it follows from (5.21) or (5.22) that $\rho$ is not implication-free; hence by (5.1), $u < \dim((\sigma^t)^-) - 1$. Now $\rho^{t+1} \neq \bar{\rho}^{t+1}$, else by (5.5)(ii) and (5.15), $\rho^{t+1} = \bar{\rho}^{t+1}$ would be a primary completion, hence (5.18)(i)(a) would hold, excluding the possibility that (5.18)(ii)(a) holds. Thus as $\sigma$ is $j$-switching, $j \leqslant t+1$. Fix $p$ and $s$ as in Lemma 3.3 ($\lambda$-Behavior) for $\sigma$. As $\sigma$ is $j$-switching, $j = p+1$, and as $\sigma^t$ requires extension and $\sigma = \mathrm{out}^0(\sigma^t)$ (else $\sigma \in S$), $t \leqslant s$. By choice of $j$ in (5.18)(ii)(a), $\bar{\rho}^j$ is the end of a primary $\rho^j$-link, so $\bar{\rho}^j$ has infinite outcome along $\rho^j$ and is not an initial derivative; hence by (2.4) and Lemma 3.3 ($\lambda$-Behavior), $s = j$ and $\bar{\rho}^j$ has finite outcome along $\sigma^j$. Hence $j \leqslant t+1 \leqslant s+1 \leqslant j+1$, so $t \leqslant j \leqslant t+1$. By hypothesis, $\bar{\rho}^t$ has infinite outcome along $\sigma^t$, so $t \neq j$. Thus $t = j-1$, completing the proof of (ii).

We now complete the proof of (iii). Suppose that $u \leqslant n$ and $\sigma^u \in U \backslash S \subseteq \{\gamma^u : u \leqslant n\ \&\ \mathrm{out}^0(\gamma^u) \nsubseteq \rho\}$, in order to obtain a contradiction. By Lemma 3.3 ($\lambda$-Behavior), it must be the case that $(\sigma^u)^- = \bar{\rho}^u$. Furthermore, as (5.18)(i)(a) and (5.18)(ii)(a) fail to hold for $\sigma$, it follows from (5.18) that $\sigma$ is nonswitching, so by Lemma 3.3 ($\lambda$-Behavior), $\rho^u = \bar{\rho}^u$. As $S \neq \langle\ \rangle$, it follows from (5.21) or (5.22) that $\rho$ is not implication-free; hence by (5.1), $u < \dim((\sigma^u)^-) - 1$. As $\sigma^u \in U \backslash S$ and $\sigma^u$ requires extension, it now follows from (5.5) and Lemma 3.3 ($\lambda$-Behavior) that $\rho^{u+1} = \bar{\rho}^{u+1}$ is a primary completion, and that $\rho$ is the 0-completion corresponding to $\rho^{u+1}$. But then (5.18)(i)(a) holds, contradicting the hypotheses of (iii).

(5.27) follows from (i)–(iii); and (5.28) follows by induction as $\sigma$ is not pseudotrue. Hence (iv) holds. ∎

The next lemma keeps track of the relationship between various nodes, as we follow the step-by-step process of going from a node which requires extension to its primary completion. At a given step in the process, we will begin with a node $\rho \in T^0$ which is completion-consistent via a sequence

$S = \langle \eta_i \colon i \leqslant m \rangle$ and extend $\rho$ to $\sigma$ such that $\sigma^- = \rho$ and $\sigma$ is completion-consistent via a sequence $U$. (We will allow $m = -1$, but only if $U \neq \langle\ \rangle$.) $U$ has been characterized by Lemma 5.9 (Completion-Consistency); let $w = |U| - 1$. For each $i \leqslant m$, fix $k(i)$ such that $\eta_i \in T^{k(i)}$, and if $\lambda^j(\sigma)$ requires extension for some $j$ such that $\lambda^j(\sigma) \neq \lambda^j(\rho)$, let $k(m+1)$ be that $j$ and let $\eta_{m+1} = \lambda^{k(m+1)}(\sigma)$. For each $i \leqslant w$, let $\delta_i = (\eta_i)^-$, and let $\eta_i$ require extension for $v_i$.

Clauses (i) and (ii) of Lemma 5.10 specify that each element of $\{\mathrm{up}(v_i) \colon i \leqslant w\}$ lies along the branch of $T^{k(i)+1}$ computed by $\sigma$, and that the inclusion ordering of elements of this set which lie on the same tree agrees with the ordering on the indices of these nodes, and so by (5.26), is the same as the ordering induced on the subset of $U$ corresponding to the same indices. And clause (v) will be shown to imply that the immediate successors of the elements in $\{\mathrm{up}(v_i) \colon i \leqslant w\}$ which lie along this branch of $T^{k(i)+1}$ require extension in the order specified by the indices which agrees with the order induced by inclusion, and none has a primary completion along the next node which requires extension. We cannot specify the ordering of $\{v_i \colon i \leqslant w\}$ lying on the same tree, but clauses (iii) and (vii) specify that each $v_i$ is shorter than $\delta_{i-1}$, causing a component for a PL set for $\delta_{i-1}$ to be formed. Clause (iv) is used to show that on this branch of $T^{k(i)+1}$, no elements of $T^{k(i)+1}$ except those in $\{\mathrm{up}(v_i) \colon i \leqslant w\}$ can require extension without having a primary completion along the path. And clause (vi) relates nodes on trees of successive dimension, and implies the property induced by (5.25), namely, that higher dimension nodes find completions before we encounter any new node on a lower dimensional tree which requires extension.

Lemma 5.10 (Component Lemma). *Fix $m \geqslant -1$ and $\rho, \sigma \in T^0$ such that $\sigma$ is preadmissible and $\sigma^- = \rho$. Assume that $\rho$ is completion-consistent via $S = \langle \eta_i \colon i \leqslant m \rangle$, that $\sigma$ is completion-consistent via $U$, and that if $m = -1$, then $|U| \neq 0$. For each $i \leqslant m$, fix $k(i)$ such that $\eta_i \in T^{k(i)}$, and if $\lambda^j(\sigma)$ requires extension for some $j$ such that $\lambda^j(\sigma) \neq \lambda^j(\rho)$, let $k(m+1)$ be that $j$ (which is unique by Lemma 5.6 (Uniqueness of Requiring Extension)) and let $\eta_{m+1} = \lambda^{k(m+1)}(\sigma)$. Let $w = m-1$ if $U \subset S$, let $w = m$ if $U = S$, and let $w = m+1$ otherwise. For each $i \leqslant w$, let $\delta_i = (\eta_i)^-$, and let $\eta_i$ require extension for $v_i$. For all $j \leqslant n$, let $\sigma^j = \lambda^j(\sigma)$, $\rho^j = \lambda^j(\rho)$, and $\bar{\rho}^j = \mathrm{up}^j(\rho)$. Then for all $i \leqslant w$:*

(i) $\mathrm{up}(v_i) \subset \rho^{k(i)+1} \wedge \sigma^{k(i)+1} = \bar{\rho}^{k(i)+1}$.

(ii) *If $0 < i$ and $k(i) = k(i-1)$, then* $\mathrm{up}(v_{i-1}) \subset \mathrm{up}(v_i)$.

(iii) *If $0 < i$ and $k(i) = k(i-1)$, then $v_i \subset \delta_{i-1}$.*

(iv) *Let $\mu^{k(i)+1} \subseteq \mathrm{up}(v_i) \subset (\xi^{k(i)+1})^- \subset \xi^{k(i)+1} \subseteq \sigma^{k(i)+1}$ be given such that $\xi^{k(i)+1}$ requires extension for $\mu^{k(i)+1}$. Then one of the following holds:*

(a) *There is a primary completion* $\kappa^{k(i)+1} \subset \sigma^{k(i)+1}$ *of* $\xi^{k(i)+1}$ *such that* $\kappa^{k(i)+1}$ *has infinite outcome along* $\sigma^{k(i)+1}$.

(b) *Either* $i < w$, $k(i+1) = k(i)$, *and* $\mathrm{up}(\nu_{i+1}) \subseteq (\xi^{k(i)+1})^-$, *or* $w = 0$ *and* $\mathrm{up}(\nu_0) = (\xi^{k(0)+1})^-$.

(c) $\sigma^{k(m)}$ *is the primary completion of* $\eta_m$ *and* $(\xi^{k(i)+1})^- = \mathrm{up}(\nu_m)$.

(v) *If* $\dim(\nu_i) > k(i) + 1$, *then the immediate successor* $\tau_i$ *of* $\mathrm{up}(\nu_i)$ *along* $\sigma^{k(i)+1}$ *requires extension for some* $\mu^{k(i)+1}$; *and if* $0 < i$ *and* $k(i) = k(i-1)$, *then* $\mu^{k(i)+1} \subset \mathrm{up}(\nu_{i-1})$.

(vi) *If* $j < i$ *and* $k(j) = k(i) - 1$, *then* $\mathrm{up}(\nu_j) \subset \nu_i$.

(vii) *If* $0 < i$ *and* $k(i) = k(i-1)$, *then* $\mathrm{PL}(\delta_i, \sigma^{k(i)})$ *is a component of* $\mathrm{PL}(\delta_{i-1}, \sigma^{k(i)})$.

*Proof.* We proceed by induction on $\mathrm{lh}(\sigma)$.

*Case* 1: $m = -1$, so $w \neq -1$ by hypothesis. By Lemma 5.9 (Completion-Consistency), $w = 0$, $\sigma = \mathrm{out}^0(\eta_0)$, and $\rho$ is completion-consistent via $\langle\ \rangle$. Conditions (ii), (iii), (vi), and (vii) are vacuous in this case since $w = 0$. We verify (i), (iv), and (v).

(i) By (5.3) and Lemma 4.3(i)(a) (Link Analysis), $\mathrm{up}(\nu_0) \subseteq \sigma^{k(0)+1}$. As $\eta_0 = \sigma^{k(0)}$, $\sigma = \mathrm{out}^0(\sigma^{k(0)})$, so as $m = -1 \neq w$, $\sigma^{k(0)} \neq \rho^{k(0)}$. Furthermore, by (5.2), $\delta_0 = (\sigma^{k(0)})^-$ has infinite outcome along $\eta_0 = \sigma^{k(0)}$. By (2.4) and as $m = -1$, $\sigma^{k(0)+1} = \lambda(\sigma^{k(0)}) = \mathrm{up}(\delta_0) \,\hat{}\, \langle \sigma^{k(0)} \rangle$, so $\sigma^{k(0)} = \mathrm{out}(\sigma^{k(0)+1})$, and $(\sigma^{k(0)+1})^- \subseteq \rho^{k(0)+1}$. Hence by Lemma 3.3 ($\lambda$-Behavior), $\mathrm{up}(\delta_0) = \rho^{k(0)+1} \wedge \sigma^{k(0)+1} = (\sigma^{k(0)+1})^-$. By (5.2), $\mathrm{up}(\nu_0) \neq (\sigma^{k(0)+1})^-$. By Definition 5.1, $\nu_0 \subset \eta_0 = \sigma^{k(0)} = \mathrm{out}(\sigma^{k(0)+1})$, so by Lemma 3.1(i) (Limit Path), $\mathrm{up}(\nu_0) \neq \sigma^{k(0)+1}$. Thus $\mathrm{up}(\nu_0) \subset (\sigma^{k(0)+1})^- = \bar{\rho}^{k(0)+1} \wedge \sigma^{k(0)+1}$, and (i) follows.

(iv) As $\eta_0 = \lambda^{k(0)}(\sigma) \in T^{k(0)}$, and $\eta_0$ requires extension, it follows from Lemma 5.6 (Uniqueness of Requiring Extension) that $\sigma^{k(0)+1} = \lambda^{k(0)+1}(\sigma)$ cannot require extension. Fix $\xi^{k(0)+1} \subseteq \sigma^{k(0)+1}$ satisfying the hypotheses of (iv), and note that $\xi^{k(0)+1} \subset \sigma^{k(0)+1}$. If $\rho = \langle\ \rangle$, then (iv) is vacuous. Thus we may assume that $\rho^-$ exists.

First suppose that $\rho^-$ is completion-consistent via $\langle\ \rangle$. As noted in the proof of (i), $(\sigma^{k(0)+1})^- \subseteq \rho^{k(0)+1}$, so $\xi^{k(0)+1} \subseteq \rho^{k(0)+1}$. Hence Lemma 5.5(iii) (Completion-Respecting), $\xi^{k(0)+1}$ has a primary completion $\kappa^{k(0)+1} \subset \rho^{k(0)+1}$ which has infinite outcome along $\rho^{k(0)+1}$. By (2.10) $\kappa^{k(0)+1} \subseteq \sigma^{k(0)+1}$, and $\kappa^{k(0)+1}$ will have infinite outcome along $\sigma^{k(0)+1}$ unless $\eta$ switches $\kappa^{k(0)+1}$. Thus if $\eta$ does not switch $\kappa^{k(0)+1}$, then (iv)(a) holds. And if $\eta$ switches $\kappa^{k(0)+1}$, then $\bar{\rho}^{k(0)+1} = \kappa^{k(0)+1}$, and by (5.5)(ii), $\mathrm{up}(\nu_0) = (\xi^{k(0)+1})^-$, so (iv)(b) holds.

Now suppose that $\rho^-$ is not completion-consistent via $\langle\ \rangle$. By the admissibility of $\rho$, $\rho$ must be a 0-completion corresponding to a primary completion $\rho^k$ for some $k$. Again by Definition 5.6, $\rho^i$ is an initial derivative of $\rho^k$ for all $i<k$, so by (2.4), $\rho^k=\bar{\rho}^k\subset\sigma^k$.

First suppose that $\bar{\rho}^k$ has finite outcome along $\sigma^k$. Then by Lemma 5.3(ii) (Implication Chain), $\sigma^{k-1}$ requires extension, so $k=k(0)+1$. Furthermore, $\bar{\rho}^{k(0)+1}=\rho^{k(0)+1}=(\sigma^{k(0)+1})^-$. Thus $\xi^{k(0)+1}\subseteq\rho^{k(0)+1}$, so (iv)(a) will follow from Lemma 5.5(ii) (Completion-Respecting) unless $\mathrm{up}(v_0)=(\xi^{k(0)+1})^-$; but this is ruled out by the hypotheses of (iv).

Now suppose that $\bar{\rho}^k$ has infinite outcome along $\sigma^k$. We compare $k$ and $k(0)$. First suppose that $k(0)<k-1$. Then by Definition 5.6, $\rho^{k(0)+1}=\bar{\rho}^{k(0)+1}$ is an initial derivative, so $\bar{\rho}^{k(0)+1}$ cannot be a primary completion. As $\bar{\rho}^k$ is a primary $k$-completion and $\mathrm{up}^k(\bar{\rho}^{k(0)})=\bar{\rho}^k$, $\dim(\bar{\rho}^{k(0)})>k(0)+1$. Hence (5.5)(ii) must hold for $\sigma^{k(0)}$, contradicting the fact that $\bar{\rho}^{k(0)+1}$ is not a primary completion.

Next suppose that $k(0)=k-1+2q$ for some $q\geqslant 0$. As $\bar{\rho}^k$ has infinite outcome along $\sigma^k$, it follows from Lemma 3.3 ($\lambda$-Behavior) that $\bar{\rho}^{k(0)}$ has finite outcome along $\sigma^{k(0)}$. But by (5.2), as $\sigma^{k(0)}$ requires extension, $\bar{\rho}^{k(0)}$ must have infinite outcome along $\sigma^{k(0)}$, a contradiction.

Finally, suppose that $k(0)=k+2q$ for some $q\geqslant 0$. Then by (5.5)(ii), (5.9), and (5.12), $\bar{\rho}^{k(0)}=\mathrm{up}^{k(0)}(\bar{\rho}^k)$ is the middle element of a triple in an implication chain, and by (5.15) or (5.16), $\bar{\rho}^{k(0)}$ must be a primary completion or an amenable pseudocompletion, contrary to Lemma 5.7(i).

(v) As $\dim(v_i)>k(i)+1$, (5.5)(ii) holds and, as $i=0$, implies (v).

*Case* 2: $m\geqslant 0$. Then $\mathrm{out}^0(\eta_0)\subseteq\rho\subset\sigma$. There are two subcases.

*Subcase* 2.1: $i\leqslant m$. (ii), (iii), and (vi) follow by induction.

(i) First suppose that $\sigma$ is not $v$-switching for any $v\leqslant k(i)+1$. By Lemma 3.3 ($\lambda$-Behavior), $\rho^{k(i)+1}\subseteq\sigma^{k(0)+1}$. By (i) inductively and Lemma 3.3 ($\lambda$-Behavior),

$$\mathrm{up}(v_i)\subset\rho^{k(i)+1}=\rho^{k(i)+1}\wedge\sigma^{k(i)+1}\subseteq\sigma^{k(i)+1}.$$

Otherwise, by (5.18) and (5.25), $\sigma$ is $(k(m)+1)$-switching and $k(i)=k(m)$. Thus by the preadmissibility of $\sigma$, (5.18)(i)(a) or (5.18)(ii)(a) must hold. Suppose that (5.18)(i)(a) holds. Then there is an $\bar{\eta}^{k(m)}\in T^{k(m)}$ such that $\rho^-$ is completion-consistent via $S^\wedge\langle\bar{\eta}^{k(m)}\rangle$. Let $\bar{\eta}^{k(m)}$ require extension for $\bar{v}^{k(m)}$. Then by (5.18)(i)(a), $\bar{\rho}^{k(m)}$ is the primary completion of $\bar{\eta}^{k(m)}$, so by (5.19), $\bar{\rho}^{k(m)+1}=\mathrm{up}(\bar{v}^{k(m)})$. By (ii) inductively and Lemma 3.3 ($\lambda$-Behavior),

$$\mathrm{up}(v_i)\subset\mathrm{up}(\bar{v}^{k(m)})=\bar{\rho}^{k(m)+1}=\rho^{k(i)+1}\wedge\sigma^{k(i)+1}\subset\sigma^{k(i)+1}.$$

Now suppose that (5.18)(ii)(a) holds. Then $\bar{\rho}^{k(m)+1}$ is the end of a primary $\rho^{k(m)+1}$-link which restrains $\mathrm{up}(v_m)$. By the case assumption, $i \leqslant m$. Hence by (ii) inductively and Lemma 3.3 ($\lambda$-Behavior),

$$\mathrm{up}(v_i) \subseteq \mathrm{up}(v_m) \subset \bar{\rho}^{k(m)+1} = \rho^{k(i)+1} \wedge \sigma^{k(i)+1} \subset \sigma^{k(i)+1}.$$

(i) now follows.

(iv) Assume the hypothesis of (iv). By (ii), it suffices to verify (iv) under the assumption that $i$ is the largest integer for which the hypotheses of (iv) hold for $\xi^{k(i)+1}$ and $\mu^{k(i)+1}$.

By (5.18) and (5.25), if $\sigma$ is $v$-switching, then $v \geqslant k(m)+1 \geqslant k(i)+1$. And by the choice of the largest $i$ in the preceding paragraph, $\xi^{k(i)+1} \neq \sigma^{k(i)+1}$ (else by Lemma 5.9 (Completion-Consistency), $w = m+1$, and we would choose $i = w$ for $\sigma^{k(i)+1}$). Now by Lemma 3.3 ($\lambda$-Behavior), $(\sigma^{k(i)+1})^- \subseteq \rho^{k(i)+1}$. We conclude that $\xi^{k(i)+1} \subseteq \rho^{k(i)+1}$. Thus by induction, one of (iv)(a–c) must hold at $\rho^{k(i)+1}$. We consider each possibility.

Assume that (iv)(a) holds at $\rho^{k(i)+1}$. If $\rho^{k(i)+1} \subseteq \sigma^{k(i)+1}$, then (iv)(a) will hold at $\sigma^{k(i)+1}$. If $\rho^{k(i)+1} \nsubseteq \sigma^{k(i)+1}$ and (iv)(a) holds at $\rho^{k(i)+1}$ but not at $\sigma^{k(i)+1}$, then by (5.18) and (5.25), $\sigma^{k(i)+1}$ is $(k(i)+1)$-switching and, by (2.10), must switch the primary completion $\bar{\rho}^{k(i)+1} = \kappa^{k(i)+1}$ of $\xi^{k(i)+1}$. Thus $\kappa^{k(i)+1}$ will have finite outcome along $\sigma^{k(i)+1}$, so by Lemma 5.3(ii) (Implication Chain) and Lemma 5.2 (Requires Extension), $\sigma^{k(i)}$ requires extension. As $\sigma = \mathrm{out}^0(\sigma^{k(i)})$, $w = m+1$ and $\eta_w = \sigma^{k(i)}$. But then by Lemma 5.2 (Requires Extension), $(\xi^{k(i)+1})^- = \mathrm{up}(v_w)$, so (iv)(b) follows from (5.25) and (ii) inductively.

If (iv)(b) holds at $\rho^{k(i)+1}$, then by the maximality of $i$ and as $k(m) = k(w)$, (iv)(b) will hold at $\sigma^{k(i)+1}$ unless $w = m-1 = i$; so assume that (iv)(b) fails and $w = m-1$. By Lemma 5.9 (Completion-Consistency), $\sigma^{k(m)}$ is the primary completion of $\eta_m$, and by Definition 5.6, $\sigma$ is nonswitching; hence $\rho^{k(i)+1} \subseteq \sigma^{k(i)+1}$. Now either (iv)(a) will hold for $\xi^{k(i)+1} \supset \mathrm{up}(v_m)$ at $\rho^{k(i)+1}$, and so at $\sigma^{k(i)+1}$, or $(\xi^{k(i)+1})^- = \mathrm{up}(v_m)$. In the latter case, (iv)(c) holds at $\sigma^{k(i)+1}$.

Suppose that (iv)(c) holds at $\rho^{k(i)+1}$. By (5.19), if $\alpha^k$ is a primary completion with corresponding 0-completion $\alpha$, then $\alpha$ is an initial derivative of $\alpha^k$ and all nodes $\beta \in (\mathrm{out}^0(\alpha^k), \alpha]$ are nonswitching; so by Lemma 3.1(i) (Limit Path), $\lambda^k(\beta) = \lambda^k(\mathrm{out}^0(\alpha^k))$ for such $\beta$. Thus (iv)(c) will hold at $\sigma$ unless $\rho$ is a 0-completion. In the latter case, by (5.18)(i), $\sigma$ switches $\bar{\rho}^{k(m)+1}$. By (5.5)(ii) and the maximality of $i$, $k(i) = k(m)$ and $\bar{\rho}^{k(m)+1} = \mathrm{up}(v_m) \subset \xi^{k(i)+1}$. But $\xi^{k(i)+1} \subseteq \rho^{k(m)+1}$ and by Lemma 3.3 ($\lambda$-Behavior), $\rho^{k(m)+1} \wedge \sigma^{k(m)+1} = \bar{\rho}^{k(m)+1}$; so $\xi^{k(i)+1} \nsubseteq \sigma^{k(i)+1}$, contrary to hypothesis.

(v) Assume the hypotheses of (v). By (i), $\mathrm{up}(\nu_i) \subset \rho^{k(i)+1} \wedge \sigma^{k(i)+1}$. Fix $\tau_i \subseteq \rho^{k(i)+1} \wedge \sigma^{k(i)+1}$ such that $(\tau_i)^- = \mathrm{up}(\nu_i)$. As $\tau_i \subseteq \rho^{k(i)+1}$, (v) follows by induction.

(vii) By (5.18) and (5.25), $\sigma$ is not $v$-switching for any $v \leqslant k(i)$. Hence by Lemma 3.3(i) ($\lambda$-Behavior), $\rho^{k(i)} \subseteq \sigma^{k(i)}$. By induction, $\mathrm{PL}(\delta_i, \rho^{k(i)})$ is a component of $\mathrm{PL}(\delta_{i-1}, \rho^{k(i)})$, and by hypothesis, $\delta_i$ has no primary completion along $\rho^{k(i)}$. It now follows from Definition 5.3 that $\mathrm{PL}(\delta_i, \sigma^{k(i)})$ is a component of $\mathrm{PL}(\delta_{i-1}, \sigma^{k(i)})$.

*Subcase* 2.2: $i = w = m+1$. By Lemma 5.9 (Completion-Consistency), (5.18)(ii)(a) holds, and hence by (5.18)(ii) and (5.25), $\sigma$ is $(k(w)+1)$-switching, $k(w)+1 \geqslant k(m)+1$, and $\bar{\rho}^{k(w)+1}$ is a primary completion. By Lemma 5.3(ii) (Implication Chain) and Lemma 5.9(ii) (Completion-Consistency), $\langle \mathrm{up}(\nu_w), \bar{\rho}^{k(w)+1}, \sigma^{k(w)+1} \rangle$ is the last triple of an $r$-implication chain for some $r$, and by the assumptions of Case 2, $m \geqslant 0$ so $\dim(\nu_w) > k(w)+1$. Hence by (5.5)(ii), if $\bar{\tau}^{k(w)+1}$ is the immediate successor of $\mathrm{up}(\nu_w)$ along $\bar{\rho}^{k(w)+1}$, then $\bar{\tau}^{k(w)+1}$ requires extension for some $\bar{\mu}^{k(w)+1}$ which we fix, and $\bar{\tau}^{k(w)+1}$ has primary completion $\bar{\rho}^{k(w)+1}$. By (5.18)(ii)(a), $[\bar{\mu}^{k(w)+1}, \bar{\rho}^{k(w)+1}]$ is a primary $\rho^{k(w)+1}$-link, and a $\rho^{k(m)+1}$-link derived from this link restrains $\mathrm{up}(\nu_m)$. By Lemma 5.9(ii) (Completions-Consistency), $\sigma^{k(w)} = \eta_w$ requires extension and by (5.1) and as $S \neq \langle\ \rangle$, (5.5)(ii) must hold: hence $\delta_w = \bar{\rho}^{k(w)}$. Furthermore, as $\sigma$ is $(k(w)+1)$-switching, $\rho^{k(w)} = \bar{\rho}^{k(w)}$. We verify (i)–(vii).

(i) As $\langle \mathrm{up}(\nu_w), \bar{\rho}^{k(w)+1}, \sigma^{k(w)+1} \rangle$ is the last triple of a $(k(w)+1)$-implication chain, it follows from (5.8)(i) that $\mathrm{up}(\nu_w) \subset \bar{\rho}^{k(w)+1}$. By Lemma 3.3 ($\lambda$-Behavior), $\bar{\rho}^{k(w)+1} = \rho^{k(w)+1} \wedge \sigma^{k(w)+1}$. (i) now follows.

(ii) We assume that $k(w) = k(m)$, else there is nothing to verify. We assume that (ii) fails, and derive a contradiction. By (i) $\mathrm{up}(\nu_w)$ and $\mathrm{up}(\nu_m)$ are comparable. First assume that $\mathrm{up}(\nu_w) = \mathrm{up}(\nu_m)$ in order to obtain a contradiction. By (i), $\bar{\tau}^{k(w)+1} \subseteq \rho^{k(w)+1} \wedge \sigma^{k(w)+1}$, and we have noted that $\bar{\rho}^{k(w)+1}$ is the primary completion of $\bar{\tau}^{k(w)+1}$. Furthermore, as $(\bar{\tau}^{k(w)+1})^- = \mathrm{up}(\nu_w) = \mathrm{up}(\nu_m)$, $\bar{\tau}^{k(w)+1}$ requires extension and $S \neq \langle\ \rangle$, it follows from (5.5) that $\dim(\mathrm{up}(\nu_m)) > k(w)+1$. Hence by (5.5)(ii) for both $\eta_m$ and $\eta_w = \sigma^{k(w)}$, it must be the case that both $\delta_m$ and $\delta_w = \rho^{k(w)}$ have infinite outcome along $\sigma^{k(w)}$, and $\mathrm{up}(\delta_m) = \mathrm{up}(\delta_w) = \bar{\rho}^{k(w)+1}$. Hence by (2.8), $\eta_m = \eta_w = \sigma^{k(w)}$. But as $k(m) = k(w)$, $\eta_m \subseteq \rho^{k(w)} = \bar{\rho}^{k(w)} \subset \sigma^{k(w)}$, yielding a contradiction.

Next suppose that $\mathrm{up}(\nu_m) \supset \mathrm{up}(\nu_m)$ in order to obtain a contradiction. As $\bar{\tau}^{k(w)+1}$ requires extension and $\bar{\rho}^{k(w)+1}$ is the primary completion of $\bar{\tau}^{k(w)+1}$, all nodes in $[\bar{\tau}^{k(w)+1}, \bar{\rho}^{k(w)+1}]$ are implication-restrained. Hence by (5.1) and (i) for $\nu_m$, $\dim(\nu_m) > k(m)+2$. By (i), if $\tau_m$ is the immediate successor of $\mathrm{up}(\nu_m)$ along $\sigma^{k(w)+1}$, then $\tau_m \subseteq \rho^{k(w)+1}$, so by (v) inductively,

$\tau_m$ requires extension. Hence by Lemma 5.5(i) (Completion Respecting), $\tau_m$ has a primary completion $\kappa_m \subset \bar{\rho}^{k(w)+1}$, and $\kappa_m$ has infinite outcome along $\bar{\rho}^{k(w)+1} \subset \sigma^{k(w)+1}$. Thus by (2.4), all derivatives of $\kappa_m$ along $\sigma^{k(w)}$ have finite outcome along $\sigma^{k(w)}$. But by (5.5)(ii) and (5.2) for $\eta_m$, $\mathrm{up}(\delta_m) = \kappa_m$ and $\delta_m$ has infinite outcome along $\eta_m \subseteq \sigma^{k(w)}$, yielding a contradiction. Condition (ii) now follows.

(iii) We assume that $k(m) = k(w)$, else there is nothing to show. Condition (5.5)(ii) specifies the relationship between requires extension situations on $T^{k(w)}$ and $(k(w)+1)$-implication chains, and specifies that $v_w$ is the initial derivative of $\mathrm{up}(v_w)$ along $\sigma^{k(w)}$. We have noted that $\bar{\tau}^{k(w)+1}$ requires extension; let $\bar{\tau}^{k(w)} = \mathrm{out}(\bar{\tau}^{k(w)+1})$, and note that $\bar{\tau}^{k(w)} \subset \sigma^{k(w)}$ by (2.5). We compare the locations of $\bar{\tau}^{k(w)}$ and $\eta_m$, noting that they are comparable as both are $\subset \sigma^{k(w)}$. First assume that $\bar{\tau}^{k(w)} \subset \eta_m$. (We will show that this is the only case which can actually occur.) Now $\bar{\rho}^{k(w)+1}$ is the primary completion of $\bar{\tau}^{k(w)+1}$ and $\bar{\rho}^{k(w)+1} \subset \sigma^{k(w)+1}$; hence by Lemma 3.1(i) (Limit Path), $\bar{\rho}^{k(w)+1}$ has an initial derivative $\bar{\mu}^{k(w)} \subset \sigma^{k(w)}$ which is the $k(w)$-completion of $\bar{\rho}^{k(w)+1}$. By (5.25) and (5.27), new nodes on lower dimension trees cannot require extension until all nodes on higher dimension trees which previously required extension have found their 0-completions; hence as $k(w) < k(w)+1$, no node in $[\bar{\tau}^{k(w)}, \bar{\mu}^{k(w)}]$ can require extension. Hence $\bar{\mu}^{k(w)} \subset \eta_m$. By Lemma 3.1(i) (Limit Path), and as $\mathrm{up}(v_w) \subset \bar{\rho}^{k(w)+1}$ by (i), it must be the case that $v_w \subset \bar{\tau}^{k(w)} \subset \bar{\mu}^{k(w)} \subset \eta_m$, so (iii) holds in this case.

$\bar{\tau}^{k(w)} \neq \eta_m$, else we would contradict Lemma 5.6 (Uniqueness of Requiring Extension).

Finally, assume that $\bar{\tau}^{k(w)} \supset \eta_m$. By (vi) inductively for $\bar{\tau}^{k(w)+1}$, $\mathrm{up}(v_m) \subset \bar{\mu}^{k(w)+1}$, contrary to (5.18)(ii)(a). Hence (iii) holds.

(iv) Assume that the hypotheses of (iv) hold for $\xi^{k(w)+1} \subseteq \sigma^{k(w)+1}$ which requires extension. By the hypotheses of (iv), $\mathrm{up}(v_w) \subset (\xi^{k(w)+1})^-$, so $\bar{\tau}^{k(w)+1} \neq \xi^{k(w)+1}$. As $\bar{\tau}^{k(w)+1} \subset \bar{\rho}^{k(w)+1} \subset \sigma^{k(w)+1}$, it must be the case that $\bar{\tau}^{k(w)+1} \subset \xi^{k(w)+1}$. As $\sigma$ is $(k(w)+1)$-switching, it follows from Lemma 3.3(ii) ($\lambda$-Behavior) and (2.4) that $(\sigma^{k(w)+1})^- = \bar{\rho}^{k(w)+1}$ and $\bar{\rho}^{k(w)+1}$ has finite outcome along $\sigma^{k(w)+1}$. Thus by (5.2), $\xi^{k(w)+1} \subseteq \bar{\rho}^{k(w)+1} \subset \rho^{k(w)+1}$. As $\bar{\rho}^{k(w)+1}$ is the primary completion of $\bar{\tau}^{k(w)+1}$, (iv)(a) follows from Lemma 5.5(i) (Completion Respecting).

(v) We have already noted that the immediate successor, $\bar{\tau}^{k(w)+1}$, of $\mathrm{up}(v_w)$ along $\bar{\rho}^{k(w)+1} \subset \sigma^{k(w)+1}$ requires extension for $\bar{\mu}^{k(w)+1}$. For the second clause of (v), we assume that $k(w) = k(m)$, else there is nothing to verify. As $\rho$ is completion-consistent via $\langle \eta_i : i \leqslant m \rangle$ and (5.18)(ii)(a) holds at $\rho$ with $\bar{\rho}^{k(w)+1}$ a primary completion, $[\bar{\mu}^{k(w)+1}, \bar{\rho}^{k(w)+1}]$ is a primary $\rho^{k(w)+1}$-link which restrains $\mathrm{up}(v_m)$, so $\bar{\mu}^{k(w)+1} \subseteq \mathrm{up}(v_m)$.

We assume that $\bar{\mu}^{k(w)+1} = \mathrm{up}(v_m)$ and derive a contradiction. As $[\bar{\mu}^{k(w)+1}, \bar{\rho}^{k(w)+1}]$ is a primary $\rho^{k(w)+1}$-link and by (i), $\bar{\mu}^{k(w)+1} \neq \bar{\rho}^{k(w)+1}$, $\bar{\mu}^{k(w)+1}$ has finite outcome along $\bar{\rho}^{k(w)+1} \subset \sigma^{k(w)+1}$, and $\mathrm{up}(\bar{\mu}^{k(w)+1}) = \mathrm{up}(\bar{\rho}^{k(w)+1})$, so $\dim(\bar{\mu}^{k(w)+1}) = \dim(v_m) > k(m)+1$. Thus by (v) for $v_m$, the immediate successor $\tau_m$ of $\mathrm{up}(v_m)$ along $\sigma^{k(w)+1}$ requires extension, so by (5.2), $\mathrm{up}(v_m) = \bar{\mu}^{k(w)+1}$ has infinite outcome along $\tau_m \subseteq \sigma^{k(w)+1}$, a contradiction. (v) now follows.

(vi) We assume that $k(w) = k(m)+1$, else there is nothing to verify. By (5.18)(ii)(a) and as $\sigma$ is $(k(w)+1)$-switching, there is a primary $\rho^{k(w)+1}$-link $[\bar{\mu}^{k(w)+1}, \bar{\rho}^{k(w)+1}]$ such that the $\rho^{k(w)}$-link $[\mu^{k(w)}, \pi^{k(w)}]$ derived from $[\bar{\mu}^{k(w)+1}, \bar{\rho}^{k(w)+1}]$ restrains $\mathrm{up}(v_m)$. Also, $\bar{\rho}^{k(w)+1}$ is the primary completion of the immediate successor $\bar{\tau}^{k(w)+1}$ of $\mathrm{up}(v_w)$ along $\bar{\rho}^{k(w)+1}$, so $\pi^{k(w)}$ is the corresponding $k(w)$-completion. By (5.2), $\mathrm{up}(v_w)$ has infinite outcome along $\bar{\tau}^{k(w)+1}$, so by (2.4), the initial derivative of $\mathrm{up}(v_w)$ along $\pi^{k(w)}$ is the principal derivative of $\mathrm{up}(v_w)$ along $\pi^{k(w)}$ and has finite outcome along $\pi^{k(w)}$; hence again by (5.2), this initial derivative must be $v_w$. So by (2.4), if $\bar{\tau}^{k(w)} = \mathrm{out}(\bar{\tau}^{k(w)+1})$, then $(\bar{\tau}^{k(w)})^- = v_w$. By (5.25) and (5.27), no node in $[\bar{\tau}^{k(w)}, \pi^{k(w)}]$ can require extension (else we would contradict the dimension ordering of (5.25)); so as, by (v), the immediate successor $\tau_m$ of $\mathrm{up}(v_m)$ along $\pi^{k(w)} \subset \sigma^{k(w)}$ requires extension, it must be the case that $\tau_m \subset \bar{\tau}^{k(w)}$. Thus $\mathrm{up}(v_m) = (\tau_m)^- \subset (\bar{\tau}^{k(w)})^- = v_w$, and (vi) holds.

(vii) By (iii) and hypothesis , if $k(m) = k(w)$, then $\eta_m$ requires extension and $v_w \subset \delta_m \subset \eta_m \subseteq \delta_w \subset \eta_w = \sigma^{k(w)}$. Condition (vii) now follows from Definition 5.3, as (5.14) holds. ▮

The next lemma provides a step-by-step analysis of the effect of extending $\rho$ to $\sigma$, as specified by (5.18), on the PL sets corresponding to each element in the sequence via which $\rho$ is completion-consistent.

Lemma 5.11 (Amenable Backtracking Lemma). *Fix hypotheses as in Lemma* 5.10 (*Component*). *Then for all* $i \leqslant m$:

(i) *If either* (5.18)(i)(a) *holds for* $\rho$ *with* $k(m+1) = k(i)$, *or* (5.18)(ii)(a) *holds for* $\rho$ *with* $j = k(i)+1$, *then*

$$\mathrm{PL}(\delta_i, \sigma^{k(i)}) = \mathrm{PL}(\delta_i, \rho^{k(i)}) \cup \{\rho^{k(i)}\}$$

*and the union is disjoint*,

$$\bar{\rho}^{k(i)+1} \in \mathrm{PL}(\mathrm{up}(v_i), \rho^{k(i)+1}),$$

*and*

$$\mathrm{PL}(\mathrm{up}(v_i), \sigma^{k(i)+1}) = \mathrm{PL}(\mathrm{up}(v_i), \rho^{k(i)+1}) \setminus \{\bar{\rho}^{k(i)+1}\}.$$

(ii) *If the hypotheses of* (i) *fail, then* $\mathrm{PL}(\delta_i, \sigma^{k(i)}) = \mathrm{PL}(\delta_i, \rho^{k(i)})$ *and* $\mathrm{PL}(\mathrm{up}(v_i), \sigma^{k(i)+1}) = \mathrm{PL}(\mathrm{up}(v_i), \rho^{k(i)+1})$.

(iii)

$$\{\mathrm{up}(\xi^{k(i)}) : \xi^{k(i)} \in \mathrm{PL}(\delta_i, \sigma^{k(i)})\} \cup \mathrm{PL}(\mathrm{up}(v_i), \sigma^{k(i)+1})$$
$$= \{\mathrm{up}(\xi^{k(i)}) : \xi^{k(i)} \in \mathrm{PL}(\delta_i, \rho^{k(i)})\} \cup \mathrm{PL}(\mathrm{up}(v_i), \rho^{k(i)+1}),$$

*and the unions are disjoint.*

(iv) *If* $w = m+1$, *then* $\mathrm{PL}(\delta_w, \sigma^{k(w)}) = \varnothing$.

(v) *If* $\sigma$ *is a* 0*-completion of* $\eta_m$, *then* $\mathrm{PL}(\mathrm{up}(v_m), \sigma^{k(m)+1}) = \varnothing$.

(vi) *If* $\sigma^-$ *is not completion-consistent via* $\langle\ \rangle$, *then* $\sigma$ *satisfies* (5.29)(i).

*Proof.* As $\mathrm{up}(\rho^{k(i)}) = \bar{\rho}^{k(i)+1}$ by (5.18), (5.25), and Lemma 3.3 ($\lambda$-Behavior), (iii) follows from (i) and (ii) by induction on $\mathrm{lh}(\rho^{k(i)})$. We first prove (iv) and (v).

If $w = m+1$, then $(\sigma^{k(w)})^- = \delta_w$ by Lemma 5.9 (Completion-Consistency), so there can be no primary $\sigma^{k(w)}$-link restraining $\delta_w$ or $\sigma^{k(w)}$. Furthermore, $\mathrm{PL}(\delta_w, \sigma^{k(w)})$ can have a component only if $\mathrm{lh}(\sigma^{k(w)}) - \mathrm{lh}(\delta_w) \geqslant 2$. (iv) now follows from Definition 5.3. Suppose that $\sigma$ is a 0-completion of $\eta_m$. By Lemma 5.9 (Component); $w = m-1$. By (5.19), $\mathrm{up}(v_m) = \mathrm{up}(\sigma^{k(m)})$, so by (2.10), $\mathrm{up}(v_m)$ is $\sigma^{k(m)+1}$-free. Suppose that elements are placed in $\mathrm{PL}(\mathrm{up}(v_m), \sigma^{k(m)+1})$ through (5.14) in order to obtain a contradiction. Then there are $\mu^{k(m)+1} \subset \mathrm{up}(v_m) \subset (\xi^{k(m)+1})^- \subset \xi^{k(m)+1} \subseteq \sigma^{k(m)+1}$ such that $\xi^{k(m)+1}$ requires extension for $\mu^{k(m)+1}$. As $w = m-1$, the hypothesis of Lemma 5.10(iv) (Component) holds for $\rho$ in place of $\sigma$, so one of conditions (iv)(a)–(iv)(c) must hold for $i = m$. (iv)(b) cannot hold, as the $w$ corresponding to $\rho$ is $m$, so $v_{m+1}$ is undefined. (iv)(c) cannot hold, else $\rho^{k(m)}$ would be a primary completion, so by (5.18), $\sigma$ would be a switching extension of $\rho$; but $\sigma$ is a 0-completion, and by Definition 5.6, 0-completions are nonswitching, yielding a contradiction. Suppose that (iv)(a) holds. Then $\xi^{k(m)+1}$ has a primary completion $\kappa^{k(m)+1}$ which has infinite outcome along $\rho^{k(m)+1}$. Thus by Definition 5.6, $[\mu^{k(m)+1}, \kappa^{k(m)+1}]$ is a primary $\rho^{k(m)+1}$-link restraining $\mathrm{up}(v_m)$, so $\mathrm{up}(v_m)$ is not $\rho^{k(m)+1}$-free, a contradiction. It now follows from Definition 5.3 that $\mathrm{PL}(\mathrm{up}(v_m), \sigma^{k(m)+1}) = \varnothing$, so (v) holds.

We now verify (i) and (ii). We proceed by induction on $\mathrm{lh}(\sigma)$. There are several cases to consider, depending on the manner in which $\sigma$ extends $\rho$. By (5.18) and (5.25), we see that if $\sigma$ is $v$-switching, then $v > k(m)$ if $m \geqslant 0$. In the first two cases, $v > k(i)+2$ or $\sigma$ is nonswitching, and $v = k(i)+2$, we show that the hypothesis and conclusion of (ii) hold. The final case is when $v = k(i)+1$, in which case (i) will be followed.

We first note the following:

*Claim.* If $\alpha^k \subset \beta^k \subset \gamma^k \in T^k$, $(\gamma^k)^- = \beta^k$, $\mathrm{out}^0(\beta^k)$ is completion-consistent via a nonempty set, and $\gamma^k$ is not $u$-switching for any $u \leqslant k+1$, then $\mathrm{PL}(\alpha^k, \beta^k) = \mathrm{PL}(\alpha^k, \gamma^k)$.

*Proof.* By Lemma 5.1(i) (PL Analysis), $\mathrm{PL}(\alpha^k, \beta^k) \subseteq \mathrm{PL}(\alpha^k, \gamma^k)$. If $\mathrm{PL}(\alpha^k, \gamma^k) \backslash \mathrm{PL}(\alpha^k, \beta^k) \neq \emptyset$, then by Lemma 5.1(iii) (PL Analysis), either $\beta^k$ is the end of a primary $\gamma^k$-link and so $\gamma^k$ is $u$-switching for some $u \leqslant k+1$, or $\gamma^k$ requires extension; and in the latter case, it follows from Lemma 5.9 (Completion-Consistency) that (5.18)(ii) holds for $\sigma$ with $j = k+1$, so $\gamma^k$ is $(k+1)$-switching. But this is contrary to our case assumption. The claim now follows. ▮

*Case* 1: $\sigma$ is $v$-switching for some $v > k(i)+2$ or is nonswitching. (i) is vacuous, and (ii) is immediate from the claim.

*Case* 2: $\sigma$ is $(k(i)+2)$-switching. By the claim, $\mathrm{PL}(\delta_i, \sigma^{k(i)}) = \mathrm{PL}(\delta_i, \rho^{k(i)})$. It follows from (5.18)(iii) that either (5.18)(i)(a) or (5.18)(ii)(a) holds. By (5.25), $k(i) \leqslant k(m)$, and by (5.18), if $\sigma$ is $v$-switching, then $v > k(m)$. Hence there are two subcases to consider, $k(i) = k(m)$, and $k(i) = k(m) - 1$. We note that in both cases, (i) is vacuous as $k(m+1) > k(i)$ if (5.18)(i) is followed, so it suffices to verify (ii).

*Subcase* 2.1: $k(i) = k(m)$. By Lemma 3.3 ($\lambda$-Behavior), $\rho^{k(m)+1} \subset \sigma^{k(m)+1}$.

*Subcase* 2.1.1: Condition (5.18)(i)(a) holds and $i = m$, and so as $\sigma$ is $(k(m)+2)$-switching, $k(m+1) = k(m)+1$. Then there are $\bar{v}_{m+1}, \bar{\eta}_{m+1} \in T^{k(m)+1}$ such that $\rho^{k(m)+1}$ is a primary completion of $\bar{\eta}_{m+1}$ for $\bar{v}_{m+1}$ and by Lemma 5.9 (Completion-Consistency), $(\rho^{k(m)+1})^-$ is completion-consistent via $S^\wedge\langle\bar{\eta}_{m+1}\rangle$. By Lemma 5.1(i)(iii), $\mathrm{PL}(\mathrm{up}(v_m), \rho^{k(m)+1}) \subseteq \mathrm{PL}(\mathrm{up}(v_m), \sigma^{k(m)+1})$ and $\mathrm{PL}(\mathrm{up}(v_m), \sigma^{k(m)+1}) \backslash \mathrm{PL}(\mathrm{up}(v_m), \rho^{k(m)+1}) \subseteq \{\rho^{k(m)+1}\}$. By Lemma 5.10(vi) (Component), $\mathrm{up}(v_m) \subset \bar{v}_{m+1}$. Now $[\bar{v}_{m+1}, \rho^{k(m)+1}]$ is the only primary $\sigma^{k(m)+1}$-link which is not a $\rho^{k(m)+1}$-link, and by Lemma 5.7(i) (Primary Completion), $\sigma^{k(m)+1}$ does not require extension. Hence neither (5.13) nor (5.14) can place $\rho^{k(m)+1}$ into $\mathrm{PL}(\mathrm{up}(v_m), \sigma^{k(m)+1})$, so $\mathrm{PL}(\mathrm{up}(v_m), \sigma^{k(m)+1}) = \mathrm{PL}(\mathrm{up}(v_m), \rho^{k(m)+1})$.

*Subcase* 2.1.2: Condition (5.18)(ii)(a) holds and $i = m$. Then there is a $\rho^{k(m)+1}$-link $[\mu^{k(m)+1}, \pi^{k(m)+1}]$ which restrains $\mathrm{up}(v_m)$ and is derived from a primary $\rho^{k(m)+2}$-link $[\mu^{k(m)+2}, \pi^{k(m)+2}]$, and $\sigma$ switches $\pi^{k(m)+2}$. Note that $[\pi^{k(m)+1}, \rho^{k(m)+1}]$ is the only primary $\sigma^{k(m)+1}$-link which is not a $\rho^{k(m)+1}$-link, and $\mathrm{up}(v_m) \subset \pi^{k(m)+1}$; hence any node placed into $\mathrm{PL}(\mathrm{up}(v_m), \sigma^{k(m)+1})$ via (5.13) is already in $\mathrm{PL}(\mathrm{up}(v_m), \rho^{k(m)+1})$. By Lemma 5.10(vi) (Component), if $\sigma^{k(m)+1}$ requires extension for some $\bar{v}_{m+1}$ which we fix, then $\mathrm{up}(v_m) \subset \bar{v}_{m+1}$; hence any node placed into

$\mathrm{PL}(\mathrm{up}(v_m), \sigma^{k(m)+1})$ via (5.14) is already in $\mathrm{PL}(\mathrm{up}(v_m), \rho^{k(m)+1})$. Thus by Definition 5.3 and Lemma 5.1(i) (PL Analysis), $\mathrm{PL}(\mathrm{up}(v_m), \sigma^{k(m)+1}) = \mathrm{PL}(\mathrm{up}(v_m), \rho^{k(m)+1})$.

*Subcase* 2.1.3: $i < m$. By Lemma 5.10(i) (Component), $\mathrm{PL}(\mathrm{up}(v_i), \rho^{k(m)+1}) \subseteq \mathrm{PL}(\mathrm{up}(v_i), \sigma^{k(m)+1})$. By Lemma 5.1(ii) (PL Analysis) and since $\rho^{k(m)+1} \subset \sigma^{k(m)+1}$, $\mathrm{PL}(\mathrm{up}(v_i), \sigma^{k(m)+1}) \backslash \mathrm{PL}(\mathrm{up}(v_i), \rho^{k(m)+1}) \subseteq \{\rho^{k(m)+1}\}$. For each $j \in [i+1, m]$, let $\tau_j^{k(m)+1}$ be the immediate successor of $\mathrm{up}(v_j)$ along $\sigma^{k(m)+1}$ and note that, by Lemma 5.10(v) (Component), $\tau_j^{k(m)+1}$ requires extension for each such $j$. For each $j \in [i+1, m]$, let $\xi_j^{k(m)+1}$ be the primary completion of $\tau_j^{k(m)+1}$ along $\sigma^{k(m)+1}$ if it exists, and let $\xi_j^{k(m)+1} = \sigma^{k(m)+1}$ otherwise. As $i+1 > 0$, it follows from (5.1) that $\dim(\mathrm{up}(v_{i+1})) > k(m)+1$. Hence by Lemma 5.10(v) (Component) and (5.14), we see that $\mathrm{PL}(\mathrm{up}(v_{i+1}), \xi_i^{k(m)+1})$ is a component of $\mathrm{PL}(\mathrm{up}(v_i), \sigma^{k(m)+1})$. Furthermore, by Lemma 5.10(iv) (Component), every component of $\mathrm{PL}(\mathrm{up}(v_i), \sigma^{k(m)+1})$ which has $\rho^{k(m)+1}$ as an element must be of the form $\mathrm{PL}(\mathrm{up}(v_j), \xi_j^{k(m)+1})$ for some $j \in [i+1, m]$. By Lemma 5.10(i, ii) (Component), if $j \in [i, m]$, then $\mathrm{up}(v_j) \subseteq \mathrm{up}(v_m) \subset \rho^{k(m)+1}$. It now follows by induction on $m-i$ that if $\rho^{k(m)+1} \in \mathrm{PL}(\mathrm{up}(v_i), \sigma^{k(m)+1})$, then $\rho^{k(m)+1} \in \mathrm{PL}(\mathrm{up}(v_m), \sigma^{k(m)+1})$, and so by Subcase 2.1.2, that $\mathrm{PL}(\mathrm{up}(v_i), \sigma^{k(m)+1}) = \mathrm{PL}(\mathrm{up}(v_i), \rho^{k(m)+1})$.

*Subcase* 2.2: $k(i) = k(m) - 1$. By Lemma 3.3 ($\lambda$-Behavior), $\rho^{k(m)} \subset \sigma^{k(m)}$.

*Subcase* 2.2.1: Condition (5.18)(i)(a) holds. Then there are $\bar{v}_{m+1}$, $\bar{\eta}_{m+1} \in T^{k(m)}$ such that $\rho^{k(m)}$ is a primary completion of $\bar{\eta}_{m+1}$ for $\bar{v}_{m+1}$ and by Lemma 5.9 (Completion-Consistency), $(\rho^{k(m)})^-$ is completion-consistent via $S^\wedge\langle\bar{\eta}_{m+1}\rangle$. By Lemma 5.10(vi) (Component), $\mathrm{up}(v_i) \subset \bar{v}_{m+1}$. Now by (5.19) and (5.18)(i), $[\bar{v}_{m+1}, \rho^{k(m)}]$ is the only primary $\sigma^{k(m)}$-link which is not a $\rho^{k(m)}$-link, and by Lemma 5.7(i) (Primary Completion), $\sigma^{k(m)}$ does not require extension. Hence by Lemma 5.1(i, iii) (PL Analysis), $\mathrm{PL}(\mathrm{up}(v_i), \sigma^{k(m)}) = \mathrm{PL}(\mathrm{up}(v_i), \rho^{k(m)})$.

*Subcase* 2.2.2: Condition (5.18)(ii)(a) holds. Then there is a primary $\rho^{k(m)+1}$-link $[\mu^{k(m)+1}, \pi^{k(m)+1}]$ which restrains $\mathrm{up}(v_m)$, so $\mathrm{up}(v_m) \subset \pi^{k(m)+1}$. Let $\pi^{k(m)}$ be the initial derivative of $\pi^{k(m)+1}$ along $\sigma^{k(m)}$. Then $[\pi^{k(m)}, \rho^{k(m)}]$ is the only primary $\sigma^{k(m)}$-link which is not a $\rho^{k(m)}$-link, and by (5.2), $v_m$ is the initial derivative of $\mathrm{up}(v_m)$ along $\sigma^{k(m)}$, hence it follows from Lemma 3.1(i) (Limit Path) that $v_m \subset \pi^{k(m)}$. If $\sigma^{k(m)}$ requires extension for some $\bar{v}_{m+1}$, then by Lemma 5.10(vi) (Component), $\mathrm{up}(v_i) \subset \bar{v}_{m+1}$. It now follows from Lemma 5.1(i) (PL Analysis) and Definition 5.3 that $\mathrm{PL}(\mathrm{up}(v_i), \sigma^{k(m)+1}) = \mathrm{PL}(\mathrm{up}(v_i), \rho^{k(m)+1})$.

*Case* 3: $\sigma$ is $(k(i)+1)$-switching. By (5.25), $k(i) \leqslant k(m)$, so $\sigma$ must be $(k(m)+1)$-switching, i.e., $k(i) = k(m)$. By Lemma 3.3 ($\lambda$-Behavior), $(\sigma^{k(m)+1})^- = \bar{\rho}^{k(m)+1}$. Now $\sigma$ is preadmissible and $m \geqslant 0$, so $\rho$ is not completion-consistent via $\langle\ \rangle$; hence (5.18)(i)(a) or (5.18)(ii)(a) must hold. We note that (ii) is vacuous in this case, and verify (i). We consider three subcases.

*Subcase* 3.1: Condition (5.18)(i)(a) holds and $i = m$. By Lemma 5.9 (Completion-Consistency), there is a node $\bar{\eta}^{k(m)} \in T^{k(m)}$ such that $(\mathrm{out}^0(\rho^{k(m)}))^-$ is completion-consistent via $S^\wedge\langle\bar{\eta}^{k(m)}\rangle$, $\rho^{k(m)}$ is the $k(m)$-completion of $\bar{\eta}^{k(m)}$ for some $\bar{v}^{k(m)}$, and $\sigma^{k(m)}$ switches $\bar{\rho}^{k(m)+1}$. Let $\bar{\delta}^{k(m)} = (\bar{\eta}^{k(m)})^-$.

By Lemma 5.10(iii) (Component), $\bar{v}^{k(m)} \subset \delta_m$; and as $(\mathrm{out}^0(\rho^{k(m)}))^-$ is completion-consistent via $S^\wedge\langle\bar{\eta}^{k(m)}\rangle$ and $\eta_m \in S$, $(\eta_m)^- = \delta_m \subset \rho^{k(m)}$. Thus $[\bar{v}^{k(m)}, \rho^{k(m)}]$ is a primary $\sigma^{k(m)}$-link which restrains $\delta_m$. It now follows from (5.13) that $\rho^{k(m)} \in \mathrm{PL}(\delta_m, \sigma^{k(m)})$, so by Lemma 5.1(i, ii) (PL Analysis), $\mathrm{PL}(\delta_m, \sigma^{k(m)}) = \mathrm{PL}(\delta_m, \rho^{k(m)}) \cup \{\rho^{k(m)}\}$.

As $\rho^{k(m)}$ is a primary completion of $\bar{\eta}^{k(m)}$ and $\bar{\eta}^{k(m)}$ requires extension for $\bar{v}^{k(m)}$, it follows from (5.19) that $\mathrm{up}(\bar{v}^{k(m)}) = \mathrm{up}(\rho^{k(m)}) = \bar{\rho}^{k(m)+1}$. Let $\tau^{k(m)+1}$ be the immediate successor of $\bar{\rho}^{k(m)+1}$ along $\rho^{k(m)+1}$. By Definition 5.6, $\bar{\eta}^{k(m)}$ and $(\rho^{k(m)})^-$ are completion-consistent via the same sequence, so by Definition 5.6 and Lemma 5.9 (Completion-Consistency), $\rho^{k(m)}$ and $\bar{\delta}^{k(m)}$ are completion-consistent via the same sequence, which is non-empty as $m \geqslant 0$. Hence as $\bar{\eta}^{k(m)}$ requires extension, it follows from (5.1) that $\dim(\bar{\delta}^{k(m)}) > k(m)+1$. By (5.3) and Lemma 4.3(i)(a) (Link Analysis), $\lambda(\bar{\eta}^{k(m)}) \supset \bar{\rho}^{k(m)+1} = \mathrm{up}(\bar{v}^{k(m)})$; so as $\sigma^{k(m)}$ switches $\bar{\rho}^{k(m)+1}$ and $\bar{\eta}^{k(m)} \subset \rho^{k(m)}$, it follows that $\tau^{k(m)+1} \subseteq \lambda(\bar{\eta}^{k(m)})$; hence by (5.5)(ii) and (5.15), $\tau^{k(m)+1}$ requires extension for $\mathrm{up}(\bar{\delta}^{k(m)})$. By Lemma 5.10(v) (Component) at $\rho^-$, $\mathrm{up}(\bar{\delta}^{k(m)}) \subset \mathrm{up}(v_m)$, and as $\mathrm{up}(\bar{v}^{k(m)}) = \bar{\rho}^{k(m)+1}$, it follows from Lemma 5.10(ii) (Component) that $\mathrm{up}(v_m) \subset \bar{\rho}^{k(m)+1}$. Thus by (5.14)(i), $\bar{\rho}^{k(m)+1} \in \mathrm{PL}(\mathrm{up}(v_m), \tau^{k(m)+1})$. As $\sigma$ switches $\bar{\rho}^{k(m)+1}$ (and so, by Definition 5.6, cannot be a primary completion), it follows from Lemma 4.5 (Free Extension) that $\bar{\rho}^{k(m)+1}$ is $\sigma^{k(m)+1}$-free; hence by Lemma 5.10(iv) (Component), we may apply Lemma 5.1(v) (PL Analysis) to conclude that $\mathrm{PL}(\mathrm{up}(v_m), \rho^{k(m)+1}) = \mathrm{PL}(\mathrm{up}(v_m), \tau^{k(m)+1})$; so by Lemma 5.1(i, ii) (PL Analysis) and Definition 5.3,

$$\mathrm{PL}(\mathrm{up}(v_m), \rho^{k(m)+1}) = \mathrm{PL}(\mathrm{up}(v_m), \bar{\rho}^{k(m)+1}) \cup \{\bar{\rho}^{k(m)+1}\}$$
$$\& \ \bar{\rho}^{k(m)+1} \notin \mathrm{PL}(\mathrm{up}(v_m), \bar{\rho}^{k(m)+1}).$$

As $\sigma$ switches $\bar{\rho}^{k(m)+1}$ and is $(k(m)+1)$-switching, it follows from Lemma 3.3 ($\lambda$-Behavior) that $(\sigma^{k(m)+1})^- = \bar{\rho}^{k(m)+1}$ and $\bar{\rho}^{k(m)+1}$ has finite outcome along $\sigma^{k(m)+1}$. Thus by Lemma 5.1(iv) (PL Analysis), $\mathrm{PL}(\mathrm{up}(v_m), \sigma^{k(m)+1}) = \mathrm{PL}(\mathrm{up}(v_m), \bar{\rho}^{k(m)+1})$. Thus

$$\mathrm{PL}(\mathrm{up}(v_m), \rho^{k(m)+1}) \backslash \mathrm{PL}(\mathrm{up}(v_m), \sigma^{k(m)+1}) = \{\bar{\rho}^{k(m)+1}\},$$

and (i) follows in this case.

*Subcase* 3.2: Condition (5.18)(ii)(a) holds and $i=m$. Then there is a primary $\rho^{k(m)+1}$-link $[\mu^{k(m)+1}, \bar{\rho}^{k(m)+1}]$ which restrains $\mathrm{up}(v_m)$ such that $\sigma$ switches $\bar{\rho}^{k(m)+1}$. By (5.13), $\bar{\rho}^{k(m)+1} \in \mathrm{PL}(\mathrm{up}(v_m), \rho^{k(m)+1})$. By induction (on $T^0$) using (i) and (ii), if $\eta_m \subset \xi^{k(m)} \subseteq \rho^{k(m)}$ then $\mathrm{PL}(\mathrm{up}(v_m), \lambda(\xi^{k(m)})) \subseteq \mathrm{PL}(\mathrm{up}(v_m), \lambda((\xi^{k(m)})^-))$. Thus $\mathrm{PL}(\mathrm{up}(v_m), \rho^{k(m)+1}) \subseteq \mathrm{PL}(\mathrm{up}(v_m), \lambda(\eta_m))$, so $\bar{\rho}^{k(m)+1} \in \mathrm{PL}(\mathrm{up}(v_m), \lambda(\eta_m))$, from which it follows that $\bar{\rho}^{k(m)+1} \subset \lambda(\eta_m)$. As $(\eta_m)^- = \delta_m$ and by (5.2), $\delta_m$ has infinite outcome along $\eta_m$, it follows from (2.4) that $(\lambda(\eta_m))^- = \mathrm{up}(\delta_m)$, and so $\bar{\rho}^{k(m)+1} \subseteq \mathrm{up}(\delta_m)$. By Lemma 3.1(i) (Limit Path), $\bar{\rho}^{k(m)+1}$ has an initial derivative $\bar{\mu}^{k(m)} \subseteq \delta_m$. Now $(\sigma^{k(m)})^- = \rho^{k(m)}$ and $\sigma$ is $(k(m)+1)$-switching, so by (2.4), $\rho^{k(m)}$ has infinite outcome along $\sigma^{k(m)}$; so as $\mathrm{up}(\rho^{k(m)}) = \bar{\rho}^{k(m)+1}$, $[\bar{\mu}^{k(m)}, \rho^{k(m)}]$ must be a primary $\sigma^{k(m)}$-link restraining $\delta_m$. By Definition 5.3, $\rho^{k(m)} \notin \mathrm{PL}(\delta_m, \rho^{k(m)})$. It now follows from (5.13) and Lemma 5.1(i, ii) (PL Analysis) that

$$\mathrm{PL}(\delta_m, \sigma^{k(m)}) \backslash \mathrm{PL}(\delta_m, \rho^{k(m)}) = \{\rho^{k(m)}\}.$$

By (5.18)(ii), $\bar{\rho}^{k(m)+1}$ is the last node of a primary $\rho^{k(m)+1}$-link which restrains $\mathrm{up}(v_m)$. Hence by (5.13), $\bar{\rho}^{k(m)+1} \in \mathrm{PL}(\mathrm{up}(v_m), \rho^{k(m)+1})$. As $\sigma$ switches $\bar{\rho}^{k(m)+1}$ (and so, by Definition 5.6, cannot be a primary completion), it follows from (2.10) that $\bar{\rho}^{k(m)+1}$ is $\rho^{k(m)+1}$-free; hence by Lemma 5.10(iv) (Component), we may apply Lemma 5.1(v) (PL Analysis) to conclude that if $\tau^{k(m)+1}$ is the immediate successor of $\bar{\rho}^{k(m)+1}$ along $\rho^{k(m)+1}$, then $\mathrm{PL}(\mathrm{up}(v_m), \rho^{k(m)+1}) = \mathrm{PL}(\mathrm{up}(v_m), \tau^{k(m)+1})$. Thus by Lemma 5.1(i, ii) (PL Analysis) and Definition 5.3,

$$\mathrm{PL}(\mathrm{up}(v_m), \rho^{k(m)+1}) = \mathrm{PL}(\mathrm{up}(v_m), \bar{\rho}^{k(m)+1}) \cup \{\bar{\rho}^{k(m)+1}\}$$
$$\& \ \bar{\rho}^{k(m)+1} \notin \mathrm{PL}(\mathrm{up}(v_m), \bar{\rho}^{k(m)+1}).$$

As $\sigma$ switches $\bar{\rho}^{k(m)+1}$ and is $(k(m)+1)$-switching, it follows from Lemma 3.3 ($\lambda$-Behavior) that $(\sigma^{k(m)+1})^- = \bar{\rho}^{k(m)+1}$ and $\bar{\rho}^{k(m)+1}$ has finite outcome along $\sigma^{k(m)+1}$. Thus by Lemma 5.1(iv) (PL Analysis), $\mathrm{PL}(\mathrm{up}(v_m), \sigma^{k(m)+1}) = \mathrm{PL}(\mathrm{up}(v_m), \bar{\rho}^{k(m)+1})$. Thus

$$\mathrm{PL}(\mathrm{up}(v_m), \rho^{k(m)+1}) \backslash \mathrm{PL}(\mathrm{up}(v_m), \sigma^{k(m)+1}) = \{\bar{\rho}^{k(m)+1}\}$$
$$\& \ \bar{\rho}^{k(m)+1} \notin \mathrm{PL}(\mathrm{up}(v_m), \sigma^{k(m)+1}),$$

and (i) follows in this case.

*Subcase* 3.3: $i<m$. Recall that $k(i)=k(m)$ and $\rho^{k(m)} \subset \sigma^{k(m)}$. By Lemma 5.10(vii) (Component) and induction using Lemma 5.1(ix) (PL Analysis),

$$\{\rho^{k(m)}\} \cup \mathrm{PL}(\delta_m, \sigma^{k(i)}) \subseteq \mathrm{PL}(\delta_i, \sigma^{k(i)}).$$

By Definition 5.3, $\rho^{k(m)} \notin \mathrm{PL}(\delta_i, \rho^{k(i)})$. Hence by Lemma 5.1(ii) (PL Analysis), $\mathrm{PL}(\delta_i, \sigma^{k(i)}) = \mathrm{PL}(\delta_i, \rho^{k(i)}) \cup \{\rho^{k(m)}\}$.

By Subcases 3.1 and 3.2, $\bar{\rho}^{k(m)+1} \in \mathrm{PL}(\mathrm{up}(v_m), \rho^{k(m)+1})$. By Definition 5.3 and Lemma 5.10(v) (Component), for all $q$ such that $i \leqslant q < m$, $\mathrm{PL}(\mathrm{up}(v_{q+1}), \rho^{k(m)+1})$ is a component of $\mathrm{PL}(\mathrm{up}(v_q), \rho^{k(m)+1})$. Hence by induction on $m-i$, $\bar{\rho}^{k(m)+1} \in \mathrm{PL}(\mathrm{up}(v_i), \rho^{k(m)+1})$. Let $\tau^{k(m)+1}$ be the immediate successor of $\bar{\rho}^{k(m)+1}$ along $\rho^{k(m)+1}$. As $\sigma$ switches $\bar{\rho}^{k(m)+1}$ (and so, by Definition 5.6, cannot be a primary completion), it follows from (2.10) that $\bar{\rho}^{k(m)+1}$ is $\rho^{k(m)+1}$-free. Hence by Lemma 5.10(iv) (Component), we may apply Lemma 5.1(v) (PL Analysis) to conclude that $\mathrm{PL}(\mathrm{up}(v_i), \rho^{k(m)+1}) = \mathrm{PL}(\mathrm{up}(v_i), \tau^{k(m)+1})$. By Lemma 5.1(i, ii) (PL Analysis) and Definition 5.3, and as $\bar{\rho}^{k(m)+1} \in \mathrm{PL}(\mathrm{up}(v_i), \rho^{k(m)+1})$,

$$\mathrm{PL}(\mathrm{up}(v_i), \tau^{k(m)+1}) = \mathrm{PL}(\mathrm{up}(v_i), \bar{\rho}^{k(m)+1}) \cup \{\bar{\rho}^{k(m)+1}\}$$
$$\& \ \bar{\rho}^{k(m)+1} \notin \mathrm{PL}(\mathrm{up}(v_i), \bar{\rho}^{k(m)+1}).$$

Now $(\sigma^{k(m)+1})^- = \bar{\rho}^{k(m)+1}$ and $\bar{\rho}^{k(m)+1}$ has finite outcome along $\sigma^{k(m)+1}$, so by Lemma 5.1(iv) (PL Analysis), $\mathrm{PL}(\mathrm{up}(v_i), \sigma^{k(m)+1}) = \mathrm{PL}(\mathrm{up}(v_i), \bar{\rho}^{k(m)+1})$. Thus

$$\mathrm{PL}(\mathrm{up}(v_i), \sigma^{k(m)+1}) = \mathrm{PL}(\mathrm{up}(v_i), \rho^{k(m)+1}) \backslash \{\bar{\rho}^{k(m)+1}\}.$$

(i) now follows.

We complete the proof of the lemma by verifying (vi). Fix $k < n$ and $\mu^k \subset v^k \subset \sigma^k$ such that $v^k$ is implication-free and $\mathrm{up}(\mu^k) \subset \mathrm{up}(v^k)$, $\sigma^{k+1}$ in order to verify (5.29)(i). (Note that we may assume, by induction, that $\eta^k = \sigma^k$ in (5.29)(i).) Fix $p$ and $s$ for $\sigma$ as in Lemma 3.3 ($\lambda$-Behavior). We proceed by cases.

*Case* 1: $\sigma$ is not $j$-switching for any $j \leqslant k+1$. Then $(\sigma^k)^- = \rho^k$ and $(\sigma^{k+1})^- = \rho^{k+1}$. If $v^k \subset \rho^k$, then $\mathrm{PL}(v^k, \rho^k) \subseteq \mathrm{PL}(v^k, \sigma^k)$ and $\mathrm{PL}(\mathrm{up}(\mu^k), \rho^{k+1}) \subseteq \mathrm{PL}(\mathrm{up}(\mu^k), \sigma^{k+1})$, so (5.29)(i) follows by induction. Otherwise, $v^k = \rho^k$. But as $v^k$ is implication-free and $\rho$ is not completion-consistent via $\langle\ \rangle$, this is impossible.

*Case* 2: $\sigma$ is $j$-switching for some $j \leqslant k$. By (5.18) and Definition 5.6, $\sigma$ switches a principal derivative which is not an initial derivative on $T^j$, so by Lemma 3.3 ($\lambda$-Behavior), $p+1 = j = s$. If $j < k$, then there is a $\delta \subset \sigma$ such that $\lambda^k(\delta) = \sigma^k$, so (5.29)(i) follows by induction. Suppose that $j = k$. If $v^k \subset \bar{\rho}^k$, then (5.29)(i) holds at $\mathrm{out}^0(\bar{\rho}^k)$, and as $j = k = s$, $\sigma^{k+1} = \lambda(\bar{\rho}^k)$. Hence (5.29)(i) follows by induction and Lemma 5.1(i) (PL Analysis). Otherwise, it must be the case that $v^k = \bar{\rho}^k$. By Lemma 4.5 (Free Extension), $\mathrm{up}(v^k) \subseteq \sigma^{k+1}$, so by Lemma 5.1(i) (PL Analysis), $\mathrm{PL}(\mathrm{up}(\mu^k), \mathrm{up}(v^k)) \subseteq \mathrm{PL}(\mathrm{up}(\mu^k), \sigma^{k+1})$, and (5.29)(i) holds.

*Case* 3: $\sigma$ is $(k+1)$-switching. Then by Lemma 5.1(i) (PL Analysis), $\mathrm{PL}(\nu^k, \rho^k) \subseteq \mathrm{PL}(\nu^k, \sigma^k)$. Suppose that $\pi^{k+1} \in (\mathrm{PL}(\mathrm{up}(\mu^k), \rho^{k+1}) \cap \mathrm{PL}(\mathrm{up}(\mu^k), \mathrm{up}(\nu^k)) \backslash \mathrm{PL}(\mathrm{up}(\mu^k), \sigma^{k+1})$. (vi) will follow by induction once we show that a derivative of $\pi^{k+1}$ lies in $\mathrm{PL}(\nu^k, \sigma^k)$. There are several subcases.

*Subcase* 3.1: $\pi^{k+1}$ is the end of a primary $\rho^{k+1}$-link restraining $\mathrm{up}(\mu^k)$. By (2.10) and as $\mathrm{up}(\mu^k) \subset \lambda(\sigma^k)$, $\sigma$ must switch $\pi^{k+1}$. If $\pi^{k+1} \supseteq \mathrm{up}(\nu^k)$, then $\mathrm{PL}(\mathrm{up}(\mu^k), \mathrm{up}(\nu^k)) \subseteq \mathrm{PL}(\mathrm{up}(\mu^k), \sigma^{k+1})$, so (vi) holds. If $\pi^{k+1} \mid \mathrm{up}(\nu^k)$, then $\pi^{k+1} \notin \mathrm{PL}(\mathrm{up}(\mu^k), \mathrm{up}(\nu^k))$, contrary to hypothesis. Hence $\pi^{k+1} \subset \mathrm{up}(\nu^k)$. By Lemma 3.1(i) (Limit Path), $\pi^{k+1}$ has an initial derivative $\pi^k \subset \nu^k$. But $\mathrm{up}(\rho^k) = \pi^{k+1}$, and as $\sigma$ is $(k+1)$-switching, $\rho^k$ has infinite outcome along $\sigma^k$. Hence by (5.13), $\rho^k \in \mathrm{PL}(\nu^k, \sigma^k)$.

*Subcase* 3.2: Condition (5.14) places $\pi^{k+1}$ into $\mathrm{PL}(\mathrm{up}(\mu^k), \rho^{k+1}) \backslash \mathrm{PL}(\mathrm{up}(\mu^k), \sigma^{k+1})$ through the component induced by some $\beta^{k+1}$ requiring extension, and the conditions of (5.13) fail. Then $\beta^{k+1}$ cannot have a primary completion with infinite outcome along $\rho^{k+1}$, else by (2.10) and Lemma 5.3(i), any component of $\mathrm{PL}(\mathrm{up}(\mu^k), \rho^{k+1})$ induced by $\beta^{k+1}$ would be a component of $\mathrm{PL}(\mathrm{up}(\mu^k), \sigma^{k+1})$. Thus by Lemma 5.10 (iv) (Component), $(\beta^{k+1})^- = \mathrm{up}(\nu_i)$ for some $i \leqslant m$, which we fix. Now by (iii), there is a derivative $\pi^k$ of $\pi^{k+1}$ such that $\pi^k \in \mathrm{PL}(\delta_i, \sigma^k) \backslash \mathrm{PL}(\delta_i, \rho^k)$, so by Lemma 5.1(ii) (PL Analysis), $\pi^k = \rho^k$. By Definition 5.3, $\rho^k$ has infinite outcome along $\sigma^k$. Now $\pi^{k+1} \in \mathrm{PL}(\mathrm{up}(\mu^k), \mathrm{up}(\nu^k))$, so again by Definition 5.3, $\pi^{k+1} \subset \mathrm{up}(\nu^k)$; thus $\pi^k \neq \nu^k$, and by Lemma 3.1(i) (Limit Path), $\pi^{k+1}$ has an initial derivative $\tilde{\pi}^k \subset \nu^k$. We now see that $[\tilde{\pi}^k, \rho^k]$ is a primary $\sigma^k$-link restraining $\nu^k$, so by (5.13), $\rho^k = \pi^k \in \mathrm{PL}(\nu^k, \sigma^k)$. ▮

Our next lemma specifies the correspondence between a PL set which is encountered when a node $\eta^k$ requires extension, and another PL set which is defined at the $k$-completion $\kappa^k$ of $\eta^k$.

LEMMA 5.12 (PL Lemma). *Fix $k \leqslant r \leqslant n$, and $\nu^k \subset \delta^k \subset \eta^k \subseteq \kappa^k \in T^k$ such that $(\eta^k)^- = \delta^k$ and $k < \dim(\nu^k)$, and let $\nu^{k+1} = \mathrm{up}(\nu^k)$. Assume that $\eta^k$ requires extension for $\nu^k$, and that $\kappa^k$ is the $k$-completion of $\eta^k$ for $\nu^k$, and is preadmissible. Then*:

(i) $\{\tau^{k+1} \subseteq \lambda(\eta^k) : \exists \rho^k(\eta^k \subset \rho^k \subseteq \kappa^k \;\&\; \rho^k$ *switches* $\tau^{k+1})\} = \mathrm{PL}(\nu^{k+1}, \lambda(\eta^k))$.

(ii) $\{\mathrm{up}(\xi^k) : \xi^k \in \mathrm{PL}(\delta^k, \kappa^k)\} = \mathrm{PL}(\nu^{k+1}, \lambda(\eta^k))$.

(iii) *If* $\langle \langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle r \geqslant j \geqslant k+1 \rangle$ *is an amenable* $(k+1)$-*implication chain*, $\mathrm{up}(\nu^k) = \sigma^{k+1}$ *and* $\mathrm{up}(\delta^k) = \hat{\sigma}^{k+1}$, *then* $\{\mathrm{up}^r(\xi^k) : \xi^k \in \mathrm{PL}(\delta^k, \kappa^k)\} = \mathrm{PL}(\sigma^r, \tau^r)$.

(iv) *Condition* (5.29)(ii) *holds at* $\kappa^k$.

*Proof.* (ii) By Lemma 5.11 (iv, v) (Amenable Backtracking), $\{\mathrm{up}(\xi^k) : \xi^k \in \mathrm{PL}(\delta^k, \eta^k)\} = \mathrm{PL}(\nu^{k+1}, \lambda(\kappa^k)) = \emptyset$, so (ii) follows by repeated applications of Lemma 5.11(iii) (Amenable Backtracking) to those $\rho^k$ such that $\eta^k \subset \rho^k \subseteq \kappa^k$.

(i) By (5.18), $\{\tau^{k+1} \subseteq \lambda(\eta^k) : \exists \rho^k(\eta^k \subset \rho^k \subseteq \kappa^k \,\&\, \rho^k \text{ switches } \tau^{k+1})\}$ is identical with $V = \{\tau^{k+1} \subseteq \lambda(\eta^k) : \exists \rho^k(\eta^k \subset \rho^k \subseteq \kappa^k \;\&\; (\mathrm{out}^0(\rho^k))^- \text{ satisfies } (5.18)(\mathrm{i})(\mathrm{a}) \text{ with } k(m) = k \text{ or } (5.18)(\mathrm{ii})(\mathrm{a}) \text{ with } j = k+1)\}$. By Lemma 5.11(i, ii, iv) (Amenable Backtracking), $V = \{\mathrm{up}(\xi^k) : \xi^k \in \mathrm{PL}(\delta^k, \kappa^k)\}$. Hence (i) follows from (ii).

(iii) Let $\hat{\sigma}^k = \kappa^k$ and $\sigma^k = \delta^k$. By induction and Lemma 5.1(iv) (PL Analysis), it suffices to show that for all $j \in [k, r)$, $\{\mathrm{up}(\xi^j) : \xi^j \in \mathrm{PL}(\sigma^j, \hat{\sigma}^j)\} = \mathrm{PL}(\sigma^{j+1}, \tau^{j+1}) = \mathrm{PL}(\sigma^{j+1}, \hat{\sigma}^{j+1})$. Fix $j \in [k, r)$, and let $\bar{\tau}^j = \mathrm{out}(\tau^{j+1})$. Note that $\bar{\tau}^k = \eta^k$. By (5.5)(ii) and Lemma 5.2 (Requires Extension), $\bar{\tau}^j$ requires extension for $\sigma^j$, and $\hat{\sigma}^j$ is the $j$-completion of $\bar{\tau}^j$. By (5.5)(ii) and (ii), $\{\mathrm{up}(\xi^j) : \xi^j \in \mathrm{PL}(\sigma^j, \hat{\sigma}^j)\} = \mathrm{PL}(\sigma^{j+1}, \tau^{j+1})$. By (5.11), $\hat{\sigma}^{j+1}$ has finite outcome along $\tau^{j+1}$, so by Lemma 5.1(iv) (PL Analysis), $\mathrm{PL}(\sigma^{j+1}, \tau^{j+1}) = \mathrm{PL}(\sigma^{j+1}, \hat{\sigma}^{j+1})$.

(iv) Immediate from (ii). ∎

Let $\Lambda^0 \in [T^0]$ be admissible, and for all $k \leqslant n$, let $\Lambda^k = \lambda^k(\Lambda^0)$. In order to show that all requirements are satisfied, we will need to show that if a node is $\Lambda^k$-free, then it is also implication-free, and so can act according to the truth of the sentence generating its action. We will be able to show this under the assumption that $\Lambda^0$ is admissible. (5.17)(i) may prevent a node which is a potential component of a 0-implication chain from acting according to the truth of the sentence generating its action, as it forces a specified outcome for certain implication-free nodes. However, we want to show that the implication-chain mechanism ensures that the action this node takes is in accordance with the truth of that sentence. The proof of this fact relies on the next lemma, which relates the implication-freeness of one of the first two nodes on $T^k$ of a $k$-implication chain to the implication-freeness of the other node.

LEMMA 5.13 (Free Amenable Implication Chain Lemma). *Suppose that $k < r \leqslant n$, $\langle\langle \sigma^j, \hat{\sigma}^j, \tau^j\rangle : r \geqslant j \geqslant k+1\rangle$ is an amenable $(k+1)$-implication-chain, that $\hat{\sigma}^k$ is the primary completion of $\bar{\tau}^k = \mathrm{out}(\tau^{k+1})$, and that $\xi \in T^0$ is preadmissible and the 0-completion corresponding to $\hat{\sigma}^k$. Let $\sigma^k = (\bar{\tau}^k)^-$. Then $\sigma^k$ is implication-free iff $\hat{\sigma}^k$ is implication-free. Furthermore, if $\xi$ is completion-consistent via $\langle\ \rangle$, then $\xi$ is implication-free.*

*Proof.* We proceed by induction on $r - k$. First suppose that $\hat{\sigma}^k$ is implication-free. By Lemma 5.2 (Requires Extension), $\bar{\tau}^k$ requires extension, so by (5.5)(ii) and (5.19), $\mathrm{up}(\hat{\sigma}^k) = \sigma^{k+1}$ and $\mathrm{up}(\sigma^k) = \hat{\sigma}^{k+1}$. As (5.23) fails to hold for $\hat{\sigma}^k$, $\sigma^{k+1}$ is implication-free. Hence by (5.1) if $r=k+1$ and by induction otherwise, $\hat{\sigma}^{k+1}$ is implication-free. As (5.21) and (5.22) fail for $\hat{\sigma}^k$, $\hat{\sigma}^k$ is completion-consistent via $\langle\ \rangle$. (We note that the completion-consistency of the admissible $\eta \in T^0$ via $S$ implies the completion-consistency of $\lambda^k(\eta)$ via the subsequence of $S$ consisting of those nodes which are on $T^j$ for some $j \geqslant k$.) By Lemma 5.9(i) (Completion-Consistency) applied to $T^k$, $(\hat{\sigma}^k)^-$ must be completion-consistent via $\langle \bar{\tau}^k \rangle$, and by (5.19), $\bar{\tau}^k$ is completion-consistent via $\langle \bar{\tau}^k \rangle$. By Lemma 5.9 (Completion-Consistency) applied on $T^k$, $\sigma^k$ is completion-consistent via $\langle\ \rangle$, so (5.21) and (5.22) fail to hold for $\sigma^k$. Thus we conclude that $\sigma^k$ is implication-free.

Now suppose that $\sigma^k$ is implication-free. By Lemma 5.2 (Requires Extension), $\bar{\tau}^k$ requires extension, so by (5.5)((ii) and (5.19), $\mathrm{up}(\hat{\sigma}^k) = \sigma^{k+1}$ and $\mathrm{up}(\sigma^k) = \hat{\sigma}^{k+1}$. As (5.23) fails to hold for $\sigma^k$, $\hat{\sigma}^{k+1}$ is implication-free. Hence by (5.1) if $r=k+1$ and by induction otherwise, $\sigma^{k+1}$ is implication-free. As (5.21) and (5.22) fail for $\sigma^k$, $\sigma^k$ is completion-consistent via $\langle\ \rangle$. (We again note that the completion-consistency of the admissible $\eta \in T^0$ via $S$ implies the completion-consistency of $\lambda^k(\eta)$ via the subsequence of $S$ consisting of those nodes which are on $T^j$ for some $j \geqslant k$.) By Lemma 5.9(ii) (Completion-Consistency) applied on $T^k$, $\bar{\tau}^k$ must be completion-consistent via $\langle \bar{\tau}^k \rangle$, and by (5.19), $(\hat{\sigma}^k)^-$ is completion-consistent via $\langle \bar{\tau}^k \rangle$. By Lemma 5.9 (Completion-Consistency) applied to $T^k$, $\hat{\sigma}^k$ is completion-consistent via $\langle\ \rangle$, so (5.21) and (5.22) fail to hold for $\hat{\sigma}^k$. We conclude that $\hat{\sigma}^k$ is implication-free.

Suppose that $\xi$ is completion-consistent via $\langle\ \rangle$ but not implication-free, in order to obtain a contradiction. By (5.10), (5.5), and Definition 5.6, for all $j$ such that $k \leqslant j \leqslant r = \dim(\hat{\sigma}^k) - 1$, $\mathrm{out}^0(\sigma^j)$ and $\mathrm{out}^0(\hat{\sigma}^j)$ are completion-consistent via the same sequence; so $\sigma^j$ is primarily implication-restrained iff $\hat{\sigma}^j$ is primarily implication-restrained. Furthermore, by (5.10), neither $\sigma^r$ nor $\hat{\sigma}^r$ is implication-restrained, and by (5.9), $\mathrm{up}^j(\hat{\sigma}^k) \in \{\sigma^j, \hat{\sigma}^j\}$ for all $j \in [k, r]$. Hence we fix the largest $j$ such that $\mathrm{up}^j(\hat{\sigma}^k)$ is implication-restrained, and note that $j < r$, and that $\mathrm{up}^j(\hat{\sigma}^k)$ is either primarily or hereditarily implication-restrained.

First suppose that $\mathrm{up}^j(\hat{\sigma}^k)$ is primarily implication-restrained. By (5.6) and as $j < r$, $\mathrm{up}^j(\hat{\sigma}^k) \in \{\sigma^j, \hat{\sigma}^j\}$, so by the preceding paragraph, both $\sigma^j$ and $\hat{\sigma}^j$ are primarily implication-restrained and completion-consistent via the same sequence. Thus by (5.8)(i), there is an $\eta^j \subseteq \sigma^j$ which requires extension but has no $j$-completion $\subseteq \hat{\sigma}^j$. Fix $\delta^j$ such that $(\eta^j)^- = \delta^j$ and $\nu^j$ such that $\eta^j$ requires extension for $\nu^j$. Now $\delta^j \subset \sigma^j$, so by Lemma 5.5(ii) (Completion-Respecting) and as $\xi$ is completion-consistent via $\langle\ \rangle$ and is not a 0-completion, $\eta^j$ has a $j$-completion $\kappa^j \subset \lambda^j(\xi)$ which has infinite outcome along $\lambda^j(\xi)$. By Definition 5.6 and (2.7), $\mathrm{up}^j(\hat{\sigma}^k) \subseteq \lambda^j(\xi)$, so as $\mathrm{up}^j(\hat{\sigma}^k) \in \{\sigma^j, \hat{\sigma}^j\}$ and $\eta^j$ has no $j$-completion $\subseteq \hat{\sigma}^j$, $\mathrm{up}^j(\hat{\sigma}^k)$ is $\lambda^j(\xi)$-restrained by the primary link $[\nu^j, \kappa^j]$, contradicting (2.10).

Now suppose that $\mathrm{up}^j(\hat{\sigma}^k)$ is hereditarily implication-restrained but not primarily implication-restrained. By (5.6) and as $j<r$, $\mathrm{up}^j(\hat{\sigma}^k)\in\{\sigma^j, \hat{\sigma}^j\}$, so by the preceding paragraph, both $\sigma^j$ and $\hat{\sigma}^j$ are hereditarily but not primarily implication-restrained and are completion-consistent via the same sequence. Thus there are $i>j$ and $\eta^i\in T^i$ such that $\eta^i$ requires extension, has no $j$-completion $\subseteq \mathrm{up}^j(\hat{\sigma}^k)$, and $\mathrm{out}^j(\eta^i)\subseteq\sigma^j$. By Lemma 5.4 (Compatibility) and as $\xi$ is completion-consistent via $\langle\ \rangle$, $\eta^i$ has a $j$-completion $\kappa^j$ such that $\hat{\sigma}^j\subseteq\kappa^j\subseteq\lambda^j(\beta)$ for some $\beta\subseteq\xi$. Let $\bar{\tau}^k\subseteq\hat{\sigma}^k$ be defined by $(\bar{\tau}^k)^-=\sigma^k$. By Definition 5.6, $\bar{\tau}^k$ requires extension; but by Lemma 3.2 (Out), $\mathrm{out}^0(\eta^i)\subset\mathrm{out}^0(\bar{\tau}^k)\subset\mathrm{out}^0(\kappa^j)$, contradicting (5.26). ∎

We are now ready to show that completions exist. We proceed as described in the example preceding Definition 5.3. The definition proceeds by induction on $n-k$, and then by induction on the cardinality of PL sets. The process used, within the proof, to construct completions is called *backtracking*.

LEMMA 5.14 (Completion Lemma). *Fix $\eta\in T^0$ such that $\eta$ is admissible, $\eta^k=\lambda^k(\eta)$ requires extension, and $\eta=\mathrm{out}^0(\eta^k)$. Then there is an effectively obtainable admissible 0-completion $\kappa\supset\eta$ of $\eta^k$.*

*Proof.* For all $j\leqslant n$, let $\eta^j=\lambda^j(\eta)$. Let $\eta^k$ require extension for $\nu^k$, set $\nu^{k+1}=\mathrm{up}(\nu^k)$, and note, by (5.3) and Lemma 4.3(i)(a) (Link Analysis), that $\nu^{k+1}\subset\eta^{k+1}$. Fix $\delta^k\subset\eta^k\in T^k$ such that $(\eta^k)^-=\delta^k$. Fix $u\geqslant 0$ and $S=\langle\alpha_i\colon i\leqslant u\rangle$ such that $\eta$ is completion-consistent via $S$. We proceed by induction on $n-k$, and then by induction on the cardinality of $\mathrm{PL}(\nu^{k+1},\eta^{k+1})$. We carry out a *backtracking process*, constructing increasing sequences $\langle\xi_i^k\in T^k\colon i\leqslant m\rangle$ and $\langle\xi_i\in T^0\colon i\leqslant m\rangle$ of strings for some $m\geqslant u$ such that each $\xi_i$ is an admissible extension of $\eta$, and $\xi_i=\mathrm{out}^0(\xi_i^k)$. $m$ will be bounded by the length of longest $\eta^{k+1}$-link restraining $\nu^{k+1}$ plus 1. We also define a map $\xi_i^k\to\bar{\xi}_i^{k+1}\in T^{k+1}$ for $i\leqslant m$, yielding a decreasing sequence of strings on $T^{k+1}$.

We begin by setting to $\xi_0=\eta$, $\xi_0^k=\eta^k$ and $\bar{\xi}_0^{k+1}=\lambda(\eta^k)$. Suppose that $\xi_i$, $\xi_i^k$, and $\bar{\xi}_i^{k+1}$ have been defined for some fixed $i<m$. We assume by induction that:

(5.31) (i) $\xi_i$ is admissible and completion-consistent via $S$.

(ii) If $i>0$, then $\xi_i$ is switching.

(5.32) (i) For all $j<i$, $\xi_j\subset\xi_i$, $\xi_j^k\subset\xi_i^k$, and $\bar{\xi}_i^{k+1}\subset\bar{\xi}_j^{k+1}$.

(ii) $\bar{\xi}_i^{k+1}\subseteq\lambda(\xi_i^k)$.

(5.33) (i) $\bar{\xi}_i^{k+1}$ is $\lambda(\xi_i^k)$-free.

(ii) (5.18)(i)(a), with $\xi_i$ in place of $\rho$, is not satisfied.

At the end, we will ensure that (5.32) and (5.33)(i) also hold for $i=m$, and in addition:

(5.34) $\bar{\xi}_m^{k+1}=\nu^{k+1}$ and $\xi_m$ is admissible and the 0-completion of $\eta^k$.

First suppose that $i=0$. Condition (5.32) is vacuous. Condition (5.31) follows by hypothesis. Condition (5.33)(i) follows from (2.10). By Definition 5.6, any node which is a 0-completion is nonswitching, and its immediate predecessor is implication-restrained. It follows from (5.1) that $\dim((\eta^k)^-)>k+1$, so (5.5)(ii) must be true when $\eta^k$ requires extension; and by (5.5)(ii) and (5.18)(i) for $(\xi_i)^-$ and $\xi_i$, if $(\xi_i)^-$ is implication-restrained and $\lambda^j(\xi_i)$ requires extension, then $\xi_i$ is switching. Hence $\eta$ cannot be a 0-completion, and (5.33)(ii) holds.

We now assume that $i\geqslant 0$ and verify (5.31)–(5.34) for $i+1$. There are two cases:

*Case* 1. There is a $\lambda(\xi_i^k)$-link $[\mu^{k+1},\pi^{k+1}]$ which restrains $\nu^{k+1}$. By Lemma 4.1 (Nesting), we can assume that $[\mu^{k+1},\pi^{k+1}]$ is the longest such link. Now there is a $t\geqslant k+1$ and a primary $\lambda^t(\xi_i^k)$-link $[\mu^t,\pi^t]$ such that $[\mu^{k+1},\pi^{k+1}]$ is derived from $[\mu^t,\pi^t]$. $\pi^t$ is $\lambda^t(\xi_i^k)$-free by Lemma 4.3(iii) (Link Analysis), and by (4.1), will be $\lambda^t(\bar{\xi}_i^k)$-free for any nonswitching extension $\bar{\xi}_i^k$ of $\xi_i^k$. And as $[\mu^t,\pi^t]$ is a primary $\lambda^t(\xi_i^k)$-link, $\pi^t$ has infinite outcome along $\lambda^t(\xi_i^k)$. Hence by Lemma 4.4 (Free Implies True Path), $\pi^t$ is $\lambda^{t-1}(\xi_i^k)$-consistent for all nonswitching extensions $\bar{\xi}_i^k$ of $\xi_i^k$. By Lemma 3.1(iii) (Limit Path), all blocks defined in Step 4 of Definition 2.8 are finite, so by repeated applications of Lemma 3.4 (Nonswitching Extension), we can keep taking nonswitching extensions of $\xi_i$, and will eventually reach the shortest such nonswitching $\beta\supset\xi_i$ such that $\mathrm{up}^t(\beta)=\pi^t$, $\mathrm{up}^{t-1}(\beta)\supset\lambda^{t-1}(\xi_i^k)$ and $\beta$ is an initial derivative of $\mathrm{up}^{t-1}(\beta)$. (When we apply the Nonswitching Extension Lemma to extend a string, and it is possible to take both activated and validated extensions and still be nonswitching, (5.18)(iii) requires that we take the activated extension, in order to uniquely define the process of taking nonswitching extensions in this induction.) As $\xi_i$ is admissible, it follows from Lemma 5.9 (Completion-Consistency) that $\beta$ will be admissible and completion-consistent via $S$ unless there is a $\rho$ such that $\xi_i\subseteq\rho\subset\beta$ and either (5.18)(i)(a) or (5.18)(ii)(a) holds for $\rho$, and that such a $\rho$ will be admissible and completion-consistent via $S$. (The clauses of (5.29) not covered by Lemma 5.9 are covered by Lemmas 5.11–5.13. In particular, (5.29)(i) is covered by Lemma 5.11(vi), (5.29)(ii) by Lemma 5.12(iv), and (5.29)(iii) by Lemma 5.13.) So assume that such a $\rho$ exists in order to obtain a contradiction.

By (4.1), for all $\gamma\in T^0$ such that $\mathrm{lh}(\gamma)>0$, if $\gamma$ is a nonswitching extension of $\gamma^-$, then for all $q\leqslant n$, the $\lambda^q(\gamma)$-links coincide with the $\lambda^q(\gamma^-)$-links.

Hence (5.18)(ii)(a) cannot hold for $\rho$, else $\rho = \beta$. As $\mathrm{up}(\nu^k)$ is not $\xi_i^k$-free, $\mathrm{up}(\nu^k)$ is not $\lambda^k(\rho)$-free. So $\rho$ cannot be the 0-completion of $\eta^k$, else by Definition 2.6, $\mathrm{up}^{k+1}(\rho) = \mathrm{up}(\nu^k)$, so by (2.10), $\mathrm{up}(\nu^k)$ would have to be $\lambda^k(\rho)$-free, which is impossible. It now follows that $\rho$ is not a 0-completion, else by Definition 5.6, $\rho$ would have to be the 0-completion of $\eta^k = \alpha_u$; hence (5.18)(i)(a) does not hold for $\rho$. The same proof shows that (5.18)(i)(a) does not hold for $\rho = \beta$.

We conclude that $\beta$ is admissible and completion-consistent via $S$, and that (5.18)(i)(a) does not hold for $\beta$. By Lemma 3.6 (Switching), we can choose an extension $\tilde{\beta}$ of $\beta$ such that $\tilde{\beta}^- = \beta$ and $\tilde{\beta}$ induces an infinite outcome for $\mathrm{up}^{t-1}(\beta)$ along $\lambda^{t-1}(\tilde{\beta})$, thus switching the outcome of $\pi^t$ to finite along $\lambda^t(\tilde{\beta})$. Note that (5.18)(ii)(a) holds for $\beta$, so by (5.18)(ii), $\tilde{\beta}$ is preadmissible. Now $t-1$ is the $p$ in Lemma 3.3 ($\lambda$-Behavior), so $\lambda^j(\beta) \subseteq \lambda^j(\tilde{\beta})$ iff $j < t$.

*Subcase* 1.1: $\pi^t$ is not a primary $t$-completion. Then by (5.18), $\xi_i$ is not a 0-completion. Set $\xi_{i+1} = \tilde{\beta}$, $\xi_{i+1}^k = \lambda^k(\tilde{\beta})$, and $\bar{\xi}_{i+1}^{k+1} = (\lambda^{k+1}(\tilde{\beta}))^-$. We note that neither the hypothesis of Lemma 5.9(i) or of Lemma 5.9(ii) (Completion-Consistency) is satisfied, so by Lemma 5.9(iii) (Completion-Consistency), $\xi_{i+1}$ is completion-consistent via $S$. Furthermore, by Lemma 5.9(iv) (Completion-Consistency) and again by Lemmas 5.11(vi), 5.12(iv), and 5.13, $\xi_{i+1}$ is admissible. Condition (5.31) now follows for $\xi_{i+1}$, and (5.32) and (5.33) follow from the properties of $\tilde{\beta}$, (2.10), and Lemma 4.5 (Free Extension).

*Subcase* 1.2: $\pi^t$ is a primary $t$-completion. First suppose that $t > k+1$. Then by (2.4), $(\lambda^t(\tilde{\beta}))^- = \pi^t$ and $\pi^t$ has finite outcome along $\lambda^t(\tilde{\beta})$. Hence by Lemma 5.3(ii) (Implication Chain) and Lemma 5.2 (Requires Extension), $\lambda^{t-1}(\tilde{\beta})$ requires extension, so by induction on $n-k$, we can find a 0-completion $\kappa$ of $\lambda^{t-1}(\tilde{\beta})$, and, by Lemma 5.9(i, iv), (Completion-Consistency) and again by Lemmas 5.11(vi), 5.12(iv), and 5.13, find an admissible $\tilde{\kappa}$ such that $\tilde{\kappa}^- = \kappa$ and $\tilde{\kappa}$ induces an infinite outcome for $(\lambda^{t-1}(\tilde{\kappa}))^-$ along $\lambda^{t-1}(\tilde{\kappa})$. By Lemma 5.7(i) (Primary Completion), $\lambda^{t-1}(\tilde{\kappa})$ does not require extension. We now set $\xi_{i+1} = \tilde{\kappa}$, $\xi_{i+1}^k = \lambda^k(\tilde{\kappa})$ and $\bar{\xi}_{i+1}^{k+1} = (\lambda^{k+1}(\tilde{\kappa}))^-$, and note that (5.31)–(5.33) follow from the properties of $\tilde{\kappa}$, (2.4), (2.10), and Lemma 4.5 (Free Extension) and Lemma 5.9 (Completion-Consistency). ((5.33)(ii) follows since $\tilde{\kappa}$ is switching, and by Definition 5.6, completions are nonswitching.) The induction step is now complete for this case.

Suppose that $t = k+1$. Then by (2.4), $(\lambda^{k+1}(\tilde{\beta}))^- = \pi^{k+1}$ and $\pi^{k+1}$ has finite outcome along $\lambda^{k+1}(\tilde{\beta})$. By Lemma 5.3(ii) (Implication Chain), there is an $r < n$ and an amenable $(k+1)$-implication chain $\langle\langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle : r \geqslant j \geqslant k+1 \rangle$ such that $\hat{\sigma}^{k+1} = \pi^{k+1}$ and the immediate successor of $\sigma^{k+1}$ along $\hat{\sigma}^{k+1}$ requires extension for $\mu^{k+1}$. Note that as $[\mu^{k+1}, \pi^{k+1}]$

restrains $\nu^{k+1}$, $\mu^{k+1} \subseteq \nu^{k+1}$. And by (5.2) and Lemma 5.10(v) (Component), $\nu^{k+1}$ has infinite outcome along $\pi^{k+1}$, so as, by (2.8), $\mu^{k+1}$ has finite outcome along $\pi^{k+1}$, $\mu^{k+1} \subset \nu^{k+1}$. By Lemma 5.2 (Requires Extension), $\widetilde{\beta}^k = \lambda^k(\widetilde{\beta})$ requires extension for some $\tilde{\gamma}^k$, and $\widetilde{\beta}^k$ is not implication-free. Furthermore, by (5.5)(ii) and (5.8), $\mathrm{up}(\tilde{\gamma}^k) = \sigma^{k+1} \subset \hat{\sigma}^{k+1} = \mathrm{up}((\widetilde{\beta}^k)^-) = \pi^{k+1} \subset \eta^{k+1}$. By Lemma 5.9(ii) (Completion-Consistency), $\widetilde{\beta}$ is completion-consistent via $S^\wedge\langle \widetilde{\beta}^k \rangle$, so by Lemma 5.10(ii) (Component), $\nu^{k+1} \subset \mathrm{up}(\tilde{\gamma}^k)$. By Lemma 3.3 ($\lambda$-Behavior), $(\lambda(\widetilde{\beta}^k))^- = \pi^{k+1}$ and $\pi^{k+1}$ has finite outcome along $\lambda(\widetilde{\beta}^k)$, so by Lemma 5.1(iv) (PL Analysis), $\mathrm{PL}(\mathrm{up}(\tilde{\gamma}^k), \lambda(\widetilde{\beta}^k)) = \mathrm{PL}(\mathrm{up}(\tilde{\gamma}^k), \pi^{k+1})$. Hence by (5.14), $\mathrm{PL}(\mathrm{up}(\tilde{\gamma}^k), \lambda(\widetilde{\beta}^k))$ is a component of $\mathrm{PL}(\nu^{k+1}, \lambda(\widetilde{\beta}^k))$, so by (5.14)(i) and Lemma 5.1(vi) (PL Analysis), $\mathrm{up}(\tilde{\gamma}^k) \in \mathrm{PL}(\nu^{k+1}, \lambda(\widetilde{\beta}^k)) \setminus \mathrm{PL}(\mathrm{up}(\tilde{\gamma}^k), \lambda(\widetilde{\beta}^k))$. Thus $\mathrm{PL}(\nu^{k+1}, \lambda(\widetilde{\beta}^k)) \supset \mathrm{PL}(\mathrm{up}(\tilde{\gamma}^k), \lambda(\widetilde{\beta}^k)) = \mathrm{PL}(\mathrm{up}(\tilde{\gamma}^k), \pi^{k+1})$. We now proceed as in the preceding paragraph to find an admissible switching extension of a 0-completion of $\widetilde{\beta}^k$, and justifying the existence of $\kappa$ by induction on the cardinality of the PL sets.

*Case* 2. Otherwise. By the case assumption, there are no $\lambda(\xi_i^k)$-links restraining $\nu^{k+1}$. Hence $\nu^{k+1}$ is $\lambda(\xi_i^k)$-free, so as in Case 1, we can keep taking nonswitching extensions of $\xi_i$, taking the activated extension when both the activated and validated extensions are nonswitching, and will eventually reach the shortest such nonswitching $\xi_{i+1} \supset \xi_i$ such that $\mathrm{up}^{k+1}(\xi_{i+1}) = \nu^{k+1}$ and $\xi_{i+1}$ is admissible and completion-consistent via $S$. By (5.31)(ii) and Definition 5.6, $\xi_i$ cannot be a 0-completion. Thus by Lemma 5.9 (Completion-Consistency) and Lemmas 5.10–5.12, $\xi_{i+1}$ is the shortest nonswitching extension of $\xi_i$ satisfying (5.18)(i)(a), and is admissible. We set $m = i+1$, $\xi_k^m = \lambda^k(\xi_{i+1})$, and $\bar{\xi}_m^{k+1} = \nu^{k+1}$. Conditions (5.32), (5.33)(i), and (5.34) now follow. ∎

We now show that admissible paths have nice properties; they are completion-respecting and do not extend amenable implication chains.

Lemma 5.15 (Admissibility Lemma). *Let an admissible path $\Lambda^0 \in [T^0]$ be given, and for all $k \leqslant n$, let $\Lambda^k = \lambda^k(\Lambda^0)$. Then for all $k \leqslant n$:*

(i) *$\Lambda^k$ does not extend an amenable $k$-implication chain.*

(ii) *Every $\eta^k \subset \Lambda^k$ which requires extension has a primary completion along $\Lambda^k$.*

*Proof.* We proceed by induction on $k$. First let $k = 0$. Condition (i) follows from (5.11)(ii) and (5.18)(i) for implication-restrained nodes, and from (5.11)(ii) and (5.17)(i) for implication-free nodes. And (ii) follows from Lemma 5.14 (Completion), the uniqueness of primary completions, and (5.18).

Suppose that $k>0$. First suppose that $\langle\langle\sigma^j, \hat{\sigma}^j, \tau^j\rangle : r \geqslant j \geqslant k\rangle$ is an amenable $k$-implication chain for some $r$, with $\tau^k \subset \Lambda^k$, in order to obtain a contradiction. By (2.5), $\bar{\tau}^{k-1} = \mathrm{out}(\tau^k) \subset \Lambda^{k-1}$ and by Lemma 5.2 (Requires Extension), $\bar{\tau}^{k-1}$ requires extension. By (ii) inductively, $\bar{\tau}^{k-1}$ has a $(k-1)$-completion $\kappa^{k-1} \subset \Lambda^{k-1}$, and by Lemma 5.3(ii) (Implication Chain) and (i) inductively, $\kappa^{k-1}$ has infinite outcome along $\Lambda^{k-1}$. Fix $\xi^{k-1} \subset \Lambda^{k-1}$ such that $(\xi^{k-1})^- = \kappa^{k-1}$. By (5.5)(ii) and (5.19), $\mathrm{up}(\kappa^{k-1}) = \sigma^k$ and so by (2.4), $\sigma^k$ has finite outcome along $\Lambda^k$. By (5.11)(i), $\sigma^k$ has infinite outcome along $\tau^k$, so $\tau^k \not\subset \Lambda^k$, yielding the desired contradiction. Hence (i) holds for $k$.

Now suppose that $\eta^k \subset \Lambda^k$ requires extension for $v^k$. By (5.18), the uniqueness of completions, and Lemma 5.14 (Completion), $\eta^k$ has a 0-completion $\subset \Lambda^0$, so by Lemma 5.4 (Compatibility), $\eta^k$ has a $(k-1)$-completion $\kappa^{k-1} \subset \Lambda^{k-1}$, so has a principal derivative $\bar{\kappa}^{k-1} \subset \Lambda^{k-1}$. First assume that $\bar{\kappa}^{k-1}$ has finite outcome along $\Lambda^{k-1}$. Fix $\bar{\tau}^{k-1} \subset \Lambda^{k-1}$ such that $(\bar{\tau}^{k-1})^- = \bar{\kappa}^{k-1}$, and let $\kappa^k = \mathrm{up}(\bar{\kappa}^{k-1})$. Then $[v^k, \kappa^k]$ is a primary $\lambda(\bar{\tau}^{k-1})$-link which restrains $\eta^k$, so by (2.6), (2.10) and as $\eta^k \subset \Lambda^k$, no $\beta^{k-1}$ such that $\bar{\tau}^{k-1} \subseteq \beta^{k-1} \subset \Lambda^{k-1}$ can switch any node $\subset \kappa^k$, and so $\kappa^k \subset \Lambda^k$. Clause (ii) now follows in this case.

Suppose that $\bar{\kappa}^{k-1}$ has infinite outcome along $\Lambda^{k-1}$, fix $\bar{\tau}^{k-1} \subset \Lambda^{k-1}$ such that $(\bar{\tau}^{k-1})^- = \bar{\kappa}^{k-1}$, and let $\kappa^k = \mathrm{up}(\bar{\kappa}^{k-1})$. Then by (2.4), $\kappa^k$ has finite outcome along $\lambda(\bar{\tau}^{k-1})$, and so by Lemma 5.3(ii) (Implication Chain), $\lambda(\bar{\tau}^{k-1})$ is the last node of the last triple of an amenable $k$-implication chain. By Lemma 5.2 (Requires Extension), $\bar{\tau}^{k-1}$ requires extension. By (ii) inductively, $\bar{\tau}^{k-1}$ has a $(k-1)$-completion $\alpha^{k-1} \subset \Lambda^{k-1}$, and by Lemma 5.3(ii) (Implication Chain) and (i) inductively, $\alpha^{k-1}$ has infinite outcome along $\Lambda^{k-1}$. Fix $\xi^{k-1} \subset \Lambda^{k-1}$ such that $(\xi^{k-1})^- = \alpha^{k-1}$. By (5.5)(ii) and (5.19), $\mathrm{up}(\alpha^{k-1}) = (\eta^k)^-$ and so by (2.4), $(\eta^k)^-$ has finite outcome along $\Lambda^k$. By (5.2), $(\eta^k)^-$ has infinite outcome along $\eta^k$, so $\eta^k \not\subset \Lambda^k$, yielding the desired contradiction. Hence (ii) holds for $k$. ∎

In order to show that all requirements are satisfied, we will need to show that if a node is $\Lambda^k$-free, then it is also implication-free, so can act in accordance with the truth of the sentence trying to generate its action. In fact, we will need to apply this lemma to $\beta^k \subset \Lambda^k$ such that $\mathrm{out}^0(\beta^k)$ is pseudotrue.

LEMMA 5.16 (Implication-Freeness Lemma). *Fix $k \leqslant n$. Suppose that $\beta^k \in T^k \cup [T^k]$ is admissible, and if $\mathrm{lh}(\beta^k) < \infty$, then $\beta = \mathrm{out}^0(\beta^k)$ is pseudotrue. For all $i \leqslant n$, let $\beta^i = \lambda^i(\beta)$. Let $\eta^{k+1} \subset \lambda(\beta^k)$ be $\lambda(\beta^k)$-free and implication-free. Then*:

(i) *For all $j \leqslant k$, the initial derivative $v^j$ of $\eta^{k+1}$ along $\beta^j$ is implication-free.*

(ii) *If* $\eta^k \subset \beta^k$, $\mathrm{up}(\eta^k) = \eta^{k+1}$, *and* $\eta^k$ *is* $\beta^k$*-free, then* $\eta^k$ *is implication--free.*

(iii) *If* $\eta^{k+1}$ *is* $(k+1)$*-completion-free, then the initial derivative* $\nu^k$ *of* $\eta^{k+1}$ *along* $\beta^k$ *is k-completion-free.*

(iv) *If* $\dim(\eta^{k+1}) = k+1$, $\nu^k$ *is the initial derivative of* $\eta^{k+1}$ *along* $\beta^k$ *and has finite outcome along* $\beta^k$*, then* $\nu = \mathrm{out}^0(\nu^k)$ *is completion-consistent via* $\langle\ \rangle$*. If, in addition,* $\delta^k$ *is the immediate successor of* $\nu^k$ *along* $\beta^k$*, then* $\delta = \mathrm{out}^0(\delta^k)$ *is pseudotrue.*

*Proof.* Recall that, if $\beta$ exists, then $\beta$ is pseudotrue, so $\beta^k$ is completion-consistent via $\langle\ \rangle$ and $\beta^j$ is $j$-completion-free for all $j \leqslant n$.

(i) Fix $j \leqslant k$. By Lemma 3.1 (Limit Path), $\nu^j \subset \beta^j$. Suppose that $\nu^j$ is not implication-free, in order to obtain a contradiction. Then one of clauses (5.21)–(5.23) must cause $\nu^j$ to be implication-restrained. If (5.21) holds, then there is a shortest $\xi^j \subseteq \nu^j$ such that $\xi^j$ requires extension but there is no $j$-completion of $\xi^j$ along $\nu^j$. Let $\xi^j$ require extension for $\mu^j$. By Lemma 5.5(ii) (Completion-Respecting), $\xi^j$ has a $j$-completion $\kappa^j \subset \beta^j$ and $\kappa^j$ has infinite outcome along $\beta^j$. Hence $\mu^j \subset \nu^j \subset \kappa^j \subset \beta^j$, and $[\mu^j, \kappa^j]$ is a primary $\beta^j$-link. By Lemma 4.3(i)(c) (Link Analysis), $\eta^{k+1} = \mathrm{up}^{k+1}(\nu^j)$ cannot be $\lambda(\beta^j)$-free, contrary to hypothesis.

Suppose that (5.22) holds in order to obtain a contradiction, and fix the largest $i$ for which (i) fails because (5.22) holds for $i$. By Lemma 5.15(ii) (Admissibility) for $\mathrm{lh}(\beta^k) = \infty$, and Lemma 5.4 (Compatibility) and since $\beta$ is completion-free if $\mathrm{lh}(\beta^k) < \infty$, there is a $q > i$ and a $\delta^q \in T^q$ such that $\delta^q$ requires extension, $\delta^q$ has an $i$-completion $\kappa^i \subset \beta^i$, and $\delta^i = \mathrm{out}^i(\delta^q) \subseteq \nu^i \subset \kappa^i$. As $q > i$, it follows from (5.18) and (5.26) that no node in $(\delta^i, \kappa^i]$ is $(i+1)$-switching. Let $\delta^{i+1} = \mathrm{out}^{i+1}(\delta^q)$, and by Lemma 5.4 (Compatibility) let $\kappa^{i+1}$ be the $(i+1)$-completion of $\delta^q$ along $\lambda(\beta^i)$, and note, by Definition 5.6, that $\mathrm{up}(\kappa^i) = \lambda(\kappa^i) = \kappa^{i+1}$ and $\kappa^i$ is an initial derivative of $\kappa^{i+1}$. As $\nu^i$ is an initial derivative of $\eta^{i+1}$, it follows from Lemma 3.1(i) (Limit Path) that $\delta^{i+1} \subseteq \mathrm{up}(\nu^i) = \nu^{i+1} \subset \kappa^{i+1}$, so $\eta^{i+1}$ is implication-restrained, contrary to the inductive hypothesis.

(5.23) cannot hold, by our induction.

(ii) Suppose that $\eta^k$ is not implication-free, in order to obtain a contradiction. Then one of clauses (5.21)–(5.23) must cause $\eta^k$ to be implication-restrained. (5.23) cannot hold by hypothesis. We assume that $\eta^k$ is primarily or hereditarily implication-restrained, and obtain a contradiction. Fix the shortest $\delta^k \subseteq \eta^k$ such that for some $j \geqslant k$ and some $\mu^j \subset \delta^j = \lambda^j(\delta^k) \subseteq \beta^j$, $\delta^j$ requires extension for $\mu^j$, but there is no $k$-completion of $\delta^j$ along $\eta^k$. By Lemma 5.5(ii) (Completion-Respecting), $\delta^j$ has a primary $j$-completion $\kappa^j \subset \beta^j$ which has infinite outcome along $\beta^j$. If $j = k$, then by Lemma 5.2(i) (Implication Chain), $[\mu^j, \kappa^j]$ is a primary $\beta^j$-link.

And if $j>k$, then as we have assumed that $\kappa^k \not\subseteq \eta^k$, it follows that $\delta^k \subseteq \eta^k \subset \kappa^k$. Now if $[\mu^k, \kappa^k]$ is the $\beta^k$-link derived from the primary $\lambda^j(\beta^k)$-link $[\mu^j, \kappa^j]$, then $\mu^k \subset \delta^k \subseteq \eta^k \subset \kappa^k$, so $\eta^k$ is not $\beta^k$-free, contrary to hypothesis.

(iii) If $\eta^{k+1}$ is $(k+1)$-completion-free, then by Lemma 3.1(i) (Limit Path), for all $j \geqslant k+1$, $\lambda^j(\eta^{k+1}) = \lambda^j(\nu^k)$ is not a primary completion. By Definition 5.6, no primary completion is an initial derivative. (iii) now follows.

(iv) For all $i \leqslant k$, let $\delta^i = \text{out}^i(\delta^k)$, let $\nu^i = (\delta^i)^-$, and note that $\nu^i$ is the principal derivative of $\nu^k$ along $\delta^i$ and that $\delta^i \subseteq \beta^i$ by (2.5). We first show that $\nu$ is completion-consistent via $\langle\ \rangle$. Suppose not in order to obtain a contradiction. Then we may fix the largest $i$ such that $\nu^i$ is implication-restrained. As $\nu^k$ is implication-free, $i<k$; note that, by choice of $i$, $\nu^i$ is either primarily or hereditarily implication-restrained. First suppose that $\nu^i$ is hereditarily implication-restrained. By Lemma 5.4 (Compatibility), $\delta^i$ lies along the $i$-completion of the node witnessing that $\nu^i$ is hereditarily implication-restrained, and by (5.18) and (5.25), $\delta^i$ is not $(i+1)$-switching. Hence $\nu^i$ is the initial derivative of $\nu^{i+1}$ along $\beta^i$. By Lemma 3.1 (Limit Path), an initial derivative can be hereditarily implication-restrained only if its immediate antiderivative is primarily or hereditarily implication-restrained; hence $\nu^{i+1}$ is primarily or hereditarily implication-restrained, contrary to the choice of $i$.

Now suppose that $\nu^i$ is primarily implication-restrained. Then there is an $\eta^i \subseteq \nu^i$ which requires extension but has no primary completion $\subseteq \nu^i$. Fix $\mu^i \subset \eta^i$ such that $\eta^i$ requires extension for $\mu^i$. By Lemma 5.5(ii) (Completion-Respecting), $\eta^i$ has a primary completion $\kappa^i \subset \beta^i$ which has infinite outcome along $\beta^i$. Thus $[\mu^i, \kappa^i]$ is a primary $\beta^i$ link restraining $\nu^i$. But then by Lemma 4.3(i)(a) (Link Analysis), $\nu^{i+1} \not\subset \beta^{i+1}$, contradicting (2.5). This completes the proof of the first part of (iv).

We now note that for all $i$, $\delta^i = \lambda^i(\delta)$ is not a primary completion. For as $\nu$ is completion-consistent via $\langle\ \rangle$ and $\delta^- = \nu$, it follows from (5.27) that if $\delta^i$ is a primary completion, then it is a primary completion of $\delta^i$, contrary to Definition 5.6.

Finally, we show by contradiction that for all $i \leqslant n$, $\delta^i$ does not require extension. Fix the largest $i$ such that $\delta^i$ requires extension in order to obtain a contradiction, and let $\delta^i$ require extension for $\mu^i$. If $i>k$, then by (5.2), $\nu^k$ is the principal derivative of $\eta^{k+1}$ along $\delta^k$, and $\text{up}^n(\nu^k)$ has a unique derivative along $\delta^j$ for all $j>k$; hence by (2.4), $(\delta^j)^- = \text{up}^j(\nu^k)$ for all $j>k$, contrary to the dimension requirements of Definition 5.1. Hence $i \leqslant k$. As $\nu^k$ has finite outcome along $\delta^k$ and $\dim(\eta^{k+1}) = k+1$, it follows from (5.2) that $i<k$. As $\beta$ is pseudotrue or $\text{lh}(\beta) = \infty$, it follows either from Lemma 5.15(ii) (Admissibility) or Lemma 5.5(ii) (Completion-Respecting)

that $\delta^i$ has a primary completion $\kappa^i \subset \beta^i$ which has infinite outcome along $\beta^i$. By Definition 5.1, $\mu^i \subset \nu^i \subset \kappa^i$ and $[\mu^i, \kappa^i]$ is a primary $\beta^i$-link. By (5.2), $\nu^i$ has infinite outcome along $\delta^i$, so is the principal derivative of $\mathrm{up}(\nu^i)$ along $\beta^i$. By Lemma 4.3(i)(c) (Link Analysis), $\mathrm{up}(\nu^i) \not\subseteq \beta^{i+1}$. But as $i < k$, $\mathrm{up}(\nu^i) = \nu^{i+1} \subset \delta^{i+1} \subseteq \beta^{i+1}$, a contradiction. Thus $\delta$ is pseudotrue. ▮

Our next lemma shows that, under the assumption that $\Lambda^0$ is admissible, every requirement $R$ is assigned to a free and implication-free node along $\Lambda^n$. Furthermore, if $R$ has dimension $k$, then we will show that $R$ is assigned to a unique free and implication-free node $\zeta^k$ along $\Lambda^k$, and that the principal derivative of $\zeta^k$ along $\Lambda^{k-1}$ is free and implication-free. We will show later that, as a result of this lemma, the construction will act to satisfy $R$ in accordance with the truth or falsity of the sentence which tries to determine the action for $R$. The implication-freeness of the nodes involved will enable us to show that sufficiently many derivatives of $\zeta^k$ will also be able to act consistently with their assigned sentences. Again we will need to apply the lemma not only to $\Lambda^0$, but to pseudotrue $\beta \subset \Lambda^0$.

LEMMA 5.17 (Assignment Lemma). *Suppose that $\beta \in T^0 \cup [T^0]$ is admissible, and if $\mathrm{lh}(\beta) < \infty$, then $\beta$ is pseudotrue. Let $R$ be a requirement of dimension $k$. For all $i \leqslant n$, let $\beta^i = \lambda^i(\beta)$. Then:*

(i) *If $\mathrm{lh}(\beta) = \infty$, then there is a $\zeta^n \subset \beta^n$ such that $\zeta^n$ is $\beta^n$-free, implication-free, and $n$-completion-free, and $R$ is assigned to $\zeta^n$.*

(ii) *If $R$ is assigned to $\zeta^n \subset \beta^n$, then there is a unique $\zeta^k \subset \beta^k$ such that $\mathrm{up}^n(\zeta^k) = \zeta^n$, and $\zeta^k$ is $\beta^k$-free, implication-free, and $k$-completion-free.*

(iii) *If $\zeta^k$ exists as in (ii), then the principal derivative $\zeta^{k-1}$ of $\zeta^k$ along $\beta^{k-1}$ is $\beta^{k-1}$-free and implication-free.*

(iv) *If $j \leqslant n$, $\xi^j \subset \beta^j$ is $\beta^j$-free and implication-free, $\delta^j \subseteq \beta^j$, and $(\delta^j)^- = \xi^j$, then $\delta = \mathrm{out}^0(\delta^j)$ is pseudotrue.*

(v) *If $\zeta^n \subset \beta^n$ and $\mathrm{lh}(\zeta^n) > 0$, then $\mathrm{out}^0(\zeta^n)$ is pseudotrue, and the initial derivative of $\zeta^n$ along $\beta$ is pseudotrue.*

*Proof.* (i) Assume that $\mathrm{lh}(\beta) = \infty$. By (5.28) or Lemma 5.15(i) (Admissibility), there are no amenable $j$-implication chains along $\beta^j$ for any $j \leqslant n$. Fix $i$ such that $R = R_i$. By Lemma 3.1(iii, iv) all blocks along $\beta^n$ are completed, so there are infinitely many blocks along $\beta^n$. Hence there is a $\zeta^n \subset \beta^n$ which completes the $(i+1)$st block. By Lemma 3.1(i) (Limit Path), $\zeta^n$ has an initial derivative along $\beta^{n-1}$, so a requirement must be assigned to $\zeta^n$. Such a requirement can only be assigned when Step 4 of Definition 2.8 is followed, and the requirement assigned is $R_i$.

As there are no $\beta^n$-links, $\zeta^n$ is $\beta^n$-free. As all requirements have dimension $\leqslant n$, it follows from (5.2) and Definition 5.7 that $\zeta^n$ is implication-free. As no nodes on $T^n$ require extension, $\zeta^n$ is $n$-completion-free.

(ii, iii) By Lemma 3.1(ii) (Limit Path) inductively, for all $i \leqslant n$, $\zeta^n$ has a principal derivative $\zeta^i \subset \beta^i$, and by (2.9) for all $i$ such that $k \leqslant i \leqslant n$, $\zeta^i$ is the unique derivative of $\zeta^n$ along $\beta^i$. For all $i \leqslant n$, it follows from Lemma 4.6(i) (Free Derivative), (i), and induction that $\zeta^i$ is $\beta^i$-free. Now by (i) and iterating Lemma 5.16(ii, iii) (Implication-Freeness) inductively, we see that $\zeta^k$ is implication-free and $k$-completion-free. Again by Lemma 5.16(ii) (Implication-Freeness), $\zeta^{k-1}$ is implication-free.

(iv) For all $i \leqslant j$, let $\delta^i = \text{out}^i(\delta^j)$, and for all $i > j$, let $\delta^i = \lambda^i(\delta^j)$. We note that by definition, for all $i \leqslant j$, $\xi^i = (\delta^i)^-$ is the principal derivative of $\xi^j$ along $\beta^j$. Fix $i \leqslant n$. By Lemma 4.6(i) (Free Derivative), $\xi^i$ is $\beta^i$-free, so by Lemma 5.16(ii) (Implication-Freeness), $\xi^i$ is implication-free; thus $\xi = \xi^0 = \text{out}^0(\xi^i)$ is completion-consistent via $\langle\ \rangle$. Hence by Lemma 5.5(iii) (Completion-Respecting) applied to $\delta$, every $\eta^i \subset \delta^i$ which requires extension has a primary completion $\subset \delta^i$. As no node can be its own primary completion, $\delta^i$ cannot be a primary completion.

We complete the proof of (iv) by assuming that $\delta^i$ requires extension, and obtaining a contradiction. As $\beta^i$ is admissible and if $\text{lh}(\beta^i) < \infty$ then $\beta^i$ is pseudotrue and so $\text{out}^0(\beta^i)$ is completion-consistent via $\langle\ \rangle$ and is not a 0-completion, it follows from Lemma 5.5(ii) (Completion-Respecting) or Lemma 5.15(ii) (Admissibility) that $\delta^i \subseteq \beta^i$ has a primary completion $\kappa^i \subseteq \beta^i$. But $\beta^i$ is pseudotrue so is not a primary completion; hence $\kappa^i \subset \beta^i$. Fix $\gamma^i \subseteq \beta^i$ such that $(\gamma^i)^- = \kappa^i$. If $\kappa^i$ has infinite outcome along $\gamma^i$, then by Lemma 5.3(i) (Implication-Chain), there is a primary $\gamma^i$-link restraining $\xi^i$; this link is then a primary $\beta^i$-link, contradicting the fact that $\xi^i$ is $\beta^i$-free. Thus $\kappa^i$ has finite outcome along $\gamma^i$, so by Lemma 5.3(ii) (Implication Chain), there is an amenable implication chain along $\beta^i$, contradicting (5.28) or Lemma 5.15(i) (Admissibility).

(v) By the proof of (i), every $\zeta^n \subset \beta^n$ is $\beta^n$-free and implication-free. The first conclusion of (v) now follows from (iv). Let $\zeta$ be the initial derivative of $\zeta^n$ along $\beta$. Then $\zeta$ is admissible, and by Lemma 3.1(i) (Limit Path) and as initial derivatives are not primary completions, for all $i \leqslant n$, $\text{up}^i(\zeta) = \lambda^i(\zeta)$ is not a primary completion. By Lemma 5.16(i) (Implication-Freeness), $\zeta$ is implication-free, hence completion-consistent via $\langle\ \rangle$. The second conclusion of (v) now follows. ∎

In Lemma 5.12 (PL), we showed that the backtracking process yielded a one-to-one correspondence between the PL sets defined for any two triples of an amenable implication chain, and that this correspondence was provided by the up function. In order to successfully correct axioms, we will need to show that if $\xi$ is pseudotrue, $\delta^1, \rho^1, \eta^1 \in T^1$, $\xi \supset \text{out}(\eta^1) = \eta$, and $\rho^1 \in \mathrm{PL}(\delta^1, \eta^1) \backslash \mathrm{PL}(\delta^1, \lambda(\xi))$, then some element of $(\eta, \xi]$ switches $\rho^1$. The next lemma will allow us to draw such a conclusion when the need to correct is due to the existence of a nonamenable implication chain (the relationship of the nonamenable implication chain on $T^{r-1}$ to the situation on $T^1$ is not readily apparent, as it is absorbed in the control machinery of Section 6). We will also need an inclusion relation between PL sets at higher levels, in order to analyze the formation of implication chains.

LEMMA 5.18 (Nonamenable Backtracking Lemma). *Fix* $k<n$, $\delta^{k+1}$, $\rho^{k+1}$, $\eta^{k+1} \in T^{k+1}$ *and* $\xi^k \supset \mathrm{out}(\eta^{k+1}) = \eta^k$, *such that* $\delta^{k+1} \subset \eta^{k+1}$, $\lambda(\xi^k)$, $\delta^{k+1}$ *is* $\lambda(\xi^k)$-*free*, $\xi = \mathrm{out}^0(\xi^k)$ *and* $\mathrm{out}^0(\eta^{k+1})$ *are admissible and pseudotrue, and* $\rho^{k+1} \in \mathrm{PL}(\delta^{k+1}, \eta^{k+1})$. *Then*:

(i) *Some element of* $(\eta^k, \xi^k]$ *switches* $\rho^{k+1}$.

(ii) $\{\mathrm{up}(\gamma^k) \colon \gamma^k \in \mathrm{PL}((\eta^k)^-, \xi^k)\} \supseteq \mathrm{PL}(\delta^{k+1}, \eta^{k+1})$.

*Proof.* We first note that $\mathrm{PL}(\delta^{k+1}, \lambda(\xi^k)) = \emptyset$. As $\delta^{k+1}$ is $\lambda(\xi^k)$-free, (5.13) cannot place any elements into $\delta^{k+1}$. Suppose that $\tau^{k+1} \supset \delta^{k+1}$ requires extension for some $\mu^{k+1} \subset \delta^{k+1}$. As $\mathrm{out}^0(\xi^k) = \mathrm{out}^0(\lambda(\xi^k))$ is pseudotrue, $\mathrm{out}^0(\lambda(\xi^k))$ is completion consistent via $\langle\ \rangle$ and is not a 0-completion. Hence by Lemma 5.5(ii) (Completion-Respecting), $\tau^{k+1}$ has a primary completion $\kappa^{k+1}$ which has infinite outcome along $\lambda(\xi^k)$. But then $[\mu^{k+1}, \kappa^{k+1}]$ is a primary $\lambda(\xi^k)$-link restraining $\delta^{k+1}$, so $\delta^{k+1}$ is not $\lambda(\xi^k)$-free, contrary to hypothesis.

By (2.6), no element of $(\eta^k, \xi^k]$ can switch any $\gamma^{k+1} \subset \delta^{k+1}$. We first consider the case in which $\rho^{k+1}$ is placed in $\mathrm{PL}(\delta^{k+1}, \eta^{k+1})$ through (5.13), and show that (i) and (ii) are satisfied. As $\delta^{k+1} \subset \eta^{k+1}$, there is a $\mu^{k+1} \subset \eta^{k+1}$ and a primary $\eta^{k+1}$-link $[\mu^{k+1}, \rho^{k+1}]$ which restrains $\delta^{k+1}$. As $\delta^{k+1} \subset \lambda(\xi^k)$ and $\mathrm{PL}(\delta^{k+1}, \lambda(\xi^k)) = \emptyset$, it follows that $\rho^{k+1} \notin \mathrm{PL}(\delta^{k+1}, \lambda(\xi^k))$; thus $[\mu^{k+1}, \rho^{k+1}]$ is not a $\lambda(\xi^k)$-link. Hence by (2.6) and (2.10), some element $\tau^k$ of $(\eta^k, \xi^k]$ must switch $\rho^{k+1}$ so (i) holds, and by (2.4), $\rho^k = (\tau^k)^-$ has infinite outcome along $\tau^k$. Now $\rho^{k+1} \subset \eta^{k+1}$, so by Lemma 3.1(i), $\rho^{k+1}$ has an initial derivative $\bar{\rho}^k \subset \eta^k$. Hence $[\bar{\rho}^k, \rho^k]$ is a primary $\xi^k$-link which restrains $(\eta^k)^-$, so by (5.13), $\rho^k \in \mathrm{PL}((\eta^k)^-, \xi^k)$ and (ii) holds.

We complete the proof of (i) by showing that if $\rho^{k+1}$ is placed into $\mathrm{PL}(\delta^{k+1}, \eta^{k+1})$ by (5.14) as an element of the component $\mathrm{PL}(\sigma^{k+1}, \zeta^{k+1})$ for some $\zeta^{k+1} \subseteq \eta^{k+1}$, or if $\rho^{k+1} = \sigma^{k+1}$ for this component, then some element of $(\eta^k, \xi^k]$ switches $\rho^{k+1}$. Let $\tau^{k+1}$ be the immediate successor of $\sigma^{k+1}$ along $\eta^{k+1}$, and note that $\tau^{k+1}$ requires extension for some $\mu^{k+1}$ which we fix. By (5.14), $\mu^{k+1} \subset \delta^{k+1} \subset \sigma^{k+1}$. As $\mathrm{out}^0(\eta^{k+1})$ is pseudotrue, $\mathrm{out}^0(\eta^{k+1})$ is completion consistent via $\langle\ \rangle$ and is not a 0-completion. Hence by Lemma 5.5(ii) (Completion-Respecting), $\tau^{k+1}$ has a primary completion $\kappa^{k+1}$ which has infinite outcome along $\eta^{k+1}$. By Lemma 5.4

(Compatibility), $\tau^{k+1}$ must have a $k$-completion $\bar{\kappa}^k \subseteq \eta^k \subset \xi^k$, so $\kappa^{k+1}$ must have a principal derivative $\kappa^k$ along $\xi^k$. Let $\alpha^k$ be the immediate successor of $\kappa^k$ along $\xi^k$. Suppose first that $\kappa^k$ has finite outcome along $\xi^k$, for the sake of obtaining a contradiction. Then by (2.4), $\kappa^k = \bar{\kappa}^k$ and $\kappa^{k+1}$ has infinite outcome along $\lambda(\alpha^k)$, so $[\mu^{k+1}, \kappa^{k+1}]$ would be a primary $\lambda(\alpha^k)$-link restraining $\rho^{k+1}$. By (2.6) and (2.10) and as $\kappa^k$ is the principal derivative of $\kappa^{k+1}$ along $\xi^k$, $[\mu^{k+1}, \kappa^{k+1}]$ must be a primary $\lambda(\xi^k)$-link restraining $\delta^{k+1}$, contrary to the hypothesis that $\delta^{k+1}$ is $\lambda(\xi^k)$-free.

We conclude that $\kappa^k$ has infinite outcome along $\xi^k$. As $\kappa^{k+1}$ has infinite outcome along $\eta^{k+1}$, it follows from (2.4) and (2.8) that $\eta^k \subseteq \kappa^k$. By Lemma 5.3(ii) Implication Chain) and Lemma 5.2 (Requires Extension), $\alpha^k$ will require extension for some $\sigma^k$ such that $\mathrm{up}(\sigma^k) = \sigma^{k+1}$. As $\xi$ is pseudotrue, $\lambda^j(\xi)$ is not a primary completion for any $j \leqslant n$. Hence by Lemma 5.9 (Completion-Consistency), $\xi^-$ is completion-consistent via $\langle\ \rangle$. Hence $\alpha^k$ must have a $k$-completion $\tilde{\kappa}^k \subseteq (\xi^k)^- \subset \xi^k$. As $\xi$ is admissible, it follows from (5.28) that there are no amenable $k$-implication chains along $\xi^k$, so by Lemma 5.3(ii) (Implication Chain), $\tilde{\kappa}^k$ must have infinite outcome along $\xi^k$. Fix $\bar{\alpha}^k \subseteq \xi^k$ such that $(\bar{\alpha}^k)^- = \tilde{\kappa}^k$. By Definition 5.3 and since $\kappa^{k+1}$ has infinite outcome along $\eta^{k+1}$, all elements of $\mathrm{PL}(\delta^{k+1}, \eta^{k+1})$ coming from a component $\mathrm{PL}(\sigma^{k+1}, \zeta^{k+1})$ for some $\zeta^{k+1} \subseteq \eta^{k+1}$ are elements of $\mathrm{PL}(\sigma^{k+1}, \kappa^{k+1})$; and as $\kappa^{k+1}$ has finite outcome along $\lambda(\alpha^k)$, it follows from Lemma 5.1(iv) (PL Analysis) that $\mathrm{PL}(\sigma^{k+1}, \kappa^{k+1}) = \mathrm{PL}(\sigma^{k+1}, \lambda(\alpha^k))$. By Lemma 5.12(i, ii) (PL) and Lemma 5.11(v) (Amenable Backtracking), $\mathrm{PL}(\sigma^{k+1}, \lambda(\tilde{\kappa}^k)) = \emptyset$, every node in $\mathrm{PL}(\sigma^{k+1}, \lambda(\alpha^k))$ is switched by some element of $(\alpha^k, \tilde{\kappa}^k]$, and $\{\mathrm{up}(\gamma^k) : \gamma^k \in \mathrm{PL}(\kappa^k, \tilde{\kappa}^k)\} = \mathrm{PL}(\sigma^{k+1}, \kappa^{k+1})$. Furthermore, $\bar{\alpha}^k$ switches $\sigma^{k+1} = \mathrm{up}(\tilde{\kappa}^k)$, so (i) holds.

By (5.2), $\sigma^k$ is the initial derivative of $\sigma^{k+1}$ along $\xi^k$. As $\sigma^{k+1} \subset (\eta^{k+1})^-$, it follows from Lemma 3.1(i) that $\sigma^k \subset \mathrm{out}((\eta^{k+1})^-) \subseteq (\eta^k)^-$. We have shown that $\eta^k \subseteq \kappa^k$. Hence by (5.14), $\mathrm{PL}(\kappa^k, \tilde{\kappa}^k)$ is a component of $\mathrm{PL}((\eta^k)^-, \xi^k)$. Furthermore, as $\sigma^{k+1} \subset (\eta^{k+1})^-$, it follows from Lemma 3.1(i) (Limit Path) that $\sigma^k \subset (\eta^k)^-$. By Definition 5.6, $\mathrm{up}(\tilde{\kappa}^k) = \sigma^{k+1}$. Hence $[\sigma^k, \tilde{\kappa}^k]$ is a primary $\xi^k$-link restraining $(\eta^k)^-$. Clause (ii) now follows. ▮

## 6. Control of Spaces

Our requirements will be of the form $(\varphi \to \psi)$ & $(\neg\varphi \to \chi)$. If $\varphi$ seems to be true at a given stage of the construction, we take action to preserve the truth of $\varphi$, to make $\psi$ true, and to preserve its truth. If $\varphi$ seems to be false, we try to satisfy $\chi$ and to preserve its truth. We will define a recursive true path $\Lambda^0 \in [T^0]$ for the construction. Action taken for $\psi$

and $\chi$ is determined by nodes $\xi \subset \Lambda^0$, which try to declare axioms for points in the space controlled by $\xi$, according to the apparent truth of $\varphi$. Thus we will assign spaces $S$ (sets of points which have geometric dimension) to the node $\xi$, define a functional $\Delta_\xi$, and try to arrange that the value $m$ for the axiom $\Delta_\xi(A; \bar{x}, x) = m$, where $\mathrm{OS}(\xi) = A$, is determined by the truth or falsity of a sentence $M_\xi$ associated with $\xi$ for sufficiently many $\langle \bar{x}, x \rangle$ such that $\langle \bar{x}, s, x \rangle \in S$. (The coordinate $s$ represents a stage of the construction rather than an argument for a functional, so we separate it.) In this case, $\xi$ will *control* $S$. The exact definition of control will vary with the type of $\xi$, but we will try to present the definitions of control for the three types of requirements in as uniform a way as possible. Fix a requirement $R = R^{j,r}_{e,b,c}$ for the remainder of this section, and so consider the type $j$ and the dimension $r$ of this requirement to be fixed.

DEFINITION 6.1. The spaces assigned to requirements of type $j$ are specified as follows. Given $\xi \in T^k$, let $\zeta = \mathrm{up}^n(\xi)$. Suppose that $R = R^{j,r}_{e,b,c}$ is assigned to $\zeta$. The space $S_\xi$ will be defined only if $k = r$, in which case we set $S_\xi = \mathbf{N}^r \times \{\mathrm{wt}(\xi)\} \times \{\xi\}$, $\mathrm{wt}(S_\xi) = \mathrm{wt}(\xi)$, and $\dim(S_\xi) = r$ if $j = 0$; we set $S_\xi = \mathbf{N}^{r+1} \times \{e\} \times \{\xi\}$, $\mathrm{wt}(S_\xi) = e$, and $\dim(S_\xi) = r + 1$ if $j = 1$; and we set $S_\xi = \mathbf{N}^r \times \{e\} \times \{\xi\}$, $\mathrm{wt}(S_\xi) = e$, and $\dim(S_\xi) = r$ if $j = 2$. Whenever we specify a *section* $S = \{\langle x_1, ..., x_{r-k} \rangle\} \times \mathbf{N}^u \times \{\langle x, \xi \rangle\}$ of $S_\xi$, we define $\dim(S) = u$, and $\mathrm{wt}(S) = x_{r-k}$ if $r > k$. For each $i \in [k, r]$, we let $\mathrm{up}^i(S) = \{\langle x_1, ..., x_{r-i} \rangle\} \times \mathbf{N}^i \times \{\langle x, \xi \rangle\}$ if $j \in \{0, 2\}$, and $\mathrm{up}^i(S) = \{\langle x_1, ..., x_{r-i} \rangle\} \times \mathbf{N}^{i+1} \times \{\langle x, \xi \rangle\}$ if $j = 1$. Given $u$ such that $S = \mathrm{up}^u(S)$, we define $\mathrm{up}(S) = \mathrm{up}^{u+1}(S)$. We identify two spaces $S_\xi$ and $S_\beta$ whenever they agree in all but the last coordinate and $\xi \equiv \beta$, in which case we write $S_\xi \equiv S_\beta$.

We will define the set of spaces *controlled by $v^k$ at $\eta^k$ with initiator $\delta^k$* (*and terminator $\tau^k$*) below. Let $S$ be a space assigned to a node of $T^k$. If $j \in \{1, 2\}$, then there may be infinitely many nodes along a given path through $T^r$ which are candidates for controlling $S$, so we may not be able to recursively identify the node which should control $S$. Thus we begin to define control on $T^{r-1}$ for requirements of types 1 or 2, spreading out the control of sections of $S$ among many nodes. Implication chains will be used for such $j$ to ensure that these nodes work together to produce the same output for the axioms they control on a subset of $S$ which is large enough to ensure a particular iterated limit. We do define control on $T^r$ when $j = 0$, as there is no ambiguity, in that case, as to which node should control the space.

Controllers for $S$ will be nodes which are derivatives of a node $\beta = \beta^r \in T^r$ such that $S$ is a section of $S_\beta$. Control of a space $S$ associated with a node of type 0 or 2 along a path $\Lambda^k \in [T^k]$ will be determined when we reach the first $\xi^k \subset \Lambda^k$ such that $\mathrm{wt}(\xi^k) \geqslant \mathrm{wt}(S)$ and $\mathrm{out}^0(\xi^k)$ is pseudotrue. We impose the latter condition in order to prevent the specification of axioms while conflicts about the value of the axiom captured by the implication chain machinery remain to be resolved; so assume that $\mathrm{out}^0(\eta^k)$ is pseudotrue. To determine control at $\eta^k \in T^k$, we see if there is such a $\xi^k \subseteq \eta^k$; if $\xi^k$ exists, then the node controlling $S$ at $\eta^k$ is the same as the node controlling $S$ at $\xi^k$. If $\mathrm{wt}(\eta^k) < \mathrm{wt}(S)$, then $S$ is not controlled at $\eta^k$. Nevertheless, in the latter case, we define a (potential) controller $v^k$ for $S$ at $\eta^k$; this node would be the controller were control to be defined. (Thus $S$ may have a controller at $\eta^k$, but may not be controlled at $\eta^k$.) The (potential) controller may be changed before we reach $\xi^k$, but will not change thereafter. (We choose this approach, rather than starting at $\xi^k$, because when we have to define control for requirements of type 1, we need to revise our determination of the controlling node beyond $\xi^k$.) As we want the controller $v^k$ for $S$ at $\eta^k$ to decide the value for axioms it controls, we require that $v^k \subset \eta^k$, so that $\eta^k$ will have a guess at $v^k$'s outcome. Initiators determine when it becomes reasonable either to first define the controller, or to define a new controller, because we see the value we want for the axioms being controlled.

Terminators for initiators will be defined if $j = 2$ and $k = r - 1$. A terminator $\tau^k$ for the initiator $\delta^k$ will be the last node of a primary link $[\mu^k, \tau^k]$ such that $\mu^k \subset \delta^k \subseteq \tau^k$ and $\mathrm{wt}(\tau^k) < \mathrm{wt}(S)$, and will have the property that elements entering the target set for the terminator will enable us to correct axioms. (We specify that $\mu^k \subset \delta^k$ in order to be able to show that, under certain circumstances, the corresponding controller is also restrained by the same link.) Terminators will help us show that the notion of control defined allows the computation of iterated limits needed to satisfy requirements. When the initiator $\delta^k$ for the controller $v^k$ and the space $S$ has a terminator $\tau^k$, then $v^k$ forfeits its eligibility to control $S$. However, if there is no controller to replace $v^k$, then we will still need to have derivatives of $v^k$ controlling sections of $S$. We say that $v^k$ *influences* $S$ in this situation.

Control for requirements of type 1 will have a slightly different flavor. In this case, we have an extra dimension for the spaces controlled at each level, so in order to compute iterated limits, we can allow finitely many axioms to produce the incorrect value on each space of dimension 2 (one of the dimensions specifies stages for the construction, so we are really computing a single limit). This will be important, as we will not have the automatic correction feature which is available for requirements of type 2. To make use of this added flexibility, we allow terminators $\tau^k$ to be defined even if $\mathrm{wt}(\tau^k) \geqslant \mathrm{wt}(S)$, but do not allow new initiators to have large weight. We will thus eventually settle on a final initiator for $S$ along any given path, or decide that no initiator exists along that path.

As mentioned above, we will have to keep track of primary links $[\mu^k, \pi^k]$ on $T^k$ which restrain $\nu^k$ and are safe for $\nu^k$, and those which are not safe. The links which may not be safe cause an element to be placed into some $A_a \in \mathrm{RS}(\nu^k)$ by switching $\pi^k$, and are called $\nu^k$-*injurious*. If such nodes also place elements into $A_c \in \mathrm{OS}(\nu^k)$, they will allow axioms to be corrected. When this is the case, $[\mu^k, \pi^k]$ will be called $\nu^k$-*correcting*. In order to remove $[\mu^k, \pi^k]$ while preserving the admissibility of strings, additional nodes may have to have their outcomes switched; these are the nodes in the set $\overline{\mathrm{PL}}(\xi^k)$ defined below, where $\xi^k$ is the immediate successor of $\pi^k$ which determines that $[\mu^k, \pi^k]$ is a primary link along the given path. $\overline{\mathrm{PL}}(\xi^k)$ is the set of nodes in $\mathrm{PL}(\nu^k, \xi^k)$ which need to be switched to make $\nu^k$ free, and which come from a specified component of $\mathrm{PL}(\nu^k, \xi^k)$, or from the end of a primary $\xi^k$-link restraining $\nu^k$.

Definition 6.2. Fix $k<n$, $\nu^k \in T^k$, and $\mu^k \subset \pi^k = (\xi^k)^- \subset \xi^k \subseteq \eta^k \in T^k$ such that $[\mu^k, \pi^k]$ is a primary $\eta^k$-link. If $\pi^k$ is the primary completion of some node $\sigma^k$, let $\overline{\mathrm{PL}}(\xi^k) = \mathrm{PL}((\sigma^k)^-, \xi^k) \cup \{(\sigma^k)^-\}$, and let $\overline{\mathrm{PL}}(\xi^k) = \{\pi^k\}$ otherwise. We say that $[\mu^k, \pi^k]$ is $\nu^k$-*injurious* if $\mathrm{RS}(\nu^k) \cap \mathrm{TS}(\beta^k) \neq \emptyset$ for some $\beta^k \in \overline{\mathrm{PL}}(\xi^k)$, and is $\nu^k$-*correcting* if $\mathrm{OS}(\nu^k) \subseteq \mathrm{TS}(\beta^k)$ for some $\beta^k \in \overline{\mathrm{PL}}(\xi^k)$.

We note that if $[\mu^k, \pi^k]$ is a $\nu^k$-injurious primary $\eta^k$-link, $\dim(\nu^k) = k$, and $\mathrm{tp}(\nu^k) = 1$, then $[\mu^k, \pi^k]$ is $\nu^k$-correcting. For as $\mu^k \neq \pi^k$ and $\mathrm{up}(\mu^k) = \mathrm{up}(\pi^k)$, it follows from (2.9) that $\dim(\mu^k) \geqslant k+1$. Hence by Lemma 2.2(iii) (Interaction), $[\mu^k, \pi^k]$ is $\nu^k$-correcting.

We will determine the spaces *controlled by* $\nu^k$ *at* $\eta^k$ below. This notion of control will have the following properties. If $\nu^k \in T^k$ is assigned the requirement $R$ and controls $S$ at $\eta^k$, then $\nu^k \subset \eta^k$, $\nu^k$ will be the unique node which controls $S$ at $\eta^k$, and if $j \in \{0, 2\}$, then $\nu^k$ will control $S$ at all $\beta^k \supseteq \eta^k$ such that $\mathrm{out}^0(\beta^k)$ is pseudotrue. The initiator for $S$ at $\eta^k$ will be the longest initiator appointed at any $\xi^k \subseteq \eta^k$ which has no terminator along $\eta^k$. Also, if $X$ is a space of the proper dimension to have sections $X^{[i]}$ controlled on $T^k$, then either only finitely many sections of $X$ will be controlled along any $\Lambda^k \in [T^k]$, or cofinitely many sections of $X$ will be controlled along $\Lambda^k$ by nodes which are derivatives of a fixed node $\nu^{k+1} \in T^{k+1}$; and if $X$ is controlled along $\Lambda^{k+1}$, then $\nu^{k+1}$ will be the controller for $X$ along $\Lambda^{k+1}$. The definition below is arranged to ensure these properties.

We proceed by induction on $r-k$ if $j=0$, and on $r-k-1$ if $j \in \{1, 2\}$, and then by induction on $\mathrm{lh}(\eta^k)$ for $\eta^k \subset \Lambda^k$. (Control will not be defined on $T^r$ if $j \in \{1, 2\}$; implication chains will ensure the existence of the iterated limit for $r=k$.) Let $X$ be a section of the space for which $R$ wants to define axioms, with $\dim(X) = k+1$ if $j \in \{0, 2\}$, and $\dim(X) = k+2$ if

$j=1$. For each $i \in \mathbf{N}$ and $\eta^k \in T^k$, we determine the node $v^k \subset \eta^k$ which is the controller for $X^{[i]}$ at $\eta^k$, the node $\delta^k \subseteq \eta^k$ which is the initiator for $X^{[i]}$ at $\eta^k$, and those nodes $\subseteq \eta^k$ which are terminators for $X^{[i]}$ and some initiator for $X^{[i]}$ at $\eta^k$.

Definition 6.3 (Initiators, Controllers, and Terminators). Fix $k \leqslant r$ if $j=0$, $k<r$ if $j \in \{1,2\}$, $\eta^k \in T^k$ such that $\mathrm{lh}(\eta^k)>0$, and a space $S$, and let $\bar{\delta}^k$ and $\bar{v}^k$ be, respectively, the initiator and controller for $S$ at $(\eta^k)^-$, if these exist. We determine whether the controller, initiator, and terminator for $S$ at $\eta^k$ exist, and if so, define those strings. We will assume by induction that

(6.1) $\bar{\delta}^k$ exists iff $\bar{v}^k$ exists.

*Case* 1. We define controllers when a new initiator is found. There are two subcases. Subcase 1.1 handles the base step, and Subcase 1.2 handles the inductive step.

*Subcase* 1.1: Either $k=r$, $j=0$, and $S=S_{(\eta^k)^-}$; or $k=r-1$, $j \in \{1,2\}$, $\mathrm{wt}(\eta^k) \leqslant \mathrm{wt}(S)$, and $\mathrm{up}(S)=S_{\mathrm{up}((\eta^k)^-)}$; and in both cases, the principal derivative $(\mathrm{out}^j(\eta^k))^-$ of $(\eta^k)^-$ along $\mathrm{out}^j(\eta^k)$ is implication-free for all $j \leqslant k$, and $\mathrm{out}^0(\eta^k)$ is pseudotrue. Then $\eta^k$ is the *initiator* for $S$ at $\eta^k$ and $(\eta^k)^-$ is the *controller* for $S$ at $\eta^k$.

*Subcase* 1.2: $k<r$ if $j=0$, $k<r-1$ if $j \in \{1,2\}$, $\mathrm{wt}(\eta^k) \leqslant \mathrm{wt}(S)$, there is an initiator $\delta^{k+1}$ for $\mathrm{up}(S)$ at $\lambda(\eta^k)$, but $\delta^{k+1}$ is not the initiator for $\mathrm{up}(S)$ at $\lambda((\eta^k)^-)$. Let $v^{k+1}$ be the controller corresponding to $\delta^{k+1}$. Then $\eta^k$ is the *initiator* for $S$ at $\eta^k$. The *controller* $v^k$ for $S$ at $\eta^k$ is the longest derivative of $v^{k+1}$ such that $v^k \subset \eta^k$. (By (6.2) below inductively, it will be the case that $v^{k+1} \subset \delta^{k+1}$, so such a derivative will exist.)

*Case* 2. (We switch controllers and initiators when a new derivative of $\mathrm{up}(\bar{v}^k)$ is found.) Case 1 is not followed, either $k<r$ and $j=0$ or $k<r-1$ and $j \in \{1,2\}$, $\mathrm{wt}(\eta^k) \leqslant \mathrm{wt}(S)$, $\mathrm{up}(\bar{v}^k)$ controls $\mathrm{up}(S)$ at $\lambda(\eta^k)$, and $\mathrm{up}((\eta^k)^-)=\mathrm{up}(\bar{v}^k)$. Then $\eta^k$ is the *initiator* for $S$ at $\eta^k$ and $(\eta^k)^-$ is the *controller* for $S$ at $\eta^k$.

*Case* 3. Neither of the previous cases is followed, $j \in \{1,2\}$, $\bar{v}^k$ and $\bar{\delta}^k$ exist, and there is a primary $\bar{v}^k$-correcting $\eta^k$-link $[\mu^k,(\eta^k)^-]$ such that $\mu^k \subset \bar{\delta}^k \subseteq (\eta^k)^-$; and if $j=2$, then $\mathrm{wt}(\eta^k) \leqslant \mathrm{wt}(S)$ and $k=r-1$. (Again note, as in the earlier description of terminators, that we require that $\mu^k \neq \bar{\delta}^k$.) We call $(\eta^k)^-$ a *terminator* for $S$ and $\bar{\delta}^k$ at $\eta^k$. (Note that if $j=1$, then we allow $\bar{v}^k$-correcting primary links to cause a change of control, even if we discover them at a node whose weight exceeds $\mathrm{wt}(S)$. This is necessary, else we would not be able to correct axioms for a thick subset of $\mathrm{up}(S)$ when control is switched.)

*Subcase* 3.1: There is no controller for $S$ at $\mu^k$. If $j=1$, then there is no controller or initiator for $S$ at $\eta^k$. If $j=2$, then $\bar{\nu}^k$ ($\bar{\delta}^k$, resp.) is the *controller* (*initiator*, resp.) for $S$ at $\eta^k$.

*Subcase* 3.2: Otherwise. By (6.1) inductively, let $\tilde{\delta}^k$ and $\tilde{\nu}^k$ be, respectively, the initiator and controller for $S$ at $\mu^k$. Then $\tilde{\nu}^k$ is the *controller* for $S$ at $\eta^k$; and the *initiator* for $S$ at $\eta^k$ is $\tilde{\delta}^k$ if $\mathrm{wt}(\eta^k) > \mathrm{wt}(S)$, and is $\eta^k$ if $\mathrm{wt}(\eta^k) \leqslant \mathrm{wt}(S)$.

*Case* 4. Otherwise. The *initiator* and *controller* for $S$ at $\eta^k$ are $\bar{\delta}^k$ and $\bar{\nu}^k$, respectively, if these exist, and fail to exist otherwise.

In all cases, we say that $\tau^k$ is a *terminator* for $S$ and $\delta^k$ *along* $\eta^k$ ($\Lambda^k \in [T^k]$, resp.) if $\tau^k$ is a terminator for $S$ and $\delta^k$ at some $\xi^k \subseteq \eta^k$ ($\xi^k \subset \Lambda^k$, resp.). ∎

The following properties are easily verified by induction on $\mathrm{lh}(\eta^k)$, as is (6.1). (6.5)(ii) follows from Lemma 4.1 (Nesting), (6.2), and Case 3 of Definition 6.3, where terminators are defined to restrain the previous initiator.

(6.2) If $\nu^k$ controls $S$ at $\eta^k$ with initiator $\delta^k$, then $\nu^k \subset \delta^k \subseteq \eta^k$.

(6.3) If $\delta^k$ is the initiator for $S$ at both $\eta^k$ and $\tilde{\eta}^k$, and $\nu^k$ and $\tilde{\nu}^k$ are the controllers for $S$ at $\eta^k$ and $\tilde{\eta}^k$. respectively, then $\nu^k = \tilde{\nu}^k$.

(6.4) If $\delta^k$ is the initiator for $S$ at $\eta^k$, then $\mathrm{wt}(\delta^k) \leqslant \mathrm{wt}(S)$.

(6.5) Suppose that $\eta^k \subset \tilde{\eta}^k$, and $\delta^k$ and $\tilde{\delta}^k$ are the initiators for $S$ at $\eta^k$ and $\tilde{\eta}^k$, respectively. Then:

(i) If $\mathrm{wt}(\tilde{\eta}^k) \leqslant \mathrm{wt}(S)$, then $\delta^k \subseteq \tilde{\delta}^k$.

(ii) If $\mathrm{wt}(S) \leqslant \mathrm{wt}(\eta^k)$, then $\tilde{\delta}^k \subseteq \delta^k$; and if $j \in \{0, 2\}$, then $\tilde{\delta}^k = \delta^k$.

We are now ready to define control. Recall that control is supported only on pseudotrue nodes, as defined in Definition 5.9. There is a corresponding notion at non-pseudotrue nodes which we call weak control. Control is replaced by influence for requirements of type 2, when the initiator has a terminator.

Definition 6.4 (Control). We say that $\nu^k$ *weakly controls* $S$ at $\eta^k$ if $\nu^k$ is the controller for $S$ at $\eta^k$ corresponding to the initiator $\delta^k$, there is no terminator for $\delta^k$ and $S$ along $\eta^k$, and

$$\mathrm{wt}(S) \leqslant \mathrm{wt}(\eta^k). \tag{6.6}$$

If $\nu^k$ is the controller for $S$ at $\eta^k$ with initiator $\delta^k$, there is a terminator for $\delta^k$ and $S$ along $\eta^k$, and (6.6) holds, then we say that $\nu^k$ *weakly influences* $S$ at $\eta^k$. $\nu^k$ *controls* (*influences*, resp.) $S$ at $\eta^k$ if $\nu^k$ weakly controls (influences, resp.) $S$ at $\eta^k$ and $\text{out}^0(\eta^k)$ is pseudotrue. Given $\Lambda^k \in T^k$, we say that $\nu^k$ *weakly controls* (*weakly influences*, resp.) $S$ ($\delta^k$ is the *initiator* for $S$, resp.) *along* $\Lambda^k$ if $\nu^k$ weakly controls $S$ ($\delta^k$ is the initiator for $S$, resp.) at all sufficiently long $\eta^k \subset \Lambda^k$; and that $\nu^k$ *controls* (*influences*, resp.) $S$ *along* $\Lambda^k$ if there are infinitely many $\eta^k \subset \Lambda^k$ such that $\text{out}^0(\eta^k)$ is pseudotrue, and $\nu^k$ controls (influences, resp.) $S$ at all sufficiently long $\eta^k \subset \Lambda^k$ such that $\text{out}(\eta^k)$ is pseudotrue. ▮

We note that control along $\Lambda^k$ and weak control along $\Lambda^k$ coincide if there are infinitely many pseudotrue $\eta^k \subset \Lambda^k$.

Suppose that $\Lambda^k \in [T^k]$. The following fact now follows easily from (2.1), Lemma 4.1 (Nesting), (6.5), and (6.6), as there must be a longest initiator along any path if there is any initiator along that path:

(6.7) Suppose that $\xi^k \subset \Lambda^k$ and $\xi^k$ extends all initiators and properly extends all terminators for $S$ at any $\eta^k \subset \Lambda^k$. (If $j \in \{0, 2\}$, this will be the case if $\text{wt}(\xi^k) \geqslant \text{wt}(S)$.) Then $\nu^k$ weakly controls (weakly influences, resp.) $S$ ($\delta^k$ is the initiator for $S$, resp.) along $\Lambda^k$ iff $\nu^k$ weakly controls (weakly influences, resp.) $S$ ($\delta^k$ is the initiator for $S$, resp.) at $\xi^k$ iff $\nu^k$ weakly controls (weakly influences, resp.) $S$ ($\delta^k$ is the initiator for $S$, resp.) at every $\eta^k$ such that $\xi^k \subseteq \eta^k \subset \Lambda^k$. Furthermore, if $\nu^k$ weakly controls $S$ along $\Lambda^k$, $\xi^k \subseteq \eta^k \subset \Lambda^k$, and $\delta^k$ is the initiator for $S$ at $\eta^k$, then $\delta^k$ is the longest node which is an initiator for $S$ at some $\gamma^k \subseteq \eta^k$ and which has no terminator along $\eta^k$.

The next lemma specifies some properties of the control process.

LEMMA 6.1 (Finite Control Lemma). *Fix $k \leqslant n$, an admissible $\Lambda^k \in [T^k]$, and a space $S$ assigned to a node working for requirement $R$, where $k \leqslant \dim(R)$ if $j = \text{tp}(R) = 0$, and $k < \dim(R)$ if $\text{tp}(R) \in \{1, 2\}$. Then:*

(i) *$\{\nu^k \in T^k : \exists \eta^k (\nu^k$ weakly controls or weakly influences $S$ at $\eta^k)\}$ is finite.*

(ii) *If $j \in \{0, 2\}$ then:*

(a) *$|\{\nu^k \subset \Lambda^k : \exists \eta^k (\eta^k \subset \Lambda^k \;\&\; \nu^k$ weakly controls or weakly influences $S$ at $\eta^k)\}| \leqslant 1$; and*

(b) *$|\{\delta^k \subset \Lambda^k : \exists \eta^k \subset \Lambda^k (\delta^k$ is an initiator for $S$ at $\eta^k \;\&\; S$ is weakly controlled or weakly influenced at $\eta^k)\}| \leqslant 1$.*

(iii) *Suppose that $k < \dim(R)$. Let $F$ be the set of initiators for $S$ on $T^k$. Then $F$ is finite and for all $\Lambda \in [T^k]$, $S$ is weakly controlled along $\Lambda$ iff there is a $\delta^k \in F$ such that $\delta^k \subset \Lambda$ and there is no terminator for $\delta^k$ and $S$ at any $\eta^k \subset \Lambda$.*

(iv) *If $\nu^k \subset \delta^k \subset \Lambda^k$, $(\delta^k)^- = \nu^k$, $k = \dim(R) - 1$, and $\nu^k$ is a controller at some $\eta^k \subset \Lambda^k$, then $\delta^k$ is an initiator at $\delta^k$.*

*Proof.* (i) If $k=\dim(R)$, then $\mathrm{tp}(R)=0$, and there is a unique node on $T^k$ which controls $S$. Suppose that $k<\dim(R)$, and that $v^k\in T^k$ and $v^k$ weakly controls or weakly influences $S$ at $\eta^k$. By (2.1), (6.2), and (6.4), $\mathrm{wt}(v^k)\leqslant \mathrm{wt}(S)$. But as the weight function is one-to-one, there are only finitely many $v^k\in T^k$ such that $\mathrm{wt}(v^k)\leqslant \mathrm{wt}(S)$.

(ii) If $k=\dim(R)$, then $\mathrm{tp}(R)=0$, and there is a unique controller $v^k$ for $S$ on $T^k$. Furthermore, for any $\eta^k\in T^k$, if $S$ is weakly controlled at $\eta^k$ with initiator $\delta^k$, then $v^k\subset\eta^k$ and $\delta^k$ is the immediate successor of $v^k$ along $\eta^k$.

Suppose that $k<\dim(R)$. By (6.6) and Definition 6.4, if $S$ is weakly controlled or weakly influenced at $\eta^k$, then $\mathrm{wt}(S)\leqslant \mathrm{wt}(\eta^k)$. Clause (ii)(b) now follows from (6.5)(ii). Clause (ii)(a) follows from (6.3).

(iii) Suppose that $k<\dim(R)$. If $\delta^k\in F$ then by (6.4), $\mathrm{wt}(\delta^k)\leqslant \mathrm{wt}(S)$. As the weight function is one-to-one, $F$ is finite. By Definitions 6.3 and 6.4, if $v^k$ weakly controls $S$ along $\Lambda\in[T^k]$ then $\Lambda$ must extend some element $\delta^k$ of $F$ such that there is no terminator for $\delta^k$ and $S$ along $\Lambda$. Conversely, suppose that $\Lambda$ extends an element $\delta^k$ of $F$ such that there is no terminator for $\delta^k$ and $S$ along $\Lambda$. By (6.7) and Definitions 6.3 and 6.4, $S$ is weakly controlled along $\Lambda$.

(iv) We note that if $\mathrm{tp}(R)=0$, then $\lambda(\delta^k)\supset \mathrm{up}(v^k)$, so $\lambda(\delta^k)$ extends an immediate successor of $\mathrm{up}(v^k)$, and so $S$ is weakly controlled along $\lambda(\delta^k)$. Thus (iv) can fail for $\mathrm{tp}(R)\leqslant 2$ only if $v^k$ is defined as the controller for some space through Case 3 of Definition 6.3. Suppose that $v^k$ is defined by that case. Then $v^k$ must be a controller at some $\xi^k\subset\eta^k$. Hence if we fix the shortest $\xi^k\subset\Lambda^k$ at which $v^k$ is a controller, then Subcase 1.1, Subcase 1.2, or Case 2 of Definition 6.3 must be followed at $\xi^k$. But then $\xi^k=\delta^k$ and $\delta^k$ is the initiator corresponding to $v^k$ at $\delta^k$. ∎

Our next lemma spells out some important relationships between initiators, terminators, and weak control for requirements of type 1.

LEMMA 6.2 (Terminator Lemma). *Fix $k<n-1$ and $\Lambda^k\in[T^k]$, and let $\Lambda^{k+1}=\lambda(\Lambda^k)$. Fix a space $X$ which is assigned to a requirement of type 1 and is weakly controlled by some node of $T^{k+1}$, and fix $i\in\mathbf{N}$. Then:*

(i) *If $\delta^k\subset\Lambda^k$ is an initiator for $X^{[i]}$ at $\delta^k$, and $u\geqslant i$, then $\delta^k$ is an initiator for $X^{[u]}$ at $\delta^k$.*

(ii) *Suppose that $\delta^{k+1}\subseteq\eta^{k+1}\subset\Lambda^{k+1}$ are given such that $\delta^{k+1}$ is the initiator for $X$ at all $\gamma^{k+1}$ such that $\eta^{k+1}\subseteq\gamma^{k+1}\subset\Lambda^{k+1}$, and there is no initiator $\tilde{\delta}^{k+1}\supset\eta^{k+1}$ for $X$ (the latter condition includes those $\tilde{\delta}^{k+1}$ which may not lie along $\Lambda^{k+1}$). Let $\eta^k=\mathrm{out}(\eta^{k+1})$. Suppose that $\eta^k\subseteq\delta^k\subset\Lambda^k$, $\delta^k$ is an initiator for $X^{[i]}$, and $\delta^{k+1}$ is not the initiator for $X$ at $\lambda(\delta^k)$. Then there is a terminator for $X^{[i]}$ and $\delta^k$ along $\Lambda^k$.*

*Proof.* (i) By (6.4), $\mathrm{wt}(\delta^k) \leqslant \mathrm{wt}(X^{[i]}) = i$; so as $u \geqslant i$, $\mathrm{wt}(\delta^k) \leqslant \mathrm{wt}(X^{[u]}) = u$. By induction on $\mathrm{lh}(\delta^k)$ and (2.1), if an initiator for one of $X^{[i]}$ or $X^{[u]}$ exists at $(\delta^k)^-$, then that node is the initiator for both $X^{[i]}$ and $X^{[u]}$ at $(\delta^k)^-$. Clause (i) now follows from Definition 6.3.

(ii) Let $\tilde{\delta}^{k+1}$ be the initiator for $X$ at $\lambda(\delta^k)$. As $\delta^k \supseteq \eta^k = \mathrm{out}(\eta^{k+1})$ and $\eta^{k+1} \subset \Lambda^{k+1}$, $\lambda(\delta^k) \supseteq \eta^{k+1}$ by (2.4) and (2.6). Hence by choice of $\eta^{k+1}$, $\tilde{\delta}^{k+1} \subseteq \eta^{k+1}$. Now by (6.7), $\bar{\delta}^{k+1}$ is the initiator for $X$ at $\gamma^{k+1}$ iff $\bar{\delta}^{k+1}$ is the longest node which is an initiator at some $\xi^{k+1} \subseteq \gamma^{k+1}$ and which does not have a terminator along $\gamma^{k+1}$. Thus $\tilde{\delta}^{k+1} \not\supset \delta^{k+1}$, else $\tilde{\delta}^{k+1}$ would have a terminator along $\eta^{k+1}$. Hence as $\lambda(\delta^k) \supseteq \eta^{k+1}$ and $\delta^{k+1}$ is not the initiator for $X$ at $\lambda(\delta^k)$, $\tilde{\delta}^{k+1} \subset \delta^{k+1}$. By (6.7), this is only possible if there is a $\gamma^{k+1} \subseteq \lambda(\delta^k)$ such that $(\gamma^{k+1})^-$ is a terminator for $X$ and $\delta^{k+1}$ along $\gamma^{k+1}$. Let $v^k$ be the initial derivative of $(\gamma^{k+1})^-$ along $\delta^k$, and note, by Lemma 3.1(i) (Limit Path), that $v^k \subset \delta^k$. As $(\gamma^{k+1})^-$ is a terminator for $X$ and $\delta^{k+1}$ along $\gamma^{k+1}$, $(\gamma^{k+1})^-$ must have infinite outcome along $\gamma^{k+1}$, so $v^k$ must have finite outcome along $\delta^k$. Now $\gamma^{k+1} \not\subset \Lambda^{k+1}$ as there is no terminator for $X$ and $\delta^{k+1}$ along $\Lambda^{k+1}$, else $\delta^{k+1}$ would not be the initiator for $X$ along $\Lambda^{k+1}$. Furthermore, as $\delta^{k+1} \subset \Lambda^{k+1}$, by (2.10), some extension of $\delta^k$ along $\Lambda^k$ must switch $(\gamma^{k+1})^-$, so there must be a derivative $\xi^k \supseteq \delta^k$ of $(\gamma^{k+1})^-$ along $\Lambda^k$ which has infinite outcome along $\Lambda^k$. It now follows that $\xi^k$ is a terminator for $\delta^k$ along $\Lambda^k$ via the primary $\Lambda^k$-link $[v^k, \xi^k]$. ∎

The next definition is notational in nature. Given $\Lambda^k \in [T^k]$, a node $v^{k+1}$ of $T^{k+1}$, and a space $X$ whose sections $X^{[i]}$ may be weakly controlled by nodes of $T^k$, we define $\mathrm{CON}(v^{k+1}, \Lambda^k, X)$ to be the set of sections of $X$ which are weakly controlled by derivatives $v^k$ of $v^{k+1}$ such that $v^k \subset \Lambda^k$. This set is partitioned into two sets, $\mathrm{ACT}(v^{k+1}, \Lambda^k, X)$ corresponding to the derivatives of $v^{k+1}$ which are activated along $\Lambda^k$, and $\mathrm{VAL}(v^{k+1}, \Lambda^k, X)$ corresponding to the derivatives of $v^{k+1}$ which are validated along $\Lambda^k$.

Definition 6.5. Let $k < n$, $v^{k+1} \in T^{k+1}$, $\Lambda^k \in [T^k]$, and a space $X$ be given. We define

$$\mathrm{CON}(v^{k+1}, \Lambda^k, X) = \bigcup \{S \subseteq X : \exists v^k \subset \Lambda^k(\mathrm{up}(v^k) = v^{k+1} \\ \& \; v^k \text{ weakly controls } S \text{ along } \Lambda^k)\},$$

$$\mathrm{VAL}(v^{k+1}, \Lambda^k, X) = \bigcup \{S \subseteq X : \exists v^k \subset \Lambda^k(\mathrm{up}(v^k) = v^{k+1} \\ \& \; v^k \text{ weakly controls } S \text{ along } \Lambda^k \\ \& \; v^k \text{ is validated along } \Lambda^k)\},$$

and

$$\mathrm{ACT}(\nu^{k+1}, \Lambda^k, X) = \bigcup \{S \subseteq X : \exists \nu^k \subset \Lambda^k(\mathrm{up}(\nu^k) = \nu^{k+1} \\ \& \ \nu^k \text{ weakly controls } S \text{ along } \Lambda^k \\ \& \ \nu^k \text{ is activated along } \Lambda^k)\}.$$

In the next definition, we introduce *thick* and *thin subsets*. Thick subsets of a space $S$ of dimension $k+1$ are the union of cofinitely many sections $S^{[i]}$ of $S$. Thin subsets are the complements of thick subsets.

DEFINITION 6.6. Fix a space $S$ of dimension $k$. We say that $\tilde{S}$ is a *thick subset* of $S$ if $\tilde{S} = \bigcup \{S^{[i]} : i \in I\}$ where $I$ is a cofinite set of natural numbers. We say that $\tilde{S}$ is a *thin subset* of $S$ if $\tilde{S} \subseteq S$ and $S \backslash \tilde{S}$ is a thick subset of $S$.

We now show that a node weakly controlling a space passes down weak control of a thick subset of that space to its derivatives.

LEMMA 6.3 (Thick Control Lemma). *Fix an admissible $\Lambda^0 \in [T^0]$, and for all $u \leqslant n$, let $\Lambda^u = \lambda^u(\Lambda^0)$. Fix $k < n$, and suppose that $\nu^{k+1} \subset \Lambda^{k+1}$ weakly controls the space $X$ along $\Lambda^{k+1}$. Then:*

(i) *If $\nu^{k+1}$ is validated along $\Lambda^{k+1}$, then $\mathrm{VAL}(\nu^{k+1}, \Lambda^k, X)$ is a thick subset of $X$.*

(ii) *If $\nu^{k+1}$ is activated along $\Lambda^{k+1}$, then $\mathrm{ACT}(\nu^{k+1}, \Lambda^k, X)$ is a thick subset of $X$.*

*Proof.* If $\mathrm{tp}(\nu^{k+1}) = 0$ and $\dim(\nu^{k+1}) = k+1$, then $\nu^{k+1}$ is the unique controller for $X$ on $T^{k+1}$, and its immediate successor $\delta^{k+1}$ along $\Lambda^{k+1}$ is the unique initiator for $X$ at any node extending $\delta^{k+1}$. Thus let $\eta^{k+1} = \delta^{k+1}$ in this case. Otherwise, we note that as $X$ is weakly controlled along $\Lambda^{k+1}$, $\dim(\nu^{k+1}) > k+1$. By (6.5)(ii), (6.7), (2.4), and Lemma 3.1 (Limit Path), we can fix the shortest $\eta^{k+1} \subset \Lambda^{k+1}$ such that $\mathrm{wt}(\eta^{k+1}) > \mathrm{wt}(X)$ and $\nu^{k+1}$ is the controller for $X$ with fixed initiator $\delta^{k+1}$ at all $\gamma^{k+1}$ such that $\eta^{k+1} \subseteq \gamma^{k+1} \subset \Lambda^{k+1}$. Note that as $\mathrm{wt}(\eta^{k+1}) > \mathrm{wt}(X)$, it follows from (6.4) that there is no initiator for $X$ (along any path through $T^{k+1}$) which extends $\eta^{k+1}$. In both cases, let $\eta^k = \mathrm{out}(\eta^{k+1})$. By Lemma 3.2(i) (Out), $\lambda(\eta^k) = \eta^{k+1}$.

We first show that for all $i \geqslant \mathrm{wt}(\eta^k)$, the controller of $X^{[i]}$ along $\Lambda^k$ is a derivative of $\nu^{k+1}$. By Definition 6.3, for all $i \geqslant \mathrm{wt}(\eta^k)$, $X^{[i]}$ will have a controller $\nu^k$ at $\eta^k$, and $\nu^k$ will be a derivative of $\nu^{k+1}$; furthermore, by Lemma 3.1(ii) (Limit Path), $\nu^k \supseteq \pi^k$, where $\pi^k$ is the principal derivative of $\nu^{k+1}$ along $\Lambda^k$. By (4.1), the initiator corresponding to $\nu^k$ is restrained by a primary link along $\Lambda^k$ iff it is restrained by that same link at $\eta^k$. Also by Lemma 6.2(ii) (Terminator) and Definition 6.3, if $i \geqslant \mathrm{wt}(\eta^k)$, $\eta^k \subset \delta^k \subset \Lambda^k$ and $\delta^k$ is an initiator for $X^{[i]}$ at $\delta^k$, then either $\delta^{k+1}$ is the initiator for $X$ at $\lambda(\delta^k)$, or $\mathrm{tp}(\nu^{k+1}) = 1$ and there is a terminator for $\delta^k$ and $X^{[i]}$ along $\Lambda^k$. Thus the controller of $X^{[i]}$ along $\Lambda^k$ must be a derivative of $\nu^{k+1}$.

Fix $i \geqslant \mathrm{wt}(\eta^k)$. By (6.7), $X^{[i]}$ is weakly controlled along $\Lambda^k$. If $\pi^k$ has infinite outcome along $\Lambda^k$, then by (2.8), $\pi^k$ weakly controls $X^{[i]}$ along $\Lambda^k$. And if $\pi^k$ has finite outcome along $\Lambda^k$, then every derivative of $\nu^{k+1}$ along $\Lambda^k$ has finite outcome along $\Lambda^k$; so if $\nu^k$ weakly controls $X^{[i]}$ along $\Lambda^k$, then $\nu^k$ has finite outcome along $\Lambda^k$. Clauses (i) and (ii) now follow, as by Definition 2.1, $\nu^k$ is validated along $\Lambda^k$ iff $\nu^{k+1}$ is validated along $\Lambda^{k+1}$. ∎

The next two lemmas combine to show that if a space $X$ is not weakly controlled along $\Lambda^{k+1}$, then either a thick subset of $X$ is weakly controlled along $\Lambda^k$, or cofinitely many sections of $X$ of dimension $k$ have only a thin subset weakly controlled along $\Lambda^{k-1}$. Also, if $X$ is weakly influenced along $\Lambda^{k+1}$, then a thick subset of $X$ is weakly controlled along $\Lambda^k$.

Lemma 6.4 (Indirect Control Lemma). *Fix $k < n$ and an admissible $\Lambda^k \in [T^k]$, and let $\Lambda^{k+1} = \lambda(\Lambda^k)$. Let $X$ be a section of the space assigned to the requirement $R$ of dimension $r$, where $r \geqslant k+1$ if $\mathrm{tp}(R) = 0$, and $r > k+1$ if $\mathrm{tp}(R) \in \{1, 2\}$. Suppose that $X$ is not weakly controlled along $\Lambda^{k+1}$, but that $X^{[i]}$ is weakly controlled along $\Lambda^k$ for infinitely many $i$. Then there is $\nu^k \subset \Lambda^k$ such that $\nu^k$ weakly controls a thick subset of $X$ along $\Lambda^k$. In particular, this will be the case if $X$ is weakly influenced along $\Lambda^{k+1}$.*

*Proof.* First suppose that $k+1 = \dim(R)$, and so, that $\mathrm{tp}(R) = 0$. By hypothesis, $X$ is not weakly controlled along $\Lambda^{k+1}$, and we note that as $\mathrm{tp}(R) = 0$, there is at most one controller for $X$ on $T^{k+1}$ and there are no terminators for $X$ along $\Lambda^{k+1}$. Hence if there is a controller for $X$ on $T^{k+1}$, then that controller is not $\subset \Lambda^{k+1}$. It thus follows from Lemma 3.1(ii) (Limit Path) that there is $\xi^k \subset \Lambda^k$ such that for all $\bar{\xi}^k \supseteq \xi^k$, if $\bar{\xi}^k \subset \Lambda^k$, then $\lambda(\bar{\xi}^k)$ does not extend an initiator for $X$.

Suppose that $k+1 < \dim(R)$. By Lemma 6.1(iii) (Finite Control), we can fix a finite subset $F$ of $T^{k+1}$ such that for all $\Lambda \in [T^{k+1}]$, $X$ is weakly controlled along $\Lambda$ iff $\Lambda$ extends some element of $F$ which does not have a terminator along $\Lambda$. As $X$ is not weakly controlled along $\Lambda^{k+1}$, it follows from the finiteness of $F$ and Lemma 3.1(ii) (Limit Path) that there is $\xi^k \subset \Lambda^k$ such that for all $\bar{\xi}^k \supseteq \xi^k$, if $\bar{\xi}^k \subset \Lambda^k$ and $\lambda(\bar{\xi}^k)$ extends an element $\delta^{k+1} \in F$, then $\delta^{k+1} \subset \Lambda^{k+1}$ and both $\lambda(\bar{\xi}^k)$ and $\Lambda^{k+1}$ properly extend the same terminator for $\delta^{k+1}$ and $X$ along $\Lambda^{k+1}$.

In either case, we conclude that there are only finitely many initiators for sections of $X$ along $\Lambda^k$. As infinitely many sections of $X$ are weakly controlled along $\Lambda^k$, there must be a $\delta^k \subset \Lambda^k$ such that $\lambda(\delta^k)$ extends an element of $F$, some $\nu^k \subset \delta^k$ weakly controls a section of $X$ at $\delta^k$, and $\delta^k$ is not restrained by any $\nu^k$-correcting primary $\Lambda^k$-link. By choice of $\xi^k$, $\delta^k \subseteq \xi^k$ for each such $\delta^k$. Fix the longest such $\delta^k$, and the unique $\nu^k$ for $\delta^k$. By Definition 6.3 and (6.7), $\nu^k$ will weakly control all but finitely many sections of $X$ along $\Lambda^k$.

We now note that if $X$ is weakly influenced along $\Lambda^{k+1}$, then $X$ has a controller $\nu^{k+1}$ and an initiator $\delta^{k+1}$ along $\Lambda^{k+1}$. By Lemma 3.1(i) (Limit Path), $\nu^{k+1}$ will have a derivative $\nu^k \subset \Lambda^k$ and $\delta^k = \mathrm{out}(\delta^{k+1})$ is an initiator for a section of $X$ at $\delta^k$. Furthermore, $\mathrm{tp}(\nu^{k+1}) = 2$, so there will be no terminators for sections of $X$ along $\Lambda^k$. Thus by Definitions 6.3 and 6.4, $\delta^k$ will witness the fact that infinitely many sections of $X$ are weakly controlled along $\Lambda^k$. The last sentence of the lemma now follows from the first part of the lemma. ∎

The next lemma shows that if $X$ is a space which is not weakly controlled along $\Lambda^{k+1}$ and no section $Y$ of $X$ is weakly controlled along $\Lambda^k$, then for cofinitely many sections $Y$ of $X$, there is very little weak control of sections of $Y$ along $\Lambda^{k-1}$. More precisely, for cofinitely many sections $Y$ of $X$, the number of sections of $Y$ which are weakly controlled at some node along $\Lambda^{k-1}$ is finite, and if $X$ is assigned to a requirement of type 0 or 2, then this number is 0 (so no section of $Y$ is weakly controlled along $\Lambda^{k-1}$). (Because of the definition of terminators, the set of sections of $X$ weakly controlled along $\Lambda^{k-1}$ will be a (possibly proper) subset of the set of sections of $X$ weakly controlled at some $\gamma^{k-1} \subset \Lambda^{k-1}$.)

LEMMA 6.5 (Non-control Lemma). *Fix an admissible $\Lambda^0 \in [T^0]$, and for all $u \leqslant n$, let $\Lambda^u = \lambda^u(\Lambda^0)$. Fix $k \in (0, n-1)$ and a requirement $R$ of dimension $r$ and type $j$, where $r \geqslant k+1$ if $j=0$, and $r > k+1$ if $j \in \{1, 2\}$. Let $X$ be a section of a space assigned to $R$ which is not weakly controlled along $\Lambda^{k+1}$. Suppose that $X^{[i]}$ is weakly controlled along $\Lambda^k$ for at most finitely many $i \in \mathbf{N}$. Then:*

(i) *For all $i \in \mathbf{N}$, either $\{u : (X^{[i]})^{[u]}$ is weakly controlled along $\Lambda^{k-1}\}$ is cofinite, or $\{u : (X^{[i]})^{[u]}$ is weakly controlled at some $\gamma^{k-1} \subset \Lambda^{k-1}\}$ is finite.*

(ii) *For cofinitely many $i \in \mathbf{N}$, $\{u : (X^{[i]})^{[u]}$ is weakly controlled at some $\gamma^{k-1} \subset \Lambda^{k-1}\}$ is finite.*

(iii) *If $j \in \{0, 2\}$, then for cofinitely many $i \in \mathbf{N}$, $\{u : (X^{[i]})^{[u]}$ is weakly controlled at some $\gamma^{k-1} \subset \Lambda^{k-1}\} = \varnothing$.*

*Proof.* By Lemma 3.7 (Infinite Injury), Lemma 6.1(iii) (Finite Control), and as, if $j=0$ and $r=k+1$, then there can be no controller for $X$ along

$\Lambda^{k+1}$ and there is at most one controller for $X$ on $T^{k+1}$, we can choose $\tilde{\eta}^{k-1} \subset \Lambda^{k-1}$ such that for all initiators $\rho^{k+1} \in T^{k+1}$ for $X$ such that $\rho^{k+1} \not\subset \Lambda^{k+1}$ and all $\xi^{k-1}$, if $\tilde{\eta}^{k-1} \subseteq \xi^{k-1} \subset \Lambda^{k-1}$ then $\lambda^{k+1}(\xi^{k-1}) \not\supseteq \rho^{k+1}$. By hypothesis, the preceding sentence, and Lemma 6.1(iii) (Finite Control), we can fix $\eta^{k+1} \subset \Lambda^{k+1}$ such that for all initiators $\rho^{k+1} \subset \Lambda^{k+1}$ for $X$, there is a terminator for $\rho^{k+1}$ and $X$ along $\eta^{k+1}$. Let $\eta^k = \text{out}(\eta^{k+1})$ and $\eta^{k-1} = \text{out}(\eta^k)$, and note that by (2.5), $\eta^k \subset \Lambda^k$ and $\eta^{k-1} \subset \Lambda^{k-1}$. Without loss of generality, we may assume that $\eta^{k-1} \supseteq \tilde{\eta}^{k-1}$. By (2.5) and (2.6), for all $\xi^{k-1}$ that $\eta^{k-1} \subseteq \xi^{k-1} \subset \Lambda^{k-1}$, $\lambda(\xi^{k-1}) \supseteq \eta^k$.

By (2.5), $\lambda(\eta^k) = \eta^{k+1}$. Now $(\eta^k)^- = (\text{out}(\eta^{k+1}))^-$ is the principal derivative of $(\eta^{k+1})^-$ along $\Lambda^k$, so by Lemma 4.3(i)(c) (Link Analysis), there is no primary $\Lambda^k$-link which restrains $(\eta^k)^-$. Hence by hypothesis, there is no initiator $\delta^k$ for any section of $X$ at $(\eta^k)^-$, else by (4.1) and Lemma 4.4 (Free Implies True Path), $\delta^k$ would have no terminator along $\Lambda^k$, so by Definition 6.3, cofinitely many sections of $X$ would have initiators along $\Lambda^k$. But then by Definition 6.4, infinitely many sections of $X$ would be weakly controlled along $\Lambda^k$, contrary to hypothesis. Furthermore, $\eta^k$ cannot be an initiator for a section of $X$, else either $X$ would be weakly controlled at $\eta^{k+1} = \lambda(\eta^k)$, or some section of $X$ would have an initiator at $(\eta^k)^-$, neither of which is possible. Hence there is no initiator for any section of $X$ at $\eta^k$. Also, $\lambda(\eta^{k-1}) = \eta^k$ and $(\eta^{k-1})^- = (\text{out}(\eta^k))^-$ is the principal derivative of $(\eta^k)^-$ along $\Lambda^{k-1}$, so again by Lemma 4.3(i)(c) (Link Analysis), there is no primary $\Lambda^{k-1}$-link which restrains $(\eta^{k-1})^-$.

Fix $i$. First assume that $i < \text{wt}(\eta^k)$. By (6.4) and (2.1), there is no initiator $\delta^k \supseteq \eta^k$ for $X^{[i]}$. Hence as there is no initiator for $X^{[i]}$ at $\eta^k$, if $\delta^{k-1} \subset \Lambda^{k-1}$ is first defined to be an initiator for a section of $X^{[i]}$ by Case 1 or Case 2 of Definition 6.3, then $\delta^{k-1} \subset \eta^{k-1}$. Also, as there is no primary $\Lambda^{k-1}$-link which restrains $(\eta^{k-1})^-$, if $\delta^{k-1} \subset \Lambda^{k-1}$ is first defined to be an initiator for a section of $X^{[i]}$ by Case 3 of Definition 6.3, then $\delta^{k-1} \subset \eta^{k-1}$. We conclude that if $\delta^{k-1}$ is an initiator for a section of $X^{[i]}$ at any $\xi^{k-1} \subset \Lambda^{k-1}$, then $\delta^{k-1} \subset \eta^{k-1}$. Now if there is a $\delta^{k-1} \subset \Lambda^{k-1}$ such that $\delta^{k-1}$ is an initiator for a section of $X^{[i]}$ and there is no terminator for $\delta^{k-1}$ along $\Lambda^{k-1}$, then by Definition 6.3, infinitely many sections of $X$ will have initiators along $\Lambda^{k-1}$, so (i) follows for $i$ from (6.7) and Definition 6.4. Otherwise, as there is no primary $\Lambda^{k-1}$-link which restrains $(\eta^{k-1})^-$, each initiator $\delta^{k-1} \subset \Lambda^{k-1}$ for a section of $X^{[i]}$ has a terminator $\tau^{k-1} \subset \eta^{k-1}$, so by Definition 6.4, for all $u \geqslant \text{wt}(\eta^{k-1})$, $(X^{[i]})^{[u]}$ is not weakly controlled at any $\xi^{k-1} \subset \Lambda^{k-1}$, and again, (i) follows for this $i$.

Suppose that $i \geqslant \text{wt}(\eta^k)$. As there is no initiator $\delta^k$ for $X^{[i]}$ at $\eta^k$ and $\lambda(\eta^{k-1}) = \eta^k$, $\eta^{k-1}$ cannot be an initiator for a section of $X^{[i]}$. Furthermore, for any $\xi^{k-1} \subset \eta^{k-1}$, it follows from (2.4) that $\lambda(\xi^{k-1}) \neq \lambda(\eta^{k-1})$, so by (2.11) and (6.6), $X^{[i]}$ is not weakly controlled at $\lambda(\xi^{k-1})$. Hence any initiator for a section of $X^{[i]}$ at some $\xi^{k-1} \subset \Lambda^{k-1}$ must properly extend $\eta^{k-1}$.

The broad outline of the verification of (ii) in this case is as follows. We first show that if $\delta^{k-1}$ is an initiator for a section of $X^{[i]}$ at some $\xi^{k-1} \subset \Lambda^{k-1}$, then $\lambda(\delta^{k-1})$ extends an initiator for $X^{[i]}$ which, in turn, extends a node which switches a terminator for $X$ along $\Lambda^{k+1}$. We then show that the node on $T^k$ which switched the terminator must have its immediate predecessor switched back by a node on $T^{k-1}$ in order to return the terminator for $X$ to $\Lambda^{k+1}$, and that this switching process can be characterized in terms of PL sets, in a way to ensure correction of axioms. The switching process will ensure that $\delta^{k-1}$ has a terminator along $\Lambda^{k-1}$, so only finitely many sections of $X^{[i]}$ are weakly controlled along $\Lambda^{k-1}$. Furthermore, we will be able to obtain a uniform bound on these terminators, so (ii) will follow.

Suppose that $\delta^{k-1} \subset \Lambda^{k-1}$ is an initiator for a section of $X^{[i]}$. We have shown that $\delta^{k-1} \supset \eta^{k-1}$, so $\lambda(\delta^{k-1}) \supseteq \eta^k$. By Definition 6.3, there must be an initiator $\delta^k \subseteq \lambda(\delta^{k-1})$ for $X^{[i]}$ at $\lambda(\delta^{k-1})$, and again by the second paragraph of the proof and (6.4), $\delta^k \supset \eta^k$. By Definition 6.3, there is an initiator $\delta^{k+1}$ for $X$ at $\lambda(\delta^k)$ with corresponding initiator $\nu^{k+1}$. But by (2.5), $\delta^{k-1} \supseteq \mathrm{out}(\delta^k) \supset \mathrm{out}(\eta^k) = \eta^{k-1}$ and by Lemma 3.2(i) (Out), $\lambda^{k+1}(\mathrm{out}(\delta^k)) = \lambda(\delta^k)$, so by choice of $\eta^{k-1}$, $\delta^{k+1} \subset \Lambda^{k+1}$ and $\delta^{k+1}$ has a terminator $\tau^{k+1} \subset \eta^{k+1} \subset \Lambda^{k+1}$. Fix $\tilde{\tau}^{k+1} \subseteq \eta^{k+1}$ such that $(\tilde{\tau}^{k+1})^- = \tau^{k+1}$, and let $\tilde{\tau}^k = \mathrm{out}(\tilde{\tau}^{k+1})$. By Definition 6.2 and Case 3 of Definition 6.3, there is a $\zeta^{k+1} \in \overline{\mathrm{PL}}(\tilde{\tau}^{k+1})$ such that $\mathrm{OS}(\nu^{k+1}) \subseteq \mathrm{TS}(\zeta^{k+1})$.

We now note that $\tau^{k+1}$ has infinite outcome along $\tilde{\tau}^{k+1} = \lambda(\tilde{\tau}^k)$, and if $\tau^{k+1} \subset \lambda(\delta^k)$, then $\tau^{k+1}$ does not have infinite outcome along $\lambda(\delta^k)$. Furthermore, $\tilde{\tau}^k \subseteq \eta^k \subset \delta^k \subseteq \lambda(\delta^{k-1})$, so by (2.4), if $\tau^{k+1}$ were to have infinite outcome along $\lambda^{k+1}(\delta^{k-1})$, then that outcome would be the same as the outcome of $\tau^{k+1}$ along $\tilde{\tau}^{k+1} = \lambda(\tilde{\tau}^k)$, and by (2.6), $\tau^{k+1}$ would have that outcome along $\lambda(\gamma^k)$ for all $\gamma^k \in [\tilde{\tau}^k, \lambda(\delta^{k-1})]$. In particular, $\tau^{k+1}$ would have that same infinite outcome along $\lambda(\delta^k)$, which we have shown not to be the case. Hence $\tau^{k+1}$ does not have infinite outcome along $\lambda^{k+1}(\delta^{k-1})$. As $\lambda(\delta^k) \supseteq \delta^{k+1}$ and there is a primary $\lambda(\eta^k)$-link $[\mu^{k+1}, \tau^{k+1}]$ which restrains $\delta^{k+1}$ with $\mu^{k+1} \subset \delta^{k+1}$, it follows from (2.10) that there is a node $\hat{\tau}^k$ such that $\tilde{\tau}^k \subseteq \eta^k \subset \hat{\tau}^k \subseteq \delta^k$ and $\hat{\tau}^k$ switches $\tau^{k+1}$. ((2.10) implies that a node can be switched only when it is free; and by (2.6), $\delta^{k+1} \subseteq \lambda(\alpha^k)$ for all $\alpha^k$ such that $\tilde{\tau}^k \subseteq \alpha^k \subseteq \delta^k$. So no node $\subset \tau^{k+1}$ can be switched by such an $\alpha^k \supset \tilde{\tau}^k$ until $\tau^{k+1}$ is switched.) Let $\bar{\tau}^k = (\hat{\tau}^k)^-$, and let $\tau^k = (\tilde{\tau}^k)^-$. Then $[\tau^k, \bar{\tau}^k]$ is a primary $\lambda(\delta^{k-1})$-link, and $\mathrm{up}(\bar{\tau}^k) = \tau^{k+1}$.

As $\eta^k \subseteq \bar{\tau}^k = (\hat{\tau}^k)^- \subset \hat{\tau}^k \subseteq \delta^k$, it follows from (2.5) and Lemma 3.1 (Limit Path) that $\bar{\tau}^k$ has an initial derivative $\bar{\tau}^{k-1}$ such that $\eta^{k-1} \subseteq \bar{\tau}^{k-1} \subset \mathrm{out}(\delta^k) \subseteq \delta^{k-1}$; fix $\hat{\tau}^{k-1} \subseteq \delta^{k-1}$ such that $(\hat{\tau}^{k-1})^- = \bar{\tau}^{k-1}$. Now $\tilde{\tau}^k = \mathrm{out}(\tilde{\tau}^{k+1})$, so $\mathrm{up}(\tau^k) = \tau^{k+1}$, and $\tau^k$ is the principal derivative of

$\tau^{k+1}$ along both $\eta^k$ and $\Lambda^k$. Furthermore, by (2.10) and as $\eta^k$ is $\Lambda^k$-free and $\tau^k \subset \eta^k \subseteq \bar{\tau}^k$, $\bar{\tau}^k$ must be switched by some proper extension $\tilde{\tau}^{k-1}$ of $\delta^{k-1}$ along $\Lambda^{k-1}$. Let $\tau^{k-1} = (\tilde{\tau}^{k-1})^-$, and note that $\tau^{k-1}$ is the principal derivative of $\bar{\tau}^k$ along $\Lambda^{k-1}$, so $[\bar{\tau}^{k-1}, \tau^{k-1}]$ is a primary $\Lambda^{k-1}$-link with $\bar{\tau}^{k-1} \subset \delta^{k-1} \subseteq \tau^{k-1}$.

We now show that $\tau^{k-1}$ is a terminator for $\delta^{k-1}$ along $\Lambda^{k-1}$. First assume that $\tau^{k+1}$ is not a primary completion. Then $\overline{\mathrm{PL}}(\tilde{\tau}^{k+1}) = \{\tau^{k+1}\}$, $\tau^{k-1} \in \overline{\mathrm{PL}}(\tilde{\tau}^{k-1})$, and $\mathrm{up}^{k+1}(\tau^{k-1}) = \tau^{k+1}$. Hence $\tau^{k-1}$ is a terminator for $\delta^{k-1}$ along $\Lambda^{k-1}$.

Now assume that $\tau^{k+1}$ is a primary completion of some $\rho^{k+1}$, which we fix, and let $\sigma^{k+1} = (\rho^{k+1})^-$. As $\tau^{k+1}$ has infinite outcome along $\tilde{\tau}^{k+1}$ but finite outcome along $\lambda(\hat{\tau}^k)$, it follows from Lemma 5.1(i, ii) (PL Analysis) and Definition 5.3 that

$$\begin{aligned}\overline{\mathrm{PL}}(\tilde{\tau}^{k+1}) &= \mathrm{PL}(\sigma^{k+1}, \tilde{\tau}^{k+1}) \cup \{\sigma^{k+1}\}\\ &= \mathrm{PL}(\sigma^{k+1}, \tau^{k+1}) \cup \{\tau^{k+1}\} \cup \{\sigma^{k+1}\},\end{aligned}$$

and by Lemma 5.1(iv) (PL Analysis),

$$\mathrm{PL}(\sigma^{k+1}, \tau^{k+1}) = \mathrm{PL}(\sigma^{k+1}, \lambda(\hat{\tau}^k)).$$

By Lemma 5.3(ii) (Implication Chain), Lemma 5.2 (Requires Extension), and (5.5)(ii), $\hat{\tau}^k$ requires extension for some derivative $\sigma^k$ of $\sigma^{k+1}$. As $\Lambda^0$ is admissible, and, by (2.5), $\mathrm{out}(\hat{\tau}^k) \subset \Lambda^{k-1}$, it follows from (5.27), Lemma 5.15(ii) (Admissibility), and Lemma 5.4 (Compatibility) that $\hat{\tau}^k$ has a $(k-1)$-completion $\beta^{k-1} \subset \Lambda^{k-1}$, and that $\kappa^k = \mathrm{up}(\beta^{k-1})$ is the primary completion of $\hat{\tau}^k$. Furthermore, by Lemma 5.12(ii) (PL),

$$\{\mathrm{up}(\zeta^k) : \zeta^k \in \mathrm{PL}(\bar{\tau}^k, \kappa^k)\} = \mathrm{PL}(\sigma^{k+1}, \lambda(\hat{\tau}^k)).$$

Fix $\tilde{\beta}^{k-1} \subset \Lambda^{k-1}$ such that $(\tilde{\beta}^{k-1})^- = \beta^{k-1}$, let $\tilde{\kappa}^k = \lambda(\tilde{\beta}^{k-1})$, and note that since $\beta^{k-1}$ is the initial derivative of $\kappa^k$, it follows from (2.4) that $(\tilde{\kappa}^k)^- = \kappa^k$. Now $\sigma^{k+1} \subset \eta^{k+1}$ and by (5.2), $\sigma^k$ is an initial derivative of $\sigma^{k+1}$; hence by Lemma 3.1(i) (Limit Path), $\sigma^k \subset \eta^k \subset \hat{\tau}^k \subset \kappa^k$. We now recall that there is no primary $\Lambda^k$-link which restrains $(\eta^k)^-$. Thus there must be a $\bar{\kappa}^{k-1} \subset \Lambda^{k-1}$ such that $\mathrm{up}(\bar{\kappa}^{k-1}) = \kappa^k$ and $\bar{\kappa}^{k-1}$ has infinite outcome along $\Lambda^{k-1}$, else by (2.6) and (2.10) $[\sigma^k, \kappa^k]$ would be a primary $\Lambda^k$-link restraining $(\eta^k)^-$. Fix $\hat{\kappa}^{k-1} \subset \Lambda^{k-1}$ such that $(\hat{\kappa}^{k-1})^- = \bar{\kappa}^{k-1}$. By Lemma 5.1(iv) (PL Analysis),

$$\mathrm{PL}(\bar{\tau}^k, \lambda(\hat{\kappa}^{k-1})) = \mathrm{PL}(\bar{\tau}^k, \kappa^k).$$

As $\bar{\tau}^{k-1}$ is the initial derivative of $\bar{\tau}^k$ along $\hat{\kappa}^{k-1}$, it follows from Lemma 5.3(ii) (Implication Chain) and Lemma 5.2 (Requires Extension) that $\hat{\kappa}^{k-1}$ requires extension for $\bar{\tau}^{k-1}$. As $\Lambda^0$ is admissible, it follows from Lemma 5.15(i, ii) (Admissibility) and Lemma 5.3(ii) (Implication Chain) that there are $\pi^{k-1} \subset \tilde{\pi}^{k-1} \subset \Lambda^{k-1}$ such that $\pi^{k-1}$ is the primary completion of $\hat{\kappa}^{k-1}$, $(\tilde{\pi}^{k-1})^- = \pi^{k-1}$ and $\pi^{k-1}$ has infinite outcome along $\tilde{\pi}^{k-1}$. By (5.19), $\mathrm{up}(\pi^{k-1}) = \bar{\tau}^k$, so by (2.8), $\pi^{k-1} = \tau^{k-1}$ and $\tilde{\pi}^{k-1} = \tilde{\tau}^{k-1}$. By Lemma 5.12(ii) (PL),

$$\mathrm{PL}(\bar{\tau}^k, \lambda(\hat{\kappa}^{k-1})) = \{\mathrm{up}(\zeta^{k-1}) : \zeta^{k-1} \in \mathrm{PL}(\bar{\kappa}^{k-1}, \pi^{k-1})\}.$$

Now by Lemma 5.1(i, ii) (PL Analysis) and Definitions 5.3 and 6.3,

$$\begin{aligned} \overline{\mathrm{PL}}(\tilde{\pi}^{k-1}) &= \mathrm{PL}(\bar{\kappa}^{k-1}, \tilde{\pi}^{k-1}) \cup \{\bar{\kappa}^{k-1}\} \\ &= \mathrm{PL}(\bar{\kappa}^{k-1}, \pi^{k-1}) \cup \{\pi^{k-1}\} \cup \{\bar{\kappa}^{k-1}\}. \end{aligned}$$

Furthermore, $\mathrm{up}(\pi^{k-1}) = \bar{\tau}^k$, $\mathrm{up}(\bar{\kappa}^{k-1}) = \kappa^k$, $\mathrm{up}(\bar{\tau}^k) = \tau^{k+1}$, and $\mathrm{up}(\kappa^k) = \sigma^{k+1}$. Hence

$$\{\mathrm{up}^{k+1}(\zeta^{k-1}) : \zeta^{k-1} \in \overline{\mathrm{PL}}(\tilde{\pi}^{k-1})\} = \overline{\mathrm{PL}}(\tilde{\tau}^{k+1}),$$

so $\pi^{k-1} = \tau^{k-1}$ is a terminator for $\delta^{k-1}$ along $\Lambda^{k-1}$.

We now verify (ii) by showing that only finitely many sections of $X^{[i]}$ are weakly controlled at nodes $\subset \Lambda^{k-1}$. By (2.11) and Lemma 3.1 (Limit Path), fix $\alpha^{k-1} \subset \Lambda^{k-1}$ such that $\mathrm{wt}(\lambda(\alpha^{k-1})) > i$, $\alpha^{k-1} \supset \eta^{k-1}$, and $\lambda(\alpha^{k-1}) \subset \Lambda^k$. By Lemma 6.1(iii) (Finite Control), there are only finitely many initiators for $X^{[i]}$ on $T^k$; since $X^{[i]}$ is not weakly controlled along $\Lambda^k$, we can assume without loss of generality that every initiator for $X^{[i]}$ at some node along $\Lambda^k$ has a terminator $\subset \lambda(\alpha^{k-1})$. Furthermore, by (2.4) and Lemma 3.1 (Limit Path), we can assume that for all $\bar{\xi}^{k-1}$ such that $\alpha^{k-1} \subseteq \bar{\xi}^{k-1} \subset \Lambda^{k-1}$, if $\lambda(\bar{\xi}^{k-1})$ extends an initiator for $X^{[i]}$, then that initiator lies along $\Lambda^{k-1}$. Suppose that $\tilde{\alpha}^{k-1}$ is given such that $\alpha^{k-1} \subseteq \tilde{\alpha}^{k-1} \subset \Lambda^{k-1}$. By (2.4) and (2.6), $\lambda(\tilde{\alpha}^{k-1}) \supseteq \lambda(\alpha^{k-1})$. As $\mathrm{wt}(\lambda(\alpha^{k-1})) > i$, it follows from the choice of $\alpha^{k-1}$, (2.1), and (6.4) that there is no initiator for $X^{[i]}$ at $\lambda(\tilde{\alpha}^{k-1})$, so $X^{[i]}$ is not weakly controlled at $\lambda(\tilde{\alpha}^{k-1})$. Hence $\tilde{\alpha}^{k-1}$ cannot be an initiator for a section of $X^{[i]}$ at any node. Thus there are only finitely many initiators for sections of $X^{[i]}$ along $\Lambda^{k-1}$. By the preceding paragraph every initiator along $\Lambda^{k-1}$ for a section of $X^{[i]}$ has a terminator along $\Lambda^{k-1}$, so we can fix $\bar{\xi}^{k-1} \subset \Lambda^{k-1}$ such that each such terminator is $\subset \bar{\xi}^{k-1}$. It now follows from Definition 6.4 that if $u \geqslant \mathrm{wt}(\bar{\xi}^{k-1})$, then $(X^{[i]})^{[u]}$ is not weakly controlled at any node $\subset \Lambda^{k-1}$, so (ii) follows.

Fix $i$ and $u$ and assume that $j \in \{0, 2\}$. Then there are no terminators for sections of $X^{[i]}$ along $\Lambda^{k-1}$. Hence if $(X^{[i]})^{[u]}$ is weakly controlled at some $\gamma^{k-1} \subset \Lambda^{k-1}$, then by Definition 6.4, there is an initiator for $(X^{[i]})^{[u]}$ which has no terminator along $\Lambda^{k-1}$. By Definition 6.3, for all $v \geqslant u$, $(X^{[i]})^{[v]}$ will have an initiator along $\Lambda^{k-1}$ which has no terminator along $\Lambda^{k-1}$, so by Definition 6.4, $(X^{[i]})^{[v]}$ will be weakly controlled along $\Lambda^{k-1}$. Clause (iii) now follows from (ii). ∎

As we extend nodes along $\Lambda^k$, the path approximation to $\Lambda^{k+1}$ via the function $\lambda$ will occasionally switch paths. We show that for requirements of types 0 and 2, the choice of initiators is invariant under switches of paths, as long as the initiator remains on the switched path and no terminators are eliminated.

LEMMA 6.6 (Constancy of Initiator Lemma). *Fix $k \leqslant n$ and $\eta^k \in T^k$. Let $S$ be a space associated with the requirement $R$ of dimension $r$ and type $j \in \{0, 2\}$, and assume that $k \leqslant r-1$ if $j=0$, and $k < r-1$ if $j=2$. Suppose that $S$ is weakly controlled at $\lambda((\eta^k)^-)$ with initiator $\delta^{k+1}$, and that $\lambda(\eta^k) \supseteq \delta^{k+1}$. Then $\delta^{k+1}$ is the initiator for $S$ at $\lambda(\eta^k)$.*

*Proof.* First assume that $j=0$ and $k=r-1$. Let $v^{k+1}$ be the controller for $S$ at $\lambda(\eta^k)$. Then $v^{k+1}$ is the only controller for $S$ on $T^{k+1}$, and the initiator for $S$ along any path properly extending $v^{k+1}$ is the immediate successor of $v^{k+1}$ along that path. The lemma now follows in this case.

Suppose that $k < r-1$. Let $\rho^{k+1} = \lambda(\eta^k) \wedge \lambda((\eta^k)^-)$, and note, by hypothesis, that $\rho^{k+1} \supseteq \delta^{k+1}$. We assume that $\rho^{k+1} \neq \lambda(\eta^k)$, else by (2.4), $\lambda(\eta^k) = \lambda((\eta^k)^-)$. Under this assumption, it follows from (2.4) that $(\lambda(\eta^k))^- = \rho^{k+1}$. As $S$ is weakly controlled along $\lambda((\eta^k)^-)$, it follows from (6.6) and (2.11) that $\mathrm{wt}(S) \leqslant \mathrm{wt}(\lambda((\eta^k)^-)) < \mathrm{wt}(\lambda(\eta^k))$, so by (6.4), $\lambda(\eta^k)$ cannot be an initiator for $S$, and by Case 3 of Definition 6.3, $\rho^{k+1}$ cannot be a terminator for $S$ at $\lambda(\eta^k)$. Hence as $\rho^{k+1} = (\lambda(\eta^k))^-$, all terminators for $S$ along $\lambda(\eta^k)$ are $\subset \rho^{k+1}$. By (6.7), $\delta^{k+1}$ is the longest initiator for $S$ along $\lambda((\eta^k)^-)$ which has no terminator along $\lambda((\eta^k)^-)$, so as $\delta^{k+1} \subseteq \rho^{k+1} \subseteq (\lambda(\eta^k))^-$, $\delta^{k+1}$ is the longest initiator for $S$ along $\lambda(\eta^k)$ which has no terminator along $\lambda(\eta^k)$. By (6.7), $\delta^{k+1}$ is the initiator for $S$ at $\lambda(\eta^k)$. ∎

In order to show later that the functionals which we define are total on certain oracles, we want to show that for requirements of types 0 and 2, if a space is weakly controlled along an approximation to $\Lambda^1$ but not along a later approximation, then that space is never weakly controlled again. This will fail to be the case only when a terminator is switched. As the proof does not depend on $\Lambda^1$, we prove the general case.

LEMMA 6.7 (Loss of Control Lemma). *Fix $k < n$, a space $S$ for a requirement $R$ of type 0 or 2 with $k+1 < \dim(R)$, and $\eta^k \in T^k$ such that*

$\mathrm{wt}(S) \leqslant \mathrm{wt}(\lambda((\eta^k)^-))$. *Suppose that $S$ has no initiator at $\lambda((\eta^k)^-)$. Then $S$ has no initiator at $\lambda(\eta^k)$.*

*Proof.* Suppose that $S$ has an initiator $\delta^{k+1}$ at $\lambda(\eta^k)$ in order to obtain a contradiction. By (2.4), $(\lambda(\eta^k))^- \subseteq \lambda((\eta^k)^-)$. As $\mathrm{wt}(S) \leqslant \mathrm{wt}(\lambda((\eta^k)^-))$, either $\lambda((\eta^k)^-) = \lambda(\eta^k)$, or by (2.11), $\mathrm{wt}(S) < \mathrm{wt}(\lambda(\eta^k))$; and in the latter case, it follows from (6.4) that $\lambda(\eta^k)$ is not an initiator for $S$ at any node. Hence $\delta^{k+1} \subseteq \lambda((\eta^k)^-)$. By Case 3 of Definition 6.3, the immediate successor $\rho^{k+1}$ of any terminator for $S$ along $\lambda((\eta^k)^-)$ is an initiator for $S$ at $\rho^{k+1}$; hence the longest node which is an initiator for $S$ at some node $\subset \lambda((\eta^k)^-)$ can have no terminator along $\lambda((\eta^k)^-)$. As $\delta^{k+1} \subseteq \lambda((\eta^k)^-)$, it follows that there is an initiator for $S$ at $\lambda((\eta^k)^-)$, contrary to hypothesis. ∎

When a node $\nu^1$ relinquishes control of a space to a node $\hat{\nu}^1$, we will need to know that, often enough, the axioms which were defined by derivatives of $\nu^1$ are either the same axioms that would have been defined by derivatives of $\hat{\nu}^1$, or are corrected. The next lemma is a key ingredient in showing that this happens. It allows us to trace the process of switching initiators, and will be used to show that we can correct axioms for requirements of types 0 and 2. We consider the case where $\eta$ switches $\kappa^1 \in T^1$, causing weak control of a space $S$ to pass from a node $\nu^1$ to a node $\hat{\nu}^1$. We will show that this can occur only when $\hat{\delta} \subseteq \kappa^1 \subset \delta^1$, where $\delta^1$ and $\hat{\delta}^1$ are, respectively, the initiators for $\nu^1$ at $\lambda(\eta^-)$ and $\hat{\nu}^1$ at $\lambda(\eta)$. By Lemma 3.3 ($\lambda$-Behavior), for all $t \geqslant 1$, $\eta$ will switch $\mathrm{up}^t(\kappa^1) = \kappa^t$. We try to carry this situation up to successive trees, by showing that $\mathrm{up}^t(\nu^1)$ weakly controls $\mathrm{up}^t(S)$ along $\lambda^t(\eta^-)$ with some initiator $\delta^t$, $\mathrm{up}^t(\hat{\nu}^1)$ weakly controls $\mathrm{up}^t(S)$ along $\lambda^t(\eta)$ with some initiator $\hat{\delta}$, and $\delta^t \wedge \hat{\delta}^t \subseteq \kappa^t \subset \delta^t \vee \hat{\delta}^t$. Furthermore, the shortest element of $\{\delta^t, \hat{\delta}^t\}$ will alternate by level, i.e., $\delta^t \subset \hat{\delta}^t$ iff $\hat{\delta}^{t+1} \subset \delta^{t+1}$. We will be able to carry this alternation up inductively through $T^p$ where $p+1$ is the smallest $j$ such that $\nu^t = \hat{\nu}^t$, and in some cases, to $T^{p+1}$. (In the other cases for $t = p+1$, we will have to resort to a different proof, as some of the arguments will fail.) The remaining lemmas of this section will then enable us to show, in the next section, that we can correct axioms when necessary.

LEMMA 6.8 (Alternating Initiator Lemma). *Fix $\eta \in T^0$ and let $S$ be a section of a space assigned to the requirement $R$ of dimension $r \geqslant 2$ and type 0 or 2. Suppose that $S$ is weakly controlled by $\nu^1$ at $\lambda(\eta^-)$ with initiator $\delta^1$, $S$ is weakly controlled by $\hat{\nu}^1$ at $\lambda(\eta)$ with initiator $\hat{\delta}^1$, and $\delta^1 \neq \hat{\delta}^1$. Let $p$ be the smallest $t$ such that $\mathrm{up}^{t+1}(\nu^1) = \mathrm{up}^{t+1}(\hat{\nu}^1)$ if such a $t$ exists, and let $p = r-1$ otherwise.* (*Note that, if $\mathrm{tp}(R) = 0$, then $t$ must exist by the definition of $\equiv$ for type 0 nodes.*) *Then for all $t \in [1, p]$, there are*

$\nu^t \subset \delta^t \subseteq \lambda^t(\eta^-)$, $\hat{\nu}^t \subset \hat{\delta}^t \subseteq \lambda^t(\eta)$, $\kappa^t = \lambda^t(\eta^-) \wedge \lambda^t(\eta)$, *and a space* $S^t$ *such that* $\nu^t = \mathrm{up}^t(\nu^1)$, $\hat{\nu}^t = \mathrm{up}^t(\hat{\nu}^1)$, $S \subseteq S^t$, *and*:

(6.8) $\nu^t$ *weakly controls* $S^t$ *at* $\lambda^t(\eta^-)$ *with initiator* $\delta^t$, *and if* $t > 1$, *then* $\lambda(\delta^{t-1}) \supseteq \delta^t$.

(6.9) $\hat{\nu}^t$ *weakly controls* $S^t$ *at* $\lambda^t(\eta)$ *with initiator* $\hat{\delta}^t$, *and if* $t > 1$, *then* $\lambda(\hat{\delta}^{t-1}) \supseteq \hat{\delta}^t$.

(6.10) $\delta^t \subseteq \kappa^t \subset \hat{\delta}^t$ *if* $t$ *is even, and* $\hat{\delta}^t \subseteq \kappa^t \subset \delta^t$ *if* $t$ *is odd.*

*Furthermore, if* $t \in [2, p]$, *then by* (6.8) *inductively and Definitions* 6.3 *and* 6.4, $\nu^t$ *weakly controls* $S^t$ *at* $\lambda(\delta^{t-1})$, *so we can fix the initiator* $\tilde{\delta}^t \subseteq \lambda(\delta^{t-1})$ *such that* $\nu^t$ *weakly controls* $S^t$ *at* $\lambda(\delta^{t-1})$ *with initiator* $\tilde{\delta}^t$. *Similarly, by* (6.9) *inductively and Definitions* 6.3 *and* 6.4, $\hat{\nu}^t$ *weakly controls* $S^t$ *at* $\lambda(\hat{\delta}^{t-1})$, *so we can fix the initiator* $\bar{\delta}^t \subseteq \lambda(\hat{\delta}^{t-1})$ *such that* $\hat{\nu}^t$ *weakly controls* $S^t$ *at* $\lambda(\hat{\delta}^{t-1})$ *with initiator* $\bar{\delta}^t$. (*We need to introduce* $\tilde{\delta}^t$ *and* $\bar{\delta}^t$ *here, as the initiators for* $S^t$ *at* $\lambda^t(\eta^-)$ *and* $\lambda^t(\eta)$ *may differ from those at* $\lambda(\delta^{t-1})$ *and* $\lambda(\hat{\delta}^{t-1})$, *respectively*). *Let* $\rho^t = \lambda(\delta^{t-1}) \wedge \lambda(\hat{\delta}^{t-1})$. *Then for all* $t \in [2, p]$:

(6.11) (i) $\tilde{\delta}^t \subseteq \rho^t \subset \bar{\delta}^t$ *if* $t$ *is even and* $\bar{\delta}^t \subseteq \rho^t \subset \tilde{\delta}^t$ *if* $t$ *is odd.*

(ii) $\tilde{\delta}^t = \delta^t$ *if* $t$ *is even, and* $\bar{\delta}^t = \hat{\delta}^t$ *if* $t$ *is odd.*

*In addition*:

(6.12) (6.8)–(6.11) *will hold for* $t = p+1$ *unless either*:

(i) $p + 1 = r$; *or*

(ii) $\nu^p$ *has finite outcome along* $\lambda^p(\eta^-)$ *iff* $\hat{\nu}^p$ *has finite outcome along* $\lambda^p(\eta)$.

*Proof.* First assume that $t = 1$. Then (6.8) and (6.9) follow by hypothesis. As $\delta^1 \neq \hat{\delta}^1$, it follows from (6.7) and Definition 6.7 that $\lambda(\eta) \mid \lambda(\eta^-)$, so by Lemma 3.3 ($\lambda$-Behavior), $\lambda(\eta)^- \subset \lambda(\eta^-)$. Thus by Lemma 6.6 (Constancy of Initiator), $\lambda(\eta^-) \wedge \lambda(\eta) \subset \delta^1$. By (6.4), (6.6), and (2.11), $\mathrm{wt}(\hat{\delta}^1) \leqslant \mathrm{wt}(S) \leqslant \mathrm{wt}(\lambda(\eta^-)) < \mathrm{wt}(\lambda(\eta))$, so $\hat{\delta}^1 \neq \lambda(\eta)$. Hence $\hat{\delta}^1 \subseteq \lambda(\eta)^- = \lambda(\eta^-) \wedge \lambda(\eta)$, and (6.10) holds.

Suppose that $t \geqslant 2$. We first verify (6.11)(i), assuming that $t$ is odd. (An analogous argument gives the proof for even $t$ by interchanging the hatted and unhatted nodes, the nodes with bars and tildes, $\eta$ and $\eta^-$, and odd and even in the proof below.) By (6.10) inductively, $\delta^{t-1} \subset \hat{\delta}^{t-1}$.

*Case* 1. $\lambda(\delta^{t-1}) \mid \lambda(\hat{\delta}^{t-1})$. We begin with the proof that $\bar{\delta}^t \subseteq \rho^t$. By (2.4) and Lemma 3.1(ii) (Limit Path), there must be a $\xi^{t-1}$ such that $\delta^{t-1} \subset \xi^{t-1} \subseteq \hat{\delta}^{t-1}$, $\lambda(\delta^{t-1}) \mid \lambda(\xi^{t-1})$, $(\xi^{t-1})^-$ is a derivative of $\rho^t$, $\lambda(\xi^{t-1}) \subseteq \lambda(\hat{\delta}^{t-1})$, and $(\lambda(\xi^{t-1}))^- = \rho^t$. As $\rho^t, \bar{\delta}^t \subseteq \lambda(\hat{\delta}^{t-1})$ by (6.2), $\rho^t$ and $\bar{\delta}^t$ are comparable. Suppose that $\rho^t \subset \bar{\delta}^t$ in order to obtain a contradiction. Then $\lambda(\xi^{t-1}) \subseteq \bar{\delta}^t$. By (6.4) and Definition 6.7,

$$\mathrm{wt}(\tilde{\delta}^t) \leqslant \mathrm{wt}(S^t) \leqslant \mathrm{wt}(\lambda(\delta^{t-1})) \tag{6.13}$$

and

$$\mathrm{wt}(\bar{\delta}^t) \leqslant \mathrm{wt}(S^t) \leqslant \mathrm{wt}(\lambda(\hat{\delta}^{t-1})). \tag{6.14}$$

(Note that (6.13) and (6.14) do not make sense when $t = r$.) As $\lambda(\delta^{t-1}) \neq \lambda(\xi^{t-1})$, it follows from (6.13), (2.11), (2.1), and (6.14) that

$$\mathrm{wt}(S^t) \leqslant \mathrm{wt}(\lambda(\delta^{t-1})) < \mathrm{wt}(\lambda(\xi^{t-1})) \leqslant \mathrm{wt}(\bar{\delta}^t) \leqslant \mathrm{wt}(S^t),$$

a contradiction. Hence $\bar{\delta}^t \subseteq \rho^t$.

We complete the proof of (6.11)(i) for Case 1 by showing that $\rho^t \subset \tilde{\delta}^t$. By (6.2), $\rho^t, \tilde{\delta}^t \subseteq \lambda(\delta^{t-1})$, so $\rho^t$ and $\tilde{\delta}^t$ must be comparable. It suffices to assume that $\tilde{\delta}^t \subseteq \rho^t$, and show that $t = p+1$ and (6.12)(ii) holds. By (6.10), $\delta^{t-1} \subset \hat{\delta}^{t-1}$, so iterating Lemma 6.6 (Constancy of Initiator) for $\delta^{t-1} \subset \hat{\delta}^{t-1}$, we see that $\tilde{\delta}^t = \bar{\delta}^t$; thus by (6.3), $v^t = \hat{v}^t$. Hence $t = p+1$. There are two cases to consider.

First consider the case in which $v^{p+1}$ has infinite outcome along $\tilde{\delta}^{p+1} = \bar{\delta}^{p+1}$. Then $v^{p+1}$ has infinite outcome along both $\lambda(\delta^p) \supseteq \tilde{\delta}^{p+1}$ and $\lambda(\hat{\delta}^p) \supseteq \bar{\delta}^{p+1}$, so all derivatives of $v^{p+1}$ along $\delta^p$ ($\hat{\delta}^p$. resp.) have finite outcome along $\delta^p$ ($\hat{\delta}^p$, resp.). In particular, by (6.2) and inductively by (6.8) and (6.9), $v^p$ has finite outcome along $\lambda^p(\eta^-) \supseteq \delta^p$ and $\hat{v}^p$ has finite outcome along $\lambda^p(\eta^-) \supseteq \hat{\delta}^p$, so (6.12)(ii) holds.

Now consider the case in which $v^{p+1}$ has finite outcome $\gamma^p$ along $\tilde{\delta}^{p+1} = \bar{\delta}^{p+1}$. Then $v^{p+1}$ has outcome $\gamma^p$ along both $\lambda(\delta^p) \supseteq \tilde{\delta}^{p+1}$ and $\lambda(\hat{\delta}^p) \supseteq \bar{\delta}^{p+1}$, so by (2.5), $\gamma^p \subseteq \delta^p, \hat{\delta}^p$. By (2.4) and (2.8), $(\gamma^p)^-$ has infinite outcome along $\gamma^p$ and is the longest (and principal) derivative of $v^{p+1}$ along either $\delta^p$ or $\hat{\delta}^p$. Now by Lemma 4.3(i)(c),(a), any primary $\delta^p$-link ($\hat{\delta}^p$-link, resp.) which restrains $(\gamma^p)^-$ restrains all derivatives of $v^{p+1}$ along $\delta^p$ ($\hat{\delta}^p$, resp.). Hence by Definitions 6.3 and 6.4, the controllers for $\mathrm{up}(S)$ corresponding to $\delta^p$ and $\hat{\delta}^p$, respectively, are the longest derivatives of $v^{p+1}$ properly contained in $\delta^p$ and $\hat{\delta}^p$, respectively, so $v^p = \hat{v}^p = (\gamma^p)^-$ and (6.12)(ii) holds. Thus $\tilde{\delta}^t \supset \rho^t$ unless (6.12)(i) or (ii) holds, concluding the proof of (6.11)(i) for this case.

*Case* 2. $\lambda(\delta^{t-1})$ and $\lambda(\hat{\delta}^{t-1})$ are comparable. By Definition 6.7, $\mathrm{wt}(\lambda(\delta^{t-1}))$, $\mathrm{wt}(\lambda(\hat{\delta}^{t-1})) \geqslant \mathrm{wt}(S^t)$, so by Case 3 of Definition 6.3 and (6.4), $\tilde{\delta}^t$ ($\bar{\delta}^t$, resp.) has a terminator along $\lambda(\delta^{t-1})$ iff $\tilde{\delta}^t$ ($\bar{\delta}^t$, resp.) has a terminator along $\lambda(\hat{\delta}^{t-1})$. Thus by Definitions 6.3 and 6.4, $\tilde{\delta}^t = \bar{\delta}^t$, so by (6.3), $v^t = \hat{v}^t$. Hence $t = p+1$. We now proceed as in the preceding two paragraphs, showing that (6.12)(ii) holds, and thus that this case is contrary to hypothesis, and concluding the proof of Case 2.

We now verify (6.8)–(6.10) and (6.11)(ii). Assume that $t$ is odd. (If $t$ is even, then an analogous proof is obtained by interchanging the hatted and unhatted nodes, the nodes with bars and tildes, $\eta$ and $\eta^-$, odd and even, and (6.8) and (6.9).) We begin by showing that $\bar{\delta}^t \subseteq \kappa^t$ (a portion of (6.10)) by eliminating the other possibilities. Let $\zeta^t = \kappa^t \wedge \bar{\delta}^t$, and assume that $\zeta^t \subset \bar{\delta}^t$ in order to obtain a contradiction. First suppose that $\zeta^t$ has finite outcome $\zeta^{t-1}$ along $\bar{\delta}^t$, and so that $(\zeta^{t-1})^-$ has infinite outcome along $\zeta^{t-1}$ and $\mathrm{up}((\zeta^{t-1})^-) = \zeta^t$. As $\bar{\delta}^t \subset \tilde{\delta}^t \subseteq \lambda(\delta^{t-1})$ by (6.11)(i) and the definition of $\tilde{\delta}^t$, it follows from (2.5) that $\zeta^{t-1} \subset \delta^{t-1}$. By (6.10) inductively, $\zeta^{t-1} \subset \lambda^{t-1}(\eta^-) \wedge \lambda^{t-1}(\eta) = \kappa^{t-1}$. Hence by (2.4) and Lemma 3.1 (Limit Path), $\kappa^t, \bar{\delta}^t \supseteq \zeta^t \hat{\ } \langle \zeta^{t-1} \rangle$, contrary to the choice of $\zeta^t$.

Suppose that $\zeta^t$ has infinite outcome $\hat{\zeta}^{t-1}$ along $\bar{\delta}^t$. By Lemma 3.3 ($\lambda$-Behavior) and as $t$ is odd and $\eta$ switches $\kappa^1$, $\kappa^t$ has finite outcome $\beta^{t-1}$ along $\lambda^t(\eta)$. Now it cannot be the case that $\kappa^t \subset \bar{\delta}^t$, else as $\kappa^{t-1} \subset \beta^{t-1} \subseteq \hat{\delta}^{t-1} \subseteq \lambda^{t-1}(\eta)$ by (6.10), it would follow from (2.4) that $\kappa^t = \zeta^t$ has finite outcome along $\bar{\delta}^t$, contrary to our assumption. Hence as $\zeta^t \subset \bar{\delta}^t$, $\kappa^t \mid \bar{\delta}^t$. As $\lambda(\hat{\delta}^{t-1}) \supseteq \bar{\delta}^t \supset \zeta^t$, we have $\hat{\zeta}^{t-1} \subseteq \hat{\delta}^{t-1}$ by (2.5), and so $(\hat{\zeta}^{t-1})^-$ is the initial derivative of $\zeta^t$ along $\hat{\delta}^{t-1}$. By Lemma 3.1 (Limit Path) and as $\zeta^t \subset \kappa^t$, $\zeta^t$ has an initial derivative along $\kappa^{t-1}$; and by (6.10) inductively, $\hat{\delta}^{t-1}$ and $\kappa^{t-1}$ are comparable; hence this initial derivative must also be $(\hat{\zeta}^{t-1})^-$. As $\zeta^t = \kappa^t \wedge \bar{\delta}^t$ and $\kappa^t \mid \bar{\delta}^t$, it follows from (2.4) that $\zeta^t$ must have finite outcome $\zeta^{t-1}$ along $\kappa^t$, so by (2.7), $\zeta^{t-1} \subseteq \kappa^{t-1}$. By (6.10) inductively, $\kappa^{t-1} \subset \hat{\delta}^{t-1}$, so $\zeta^t \hat{\ } \langle \zeta^{t-1} \rangle \subseteq \lambda(\hat{\delta}^{t-1})$ by (2.4). Thus $\zeta^t = \kappa^t \wedge \lambda(\hat{\delta}^{t-1})$ and $\zeta^t \hat{\ } \langle \zeta^{t-1} \rangle \subseteq \kappa^t, \lambda(\hat{\delta}^{t-1})$, a contradiction. We thus conclude that $\bar{\delta}^t \subseteq \kappa^t$.

We next verify (6.11)(ii) and (6.9). By Definition 6.4, we noted in the hypothesis of the lemma that $\hat{\nu}^t$ weakly controls $S^t$ at $\lambda(\hat{\delta}^{t-1})$ with initiator $\bar{\delta}^t$, and $\lambda(\hat{\delta}^{t-1}) \supseteq \bar{\delta}^t$. By (6.9) inductively, $\hat{\delta}^{t-1} \subseteq \lambda^{t-1}(\eta)$. As $S^t$ is weakly controlled at $\lambda(\hat{\delta}^{t-1})$, $\mathrm{wt}(S^t) \leqslant \mathrm{wt}(\lambda(\hat{\delta}^{t-1}))$ by Definition 6.7. By (2.11), for all $\mu^{t-1}$ such that $\hat{\delta}^{t-1} \subset \mu^{t-1} \subseteq \lambda^{t-1}(\eta)$ and $\lambda(\hat{\delta}^{t-1}) \neq \lambda(\mu^{t-1})$, $\mathrm{wt}(\lambda(\mu^{t-1})) > \mathrm{wt}(\lambda(\hat{\delta}^{t-1})) \geqslant \mathrm{wt}(S^t)$, so by (6.4), $\lambda(\mu^{t-1})$ cannot be an initiator for $S^t$, and $(\lambda(\mu^{t-1}))^-$ cannot be a terminator for $S^t$ along $\lambda(\mu^{t-1})$. Hence by (6.5)(ii), we have $\hat{\delta}^t = \bar{\delta}^t$, verifying (6.11)(ii). Also note, by (6.7), that $\hat{\delta}^t$ is the longest initiator for $S^t$ at $\lambda^t(\eta)$ which has no terminator along $\lambda^t(\eta)$. Clause (6.9) now follows from Definition 6.3 and (6.3).

We now verify (6.8). By (6.8) inductively, $\delta^{t-1} \subseteq \lambda^{t-1}(\eta^-)$, so by Definition 6.7 and (2.11), $\mathrm{wt}(S^t) \leqslant \mathrm{wt}(\lambda(\delta^{t-1})) \leqslant \mathrm{wt}(\lambda^t(\eta^-))$. As $\hat{\delta}^t = \bar{\delta}^t \subseteq \kappa^t \subseteq \lambda^t(\eta^-)$ and $\hat{\delta}^t$ is the initiator for $S^t$ at $\lambda^t(\eta) \supseteq \kappa^t$, it follows from Definition 6.3 that $\hat{\delta}^t$ has no terminator along $\kappa^t$; and as $\kappa^t$ is $\lambda^t(\eta^-)$-free by (2.10), there is no primary $\lambda^t(\eta^-)$-link which restrains $\kappa^t$. It thus follows from (6.5) that there is an initiator $\delta^t$ for $S^t$ at $\lambda^t(\eta^-)$. Hence by Definition 6.4 and as $\mathrm{wt}(S^t) \leqslant \mathrm{wt}(\lambda^t(\eta^-))$, $S^t$ is weakly controlled at $\lambda^t(\eta^-)$. We complete the proof that (6.8) holds by showing that $\lambda(\delta^{t-1}) \supseteq \delta^t$. Assume to the contrary, i.e., that $\delta^t \not\subseteq \lambda(\delta^{t-1})$, in order to obtain a contradiction. As

$\delta^{t-1} \subseteq \lambda^{t-1}(\eta^-)$, $\delta^t \nsubseteq \lambda(\delta^{t-1})$, and $\delta^t \subseteq \lambda^t(\eta^-)$, it follows from Lemma 3.1 (Limit Path) that there must be a $\mu^{t-1}$ such that $\delta^{t-1} \subset \mu^{t-1} \subseteq \lambda^{t-1}(\eta^-)$ and $\lambda(\mu^{t-1}) = \delta^t$. But then by Definition 6.7, (2.11), and (6.4), $\mathrm{wt}(S^t) \leqslant \mathrm{wt}(\lambda(\delta^{t-1})) < \mathrm{wt}(\lambda(\mu^{t-1})) = \mathrm{wt}(\delta^t) \leqslant \mathrm{wt}(S^t)$, a contradiction. Hence (6.8) holds.

Finally, we complete the verification of (6.10). Since we have already shown that $\hat{\delta}^t = \bar{\delta}^t \subseteq \kappa^t$, it remains only to show that $\kappa^t \subset \delta^t$. As $\delta^t$ is an initiator at $\lambda^t(\eta^-) \supseteq \kappa^t$, it follows from (6.2) that $\kappa^t$ and $\delta^t$ are comparable. We assume that $\delta^t \subseteq \kappa^t$, and obtain a contradiction. By (6.7), the initiator for a space at a node $\gamma$ is the longest initiator for that space at any node $\alpha \subseteq \gamma$ which has no terminator along $\gamma$. We showed earlier that $\hat{\delta}^t = \bar{\delta}^t \subseteq \kappa^t$. Now $\delta^t, \hat{\delta}^t \subseteq \kappa^t = \lambda^t(\eta^-) \wedge \lambda^t(\eta)$, $\delta^t$ is the initiator for $S^t$ at $\lambda^t(\eta^-)$, and $\hat{\delta}^t$ is the initiator for $S^t$ at $\lambda^t(\eta)$. By (2.10) or Lemma 4.5 (Free Extension), any terminator $\gamma^t$ for $\delta^t$ along $\lambda^t(\eta^-)$ ($\hat{\delta}^t$ along $\lambda^t(\eta)$, resp.) must be $\subseteq \kappa^t$. If $\gamma^t = \kappa^t$, then by Definition 6.3, the immediate successor $\beta^t$ of $\kappa^t$ along $\lambda^t(\eta^-)$ ($\lambda^t(\eta)$, resp.) must be an initiator for $S^t$ at $\beta^t$, so by (6.7), must have a terminator along $\lambda^t(\eta^-)$ ($\lambda^t(\eta)$, resp.). But this would imply that there is a primary $\lambda^t(\eta^-)$-link ($\lambda^t(\eta)$-link, resp.) restraining $\kappa^t$, contradicting (2.10) or Lemma 4.5 (Free Extension). Hence, $\gamma^t \subset \kappa^t$, so $\gamma^t$ is the terminator for $\delta^t$ ($\hat{\delta}^t$, resp.) along both $\lambda^t(\eta^-)$ and $\lambda^t(\eta)$. By (6.7), it must then be the case that $\delta^t = \hat{\delta}^t$. But then by (6.3), $v^t = \hat{v}^t$, so $t = p+1$ and (6.12)(ii) follows from (6.2).

As we have noted above throughout the proof, (6.12) also holds. ∎

We now show that, under the hypotheses and notation of the Alternating Initiator Lemma, activation (validation, resp.) for $v^t$ along $\lambda^t(\eta^-)$ corresponds to activation (validation, resp.) for $v^{t+1}$ along $\lambda^{t+1}(\eta^-)$ for $t \in [1, p]$; and activation (validation, resp.) for $\hat{v}^t$ along $\lambda^t(\eta)$ corresponds to activation (validation, resp.) for $\hat{v}^{t+1}$ along $\lambda^{t+1}$ for $t \in [1, p]$. Furthermore, the same will be true for $t = p+1$ if $v^p$ is activated along $\lambda^p(\eta^-)$ iff $\hat{v}^p$ is validated along $\lambda^p(\eta)$ and $\mathrm{up}(v^p) = \mathrm{up}(\hat{v}^p)$. (If the latter fails, then we will not need the lemma, as correction of axioms will be unnecessary.) We need to add the hypothesis that no $\xi^1 \in T^1$ such that $\xi^1 \equiv v^1$ is switched at $\eta$; if some $\xi^1 \equiv v^1$ is switched at $\eta$, axioms which are newly weakly controlled by $\hat{v}^1$ at $\eta$ are corrected, so we will not have to use the Outcome Lemma below. For requirements of type 0, we only need the simpler, but equivalent condition that no $\xi$ such that $\mathrm{up}^t(\xi^1) = \mathrm{up}^t(v^1)$ for some $t \leqslant \dim(v^1)$ is switched at $\eta$. The more general condition is needed for requirements of type 2.

Lemma 6.9 (Outcome Lemma). *Fix $\eta \in T^0$. Suppose that $S$ is weakly controlled by $v^1$ at $\lambda(\eta^-)$ with initiator $\delta^1$, $S$ is weakly controlled by $\hat{v}^1$ at $\lambda(\eta)$ with initiator $\hat{\delta}^1$, no $\xi^1 \in T^1$ such that $\xi^1 \equiv v^1$ is switched at $\eta$, $\delta^1 \neq \hat{\delta}^1$,* *and* $\mathrm{tp}(v^1) \in \{0, 2\}$. *For all* $t \in [1, n]$, *let* $v^t = \mathrm{up}^t(v^1)$ *and* $\hat{v}^t = \mathrm{up}^t(\hat{v}^1)$. *Let* $p$ *be the smallest* $t$ *such that* $v^{t+1} = \hat{v}^{t+1}$ *if such a* $t$ *exists, and let* $p = \dim(v^1) - 1$ *otherwise. Then for all* $t \in [1, p]$, $v^t$ *is activated along* $\lambda^t(\eta^-)$ *iff* $v^1$ *is activated along* $\lambda(\eta^-)$; *and* $\hat{v}^t$ *is activated along* $\lambda^t(\eta)$ *iff* $\hat{v}^1$ *is activated along* $\lambda(\eta)$. *If, furthermore,* $v^p$ *is activated along* $\lambda^p(\eta^-)$ *iff* $\hat{v}^p$ *is validated along* $\lambda^p(\eta)$ *and* $v^{p+1} = \hat{v}^{p+1}$, *then* $v^{p+1}$ *is activated along* $\lambda^{p+1}(\eta^-)$ *iff* $v^1$ *is activated along* $\lambda(\eta^-)$, *and* $\hat{v}^{p+1}$ *is activated along* $\lambda^{p+1}(\eta)$ *iff* $\hat{v}^1$ *is activated along* $\lambda(\eta)$.

*Proof.* We proceed by induction on $t$. We will prove the lemma for $v^t$ only (a similar argument yields a proof for $\hat{v}^t$). The lemma is vacuous for $t = 1$. Fix notation as in Lemma 6.8 (Alternating Initiator). Let $q = p + 1$. As the Alternating Initiator Lemma cannot be applied if $t = q = \dim(v^1)$, we first prove a weak version, (6.15), of (6.8) to cover the case in which $t = q = \dim(v^1)$, $v^q = \mathrm{up}(v^p) = \mathrm{up}(\hat{v}^p) = \hat{v}^q$, and $v^p$ is activated along $\lambda^p(\eta^-)$ iff $\hat{v}^p$ is validated along $\lambda^p(\eta)$. This weak version of (6.8) will suffice for this case. (Note that a similar proof will also yield a weak version, (6.16), of (6.9).) By hypothesis, (6.12)(ii) will not preclude the use of Lemma 6.8 (Alternating Initiator).

Suppose that $t = q = p + 1 = \dim(v^1)$ and $v^q = \hat{v}^q$. By hypothesis, $v^q \subset \lambda^q(\eta^-), \lambda^q(\eta)$. Fix $\delta^q \subseteq \lambda^q(\eta^-)$ such that $(\delta^q)^- = v^q$ and $\hat{\delta}^q \subseteq \lambda^q(\eta)$ such that $(\hat{\delta}^q)^- = \hat{v}^q$. We will show that:

(6.15) If $v^q$ has finite outcome along $\lambda^q(\eta^-)$, then $v^q \subset \delta^q \subseteq \lambda(\delta^{q-1})$.

We leave it to the reader to verify with a similar proof that:

(6.16) If $\hat{v}^q$ has finite outcome along $\lambda^q(\eta)$, then $\hat{v}^q \subset \hat{\delta}^q \subseteq \lambda(\hat{\delta}^{q-1})$.

We have noted that:

(6.17) $v^q \subseteq \kappa^q = \lambda^q(\eta^-) \wedge \lambda^q(\eta)$.

We note that, in the notation of Lemma 6.8 (Alternating Initiator), $\eta$ switches $\kappa^1 = \lambda(\eta^-) \wedge \lambda(\eta)$. By hypothesis, $(\hat{\delta}^q)^- = \hat{v}^q = v^q = (\delta^q)^-$. By (6.17), $\lambda^q(\eta^-) \wedge \lambda^q(\eta) = \kappa^q \supseteq v^q$. Fix $\beta^{q-1}$ such that that $v^q$ has finite outcome $\beta^{q-1}$ along $\lambda^q(\eta^-)$, let $\pi^{q-1} = (\beta^{q-1})^-$, and let $\mu^{q-1}$ be the initial derivative of $v^q$ along $\beta^{q-1}$. As $\delta^q \subseteq \lambda^q(\eta^-)$ and $(\delta^q)^- = v^q$, it follows from (2.4) that $v^q \wedge \langle \beta^{q-1} \rangle = \delta^q$. By Definition 2.1, $\pi^{q-1}$ has infinite outcome along $\beta^{q-1} \subseteq \lambda^{q-1}(\eta^-)$, and by (6.2) and (6.10), $\mu^{q-1} \subset \kappa^{q-1}$. As $\beta^{q-1}$, $\kappa^{q-1} \subseteq \lambda^{q-1}(\eta^-)$, $\beta^{q-1}$ and $\kappa^{q-1}$ are comparable. It cannot be the case that $\pi^{q-1} \supset \kappa^{q-1}$, else $[\mu^{q-1}, \pi^{q-1}]$ would be a primary $\lambda^{q-1}(\eta^-)$-link restraining $\kappa^{q-1}$, so by (2.10), $\eta$ could not switch $\kappa$, contrary to assumption. By hypothesis, $\eta$ does not switch any node $\equiv v^1$, so $\pi^{q-1} \neq \kappa^{q-1}$. Hence $\pi^{q-1} \subset \kappa^{q-1}$ and so $\beta^{q-1} \subseteq \kappa^{q-1}$. By (2.8), $\pi^{q-1}$ is the longest derivative of $v^q$ along $\lambda^{q-1}(\eta^-)$, so all initiators for $S^{q-1}$ at nodes

$\subseteq \lambda^{q-1}(\eta^-)$ whose corresponding controller is a derivative of $\nu^q$ are $\subseteq \kappa^{q-1}$. Now no initiator for $S^{q-1}$ at any node along $\lambda^{q-1}(\eta^-)$ can have $\pi^{q-1}$ as its controller via Case 3 of Definition 6.3 unless there is a shorter initiator for $S^{q-1}$ which has $\pi^{q-1}$ as its controller via Subcase 1.1 of Definition 6.3; and by Case 1 of Definition 6.3, that shorter initiator must be $\beta^{q-1}$. As $\nu^{q-1}$ is activated along $\lambda^{q-1}(\eta^-)$ iff $\hat{\nu}^{q-1}$ is validated along $\lambda^{q-1}(n)$ and $\nu^{q-1}$ and $\hat{\nu}^{q-1}$ are derivatives of $\nu^q$ and are controllers for sections of $S^{q-1}$, it must therefore be the case that $\beta^{q-1}$ is an initiator for $S^{q-1}$ at $\beta^{q-1}$. As $\lambda^{q-1}(\eta^-) \supseteq \kappa^{q-1} \supseteq \beta^{q-1}$, we have $\delta^{q-1} \supseteq \beta^{q-1}$ by Definition 6.3. So as $\nu^q \subseteq \lambda(\delta^{q-1})$, $\text{up}(\nu^{q-1}) = \nu^q$ and $(\beta^{q-1})^-$ has infinite outcome along $\beta^{q-1}$, it follows that $\lambda(\delta^{q-1}) \supseteq \nu^q \wedge \langle \beta^{q-1} \rangle = \delta^q$, so (6.15) holds.

Now consider any $t$ such that $2 \leqslant t \leqslant q$. First consider the case where $\nu^t$ has infinite outcome along $\lambda^t(\eta^-)$. Then all derivatives of $\nu^t$ along $\lambda^{t-1}(\eta^-)$ must have finite outcome along $\lambda^{t-1}(\eta^-)$. In particular, $\nu^{t-1}$ has finite outcome along $\lambda^{t-1}(\eta^-)$, so the lemma follows by induction in this case.

Next consider the case where $\nu^t$ has finite outcome $\beta^{t-1}$ along $\lambda^t(\eta^-)$. By (2.5), $\beta^{t-1} \subseteq \lambda^{t-1}(\eta^-)$, so by (2.8), $(\beta^{t-1})^-$ is the longest (and principal) derivative of $\nu^t$ along $\lambda^{t-1}(\eta^-)$; hence $\nu^{t-1} \subseteq (\beta^{t-1})^-$. As $\delta^t$ is an initiator at $\lambda^t(\eta^-)$ if $t < q$, and by choice of $\delta^t$ if $t = q$, $\nu^t \subset \delta^t \subseteq \lambda^t(\eta^-)$, so $\nu^t \wedge \langle \beta^{t-1} \rangle \subseteq \delta^t$. By (6.8) or (6.15), $\delta^t \subseteq \lambda(\delta^{t-1})$, so by (2.5), $\beta^{t-1} \subseteq \delta^{t-1}$. Now $\delta^{t-1}$ is an initiator for $S^{t-1}$ and $\nu^{t-1}$ at $\lambda^{t-1}(\eta^-)$, $\nu^{t-1} \subseteq (\beta^{t-1})^-$, and $\text{up}((\beta^{t-1})^-) = \text{up}(\nu^{t-1})$. By Lemma 4.3(i)(c),(a), $(\beta^{t-1})^-$ is restrained by a primary $\delta^{t-1}$-link iff every derivative of $\nu^t$ is restrained by the same primary $\delta^{t-1}$-link; hence by Definition 6.3, the controller $\nu^{t-1}$ chosen for the initiator $\delta^{t-1}$ is the longest derivative of $\nu^t$ along $\delta^{t-1}$, so $\nu^{t-1} = (\beta^{t-1})^-$. Thus the lemma follows by induction. ∎

We now want to show that when the controlling node on $T^1$ is changed, then either the new controller inherits axioms with the value it desires, or the axioms are corrected, allowing the new controller to redefine those axioms. The situation differs with the type of the requirement, so we prove different lemmas for each type.

We begin with a requirement $R$ of type 0. The situation will be as follows. $\eta$ will be 1-switching, causing weak control of a space to pass from $\nu^1$ to $\hat{\nu}^1$ on $T^1$. If $\eta^- \equiv \nu^1$, then $\eta^-$ will cause something to be placed into the oracle set for the axioms newly weakly controlled by $\hat{\nu}^1$, thus allowing $\hat{\nu}^1$ to correct the axioms to the value which it predicts. Otherwise, we will show that both $\nu^1$ and $\hat{\nu}^1$ predict the same value for those axioms, so no correction is necessary. To show that the predictions by $\nu^1$ and $\hat{\nu}^1$ agree, we need to go up to the smallest $q$ such that $\text{up}^q(\nu^1) = \text{up}^q(\hat{\nu}^1)$. An analysis of the situation on $T^q$ will enable us to go down to $T^{q-1}$ and show that $\text{up}^{q-1}(\nu^1)$ is activated along $\lambda^{q-1}(\eta^-)$ iff $\text{up}^{q-1}(\hat{\nu}^1)$ is activated along

$\lambda^{q-1}(\eta)$. It will then follow from Lemma 6.9 (Outcome) that $v^1$ is activated along $\lambda(\eta^-)$ iff $\hat{v}^1$ is activated along $\lambda(\eta)$.

LEMMA 6.10 (0-Correction Lemma). *Fix $\eta \in T^0$. Suppose that $S$ is weakly controlled by $v^1$ at $\lambda(\eta^-)$ with initiator $\delta^1$, $S$ is weakly controlled by $\hat{v}^1$ at $\lambda(\eta)$ with initiator $\hat{\delta}^1$, $\delta^1 \neq \hat{\delta}^1$, and* $\mathrm{tp}(v^1) = 0$. *Let $\kappa^1 = \lambda(\eta^-) \wedge \lambda(\eta)$. Then one of the following holds:*

(i) *$v^1$ is activated along $\delta^1$ iff $\hat{v}^1$ is activated along $\hat{\delta}^1$.*

(ii) *$\eta$ switches $\kappa^1 \subset \delta^1$ and $\kappa^1 \equiv v^1$.*

*Proof.* Fix notation as in Lemma 6.8 (Alternating Initiator), and fix the least $q$ such that $v^q = \hat{v}^q$. If $q = 1$, then by hypothesis, either (i) holds or $\eta$ must switch $v^1$ and (ii) will hold. So we may assume that $q > 1$. Let $p = q - 1$.

If $v^p$ is activated along $\lambda^p(\eta^-)$ iff $\hat{v}^p$ is activated along $\lambda^p(\eta)$, then (i) follows from Lemma 6.9 (Outcome) if $\eta^- \not\equiv v^1$, and (ii) follows if $\eta^- \equiv v^1$. So we assume that $v^p$ is activated along $\lambda^p(\eta^-)$ iff $\hat{v}^p$ is validated along $\lambda^p(\eta)$.

Suppose that $q < \dim(v^1)$. By our assumptions, the conditions of (6.12) fail, so we can apply Lemma 6.8 (Alternating Initiator) with $t = q$. By (6.10), $\kappa^1 \subset \delta^1$. As $v^q = \hat{v}^q$, it follows from (6.10) that $v^q = v^q \wedge \hat{v}^q \subset \delta^q \wedge \hat{\delta}^q \subseteq \kappa^q = \lambda^q(\eta^-) \wedge \lambda^q(\eta)$. Thus the outcome of $v^q$ along $\lambda^q(\eta^-)$ is the same as the outcome of $\hat{v}^q$ along $\lambda^q(\eta)$, so $v^q$ is activated along $\lambda^q(\eta^-)$ iff $\hat{v}^q$ is activated along $\lambda^q(\eta)$. Clause (i) now follows from the second conclusion of Lemma 6.9 (Outcome) if $\eta^- \not\equiv v^1$, and clause (ii) follows if $\eta^- \equiv v^1$.

Suppose that $q = \dim(v^1)$. By (6.10), $\delta^p \wedge \hat{\delta}^p \subseteq \kappa^p \subset \delta^p \vee \hat{\delta}^p$. Furthermore, as $\mathrm{tp}(v^1) = 0$, Subcase 1.2 or Case 2 of Definition 6.3 must be followed to define controllers and initiators on $T^p$, so $v^p = (\delta^p)^-$ and $\hat{v}^p = (\hat{\delta}^p)^-$. (We note that if Subcase 1.2 is followed, then as, by Subcase 1.1, all initiators for $\mathrm{up}(v^p) = \mathrm{up}(\hat{v}^p)$ are immediate successors of $\mathrm{up}(\hat{v}^p)$, it follows from Lemma 3.1 (Limit Path) and Lemma 3.3 ($\lambda$-Behavior) that $v^p = (\delta^p)^-$ and $\hat{v}^p = (\hat{\delta}^p)^-$.) Thus $v^p \wedge \hat{v}^p \subset \kappa^p$. It cannot be the case that $v^p \vee \hat{v}^p \supset \kappa^p$, else $v^p \vee \hat{v}^p$ would be the last node of a primary $\lambda^p(\eta^-)$-link or $\lambda^p(\eta)$-link which restrains $\kappa^p$, contrary to (2.10) or Lemma 4.5 (Free Derivative). It cannot be the case that $v^p \vee \hat{v}^p \subset \kappa^p$, else $\delta^p \vee \hat{\delta}^p \subseteq \kappa^p$. Hence $v^p \vee \hat{v}^p = \kappa^p$, and (ii) holds. ∎

Suppose $R$ is a requirement of dimension $r$ and type 1 and that the space $X$ is assigned to $R$. Control of sections of $X$ along a path $\Lambda^{r-1}$ is divided among derivatives of many different nodes of $T^n$. The following lemma, together with the requirement that the construction of Section 7 respect implication chains, will ensure that all but finitely many of these sections are controlled by nodes which are activated along $\Lambda^{r-1}$, or all but finitely many of these sections are controlled by nodes which are validated along $\Lambda^{r-1}$. The lemma will be used to analyze the situations which can occur when control of a space is relinquished by $\hat{\sigma}^{r-1}$ to $\sigma^{r-1}$. Condition (i) says that both $\hat{\sigma}^{r-1}$ and $\sigma^{r-1}$ want to declare axioms with the same value, so the axioms declared by derivatives of $\sigma^{r-1}$ are safe for $\hat{\sigma}^{r-1}$. Conditions (ii) and (iv) will be used to show that enough of the axioms declared by derivatives of $\sigma^{r-1}$ are corrected when control is interchanged. And condition (iii) will allow us to show that the set of conflicting axioms is sufficiently thin, and so will not interfere with the existence of the desired limit. The hypotheses placed on the lemma are chosen to capture exactly the cases for which the lemma is used.

LEMMA 6.11 (1-Similarity Lemma). *Fix an admissible $\Lambda^0 \in [T^0]$ and for all $t \leqslant n$, let $\Lambda^t = \lambda^t(\Lambda^0)$. Fix $r \leqslant n$ and $\sigma^{r-1} \subset \hat{\sigma}^{r-1} \subset \tau^{r-1} \subset \Lambda^{r-1}$, such that $\sigma^{r-1} \equiv \hat{\sigma}^{r-1}$, $\mathrm{up}(\sigma^{r-1}) \neq \mathrm{up}(\hat{\sigma}^{r-1})$, $(\tau^{r-1})^- = \hat{\sigma}^{r-1}$, $\mathrm{tp}(\sigma^{r-1}) = 1$, $\dim(\sigma^{r-1}) = r$, and $\sigma^{r-1}$ and $\hat{\sigma}^{r-1}$ control (different) sections of a space $X$ at $\tau^{r-1}$. Fix $\bar{\tau}^{r-1} \subset \tau^{r-1}$ such that $(\bar{\tau}^{r-1})^- = \sigma^{r-1}$ and assume that if $\sigma^{r-1}$ has infinite outcome along $\bar{\tau}^{r-1}$ then there is no derivative of $\mathrm{up}(\hat{\sigma}^{r-1})$ along $\sigma^{r-1}$. Then one of the following conditions holds*:

(i) *$\sigma^{r-1}$ has finite outcome along $\tau^{r-1}$ iff $\hat{\sigma}^{r-1}$ has finite outcome along $\tau^{r-1}$.*

(ii) *$\sigma^{r-1}$ has infinite outcome along $\tau^{r-1}$, $\hat{\sigma}^{r-1}$ has finite outcome along $\tau^{r-1}$, and there is a $\sigma^{r-1}$-injurious primary $\tau^{r-1}$-link $[\mu^{r-1}, \pi^{r-1}]$ such that $\pi^{r-1} \in \mathrm{PL}(\sigma^{r-1}, \tau^{r-1})$.*

(iii) *$\sigma^{r-1}$ has finite outcome along $\tau^{r-1}$, $\hat{\sigma}^{r-1}$ has infinite outcome along $\tau^{r-1}$, and there is a primary $\tau^{r-1}$-link which restrains $\sigma^{r-1}$.*

(iv) *$\sigma^{r-1}$ has finite outcome along $\tau^{r-1}$, $\hat{\sigma}^{r-1}$ has infinite outcome along $\tau^{r-1}$, $\mathrm{up}(\sigma^{r-1}) \subset \mathrm{up}(\hat{\sigma}^{r-1})$, there is no primary $\tau^{r-1}$-link which restrains $\sigma^{r-1}$, but there is a $\pi^r \in \mathrm{PL}(\mathrm{up}(\sigma^{r-1}), \lambda(\tau^{r-1}))$ such that* $\mathrm{OS}(\sigma^{r-1}) \subseteq \mathrm{TS}(\pi^r)$.

*Proof.* Suppose that (i)–(iv) fail, in order to obtain a contradiction. By choice of $r$, as $\sigma^{r-1}$ and $\hat{\sigma}^{r-1}$ control spaces at $\tau^{r-1}$, and by Subcase 1.1 of Definition 6.3, for all $i \leqslant r-1$, the principal derivatives of $\sigma^{r-1}$ along $\mathrm{out}^i(\bar{\tau}^{r-1})$ and $\hat{\sigma}^{r-1}$ along $\mathrm{out}^i(\tau^{r-1})$ must be implication-free.

First suppose that $\hat{\sigma}^{r-1}$ has finite outcome along $\tau^{r-1}$. We can assume, without loss of generality, that $\sigma^{r-1}$ is the shortest string satisfying the hypotheses, but not the conclusion of the lemma for $\hat{\sigma}^{r-1}$. By the failure of (i), $\sigma^{r-1}$ has infinite outcome along $\tau^{r-1}$. As $\sigma^{r-1}$ controls a space at $\tau^{r-1}$, it follows from Definitions 6.3 and 6.4 that $\sigma^{r-1}$ controls a space at $\bar{\tau}^{r-1}$, and so that $\mathrm{out}^0(\bar{\tau}^{r-1})$ is pseudotrue. Thus $\bar{\tau}^{r-1}$ must be implication-free, and cannot require extension.

Let $\tilde{\sigma}^{r-1}$ be the initial derivative of $\mathrm{up}(\hat{\sigma}^{r-1})$ along $\tau^{r-1}$, and let $\tilde{\tau}^{r-1}$ be the immediate successor of $\tilde{\sigma}^{r-1}$ along $\tau^{r-1}$. We show that $\tilde{\sigma}^{r-1}$ controls a section of $X$ at $\tilde{\tau}^{r-1}$. If $\tilde{\sigma}^{r-1}=\hat{\sigma}^{r-1}$, then this follows by hypothesis. Otherwise, it follows from (2.8) that $\tilde{\sigma}^{r-1}$ has finite outcome along $\tau^{r-1}$. Now by Lemma 4.5 (Free Extension), $\mathrm{up}(\tilde{\sigma}^{r-1})=\mathrm{up}(\hat{\sigma}^{r-1})\subset\lambda(\tau^{r-1})$ and $\mathrm{up}(\hat{\sigma}^{r-1})$ is $\lambda(\tau^{r-1})$-free. Furthermore, $\mathrm{up}(\hat{\sigma}^{r-1})$ must be implication-free, else by (5.23), $\hat{\sigma}^{r-1}$ would not be implication-free and would not control a section of $X$ at $\tau^{r-1}$. Hence by Lemma 5.16(iv) (Implication-Freeness), $\mathrm{out}^0(\tilde{\tau}^{r-1})$ is pseudotrue. Now by Lemma 4.5 (Free Extension), $\mathrm{up}^n(\tilde{\sigma}^{r-1})\subset\lambda^n(\tilde{\tau}^{r-1})$ must be $\lambda^n(\tilde{\tau}^{r-1})$-free, and by (2.9) $\tilde{\sigma}^{r-1}$ is both the initial and principal derivative of $\mathrm{up}^n(\tilde{\sigma}^{r-1})$ along $\tilde{\tau}^{r-1}$. By Lemma 5.17(iii) (Assignment), $\tilde{\sigma}^{r-1}$ is $\tilde{\tau}^{r-1}$-free and implication-free. Now iterating Lemma 4.6(i) (Free Derivative) and Lemma 5.16(ii) (Implication-Freeness), we see that for all $i\leqslant r-1$, the principal derivative of $\tilde{\sigma}^{r-1}$ along $\tilde{\tau}^{r-1}$ is implication-free. It follows from Definitions 6.3 and 6.4 that $\tilde{\sigma}^{r-1}$ controls a section of $X$ at $\tilde{\tau}^{r-1}$. Hence without loss of generality, we may assume that $\hat{\sigma}^{r-1}=\tilde{\sigma}^{r-1}$.

As $\mathrm{up}(\hat{\sigma}^{r-1})$ has no derivative along $\sigma^{r-1}$ and (ii) fails, (5.16) holds; hence as $\sigma^{r-1}$ controls a space at $\tau^{r-1}$, it follows from Definition 5.2 and Subcase 1.1 of Definition 6.3 that for some $\bar{\sigma}^{r-1}\subseteq\sigma^{r-1}$, $\langle\langle\bar{\sigma}^{r-1},\hat{\sigma}^{r-1},\tau^{r-1}\rangle\rangle$ is an amenable $(r-1)$-implication chain along $\Lambda^{r-1}$. But this contradicts Lemma 5.15(i) (Admissibility).

Now suppose that $\hat{\sigma}^{r-1}$ has infinite outcome along $\tau^{r-1}$. As (i) fails, $\sigma^{r-1}$ has finite outcome along $\tau^{r-1}$. As (iii) fails, it follows from Lemma 4.3(i)(a) (Link Analysis) that $\mathrm{up}(\sigma^{r-1})\subseteq\lambda(\tau^{r-1})$. As $\hat{\sigma}^{r-1}=(\tau^{r-1})^-$, it follows from Lemma 4.5 (Free Extension) that $\mathrm{up}(\hat{\sigma}^{r-1})\subseteq\lambda(\tau^{r-1})$. Hence $\mathrm{up}(\hat{\sigma}^{r-1})$ and $\mathrm{up}(\sigma^{r-1})$ are comparable. Now $\hat{\sigma}^{r-1}$ has infinite outcome along $\tau^{r-1}$, so $\hat{\sigma}^{r-1}$ is the principal derivative of $\mathrm{up}(\hat{\sigma}^{r-1})$ along $\tau^{r-1}$. It cannot be the case that $\mathrm{up}(\hat{\sigma}^{r-1})\subset\mathrm{up}(\sigma^{r-1})$, else by Lemma 3.1 (Limit Path) there would be no derivative of $\mathrm{up}(\sigma^{r-1})$ which is $\subset\hat{\sigma}^{r-1}$, contrary to the hypothesis that $\sigma^{r-1}\subset\hat{\sigma}^{r-1}$. Thus by the above, $\mathrm{up}(\sigma^{r-1})\subset\mathrm{up}(\hat{\sigma}^{r-1})$.

We now show that $\tau^{r-1}$ requires extension for $\sigma^{r-1}$. (5.1), (5.2), and (5.5)(i) follow easily from hypothesis and the observations already made. The failure of (iii) implies (5.3). We noted, following Definition 6.2, that every $\sigma^{r-1}$-injurious primary $\lambda(\tau^{r-1})$-link $[\mu^r,\pi^r]$ is $\sigma^{r-1}$-correcting. Suppose that $\gamma^r=\mathrm{PL}(\mathrm{up}(\sigma^{r-1}),\lambda(\tau^{r-1}))$ and $\mathrm{TS}(\gamma^r)\cap\mathrm{RS}(\sigma^{r-1})\neq\emptyset$, in order to obtain a contradiction. First suppose that (5.13) causes $\gamma^r$ to enter $\mathrm{PL}(\mathrm{up}(\sigma^{r-1}),\lambda(\tau^{r-1}))$. Then there is a $\mu^r$ such that $[\mu^r,\gamma^r]$ is a primary $\lambda(\tau^{r-1})$-link restraining $\mathrm{up}(\sigma^{r-1})$. Let $\xi^r$ be the immediate successor of $\gamma^r$ along $\lambda(\tau^{r-1})$. Then $\gamma^r\in\overline{\mathrm{PL}}(\xi^r)$, so $[\mu^r,\gamma^r]$ is $\mathrm{up}(\sigma^{r-1})$-injurious and restrains $\mathrm{up}(\sigma^{r-1})$. But then $[\mu^r,\gamma^r]$ is $\mathrm{up}(\sigma^{r-1})$-correcting, contrary to our assumption that (iv) fails.

Now suppose that (5.14) causes $\gamma^r$ to enter $\mathrm{PL}(\mathrm{up}(\sigma^{r-1}), \lambda(\tau^{r-1}))$, but (5.13) does not. Then there are $\mu^r \subset \mathrm{up}(\sigma^{r-1}) \subset \delta^r = (\sigma^r)^- \subset \sigma^r \subseteq \xi^r$ such that $\sigma^r$ requires extension but has no primary completion with infinite outcome along $\xi^r$, and as (5.13) did not apply, $\gamma^r \in \mathrm{PL}(\delta^r, \xi^r) \cup \{\delta^r\}$. As $\mathrm{out}^0(\tau^{r-1})$ is pseudotrue, it follows from Lemma 5.5(ii) (Completion-Respecting) that $\sigma^r$ has a primary completion $\kappa^r$ along $\lambda(\tau^{r-1})$ which has infinite outcome along $\lambda(\tau^{r-1})$. Fix $\alpha^r \subseteq \lambda(\tau^{r-1})$ such that $(\alpha^r)^- = \kappa^r$. By Definition 5.3 and Lemma 5.1(i), (PL Analysis), $\gamma^r \in \mathrm{PL}(\delta^r, \kappa^r) \cup \{\delta^r\} \subseteq \overline{\mathrm{PL}}(\xi^r)$. Thus $[\mu^r, \kappa^r]$ is $\mathrm{up}(\sigma^{r-1})$-injurious and restrains $\mathrm{up}(\sigma^{r-1})$. But then $[\mu^r, \kappa^r]$ is $\mathrm{up}(\sigma^{r-1})$-correcting, contrary to our assumption that (iv) fails.

We conclude that (5.4) holds, and so that $\tau^{r-1}$ requires extension for some $\bar{\sigma}^{r-1} \subseteq \sigma^{r-1}$. By Definition 5.6, $\tau^{r-1}$ is not the completion of $\tau^{r-1}$ for $\bar{\sigma}^{r-1}$. Hence by (5.21), $\tau^{r-1}$ is implication-restrained, and so $\mathrm{out}^0(\tau^{r-1})$ is not pseudotrue. But then by Subcase 1.1 of Definition 6.3, $\hat{\sigma}^{r-1}$ is not a controller for a section of $X$ at $\tau^{r-1}$, contrary to hypothesis. ▮

Because of the finiteness of the number of initiators for a given space $X$, we can settle on an initiator which will control a given space along a path. However, it is possible to have comparable initiators along a given path, each determining control of sections of the same space at infinitely many nodes along the approximation to the path. The switching of control is determined by the terminators. The next lemma will allow us to show that all but finitely many axioms declared for a space controlled by a node of type 1 along $\Lambda^1$ will have the correct value.

LEMMA 6.12 (1-Correction Lemma). *Fix an admissible $\Lambda^0 \in T^0$ and let $\Lambda^1 = \lambda(\Lambda^0)$. Suppose that $v^1 \subset \Lambda^1$ controls the space $S$ along $\Lambda^1$ with initiator $\delta^1$, and that $\mathrm{tp}(v^1) = 1$. Assume that $\eta \subset \kappa \subset \Lambda^0$, $\mathrm{wt}(\eta^-) \geqslant \mathrm{wt}(S)$, $\delta^1$ is the initiator for $S$ at $\lambda(\eta^-)$ and at $\lambda(\kappa)$, but not at any $\lambda(\gamma)$ such that $\eta \subseteq \gamma \subset \kappa$. Then there is a $\mu^1$ such that for all $\gamma \in [\eta, \kappa)$, $[\mu^1, (\lambda(\eta))^-]$ is a $v^1$-correcting primary $\lambda(\gamma)$-link with $\mu^1 \subset \delta^1 \subseteq (\lambda(\eta))^-$, and $\kappa$ switches $(\lambda(\eta))^-$. Furthermore, if $\xi$ is the shortest pseudotrue node such that $\kappa \subseteq \xi \subset \Lambda^0$, then for every node $\beta^1 \in \overline{\mathrm{PL}}(\lambda(\eta))$, there is a $\beta$ such that $\kappa \subseteq \beta \subseteq \xi$ and $\beta$ switches $\beta^1$.*

*Proof.* By hypothesis, for all $\gamma$ such that $\eta \subseteq \gamma \subset \kappa$, $\lambda((\eta)^-) \neq \lambda(\gamma)$. As $\mathrm{wt}(\eta^-) \geqslant \mathrm{wt}(S)$, it follows from (2.11) that $\mathrm{wt}(\lambda(\gamma)) > \mathrm{wt}(S)$ for all $\gamma$ such that $\eta \subseteq \gamma \subset \kappa$. Thus Case 3 of Definition 6.3 must be followed at $\lambda(\eta)$ to define $(\lambda(\eta))^-$ as a terminator for $\delta^1$, so there is a $v^1$-correcting primary $\lambda(\eta)$-link $[\mu^1, (\lambda(\eta))^-]$ with $\mu^1 \subset \delta^1 \subseteq (\lambda(\eta))^-$, and $(\lambda(\eta))^-$ has infinite outcome along $\lambda(\eta)$.

As $\delta^1 \subset \Lambda^1$, it follows from (2.6) that no $\gamma$ such that $\eta \subseteq \gamma \subset \kappa$ can switch any $\rho^1 \subset \delta^1$. By (2.10), no such $\gamma$ can switch any $\rho^1$ such that $\mu^1 \subseteq \rho^1 \subset (\lambda(\eta))^-$. Hence by (2.10), $\lambda(\kappa)$ and $(\lambda(\eta))^-$ must be comparable.

Also, no $\gamma$ such that $\eta \subseteq \gamma \subset \kappa$ can switch $(\lambda(\eta))^-$, else by Lemma 3.3 ($\lambda$-Behavior), $(\lambda(\gamma))^- = (\lambda(\eta))^-$ and $(\lambda(\gamma))^-$ would have finite outcome along $\lambda(\gamma)$, so $\delta^1$ would be the initiator for $S$ at $\lambda(\gamma)$. Hence for all $\gamma$ such that $\eta \subseteq \gamma \subset \kappa$, $[\mu^1, (\lambda(\eta))^-]$ is a $\nu^1$-correcting primary $\lambda(\gamma)$-link which restrains $\delta^1$.

Now as $\kappa$ cannot switch any $\rho^1 \subset (\lambda(\eta))^-$, as $\lambda(\kappa)$ and $(\lambda(\eta))^-$ are comparable, and as $\delta^1$ is the initiator for $S$ at $\lambda(\kappa)$, $[\mu^1, (\lambda(\eta))^-]$ cannot be a primary $\lambda(\kappa)$-link, so $\kappa$ must switch $(\lambda(\eta))^-$. If $(\lambda(\eta))^-$ is not a primary completion, then $\overline{\mathrm{PL}}(\lambda(\eta)) = \{(\lambda(\eta))^-\}$. Otherwise, let $(\lambda(\eta))^-$ be the primary completion of the immediate successor $\gamma^1$ of a node $\sigma^1$ along $\lambda(\eta)$. Then $\overline{\mathrm{PL}}(\lambda(\eta)) = \mathrm{PL}(\sigma^1, \lambda(\eta)) \cup \{\sigma^1\}$. By Lemma 5.3(ii) (Implication Chain) and Lemma 5.2 (Requires Extension), $\kappa$ must require extension for a derivative of $\sigma^1$, and so as $\xi$ is pseudotrue, it follows from Lemma 5.5(ii) (Completion-Respecting) that $\kappa$ must have a primary completion $\bar{\kappa} \subset \xi$ which has infinite outcome along $\xi$. By (5.19), $\mathrm{up}(\bar{\kappa}) = \sigma^1$. Hence the immediate successor of $\bar{\kappa}$ along $\xi$ switches $\sigma^1$. By Lemma 5.1(ii) (PL Analysis), $\mathrm{PL}(\sigma^1, \lambda(\eta)) \subseteq \mathrm{PL}(\sigma^1, (\lambda(\eta))^-) \cup \{(\lambda(\eta))^-\}$. As $\kappa$ switches $(\lambda(\eta))^-$, it follows from (2.4) that $\lambda(\kappa) = (\lambda(\eta))^- {}^\wedge \langle \kappa \rangle$, and that $(\lambda(\eta))^-$ has finite outcome along $\lambda(\kappa)$. Hence by Lemma 5.1(iv) (PL Analysis), $\mathrm{PL}(\sigma^1, \lambda(\eta)) = \mathrm{PL}(\sigma^1, (\lambda(\eta))^-)$. It now follows from Lemma 5.12(i) (PL) and as $\kappa$ switches $(\lambda(\eta))^-$ that every node in $\mathrm{PL}(\sigma^1, \lambda(\eta))$ must be switched by some node in $[\kappa, \bar{\kappa}]$. ∎

Suppose that $X$ is a space assigned to a requirement of dimension $r$ and type 2. When $k = r-1$, control of sections of $X$ along a path $\Lambda^{r-1}$ is divided among derivatives of many different nodes of $T^n$. The following lemma will allow us to use implication chains to ensure that all but finitely many of these sections are controlled by nodes which are activated along $\Lambda^{r-1}$, or all but finitely many of these sections are controlled by nodes which are validated along $\Lambda^{r-1}$.

LEMMA 6.13 (2-Similarity Lemma). *Fix an admissible $\eta \in T^0$ and $\sigma^{r-1} \subset \bar{\tau}^{r-1} \subseteq \hat{\sigma}^{r-1} \subset \tau^{r-1} \subseteq \lambda^{r-1}(\eta)$ such that $\sigma^{r-1}$ and $\hat{\sigma}^{r-1}$ are nodes to which the requirement $R$ of dimension $r$ and type 2 has been assigned. Assume that $\sigma^{r-1} \equiv \hat{\sigma}^{r-1}$, $\mathrm{up}(\sigma^{r-1}) \neq \mathrm{up}(\hat{\sigma}^{r-1})$, $(\tau^{r-1})^- = \hat{\sigma}^{r-1}$, $(\bar{\tau}^{r-1})^- = \sigma^{r-1}$, and $\sigma^{r-1}$ and $\hat{\sigma}^{r-1}$ are controllers at $\bar{\tau}^{r-1}$ and $\tau^{r-1}$, respectively. Then one of the following conditions holds:*

(i) *$\sigma^{r-1}$ has finite outcome along $\tau^{r-1}$ iff $\hat{\sigma}^{r-1}$ has finite outcome along $\tau^{r-1}$.*

(ii) *$\sigma^{r-1}$ has finite outcome along $\tau^{r-1}$, $\hat{\sigma}^{r-1}$ has infinite outcome along $\tau^{r-1}$, and there is a primary $\tau^{r-1}$-link which restrains $\sigma^{r-1}$.*

*Proof.* We assume that (i) and (ii) fail, and obtain a contradiction. We will be showing, under additional assumptions, either that $\langle\langle\sigma^{r-1}, \hat{\sigma}^{r-1}, \tau^{r-1}\rangle\rangle$ is an amenable implication chain, or that $\tau^{r-1}$ requires extension for $\sigma^{r-1}$. We begin by showing that certain clauses from (5.1)–(5.12), (5.15) and (5.16) hold without any additional assumptions. (5.5)–(5.9) and (5.12) follow from hypothesis.

As $\sigma^{r-1}$ and $\hat{\sigma}^{r-1}$ are controllers at $\bar{\tau}^{r-1}$ and $\tau^{r-1}$, respectively, it follows from Subcase 1.1 of Definition 6.3 that for all $i \leqslant r-1$, the principal derivatives of $\sigma^{r-1}$ along $\mathrm{out}^i(\bar{\tau}^{r-1})$ and $\hat{\sigma}^{r-1}$ along $\mathrm{out}^i(\tau^{r-1})$, are implication-free, and that $\mathrm{out}^0(\bar{\tau}^{r-1})$ and $\mathrm{out}^0(\tau^{r-1})$ are pseudotrue. Hence (5.1) and (5.10) hold.

We next show that we may assume, without loss of generality, that $\sigma^{r-1}$ ($\hat{\sigma}^{r-1}$, resp.) is the principal derivative of $\mathrm{up}(\sigma^{r-1})$ ($\mathrm{up}(\hat{\sigma}^{r-1})$, resp.) along $\bar{\tau}^{r-1}$ ($\tau^{r-1}$, resp.). This is clearly the case if $\sigma^{r-1}$ ($\hat{\sigma}^{r-1}$, resp.) has infinite outcome along $\bar{\tau}^{r-1}$ ($\tau^{r-1}$, resp.). Suppose that $\sigma^{r-1}$ ($\hat{\sigma}^{r-1}$, resp.) has finite outcome along $\bar{\tau}^{r-1}$ ($\tau^{r-1}$, resp.), and let $\tilde{\sigma}^{r-1}$ be the initial derivative of $\mathrm{up}(\sigma^{r-1})$ ($\mathrm{up}(\hat{\sigma}^{r-1})$, resp.) along $\bar{\tau}^{r-1}$ ($\tau^{r-1}$, resp.). By Lemma 5.15(iv) (Implication-Freeness), one of the conclusions of the lemma must hold for $\tilde{\sigma}^{r-1}$ in place of $\sigma^{r-1}$ ($\hat{\sigma}^{r-1}$, resp.). If (i) holds for $\tilde{\sigma}^{r-1}$ then (i) also holds for $\sigma^{r-1}$ ($\hat{\sigma}^{r-1}$, resp.). Suppose that (ii) holds for $\tilde{\sigma}^{r-1}$ and let $[\mu^{r-1}, \pi^{r-1}]$ be the associated primary $\tau^{r-1}$-link. If $\mathrm{up}(\tilde{\sigma}^{r-1}) \not\subseteq \lambda(\tau^{r-1})$, then by Lemma 4.3(i)(a) (Link Analysis) $[\mu^{r-1}, \pi^{r-1}]$ restrains $\sigma^{r-1}$ ($\hat{\sigma}^{r-1}$, resp.). Otherwise, by Lemma 4.3(i)(d) (Link Analysis), $\mu^{r-1} = \tilde{\sigma}^{r-1}$, so by (2.8) $[\mu^{r-1}, \pi^{r-1}]$ restrains $\sigma^{r-1}$ ($\hat{\sigma}^{r-1}$, resp.).

We next note that $\mathrm{tp}(\sigma^{r-1}) = \mathrm{tp}(\hat{\sigma}^{r-1}) = 2$, so (5.4) holds, and if $\hat{\sigma}^{r-1}$ is a pseudocompletion of $\sigma^{r-1}$, then $\hat{\sigma}^{r-1}$ is an amenable pseudocompletion of $\sigma^{r-1}$, so (5.16) will follow once the appropriate clauses of (5.6)–(5.12) are verified.

We now proceed by cases.

*Case* 1. $\hat{\sigma}^{r-1}$ has finite outcome along $\tau^{r-1}$. Then by the failure of (i), $\sigma^{r-1}$ has infinite outcome along $\tau^{r-1}$. We will obtain a contradiction in this case, so may assume without loss of generality that $\sigma^{r-1}$ has shortest possible length satisfying the properties of the lemma. Condition (5.11) follows from the case assumption, so $\langle\langle\sigma^{r-1}, \bar{\sigma}^{r-1}, \bar{\tau}^{r-1}\rangle\rangle$ is an implication chain. Now we have assumed that $\hat{\sigma}^{r-1}$ is the principal derivative of $\mathrm{up}(\hat{\sigma}^{r-1})$ along $\tau^{r-1}$, so as $\hat{\sigma}^{r-1}$ has finite outcome along $\tau^{r-1}$, $\hat{\sigma}^{r-1}$ is an initial derivative. Hence $\mathrm{up}(\hat{\sigma}^{r-1})$ has no derivative $\subset \sigma^{r-1}$. We have already noted that (5.16) holds, so $\langle\langle\sigma^{r-1}, \hat{\sigma}^{r-1}, \tau^{r-1}\rangle\rangle$ is an amenable $(r-1)$-implication chain. But then by Lemma 5.2 (Requires Extension), $\mathrm{out}(\tau^{r-1})$ requires extension, so $\mathrm{out}^0(\tau^{r-1})$ is implication-restrained, hence cannot be pseudotrue, yielding a contradiction.

*Case* 2. $\hat{\sigma}^{r-1}$ has infinite outcome along $\tau^{r-1}$. We first show that $\tau^{r-1}$ requires extension for some $\bar{\sigma}^{r-1} \subseteq \sigma^{r-1}$, by showing that (5.1)–(5.5) hold for $\sigma^{r-1}$ in place of $v^k$, $\hat{\sigma}^{r-1}$ in place of $\delta^k$, and $\tau^{r-1}$ in place of $\eta^k$. Conditions (5.2) and (5.5)(i) follow easily from hypothesis. Condition (5.1) follows from Case 1 of Definition 6.3 and the comments at the beginning of the proof. Condition (5.3) follows from the failure of (ii). And we have already noted that (5.4) holds. Thus $\tau^{r-1}$ requires extension for some $\bar{\sigma}^{r-1} \subseteq \sigma^{r-1}$. But then, by Definition 5.6, $\tau^{r-1}$ is implication-restrained, so $\mathrm{out}^0(\tau^{r-1})$ cannot be pseudotrue. Hence by Subcase 1.1 of Definition 6.3, $\hat{\sigma}^{r-1}$ cannot be a controller at $\tau^{r-1}$, contradicting our assumption. ∎

The next lemma will be used to show that whenever necessary, axioms for type 2 requirements which need to be corrected when control is changed, will be corrected.

LEMMA 6.14 (2-Correction Lemma). *Fix an admissible* $\eta \in T^0$. *Suppose that S is weakly controlled by* $v^1$ *at* $\lambda(\eta^-)$ *with initiator* $\delta^1$, *S is weakly controlled by* $\hat{v}^1$ *at* $\lambda(\eta)$ *with initiator* $\hat{\delta}^1$, $\delta^1 \neq \hat{\delta}^1$, *and* $\mathrm{tp}(v^1) = 2$. *Let* $\kappa^1 = \mathrm{up}(\eta^-)$. *Then one of the following holds*:

(i) $v^1$ *is activated along* $\delta^1$ *iff* $\hat{v}^1$ *is activated along* $\hat{\delta}^1$.

(ii) $\eta$ *switches* $\kappa^1 \subset \delta^1$ *and* $\dim(\kappa^1) \geqslant \dim(v^1)$.

*Proof.* Let $r = \dim(v^1)$. Fix notation as in Lemma 6.8 (Alternating Initiator). If $v^{r-1} = \hat{v}^{r-1}$, then the proof follows as in the third paragraph of the proof of Lemma 6.10 (0-Correction). Suppose that $v^{r-1} \neq \hat{v}^{r-1}$. We assume that (i) and (ii) fail, and derive a contradiction. As (i) and (ii) fail, it follows from Lemma 6.9 (Outcome) that $v^{r-1}$ is activated along $\lambda^{r-1}(\eta^-)$ iff $\hat{v}^{r-1}$ is validated along $\lambda^{r-1}(\eta)$.

We assume that $r$ is even. A similar proof holds when $r$ is odd. By (6.10) and Definition 6.3, $v^{r-1} \wedge \hat{v}^{r-1} \subset v^{r-1} \vee \hat{v}^{r-1} \subset \lambda^{r-1}(\eta)$. Fix $\bar{\tau}^{r-1}$, $\tau^{r-1} \subseteq \lambda^{r-1}(\eta)$ such that $(\bar{\tau}^{r-1})^- = v^{r-1} \wedge \hat{v}^{r-1}$, and $(\tau^{r-1})^- = v^{r-1} \vee \hat{v}^{r-1}$. It follows by an easy induction that $\bar{\tau}^{r-1}$ and $\tau^{r-1}$ are initiators for $\mathrm{up}^{r-1}(S)$, else either $v^{r-1} \wedge \hat{v}^{r-1}$ or $v^{r-1} \vee \hat{v}^{r-1}$ would not be a controller for $\mathrm{up}^{r-1}(S)$. There are two cases.

*Case* 1. $v^r \neq \hat{v}^r$. By the preceding paragraph, we can apply Lemma 6.13 (2-Similarity), to conclude that there is a primary $\lambda^{r-1}(\eta)$-link $[\mu^{r-1}, \pi^{r-1}]$ restraining $v^{r-1} \wedge \hat{v}^{r-1}$. By (2.10) and Lemma 4.5 (Free Derivative), $\kappa^{r-1}$ is both $\lambda^{r-1}(\eta^-)$-free and $\lambda^{r-1}(\eta)$-free; and by (6.10), $v^{r-1} \wedge \hat{v}^{r-1} \subset \kappa^{r-1}$. Hence $\pi^{r-1} \subseteq \kappa^{r-1}$.

By (6.10), $\delta^{r-1} \vee \hat{\delta}^{r-1} \supset \kappa^{r-1}$, so by (2.1), $\mathrm{wt}(\kappa^{r-1}) < \mathrm{wt}(\delta^{r-1} \vee \hat{\delta}^{r-1})$. Now $\pi^{r-1} \not\subset \kappa^{r-1}$, else $\pi^{r-1}$ would be a terminator for $\bar{\tau}^{r-1}$ along both

$\lambda^{r-1}(\eta^-)$ and $\lambda^{r-1}(\eta)$, so by (6.19) $v^{r-1} \wedge \hat{v}^{r-1}$ could not be a controller at either of these nodes. Thus $\pi^{r-1} = \kappa^{r-1}$, so by Lemma 3.3 ($\lambda$-Behavior), $\eta$ switches $\pi^{r-1}$. But then $\pi^{r-1}$ is not an initial derivative, so by (2.9), $\dim(\pi^{r-1}) > r-1$; so (ii) must hold, yielding a contradiction.

*Case* 2. $v^r = \hat{v}^r$. By (6.10), $v^{r-1} \wedge \hat{v}^{r-1} \subset \delta^{r-1} \wedge \hat{\delta}^{r-1} \subseteq \kappa^{r-1} \subset \delta^{r-1} \vee \hat{\delta}^{r-1}$. By the case assumption and as (i) fails, $[v^{r-1} \wedge \hat{v}^{r-1}, v^{r-1} \vee \hat{v}^{r-1}]$ must form a primary $(\delta^{r-1} \wedge \hat{\delta}^{r-1})$-link, so by (2.10) or Lemma 4.5 (Free Extension), $\kappa^{r-1} \supseteq v^{r-1} \vee \hat{v}^{r-1}$. We now set $\pi^{r-1} = v^{r-1} \vee \hat{v}^{r-1}$, and proceed as in the last paragraph of Case 1. ∎

Our final lemma shows that nodes coming from the true path of the construction control spaces.

LEMMA 6.15 (Initial Control Lemma). *Fix an admissible $\Lambda^0 \in [T^0]$ and for all $k \leqslant n$, let $\Lambda^k = \lambda^k(\Lambda^0)$. Fix $\zeta^n \subset \Lambda^n$ and $r \leqslant n$ such that* $\dim(\zeta^n) = r$ *and* $\mathrm{tp}(\zeta^n) \in \{1, 2\}$, *let $\zeta^{r-1}$ be the principal derivative of $\zeta^n$ along $\Lambda^{r-1}$, and let $\zeta^r = \mathrm{up}(\zeta^{r-1})$. Let $S$ be the space, $S_{\zeta^r}$, assigned to* $\mathrm{up}(\zeta^{r-1})$, *and fix $\delta^{r-1} \subset \Lambda^{r-1}$ such that $(\delta^{r-1})^- = \zeta^{r-1}$. Then*:

(i) *$\zeta^{r-1}$ controls $S^{[\mathrm{wt}(\delta^{r-1})]}$ along $\Lambda^{r-1}$ with initiator $\delta^{r-1}$.*

(ii) *If $\zeta^r$ has infinite outcome along $\Lambda^r$, then infinitely many derivatives of $\zeta^r$ control spaces along $\Lambda^{r-1}$.*

*Proof.* By Lemma 5.17(ii, iii) (Assignment), $\zeta^r$ and $\zeta^{r-1}$ are implication-free, $\zeta^r$ is $\Lambda^r$-free, and $\zeta^{r-1}$ is $\Lambda^{r-1}$-free.

By Lemma 4.6(ii) (Free Derivative) and Lemma 5.16(ii) (Implication-Freeness), we see that if $\zeta^r$ has infinite outcome along $\Lambda^r$, then $\zeta^r$ has infinitely many implication-free derivatives which are $\Lambda^{r-1}$-free. Fix a $\Lambda^{r-1}$-free and implication-free derivative $\bar{\zeta}^{r-1}$ of $\zeta^r$ along $\Lambda^{r-1}$, and fix $\xi^{r-1} \subset \Lambda^{r-1}$ such that $(\xi^{r-1})^- = \bar{\zeta}^{r-1}$. Note that, by definition, for all $i \leqslant r-1$, the principal derivative of $\bar{\zeta}^{r-1}$ along $\Lambda^i$ is $\bar{\zeta}^i = (\mathrm{out}^i(\xi^{r-1}))^-$. By repeated applications of Lemma 4.6(i) (Free Derivative) and Lemma 5.16(ii) (Implication-Freeness), we see that for all $i \leqslant r-1$, $\bar{\zeta}^i$ is $\Lambda^i$-free and implication-free. By Lemma 5.17(iv) (Assignment), $\xi = \mathrm{out}^0(\xi^{r-1})$ is pseudotrue.

By Lemma 6.1(iv) (Finite Control), $\xi^{r-1}$ is an initiator for $S^{[\mathrm{wt}(\xi^{r-1})]}$ at $\xi^{r-1}$, with corresponding controller $\bar{\zeta}^{r-1}$. As $\xi$ is pseudotrue, it follows from Definition 6.4 that $\bar{\zeta}^{r-1}$ controls $S^{[\mathrm{wt}(\xi^{r-1})]}$ at $\xi^{r-1}$ with initiator $\xi^{r-1}$. As $\bar{\zeta}^{r-1}$ is $\Lambda^{r-1}$-free and $(\xi^{r-1})^- = \bar{\zeta}^{r-1}$, $\xi^{r-1}$ cannot have a terminator along $\Lambda^{r-1}$, else $\bar{\zeta}^{r-1}$ would be restrained by a primary $\Lambda^{r-1}$-link. Hence by Definition 6.4, $\bar{\zeta}^{r-1}$ controls $S^{[\mathrm{wt}(\xi^{r-1})]}$ along $\Lambda^{r-1}$. ∎

## 7. Construction and Proof

Fix $k \leqslant n$ and $\langle b, c\rangle \in Z_{0,k}$. In order to show that $A_c^{(k-1)} \not\leqslant_T A_b^{(k-1)}$, we wish to define a partial recursive functional $\Delta_{b,c}^{0,k}$ which is total on domain $\mathbf{N}^k$ from oracle $A_c$ such that for each $e \in \mathbf{N}$. there is an $x$ such that $\lim_{\bar{u}} \Phi_e(A_b; \bar{u}, x) \neq \lim_{\bar{v}} \Delta_{b,c}^{0,k}(A_c; \bar{v}, x)$, and for all $y$, $\lim_{\bar{v}} \Delta_{b,c}^{0,k}(A_c; \bar{v}, y)$ exists. $\Delta_{b,c}^{0,k}(A_c; \bar{v}, x)$ will be the value defined by some $\zeta$ controlling $\langle \bar{v}, s, x, \zeta\rangle$ along $\Lambda^0$ for some $s$ whenever such a $\zeta$ exists, where $R_\zeta = R_{e,b,c}^{0,k}$ for some $e$. (We recall that there is an additional limit which enters into the computation, namely, the limit over *stages* at which we place elements into $A_c$ and declare axioms, which we must also take into account.) Thus all axioms declared for such $A_\zeta$ will be axioms for $\Delta_{b,c}^{0,k}$. We will take additional steps to ensure that $\Delta_{b,c}^{0,k}$ is total on oracle $A_c$ by defining this functional on arguments which are not in spaces being controlled, and will prove that $\Delta_{b,c}^{0,k}$ is a well-defined partial recursive functional and $\Delta_{b,c}^{0,k}(A_c)$ is total in Lemma 7.2 (Well-Definedness and Totality). Similarly, for $j \in \{1, 2\}$, the requirement $R_{e,b,c}^{j,k}$ requires us to define a functional $\Delta_{b,c}^{j,k}$ for each $\langle b, c\rangle \in Z_{j,k}$, uniformly in $e$. We define this functional to contain the union of all functionals $\Delta_\zeta$ such that $\zeta$ deals with a requirement for this fixed $\langle b, c\rangle \in Z_{j,k}$, and take additional steps to ensure that $\Delta_{b,c}^{j,k}$ is total on oracle $A_c$ by defining this functional on arguments which are not in spaces being controlled. We identify $\Delta_\eta$ with $\Delta_\xi$ whenever $\Delta_\eta$ and $\Delta_\xi$ are components of the same functional $\Delta_{b,c}^{j,k}$. (Thus if $\eta$ defines an axiom for $\Delta_\eta$, then that axiom is in existence for $\Delta_\xi$ as well.)

The decision about the action taken for a requirement associated with $\eta \in T^0$ is based on our ability to force $M_\eta$ to be true. $M_\eta$ will be equivalent to a $\Pi_1$-sentence with a single unbounded (universal) quantifier which will be part of a quantifier block $\exists s \leqslant \mathrm{wt}(\eta)\, \forall t \geqslant s$, which is equivalent to $\forall t \geqslant \mathrm{wt}(\eta)$. (This quantifier will range over stages.)

Definition 7.1. For $\eta \in T^0$, we say that $M_\eta$ is *potentially true* if the sentence $M_\eta^{[\mathrm{wt}(\eta)]}$, obtained from $M_\eta$ by dropping the quantifier block $\exists s \leqslant \mathrm{wt}(\eta)\, \forall t \geqslant s$ and replacing all occurrences of $s$ and $t$ with $\mathrm{wt}(\eta)$, is true.

*The Construction*

We define an admissible path $\Lambda^0 \in [T^0]$ by induction on $\mathrm{lh}(\eta)$ for $\eta \subset \Lambda^0$. We begin by specifying that $\langle\ \rangle \subset \Lambda^0$. Fix $\eta \subset \Lambda^0$. If $\mathrm{lh}(\eta) = 0$, then no axioms are declared and all sets $A^s$ are empty for $s \leqslant \mathrm{wt}(\eta)$. Assume that $\mathrm{lh}(\eta) \geqslant 0$. We assume, by induction, that $\eta$ is admissible and completion-consistent via $\langle\ \rangle$. In Step 1, we will determine an admissible node $\hat{\eta}$ such that $\eta \subset \hat{\eta} \subset \Lambda^0$. We begin, in Step 1.1, by determining an immediate successor $\beta$ of $\eta$. There will be three cases to the definition of $\beta$, designed to ensure that $\beta$ is preadmissible. If $\beta$ is completion-consistent via $\langle\ \rangle$, then we will set $\hat{\eta} = \beta$. Otherwise, $\lambda^k(\eta)$ will require extension for a unique $k$, and we will define $\hat{\eta}$ to be the 0-completion of $\beta$ in Step 1.2. We will determine which elements are placed into sets in Step 2, and this will depend on the path chosen in Step 1. New axioms for our functionals are declared in Step 3.

*Step* 1. (Path Definition). We note, by induction, that $\eta$ is admissible and completion-consistent via $\langle\ \rangle$.

*Step* 1.1. There are three cases.

*Case* 1. $\eta$ is a primary 0-completion or a pseudocompletion. Set $\beta = \eta\hat{\ }\langle\infty\rangle$.

*Case* 2. The previous case is not followed and $\eta$ is implication-restrained. Let $\beta$ be a nonswitching extension of $\eta$. (We take the activated extension if both possible extensions are nonswitching, in order to satisfy (5.17)(ii).)

*Case* 3. Otherwise. Set $\beta = \eta\hat{\ }\langle\infty\rangle \subseteq \Lambda^0$ if $M_\eta$ is potentially true, and $\beta = \eta\hat{\ }\langle 0\rangle \subseteq \Lambda^0$ otherwise.

It follows from (5.17) and (5.18) that $\beta$ is preadmissible, and from Lemma 5.8 (Completion-Respecting Admissible Extension) that $\beta$ is admissible. If $\beta$ is completion-consistent via $\langle\ \rangle$, then the induction hypothesis holds at $\beta$, and we set $\hat{\eta} = \beta$ and go to Step 2. Otherwise, by Lemma 5.8 (Completion-Respecting Admissible Extension) and Lemma 5.6 (Uniqueness of Requiring Extension), there is a unique $k$, which we fix, such that $\lambda^k(\beta)$ requires extension. We now go to Step 1.2.

*Step* 1.2. By Lemma 5.14 (Completion) we can effectively obtain the 0-completion $\hat{\eta}$ of $\lambda^k(\beta)$. By (5.19) and Lemma 5.14 (Completion), $\hat{\eta}$ is admissible and completion-consistent via $\langle\ \rangle$, so the induction condition holds. Now go to Step 2.

*Step* 2. (Set Definition). For each node $\pi$ such that $\eta \subseteq \pi \subset \hat{\eta}$, $\pi$ is validated along $\hat{\eta}$, and $\pi$ is not the initial derivative of $\mathrm{up}(\pi)$ along $\hat{\eta}$, place $\mathrm{wt}(\mathrm{up}(\pi))$ into $A^{\mathrm{wt}(\pi)+1}$ for all $A \in \mathrm{TS}(\pi)$. For each set $A$ and all $s$ such that $\mathrm{wt}(\eta) < s \leqslant \mathrm{wt}(\hat{\eta})$, we let $A^s = A^{\mathrm{wt}(\eta)} \cup \{x\colon x \text{ is placed in } A^{\mathrm{wt}(\pi)+1} \text{ for some } \pi \text{ such that } \eta \subseteq \pi \subset \hat{\eta} \text{ and } \mathrm{wt}(\pi) < s\}$.

*Step* 3. (Declaration of Axioms). We carry out this step only if $\hat{\eta}$ is pseudotrue. Let $\alpha = \hat{\eta}$. This step is carried out for each functional $\Delta = \Delta^{j,k}_{b,c}$ and each $\langle \bar{x}, s, x\rangle$ which is potentially in the domain of $\Delta$ such that $x < \mathrm{wt}(\lambda(\alpha))$, and $x_i < \mathrm{wt}(\lambda(\alpha))$ for all coordinates $x_i$ of $\bar{x}$. (Note that we identify functionals whose last coordinates are $\equiv$, so choose to ignore the last coordinate. If such an $\langle \bar{x}, t, x\rangle$ is not controlled at $\alpha$ for any $t$ and

$\mathrm{tp}(R) \in \{0, 2\}$, then we will show in Lemma 7.2 (Well-Definedness and Totality) that $\langle \bar{x}, t, x \rangle$ will not be controlled at any $\rho \subset \Lambda^0$ for any $t$; hence it is safe to declare an axiom $\Delta_{\mathrm{wt}(\alpha)}(A^{\mathrm{wt}(\alpha)}; \bar{x}, x) = 0$, and we do so in Case 3.3. And if $\mathrm{tp}(R) = 1$, then terminators will let us correct such axioms as required.) Let $A = A_c$ be the oracle for $\Delta$.

*Case* 1. $\Delta_{\mathrm{wt}(\gamma)}(A^{\mathrm{wt}(\alpha)}; \bar{x}, x)\downarrow = q$ for some $q$ and $\gamma \subset \alpha$. Set $\Delta_t(A^{\mathrm{wt}(\alpha)}; \bar{x}, x) = \Delta_{\mathrm{wt}(\gamma)}(A^{\mathrm{wt}(\alpha)}; \bar{x}, x)$ for all $t$ such that $\mathrm{wt}(\gamma) < t \leqslant \mathrm{wt}(\alpha)$. The use of all such axioms is the use of the axiom $\Delta_{\mathrm{wt}(\gamma)}(A^{\mathrm{wt}(\alpha)}; \bar{x}, x) = q$.

*Case* 2. Case 1 does not apply, and there is a $t < \mathrm{wt}(\alpha)$ such that $\langle \bar{x}, t, x \rangle$ is in the space controlled at $\alpha$. (Note that we identify functionals whose last coordinates are $\equiv$, so choose to ignore the last coordinate.) Fix the largest such $t$, and let $\langle \bar{x}, t, x \rangle$ be in the space controlled by $v$ at $\alpha$ with initiator $\delta$. We *declare* the axiom $\Delta_{\mathrm{wt}(\alpha)}(A^{\mathrm{wt}(\alpha)}; \bar{x}, x) = 1$ if $\delta \supseteq v\hat{\ }\langle \infty \rangle$ and $\Delta_{\mathrm{wt}(\alpha)}(A^{\mathrm{wt}(\alpha)}; \bar{x}, x) = 0$ if $\delta \supseteq v\hat{\ }\langle 0 \rangle$. The use of each axiom so defined is $\mathrm{wt}(\lambda(\alpha)) - 1$.

*Case* 3. Otherwise. *Declare* the axiom $\Delta_{\mathrm{wt}(\alpha)}(A^{\mathrm{wt}(\alpha)}; \bar{x}, x) = 0$ with use $\mathrm{wt}(\lambda(\alpha)) - 1$.

The construction is now complete. For all $r \leqslant n$, let $\Lambda^r = \lambda^r(\Lambda^0)$. We note that as the induction hypotheses are satisfied, $\Lambda^0$ is admissible. ∎

Our first lemma provides upper and lower bounds on the use of any axiom on a point controlled by some $\xi \in T^0$. The upper bound is used to prove that all functionals are total on the required oracles. The lower bound is obtained only if $\mathrm{tp}(\xi) \in \{0, 2\}$, and is used to show that axioms are corrected when necessary. (Recall that correction of axioms is unnecessary on a thin subspace of the space assigned to a requirement of type 1, so a lower bound is unnecessary in that case.)

Lemma 7.1 (Use Lemma). *Let $\xi \subset \Lambda^0$ be given such that $\xi$ is pseudotrue, and let $s = \mathrm{wt}(\xi)$. Let $\Delta = \Delta^{j,k}_{b,c}$ be a functional, and fix $\langle \bar{x}, \mathrm{wt}(\xi), x \rangle$ potentially in the domain of $\Delta$ such that $x < \mathrm{wt}(\lambda(\xi))$ and for all coordinates $x_i$ of $\bar{x}$, $x_i < \mathrm{wt}(\lambda(\xi))$. Then*:

(i) *$\Delta_s(A^s_c; \bar{x}, x)$ converges with some use $u < \mathrm{wt}(\lambda(\xi))$.*

(ii) *If $\lambda(\xi) \subset \lambda(\Lambda^0)$, then $A_c \upharpoonright \mathrm{wt}(\lambda(\xi)) = A^s_c \upharpoonright \mathrm{wt}(\lambda(\xi))$.*

(iii) *If $j \in \{0, 2\}$, $v^1 \subset \delta^1 \subseteq \lambda(\xi)$ and $\langle \bar{x}, s, x \rangle$ is in the space $S$ such that $v^1$ controls $S$ at $\lambda(\xi)$ with initiator $\delta^1$, then $\mathrm{wt}(v^1) < \mathrm{wt}(\delta^1) \leqslant u$, where $u$ is the use determined in* (i).

*Proof.* (i) By (2.11) and Step 3 of the construction, $\Delta_s(A^s_c; \bar{x}, x)\downarrow$ with some use $u < \mathrm{wt}(\lambda(\xi))$.

(ii) By Step 2 of the construction, if $z$ enters $A_c$, there is a $\pi \subset \Lambda^0$ such that $z \in A_c^{\mathrm{wt}(\pi)} \backslash A_c^{\mathrm{wt}(\pi^-)}$, $\pi^-$ is validated along $\pi$, and $z = \mathrm{wt}(\mathrm{up}(\pi^-))$. If $\mathrm{wt}(\mathrm{up}(\pi^-)) < \mathrm{wt}(\lambda(\xi))$, then as $\lambda(\xi) \subset \lambda(\Lambda^0)$, it follows from (2.1), (2.4), and (2.6) that $\pi^- \subset \xi$ and so that $\mathrm{wt}(\pi^-) < \mathrm{wt}(\xi)$. Hence $z \in A_c^{\mathrm{wt}(\xi)} \upharpoonright \mathrm{wt}(\lambda(\xi))$.

(iii) Suppose that $\nu^1 \subset \delta^1 \subseteq \lambda(\xi)$ and $\langle \bar{x}, s, x \rangle$ is in the space $S$ such that $\nu^1$ controls $S$ at $\lambda(\xi)$ with initiator $\delta^1$. (Note that we identify functionals whose last coordinates are $\equiv$, so choose to ignore the last coordinate.) By (2.1), $\mathrm{wt}(\nu^1) < \mathrm{wt}(\delta^1)$. Let $y = x$ if $k = \dim(\nu^1) = 1$ and $j = \mathrm{tp}(\nu^1) = 0$, and let $y = x_{k-1}$ if $k = \dim(\nu^1) > 1$. By (6.4) and (6.6), $\mathrm{wt}(\delta^1) \leqslant y \leqslant \mathrm{wt}(\lambda(\xi))$. By Step 3 of the construction, $\Delta_t(A_c^t; \bar{x}, x)$ diverges unless $t \geqslant \mathrm{wt}(\mu)$ for some $\mu \subset \Lambda^0$ such that $\mathrm{wt}(\lambda(\mu)) > y$. Hence by (2.11) and Step 3 of the construction, all axioms $\Delta_t(A_c^t; \bar{x}, x) = q$ which are ever declared have use $u \geqslant \mathrm{wt}(\lambda(\mu)) - 1$ for some such $\mu$, so $u \geqslant y \geqslant \mathrm{wt}(\delta^1)$. ∎

We now begin to show that all requirements are satisfied. We first show that the functionals which we define are partial recursive, total on the appropriate oracles, and well-defined.

Lemma 7.2 (Well-Definedness and Totality Lemma). *For all* $j \leqslant 2$, $k \leqslant n$ *and* $\langle b, c \rangle \in Z_{j,k}$, $\Delta_{b,c}^{j,k}(A_c)$ *is total and* $\Delta_{b,c}^{j,k}$ *is a well-defined partial recursive functional.*

*Proof.* By Step 3 of the construction, all functionals are partial recursive, and new axioms are not defined when an axiom from an oracle compatible with $A_c$ already exists, so $\Delta_{b,c}^{j,k}(A_c)$ is well-defined. Fix $x$ and $\bar{x}$. Any axiom $\Delta_{b,c}^{j,k}(A_c; \bar{x}, x) = q$ which is ever declared at $\pi \subset \Lambda^0$ has use $< \mathrm{wt}(\lambda(\pi))$, and furthermore, $\mathrm{wt}(\lambda(\pi)) > x$ and $\mathrm{wt}(\lambda(\pi)) > x_i$ for all coordinates $x_i$ of $\bar{x}$. By Lemma 5.17(v) (Assignment), there are infinitely many nodes $\pi \subset \Lambda^0$ such that $\pi$ is $\Lambda^0$-true and pseudotrue, $x < \mathrm{wt}(\lambda(\pi))$, and $x_i < \mathrm{wt}(\lambda(\pi))$ for all coordinates $x_i$ of $\bar{x}$. By Lemma 7.1(ii) (Use), $A_c \upharpoonright \mathrm{wt}(\lambda(\pi)) = A_c^{\mathrm{wt}(\pi)} \upharpoonright \mathrm{wt}(\lambda(\pi))$, so as the use of $\Delta_{b,c}^{j,k}(A_c; \bar{x}, x) = q$ is $< \mathrm{wt}(\lambda(\pi))$, $\Delta_{b,c}^{j,k}(A_c; \bar{x}, x) = \Delta_{b,c}^{j,k}(A_c^{\mathrm{wt}(\pi)}; \bar{x}, x)$. Thus $\Delta_{b,c}^{j,k}(A_c)$ is total. ∎

The next lemma establishes the existence of all (iterated) limits except for the outermost limit, and relates the limiting value to the outcome of a controller, should the latter exist.

Lemma 7.3 (Convergence and Correctness Lemma). *Fix a requirement* $R = R_{e,b,c}^{j,r}$, *and let* $\Delta = \Delta_{b,c}^{j,r}$ *be the functional associated with* $R$. *Fix* $k \in [1, r-1]$. (*Thus we explicitly exclude the case where* $\dim(R) = 1$.) *Let* $p = r - k + 1$. *Fix* $u_1, \ldots, u_{p-1}$, $x \in \mathbf{N}$, *and let* $S = \{\langle u_1, \ldots, u_{p-1} \rangle\} \times \mathbf{N}^k \times \{\langle x \rangle\}$ *if* $j \in \{0, 2\}$ *and* $S = \{\langle u_1, \ldots, u_{p-1} \rangle\} \times \mathbf{N}^{k+1} \times \{\langle x \rangle\}$ *if* $j = 1$. (*Note* *that we use identification of axioms here, so that* $S = \{\langle u_1, ..., u_{p-1}\rangle\} \times \mathbf{N}^k \times \{\langle x, \xi\rangle\}$ *or* $\{\langle u_1, ..., u_{p-1}\rangle\} \times \mathbf{N}^{k+1} \times \{\langle x, \xi\rangle\}$ *for some* $\xi$.) *Then*:

(i) *If* $\operatorname{tp}(R) \in \{0, 2\}$, *then* $\lim_{u_p} \cdots \lim_{u_{r-1}} \Delta(A_c; u_1, ..., u_{r-1}, x) \downarrow \in \{0, 1\}$, *and if* $\operatorname{tp}(R) = 1$, *then* $\lim_{u_p} \cdots \lim_{u_r} \Delta(A_c; u_1, ..., u_r, x) \downarrow \in \{0, 1\}$. *In both cases, define this value to be* $L(u_1, ..., u_{p-1}, x)$.

(ii) *If* $\nu^k$ *controls* $S$ *along* $\Lambda^k$, *then* $L(u_1, ..., u_{p-1}, x) = 1$ *iff* $\nu^k$ *is validated along* $\Lambda^k$.

(iii) *If* $S$ *is not controlled along* $\Lambda^k$ *and only finitely many sections of* $S$ *are controlled along* $\Lambda^{k-1}$, *then* $L(u_1, ..., u_{p-1}, x) = 0$.

*Proof.* We proceed by induction on $k$, considering various cases.

*Case* 1. $k = 1$ (so $p = r$).

*Subcase* 1.1: $j = 1$. By clause (iii) of Lemma 6.1 (Finite Control), there are only finitely many initiators for $S$ on $T^1$. Suppose first that $S$ is controlled along $\Lambda^1$. By (6.7), we can fix $\nu^1 \subset \delta^1 \subseteq \tau^1 \subset \Lambda^1$ such that for all $\rho^1 \subset \Lambda^1$ with $\rho^1 \supseteq \tau^1$, $S$ has controller $\nu^1$ and initiator $\delta^1$ at $\rho$. By Lemma 3.1 (Limit Path), we can fix $\eta \subset \Lambda^0$ such that $\lambda(\eta) = \tau^1$. Suppose that $u_r \geqslant \operatorname{wt}(\eta)$ and an axiom $\Delta_{\operatorname{wt}(\xi)}(A_c^{\operatorname{wt}(\xi)}; u_1, ..., u_r, x) = q$ is declared at $\xi$ where $\eta \subseteq \xi \subset \Lambda^0$. If $S$ has controller $\nu^1$ and initiator $\delta^1$ at $\lambda(\xi)$, then we set $q = 0$ if $\nu^1$ is activated along $\delta^1$, and $q = 1$ if $\nu^1$ is validated along $\delta^1$.

If the controller of $S$ at $\lambda(\xi)$ is not $\nu^1$ or the initiator for $S$ at $\lambda(\xi)$ is not $\delta^1$, then by Lemma 6.12 (1-Correction), there is $\nu^1$-correcting $\lambda(\xi)$-link $[\mu^1, \pi^1]$ such that $\mu^1 \subset \delta^1 \subseteq \pi^1$. By the construction and (2.1), any axiom $\Delta_{\operatorname{wt}(\xi)}(A_c^{\operatorname{wt}(\xi)}; u_1, ..., u_r, x) = q$ declared at $\xi$ (but not in existence at $\xi^-$) has use $\operatorname{wt}(\lambda(\xi)) - 1 \geqslant \operatorname{wt}(\pi^1)$. As $S$ is controlled by $\nu^1$ with initiator $\delta^1$ along $\Lambda^1$, it follows from Lemma 3.1 (Limit Path) that there is a shortest $\rho \supset \xi$ such that $S$ is controlled by $\nu^1$ with initiator $\delta^1$ at $\lambda(\rho)$, and note $\rho$ that is pseudotrue. By Lemma 6.12 (1-Correction) and the construction, as $[\mu^1, \pi^1]$ is a primary $\nu^1$-correcting link, there is a $\beta^1 \subseteq \pi^1$ such that $A_c \in \operatorname{TS}(\beta^1)$ and $\operatorname{wt}(\beta^1)$ is placed in $A_c$ at some $\gamma$ such that $\xi \subset \gamma \subseteq \rho$. Furthermore, when axioms are changed on a fixed argument at any node $\tilde{\eta} \subset \Lambda^0$, the use of the axiom declared at $\tilde{\eta}$ is $\operatorname{wt}(\lambda(\tilde{\eta})) - 1$, so by (2.11) and (2.1), if an axiom $\Delta_{\operatorname{wt}(\xi)}(A_c^{\operatorname{wt}(\xi)}; u_1, ..., u_r, x) = q$ is in existence at $\gamma^-$, then it has use $\geqslant \operatorname{wt}(\pi^1) \geqslant \operatorname{wt}(\beta^1)$. But this allows us to define a new axiom $\Delta_{\operatorname{wt}(\rho)}(A_c^{\operatorname{wt}(\rho)}; u_1, ..., u_r, x) = q$, where $q = 0$ if $\nu^1$ is activated along $\delta^1$, and $q = 1$ if $\nu^1$ is validated along $\delta^1$. By Lemma 7.2 (Well-Definedness and Totality), we see that (i) and (ii) hold in this case.

Suppose that $S$ is not controlled along $\Lambda^1$ and only finitely many sections of $S$ are controlled along $\Lambda^0$. We note that by Lemma 5.17(v) (Assignment), there are infinitely many pseudotrue nodes $\subset \Lambda^0$. By Lemma 6.1(iii) (Finite Control), there are only finitely many initiators for

$S$ on $T^1$, and as $S$ is not controlled along $\Lambda^1$, every initiator for $S$ at some node $\subset \Lambda^1$ has a terminator along $\Lambda^1$. Thus there is an $\eta \subset \Lambda^0$ such that for all $\alpha \subset \Lambda^0$ such that $\alpha \supset \eta$, $S$ has no controller at $\lambda(\alpha)$; so every initiator $\bar{\delta} \subset \Lambda^0$ for a section of $S$ at any node along $\Lambda^0$ must satisfy $\bar{\delta} \subseteq \eta$. As only finitely many sections of $S$ are controlled along $\Lambda^0$ and there are infinitely many pseudotrue nodes along $\Lambda^0$, each such $\bar{\delta}$ has a terminator along $\Lambda^0$. If $\eta \subset \bar{\eta} \subset \Lambda^0$ and $\bar{\eta}$ properly extends each such terminator, then no section of $S$ is controlled at any node along $\Lambda^0$ which extends $\bar{\eta}$, so by (6.6), if $S^{[i]}$ is controlled along $\Lambda^0$, then $i \leqslant \mathrm{wt}(\bar{\eta})$. Clauses (i) and (iii) now follow from Case 3 of Step 3 of the construction.

Suppose that $S$ is not controlled along $\Lambda^1$ but infinitely many sections of $S$ are controlled along $\Lambda^0$. As in the preceding paragraph, we see that there are only finitely many initiators for sections of $X$ along $\Lambda^0$. As infinitely many sections of $S$ are controlled along $\Lambda^0$, there is a longest initiator, for a section of $S$, along $\Lambda^0$ which has no terminator along $\Lambda^0$. Let $v$ be the controller corresponding to this initiator. Then by (6.7), for all but finitely many sections $Y$ of $S$, $v$ will control $Y$ at all sufficiently long pseudotrue $\rho \subset \Lambda^0$. So for all but finitely many $u_r$, the axioms $\Delta_{\mathrm{wt}(\xi)}(A_c^{\mathrm{wt}(\xi)}; u_1, ..., u_r, x) = q$ which are declared have value $q$ determined by the outcome of $v$ along $\Lambda^0$. Clause (i) now follows.

*Subcase* 1.2: $j \in \{0, 2\}$. (Note that no limit is being computed, and $L(u_1, ..., u_{r-1}, x)$ just gives the value of an axiom.) Recall that, by (6.7), a space is controlled by a node along a path iff it is controlled by that node at all sufficiently long pseudotrue nodes along the path. If $S$ is not controlled along $\Lambda^1$ and no section of $S$ is controlled along $\Lambda^0$, then as controllers are never terminated along $\Lambda^0$, all axioms $\Delta_{\mathrm{wt}(\xi)}(A_c^{\mathrm{wt}(\xi)}; u_1, ..., u_{r-1}, x) = q$ will be declared in Case 3 of Step 3 of the construction and will set $q = 0$, so (i) and (iii) follow from Lemma 7.2 (Well-Definedness and Totality). If $S$ is not controlled along $\Lambda^1$ but some section of $S$ is controlled along $\Lambda^0$, then (i) follows from Lemma 7.2 (Well-Definedness and Totality). As controllers are never terminated along $\Lambda^0$, infinitely many sections of $S$ will be controlled along $\Lambda^0$, so the hypothesis of (iii) fails.

In order to complete the verification of (i) and (ii) for $j \neq 1$, it suffices to verify the following condition, under the assumption that $S$ is controlled along $\Lambda^1$:

(7.1) For all $\eta$ and $v^1$, if $\eta \subset \Lambda^0$ is pseudotrue and $v^1$ controls $S$ at $\lambda(\eta)$, then $\Delta_{\mathrm{wt}(\eta)}(A_c^{\mathrm{wt}(\eta)}; u_1, ..., u_{r-1}, x) = 1$ iff $v^1$ is validated along $\Lambda^1$.

We proceed by induction on $\mathrm{lh}(\eta)$ for $\eta$ pseudotrue. Given $u_{r-1}$, let $\eta_0$ be the shortest string for which $\Delta_{\mathrm{wt}(\eta_0)}(A_c^{\mathrm{wt}(\eta_0)}; u_1, ..., u_{r-1}, x)\downarrow$, and note that by Step 3 of the construction, $\eta_0$ is pseudotrue. If $\eta = \eta_0$, then by the construction, we define $\Delta_{\mathrm{wt}(\eta)}(A_c^{\mathrm{wt}(\eta)}; u_1, ..., u_{r-1}, x) = q$ for some $q$, and the value chosen for $q$ is the one satisfying (7.1) if there is a $\nu^1$ which controls $S$ at $\lambda(\eta)$. Suppose, by induction on $\mathrm{lh}(\eta)$ with $\eta$ pseudotrue, that (7.1) holds for $\rho$, where $\eta_0 \subseteq \rho$ and $\rho$ is the longest pseudotrue node $\subset \eta$. By Lemma 6.7 (Loss of Control), (7.1) will hold at $\eta$ through the absence of a controller, unless there is a controller $\nu^1$ and initiator $\delta^1$ for $S$ at $\lambda(\rho)$; so we may fix such $\nu^1$ and $\delta^1$. Let $u$ be the use of the axiom $\Delta_{\mathrm{wt}(\rho)}(A_c^{\mathrm{wt}(\rho)}; u_1, ..., u_{r-1}, x) = q$, and note that by Lemma 7.1(iii) (Use), $\mathrm{wt}(\delta^1) \leqslant u$.

If $\delta^1 \subseteq \lambda(\eta)$, then by Lemma 6.6 (Constancy of Initiator), $\nu^1$ controls $S$ at $\lambda(\eta)$ with initiator $\delta^1$, so (7.1) follows by induction. Suppose that $\delta^1 \nsubseteq \lambda(\eta)$, and fix the shortest $\beta$ such that $\rho \subset \beta \subseteq \eta$ and $\delta^1 \nsubseteq \lambda(\beta)$, and fix $\kappa^1$ such that $\beta$ switches $\kappa^1$. By Lemma 6.7 (Loss of Control), (7.1) will hold for $\eta$ if $S$ does not have an initiator and controller at $\lambda(\beta)$; thus we may fix an initiator $\hat{\delta}^1$ and controller $\hat{\nu}^1$ for $S$ at $\lambda(\beta)$, and note that, by our assumption, $\delta^1 \neq \hat{\delta}^1$. By (6.10), $\hat{\delta}^1 \subseteq \lambda(\beta)^- \subset \lambda(\beta^-)$, so $\hat{\delta}^1 \subseteq \kappa^1 \subset \delta^1$. Hence we may apply Lemmas 6.10 or 6.14 (Correction).

If conclusion (i) of the relevant Correction Lemma holds, then (7.1) follows by induction. If conclusion (ii) of the Correction Lemma holds and $\mathrm{tp}(\nu^1) = 0$, then $\beta^- \equiv \nu^1$ and we place $\mathrm{wt}(\kappa^1) \in A_c^{\mathrm{wt}(\beta)} \backslash A_c^{\mathrm{wt}(\beta^-)}$. And if conclusion (ii) of the Correction Lemma holds and $\mathrm{tp}(\nu^1) = 2$, then $\dim(\beta^-) \geqslant \dim(\nu^1)$ and by Lemma 2.2(iv) (Interaction), we place $\mathrm{wt}(\kappa^1) \in A_c^{\mathrm{wt}(\beta)} \backslash A_c^{\mathrm{wt}(\beta^-)}$. As $\kappa^1 \subset \delta^1$, it follows from (2.1) that $\mathrm{wt}(\kappa^1) < \mathrm{wt}(\delta^1) \leqslant u$, and so that $A_c^{\mathrm{wt}(\eta)} \upharpoonright u \neq A_c^{\mathrm{wt}(\rho)} \upharpoonright u$. Now axioms are only defined at pseudotrue nodes, so the construction declares a new axiom $\Delta_{\mathrm{wt}(\eta)}(A_c^{\mathrm{wt}(\eta)}; u_1, ..., u_{r-1}, x)\downarrow$ to satisfy (7.1).

*Case* 2. $k > 1$. By induction, the lemma holds for $k-1$.

*Subcase* 2.1: $S$ is controlled by $\nu^k$ along $\Lambda^k$. By Lemma 6.3 (Thick Control), a thick subset of $S$ is controlled along $\Lambda^{k-1}$ by derivatives of $\nu^k$ which are validated along $\Lambda^{k-1}$ if $\nu^k$ is validated along $\Lambda^k$, and are activated along $\Lambda^{k-1}$ if $\nu^k$ is activated along $\Lambda^k$. Clauses (i) and (ii) now follow by induction.

*Subcase* 2.2: $S$ is not controlled along $\Lambda^k$ and only finitely many sections of $S$ are controlled along $\Lambda^{k-1}$. By Lemma 6.5(iii) (Non-Control), there are only finitely many $i$ such that a section of $S^{[i]}$ is controlled along $\Lambda^{k-2}$. Clauses (i) and (iii) now follow inductively from (i) and (iii) for $k-1$.

*Subcase* 2.3: $S$ is not controlled along $\Lambda^k$, but infinitely many sections of $S$ are controlled along $\Lambda^{k-1}$. By Lemma 6.4 (Indirect Control), all but finitely many sections of $S$ are controlled by a fixed node along $\Lambda^{k-1}$. Clause (i) now follows from (i) and (ii) inductively. ∎

The next lemma relates the outcomes of nodes which are critical for axiom definition, to the truth of the sentences assigned to those nodes.

LEMMA 7.4 (Accuracy Lemma). *Fix $k \in n$ and $\xi^k \subset \Lambda^k$ such that $k \leqslant \dim(\xi^k)$ and $\xi^k$ is $\Lambda^k$-free and implication-free. Then $\xi^k$ is validated along $\Lambda^k$ iff $M_{\xi^k}$ is true.*

*Proof.*

*Case* 1. $k=0$. Let $\xi=\xi^0$. Recall that $M_\xi$ is a $\Pi_1$-sentence beginning with a block of bounded quantifiers and followed by $\exists s \leqslant \mathrm{wt}(\eta^1)\ \forall t \geqslant s\ S$, where $S$ is quantifier-free and $\eta^1 = \mathrm{up}(\xi)$.

*Case* 1.1. $\xi$ is validated along $\Lambda^0$. We first show that $M_\xi$ is potentially true, and all uses in $M_\xi$ are $<\mathrm{wt}(\xi)$, under the weaker assumption that $\xi \subset \Lambda^0$ is implication-free. We proceed by induction on $\mathrm{lh}(\xi)$. There are two cases.

*Case* 1.1.1. $\xi$ is not a primary 0-completion or an amenable pseudocompletion. Then by the construction, $M_\xi$, is potentially true, and by (0.1), all uses in $M_\xi$ are $<\mathrm{wt}(\xi)$.

*Case* 1.1.2. $\xi$ is a primary 0-completion or an amenable pseudocompletion. Thus $\mathrm{tp}(\xi) \in \{1, 2\}$. If $\xi$ is a primary completion, fix $\eta$ such that $\xi$ is a primary completion of $\eta$, and let $\gamma = \eta^-$. And if $\xi$ is a pseudocompletion, fix the shortest $\gamma$ such that $\xi$ is a pseudocompletion of $\gamma$, and fix $\eta \subseteq \xi$ such that $\eta^- = \gamma$. By (5.5)(ii) and Lemma 5.13 (Amenable Implication Chain) if $\dim(\xi) > 1$ and by (5.1) or (5.10)(i) if $\dim(\xi) = 1$, $\gamma$ is implication-free.

By (5.2) or (5.11)(i), $\gamma$ is validated along $\xi$, so it follows by induction that $M_\gamma$ is potentially true, and by (0.1) and (2.1), all uses in $M_\gamma$ are $\leqslant \mathrm{wt}(\gamma) < \mathrm{wt}(\xi)$. First suppose that $\xi$ is a primary 0-completion. By Lemma 5.12(i) (PL) and (5.19), all nodes $\beta^1$ of $T^1$ which are switched by nodes in $(\eta, \xi]$ are in $\mathrm{PL}(\mathrm{up}(\xi), \lambda(\eta))$. If $\dim(\xi) = 1$ (and hence $\mathrm{tp}(\xi) = 1$), it follows from (5.4) that $\mathrm{TS}(\beta^1) \cap \mathrm{RS}(\xi) = \varnothing$ for each such $\beta^1$. Suppose that $\dim(\xi) = r > 1$. Then by (5.5)(ii), there is an amenable 1-implication chain $\langle \langle \sigma^j, \hat{\sigma}^j, \tau^j \rangle : r-1 \geqslant j \geqslant 1 \rangle$ such that $\mathrm{out}(\tau^1) = \eta$. By Lemma 5.12 (ii, iii) (PL), $\{\mathrm{up}^{r-1}(\beta^1) : \beta^1 \in \mathrm{PL}(\mathrm{up}(\xi), \lambda(\eta))\} = \mathrm{PL}(\sigma^{r-1}, \tau^{r-1})$, and if we fix $\bar{\tau}^{r-1} \subset \tau^{r-1}$ such that $(\bar{\tau}^{r-1})^- = \sigma^{r-1}$, then either $\hat{\sigma}^{r-1}$ is a pseudocompletion of $\sigma^{r-1}$, or $\bar{\tau}^{r-1}$ requires extension. If $\bar{\tau}^{r-1}$ requires extension, then by (5.11)(ii), $\hat{\sigma}^{r-1}$ has finite outcome along $\tau^{r-1}$, so by Lemma 5.1(iv) (PL Analysis) and Lemma 5.12(ii) (PL) and (5.19), $\{\mathrm{up}(\pi^{r-1}) : \pi^{r-1} \in \mathrm{PL}(\sigma^{r-1}, \tau^{r-1})\} = \{\mathrm{up}(\pi^{r-1}) : \pi^{r-1} \in \mathrm{PL}(\sigma^{r-1}, \hat{\sigma}^{r-1})\} = \mathrm{PL}(\mathrm{up}(\hat{\sigma}^{r-1}), \lambda(\bar{\tau}^{r-1}))$. Hence by Definition 5.4 if $\hat{\sigma}^{r-1}$ is an amenable pseudocompletion and by (5.4) otherwise, $\mathrm{TS}(\beta^1) \cap \mathrm{RS}(\xi) = \varnothing$ for each such $\beta^1$. Thus by

Lemma 2.2(i) Interaction) and the construction, $M_\xi$ must be potentially true, and all uses in $M_\xi$ are $<\mathrm{wt}(\xi)$.

Now suppose that $\xi$ is an amenable pseudocompletion. By (5.11)(i), $\gamma$ is the principal derivative of $\mathrm{up}(\gamma)$ along $\xi$. Hence by (2.11), (2.2), and (2.4), any element $\leqslant \mathrm{wt}(\gamma)$ placed in a set at any $\pi \in (\eta, \xi]$ is of the form $\mathrm{wt}(\mathrm{up}(\pi))$ with $\mathrm{up}(\pi) \subset \mathrm{up}(\gamma)$, and $\pi$ is validated along $\xi$. By Lemma 3.1(i) (Limit Path), there must be a $\mu \subset \xi$ such that $[\mu, \pi]$ is a primary $\xi$-link which restrains $\gamma$. As $\xi$ is an amenable pseudocompletion of $\gamma$, $\mathrm{TS}(\pi) \cap \mathrm{RS}(\xi) = \varnothing$ for each such $\pi$. Thus by Lemma 2.2(i) (Interaction) and the construction, $M_\xi$ must be potentially true, and all uses in $M_\xi$ are $<\mathrm{wt}(\xi)$.

For both Subcase 1.1.1 and Subcase 1.1.2, we note that elements placed into sets are of the form $z = \mathrm{wt}(\mathrm{up}(v))$ for $v \subset \Lambda^0$, and $z$ is first placed in a set $A^{s+1}$ when $s = \mathrm{wt}(\delta)$ and $\mathrm{up}(v)$ is validated along $\lambda(\delta)$ but not along $\lambda(\delta^-)$. Hence by Lemma 3.1 (Limit Path), $M_\xi$ will be true if no element $<\mathrm{wt}(\xi)$ is first placed in any $A \in \mathrm{RS}(\xi)$ by any $v \supseteq \xi$ such that $v \subset \Lambda^0$. By Lemma 2.2(i) (Interaction), $\xi$ does not place elements into any set in $\mathrm{RS}(\xi)$. Fix $\pi \subset \Lambda^0$ such that $\pi^- = \xi$. By Lemma 3.1 (Limit Path), it follows that $\lambda(\pi)^- = \mathrm{up}(\xi)$ and for all $v$ such that $\pi \subseteq v \subset \Lambda^0$, $\lambda(v) \supseteq \lambda(\pi)$. Hence the elements placed into sets by $v \supset \xi$ are of the form $\mathrm{wt}(\alpha)$, where $\mathrm{up}(v) = \alpha \supseteq \lambda(\pi)$. By (2.1) and (2.2), $\mathrm{wt}(\mathrm{up}(v)) \geqslant \mathrm{wt}(\lambda(\pi)) > \mathrm{wt}(\mathrm{out}(\lambda(\pi))) = \mathrm{wt}(\pi) > \mathrm{wt}(\xi)$. Hence $M_\xi$ is true.

*Case* 1.2. $k = 0$ and $\xi$ is activated along $\Lambda^0$. $M_\xi$ cannot be potentially true, else the action taken for $\xi$ would force $\xi$ to be validated along $\Lambda^0$. Hence $M_\xi$ cannot be true.

*Case* 2. $k > 0$. By induction, we may assume that the lemma holds for $k-1$. Let $v$ be the principal derivative of $\xi$ along $\Lambda^{k-1}$. It follows from Lemma 4.6 (Free Derivative) and Lemma 5.16(ii) (Implication-Freeness), that $v$ is $\Lambda^{k-1}$-free and implication-free, and if $\xi$ has infinite outcome along $\Lambda^k$, then $\xi$ has infinitely many $\Lambda^{k-1}$-free, implication-free derivatives $\mu$ along $\Lambda^{k-1}$.

Suppose that $k$ is odd. By Definitions 2.9 and 2.10, $M_\xi$ is a sentence of the form $Q_1 y_1 \cdots Q_p y_p \exists \bar{z} P(\bar{y}, \bar{z})$ where $P$ is $\Pi_k$, and the $Q_j$ are bounded quantifiers, and $M_v$ is $Q_1 y_1 \cdots Q_p y_p \exists \bar{z} \leqslant \mathrm{wt}(v)\, P(\bar{y}, \bar{z})$. If $M_v$ is true, then $M_\xi$ is true. But then by induction, $v$ is validated along $\Lambda^{k-1}$, i.e., $v$ has infinite outcome along $\Lambda^{k-1}$, so by the definition of the function $\lambda$, $\xi$ has finite outcome along $\Lambda^k$ and $\xi$ is validated along $\Lambda^k$. If $M_v$ is not true, then as $v$ is the principal derivative of $\xi$ along $\Lambda^{k-1}$, it follows from (2.4) that all derivatives of $\xi$ along $\Lambda^{k-1}$ are activated along $\Lambda^{k-1}$, i.e., have finite outcome along $\Lambda^{k-1}$. Hence by induction, for every derivative $\mu$ of $\xi$ along $\Lambda^{k-1}$ which is $\Lambda^{k-1}$-free and implication-free, $M_\mu$ is not true. For each such $\mu$, $M_\mu$ is $Q_1 y_1 \cdots Q_p y_p \exists \bar{z} \leqslant \mathrm{wt}(\mu)\, P(\bar{y}, \bar{z})$ As there are infinitely many such $\mu$, $\mathrm{wt}(\mu)$ is unbounded as we range over these $\mu$. Thus $M_\xi$ is not true. By induction, $\mu$ has finite outcome along $\Lambda^{k-1}$ for each such $\mu$, so by the definition of the function $\lambda$, $\xi$ has infinite outcome along $\Lambda^k$, so $\xi$ is activated along $\Lambda^k$.

Suppose that $k$ is even. We proceed as in the preceding paragraph, interchanging universal and existential quantifiers, $\Pi$ and $\Sigma$, and true and not true. ▌

We now show that all requirements are satisfied.

Lemma 7.5 (0-Satisfaction Lemma). *Every requirement of type* 0 *is satisfied.*

*Proof.* Fix a requirement $R = R^{0,r}_{e,b,c}$ of type 0, and let $\Delta = \Delta^{0,r}_{b,c}$ be the functional for the requirement $R$ as described at the beginning of this section. By Lemma 5.17(i, ii, iv) (Assignment), $R$ is assigned to a unique $\sigma^r \subset \Lambda^r$ such that $\sigma^r$ is $\Lambda^r$-free and implication-free, and if $\tau^r$ is the immediate successor of $\sigma^r$ along $\Lambda^r$, then $\mathrm{out}^0(\tau^r)$ is pseudotrue.

First assume that $r = 1$. Let $x = \mathrm{wt}(\sigma^1)$. By Lemma 7.2 (Well-Definedness and Totality), we can fix $q$ such that $\Delta(A_c; x) = q$. Let $\nu$ ($\pi$, resp.) be the initial (principal, resp.) derivative of $\sigma^1$ along $\Lambda^1$ and let $\beta$ ($\delta$, resp.) be the immediate successor of $\nu$ ($\pi$, resp.) along $\Lambda^0$. By Lemma 5.17(iv) (Assignment), $\delta$ is pseudotrue, and by Lemma 5.17(iii) (Assignment), $\pi$ is $\delta$-free and implication-free. By Lemma 5.16(iv) (Implication-Freeness), $\beta$ is pseudotrue and $\nu$ is implication-free, and by Lemma 4.5 (Free Extension), $\nu$ is $\beta$-free. By Definition 6.4 and the construction, we declare an axiom $\Delta_{\mathrm{wt}(\beta)}(A_c^{\mathrm{wt}(\beta)}; x) = z$ for some $z \in \{0, 1\}$ with use $\mathrm{wt}(\lambda(\beta)) - 1$, where $z = 0$ iff $\nu$ is activated along $\beta$. As $\sigma^1 \subset \Lambda^1$, it follows from (2.6) that no $\alpha$ such that $\beta \subset \alpha \subset \Lambda^0$ can switch any $\rho^1 \subset \sigma^1$. Hence by Lemma 7.1(ii) (Use) and (2.1), $\Delta(A_c; x) = z$ unless $\pi \supset \nu$, i.e., $\lambda(\beta) \not\subset \Lambda^1$. Suppose this to be the case. Then the construction places $\mathrm{wt}(\sigma^1)$ into $A_c^{\mathrm{wt}(\delta)}$. By (2.1), $\mathrm{wt}(\sigma^1) \leqslant \mathrm{wt}(\lambda(\beta)) - 1$, so we define a new axiom $\Delta_{\mathrm{wt}(\delta)}(A_c^{\mathrm{wt}(\delta)}; x) = 1$ with use $\mathrm{wt}(\lambda(\delta)) - 1$, and $\nu$ is activated along $\delta \subset \Lambda^1$. As $\sigma^1 \subset \Lambda^1$, it follows from (2.8) and (2.6) that no $\alpha$ such that $\delta \subset \alpha \subset \Lambda^0$ can switch any $\rho^1 \subseteq \sigma^1$, so $\lambda(\delta) \subset \Lambda^1$. Hence by Lemma 7.1(ii) (Use) and (2.1), $\Delta(A_c; x) = 1$. Hence $\sigma^1$ is activated along $\Lambda^1$ if $z = 0$, and $\sigma^1$ is validated along $\Lambda^1$ if $z = 1$. By Lemma 7.4 (Accuracy), $\sigma^1$ is validated along $\Lambda^1$ iff $M_{\sigma_1}$ is true. Hence if $M_{\sigma_1}$ is true then $z = 1$, and if $M_{\sigma_1}$ is not true then $z = 0$. Thus $R$ is satisfied in this case.

Now assume that $r > 1$. Fix a space $S = \mathbf{N}^r \times \{x\}$ in the domain of the functional $\Delta$. First suppose that $S$ is not controlled along $\Lambda^r$. If infinitely many sections of $S$ are controlled along $\Lambda^{r-1}$, then by Lemma 6.4 (Indirect Control), cofinitely many sections of $S$ are controlled along $\Lambda^{r-1}$ by the same node $\nu^{r-1}$, so by Lemma 7.3(i, ii) (Convergence and Correctness) applied separately to each section of $S$, $\lim_{u_1} \cdots \lim_{u_{r-1}} \Delta(A_c; u_1, ..., u_{r-1}, x) = L$ exists, $L = 0$ if $v^{r-1}$ is activated along $\Lambda^{r-1}$, and $L = 1$ if $v^{r-1}$ is validated along $\Lambda^{r-1}$. Otherwise, by Lemma 6.5(iii) (Non-Control) and Lemma 7.3(iii) (Convergence and Correctness) applied separately to each section of $S$, $\lim_{u_1} \cdots \lim_{u_{r-1}} \Delta(A_c; u_1, ..., u_{r-1}, x) = 0$.

Now suppose that $S = S_{\gamma^r}$ for some $\gamma^r \subset \Lambda^r$ associated with $\Delta$ such that $\gamma^r$ controls $S$ along $\Lambda^r$. Then by Lemma 6.3 (Thick Control) either cofinitely many sections of $S$ are controlled, along $\Lambda^{r-1}$, by derivatives of $\gamma^r$ which are activated along $\Lambda^{r-1}$, or cofinitely many sections of $S$ are controlled, along $\Lambda^{r-1}$, by derivatives of $\gamma^r$ which are validated along $\Lambda^{r-1}$. It now follows from Lemma 7.3(i, ii) (Convergence and Correctness) applied separately to each section $X$ of $S_{\gamma^r}$ that $\lim_{u_1} \cdots \lim_{u_{r-1}} \Delta(A_c; u_1, ..., u_{r-1}, \mathrm{wt}(\gamma^r)) = L(\mathrm{wt}(\gamma^r))$ exists, and that $\gamma^r$ is validated along $\Lambda^r$ iff $L(\mathrm{wt}(\gamma^r)) = 1$.

Recall that $R$ is assigned to a $\sigma^r \subset \Lambda^r$ such that $\sigma^r$ is $\Lambda^r$-free and implication-free, and that if $\tau^r$ is the immediate successor of $\sigma^r$ along $\Lambda^r$, then $\mathrm{out}^0(\tau^r)$ is pseudotrue. Hence by Definition 6.4, $\sigma^r$ controls $S$ along $\Lambda^r$. By the preceding paragraph, $\lim_{u_1} \cdots \lim_{u_{r-1}} \Delta(A_c; u_1, ..., u_{r-1}, \mathrm{wt}(\sigma^r)) = L(\mathrm{wt}(\sigma^r))$ exists, and $\sigma^r$ is validated along $\Lambda^r$ iff $L(\mathrm{wt}(\sigma^r)) = 1$. By Lemma 7.4 (Accuracy), $\sigma^r$ is validated along $\Lambda^r$ iff $M_{\sigma^r}$ is true. Hence if $M_{\sigma^r}$ is true then $L(\mathrm{wt}(\sigma^r)) = 1$, and if $M_{\sigma^r}$ is not true then $L(\mathrm{wt}(\sigma^r)) = 0$. Thus $R$ is satisfied. ∎

LEMMA 7.6 (1-Satisfaction Lemma). *Every requirement of type* 1 *is satisfied.*

*Proof.* Fix a requirement $R = R^{1,r}_{e,b,c}$ of type 1, and let $\Delta = \Delta^{1,r}_{b,c}$ be the functional for the requirement $R$ as described at the beginning of this section. By Lemma 7.3(i) (Convergence and Correctness) for $r > 1$, $L(i, e) = \lim_{u_2} \cdots \lim_{u_r} \Delta(A_c; i, u_2, ..., u_r, e)$ exists and takes a value in $\{0, 1\}$ for all $e, i \in \mathbf{N}$.

By Lemma 5.17(i, ii) (Assignment), $R$ is assigned to a unique $\kappa^r \subset \Lambda^r$ such that $\kappa^r$ is $\Lambda^r$-free and implication-free. Let $v^{r-1}$ be the principal derivative of $\kappa^r$ along $\Lambda^{r-1}$, and fix $\delta^{r-1} \subset \Lambda^{r-1}$ such that $(\delta^{r-1})^- = v^{r-1}$. By Lemma 6.15(i) (Initial Control), $v^{r-1}$ controls $\{\mathrm{wt}(\delta^{r-1})\} \times \mathbf{N}^r \times \{e\}$ with initiator $\delta^{r-1}$ along $\Lambda^{r-1}$. By Case 1.1 of Definition 6.3, $\delta^{r-1}$ is also the initiator for $\{i\} \times \mathbf{N}^r \times \{e\}$ at $\delta^{r-1}$ for all $i \geqslant \mathrm{wt}(\delta^{r-1})$. Now if $i \geqslant \mathrm{wt}(\delta^{r-1})$, then $\{i\} \times \mathbf{N}^r \times \{e\}$ is controlled along $\Lambda^{r-1}$ iff there is an initiator $\gamma^{r-1} \subset \Lambda^{r-1}$ for $\{i\} \times \mathbf{N}^r \times \{e\}$ such that there is no $v^{r-1}$-correcting primary $\Lambda^{r-1}$-link $[\mu^{r-1}, \pi^{r-1}]$ with $\mu^{r-1} \subset \gamma^{r-1} \subseteq \pi^{r-1}$; and by (6.7), if $\{i\} \times \mathbf{N}^r \times \{e\}$ is controlled along $\Lambda^{r-1}$, then the initiator for $\{i\} \times \mathbf{N}^r \times \{e\}$ along $\Lambda^{r-1}$ is the longest such $\gamma^{r-1}$. As $v^{r-1}$ is $\Lambda^{r-1}$-free, $\delta^{r-1}$ is such a $\gamma^{r-1}$. Hence for all $i \geqslant \mathrm{wt}(\delta^{r-1})$, $\{i\} \times \mathbf{N}^r \times \{e\}$ is controlled along $\Lambda^{r-1}$, and if $\{i\} \times \mathbf{N}^r \times \{e\}$ is controlled at any $\gamma^{r-1} \subset \Lambda^{r-1}$ with initiator $\delta_i^{r-1}$, then $\delta_i^{r-1} \supseteq \delta^{r-1}$.

Fix $i$ and $\delta_i^{r-1}$ as in the preceding paragraph such that $\delta_i^{r-1}$ has no terminator along $\Lambda^{r-1}$. Let $v_i^{r-1}$ be the controller corresponding to $\delta_i^{r-1}$. As $v^{r-1}$ is $\Lambda^{r-1}$-free and implication-free, it follows from (4.1) and Case 3 of Definition 6.3 that $v_i^{r-1} \supseteq v^{r-1}$.

First suppose that $v^{r-1}$ has infinite outcome along $\Lambda^{r-1}$. We note that as $v^{r-1}$ is $\Lambda^{r-1}$-free, there is no primary $\Lambda^{r-1}$-link restraining $v^{r-1}$. Furthermore, by Lemma 5.17(v) (Assignment), there are infinitely many $\tau^{r-1} \subset \Lambda^{r-1}$, such that $\mathrm{out}^0(\tau^{r-1})$ is pseudotrue, so by (5.28), every node along $\Lambda^{r-1}$ which requires extension has a primary completion along $\Lambda^{r-1}$ which has infinite outcome along $\Lambda^{r-1}$; hence every component of $\mathrm{PL}(v^{r-1}, \xi^{r-1})$ for some $\xi^{r-1} \subset \Lambda^{r-1}$ gives rise to a primary $\Lambda^{r-1}$-link which restrains $v^{r-1}$, so no such component can exist. If $\mathrm{up}(v^{r-1}) = \mathrm{up}(v_i^{r-1})$, then by (2.8), $v^{r-1} = v_i^{r-1}$; so $v_i^{r-1}$ has infinite outcome along $\Lambda^{r-1}$. And if $\mathrm{up}(v^{r-1}) \neq \mathrm{up}(v_i^{r-1})$, then as $v^{r-1}$ is $\Lambda^{r-1}$-free, it follows from Lemma 6.11 (1-Similarity, with $\sigma^{r-1} = v^{r-1}$ and $\hat{\sigma}^{r-1} = v_i^{r-1}$) that $v_i^{r-1}$ has infinite outcome along $\Lambda^{r-1}$.

Suppose that $v^{r-1}$ has finite outcome along $\Lambda^{r-1}$. If $\mathrm{up}(v^{r-1}) = \mathrm{up}(v_i^{r-1})$, then as $v^{r-1}$ is the principal derivative of $\kappa^r$ along $\Lambda^{r-1}$, it follows from (2.4) that $v_i^{r-1}$ has finite outcome along $\Lambda^{r-1}$. Suppose that $\mathrm{up}(v^{r-1}) \neq \mathrm{up}(v_i^{r-1})$, and let $\eta_i^{r-1}$ be the immediate successor of $v_i^{r-1}$ along $\Lambda^{r-1}$. By Subcase 1.2 of Definition 6.3, $\mathrm{out}^0(\eta_i^{r-1})$ must be pseudotrue, else $v_i^{r-1}$ would not be a controller at $\eta_i^{r-1}$, so could not be a controller at any node extending $\eta_i^{r-1}$. We note that as $v^{r-1}$ is $\Lambda^{r-1}$-free, there is no primary $\Lambda^{r-1}$-link restraining $v^{r-1}$. Furthermore, by Lemma 5.17(v) (Assignment), there is a $\Lambda^{r-1}$-free node $\xi^{r-1} \subset \Lambda^{r-1}$ such that $\mathrm{out}^0(\xi^{r-1})$ is pseudotrue and $\eta_i^{r-1} \subseteq \xi^{r-1}$. Fix the shortest such $\xi^{r-1}$. We show that there is no $\rho^r \in \mathrm{PL}(\mathrm{up}(v^{r-1}), \lambda(\eta_i^{r-1}))$ such that $\mathrm{OS}(v^{r-1}) \subseteq \mathrm{TS}(\rho^r)$. For suppose that such a $\rho^r$ exists, in order to obtain a contradiction. By hypothesis, $v^{r-1}$ is $\Lambda^{r-1}$-free, so $\mathrm{up}(v^{r-1})$ is $\Lambda^r$-free. By (4.1) and Lemma 4.3(iii) (Link Analysis), there are no primary $\lambda(\xi^{r-1})$-links restraining $\mathrm{up}(v^{r-1})$. Hence we may apply Lemma 5.18(ii) (Nonamenable Backtracking) (with $\xi^k = \xi^{r-1}$, $(\eta^k)^- = v_i^{r-1}$, $\eta^{k+1} = \lambda(\eta_i^{r-1})$, $\delta^{k+1} = \mathrm{up}(v^{r-1})$, and $\eta^k = \eta_i^{r-1}$) to conclude that $\mathrm{PL}(\mathrm{up}(v^{r-1}), \lambda(\eta_i^{r-1})) \subseteq \{\mathrm{up}(\gamma^{r-1})\colon \gamma^{r-1} \in \mathrm{PL}(v_i^{r-1}, \xi^{r-1})\}$. Hence we may fix $\rho^{r-1} \in \mathrm{PL}(v_i^{r-1}, \xi^{r-1})$ such that $\mathrm{up}(\rho^{r-1}) = \rho^r$.

As $\mathrm{out}^0(\xi^{r-1})$ is pseudotrue and by Definition 5.3, there are $\mu_i^{r-1} \subset \rho^{r-1} \subseteq \pi_i^{r-1} \subset \beta_i^{r-1} \subseteq \xi^{r-1}$ such that $(\beta_i^{r-1})^- = \pi_i^{r-1}$, $[\mu_i^{r-1}, \pi_i^{r-1}]$ is a primary $\xi^{r-1}$-link, and $\rho^{r-1} \in \overline{\mathrm{PL}}(\beta_i^{r-1}) \subseteq \mathrm{PL}(v_i^{r-1}, \xi^{r-1})$. Furthermore, either $\overline{\mathrm{PL}}(\beta_i^{r-1}) = \{\pi_i^{r-1}\}$ and $[\mu_i^{r-1}, \pi_i^{r-1}]$ restrains $v_i^{r-1}$, or by Definitions 5.3 and 6.2, $\pi_i^{r-1}$ is the primary completion of some node for $\mu_i^{r-1}$ and $\mu_i^{r-1} \subset v_i^{r-1} \subset \pi_i^{r-1}$. As $v_i^{r-1}$ is a principal derivative along $\xi^{r-1}$, it follows that $\mu_i^{r-1} \subset \nu_i^{r-1}$ in both cases. Hence as $\mathrm{OS}(\nu^{r-1}) = \mathrm{OS}(\nu_i^{r-1}) \subseteq \mathrm{TS}(\rho^r) = \mathrm{TS}(\rho^{r-1})$, $[\mu_i^{r-1}, \pi_i^{r-1}]$ is a $\nu_i^{r-1}$-injurious link. By the comments following Definition 6.2, $[\mu_i^{r-1}, \pi_i^{r-1}]$ is a $\nu_i^{r-1}$-correcting link.

Recall that $\eta_i^{r-1}$ is the immediate successor of $\nu_i^{r-1}$ along $\xi^{r-1}$. Now $\pi_i^{r-1}$ is a terminator for $\eta_i^{r-1}$ along $\xi^{r-1}$. By Case 3 of Definition 6.3, when a terminator for $\eta_i^{r-1}$ is found at $\alpha^{r-1} \subset \Lambda^{r-1}$, it is a terminator for all initiators for $\nu_i^{r-1}$ which are $\subset \alpha^{r-1}$, and so $\nu_i^{r-1}$ cannot be a controller at any $\tilde{\alpha}^{r-1}$ such that $\alpha^{r-1} \subseteq \tilde{\alpha}^{r-1} \subset \Lambda^{r-1}$. Thus by Case 3 of Definition 6.3, $\nu_i^{r-1}$ cannot control a space along $\Lambda^{r-1}$ contrary to assumption. This contradiction shows that there is no $\rho^r \in \mathrm{PL}(\mathrm{up}(\nu^{r-1}), \lambda(\eta_i^{r-1}))$ such that $\mathrm{OS}(\nu^{r-1}) \subseteq \mathrm{TS}(\rho^r)$.

It now follows from Lemma 6.11 (1-Similarity, with $\sigma^{r-1} = \nu^{r-1}$ and $\hat{\sigma}^{r-1} = \nu_i^{r-1}$) that $\nu_i^{r-1}$ has finite outcome along $\Lambda^{r-1}$. We thus conclude that for all $i \geqslant \mathrm{wt}(\eta_i^{r-1})$, $\nu_i^{r-1}$ is validated along $\Lambda^{r-1}$ iff $\nu^{r-1}$ is validated along $\Lambda^{r-1}$. There are two cases:

*Case* 1. $r > 1$. By Lemma 7.3(ii) (Convergence and Correctness), $\nu_i^{r-1}$ is validated along $\Lambda^{r-1}$ iff $L(i, e) = 1$. But as $\nu^{r-1}$ is the principal derivative of $\kappa^r$ along $\Lambda^{r-1}$, it follows from (2.4) that $\nu^{r-1}$ is validated along $\Lambda^{r-1}$ iff $\kappa^r$ is validated along $\Lambda^r$. By Lemma 7.4 (Accuracy), $\kappa^r$ is validated along $\Lambda^r$ iff $M_{\kappa^r}$ is true. Hence if $M_{\kappa^r}$ is true then $L(i, e) = 1$ for cofinitely many $i$, and if $M_{\kappa^r}$ is not true then $L(i, e) = 0$ for cofinitely many $i$. Thus $R$ is satisfied.

*Case* 2. $r = 1$. First suppose that $M_{\kappa^1}$ is true. For all $\sigma^1, \tau^1 \in T^1$, if $\sigma^1 \equiv \tau^1 \equiv \kappa^1$ then $M_{\sigma^1} = M_{\tau^1}$. Hence for all sufficiently long $\xi \subset \Lambda^0$, if $\xi \equiv \kappa^1$ then $M_\xi$, is potentially true, so $\Delta(A_c; i, e) = 1$ for cofinitely many $i$.

Suppose that $M_{\kappa^1}$ is not true. By Lemma 7.4 (Accuracy), $\nu = \nu^0$ is the initial derivative of $\kappa^1$ along $\Lambda^0$ and $\nu$ has finite outcome along $\Lambda^0$. By the last sentence of the paragraph preceding Case 1, $\nu_i = \nu_i^0$ has finite outcome along $\Lambda^0$. But then by the construction, $\Delta(A_c; i, e) = 0$ for cofinitely many $i$, and $R$ is satisfied. ∎

Lemma 7.7 (2-Satisfaction Lemma). *Every requirement of type* 2 *is satisfied.*

*Proof.* Fix a requirement $R = R^{2, r}_{e, 1, c}$ of type 2, and let $\Delta = \Delta^{2, r}_{1, c}$ be the functional for the requirement $R$ as described at the beginning of this section. By Lemma 7.3(i) (Convergence and Correctness), $L(i, e) = \lim_{u_2} \cdots \lim_{u_{r-1}} \Delta(A_c; i, u_2, ..., u_{r-1}, e)$ exists and takes a value in $\{0, 1\}$ for all $i \in \mathbf{N}$.

By Lemma 5.17(i, ii) (Assignment), $R$ is assigned to a unique $\kappa^r \subset \Lambda^r$ such that $\kappa^r$ is $\Lambda^r$-free and implication-free. Let $\nu^{r-1}$ be the principal derivative of $\kappa^r$ along $\Lambda^{r-1}$, and fix $\delta^{r-1} \subset \Lambda^{r-1}$ such that $(\delta^{r-1})^- = \nu^{r-1}$.

By Lemma 6.15(i) (Initial Control) and Definition 6.3, $v^{r-1}$ controls $\{\mathrm{wt}(\delta^{r-1})\} \times \mathbf{N}^{r-1} \times \{e\}$ with initiator $\delta^{r-1}$ along $\Lambda^{r-1}$ and $\delta^{r-1}$ is also the initiator for $\{i\} \times \mathbf{N}^{r-1} \times \{e\}$ at $\delta^{r-1}$ for all $i \geqslant \mathrm{wt}(\delta^{r-1})$. Now if $i \geqslant \mathrm{wt}(\delta^{r-1})$, then $\{i\} \times \mathbf{N}^{r-1} \times \{e\}$ is controlled along $\Lambda^{r-1}$ iff there is an initiator $\gamma^{r-1} \subset \Lambda^{r-1}$ for $\{i\} \times \mathbf{N}^{r-1} \times \{e\}$ such that there is no primary $\Lambda^{r-1}$-link $[\mu^{r-1}, \pi^{r-1}]$ with $\mu^{r-1} \subset \gamma^{r-1} \subseteq \pi^{r-1}$; and if $\alpha^{r-1} \subset \Lambda^{r-1}$ and $(\alpha^{r-1})^{-} = \pi^{r-1}$, then $\mathrm{wt}(\alpha^{r-1}) \leqslant i$. (By Lemma 2.2(iv) (Interaction), every primary $\Lambda^{r-1}$-link is $v^{r-1}$-correcting.) Also, if $\{i\} \times \mathbf{N}^{r-1} \times \{e\}$ is controlled along $\Lambda^{r-1}$, then the initiator for $\{i\} \times \mathbf{N}^{r-1} \times \{e\}$ along $\Lambda^{r-1}$ is the longest such $\gamma^{r-1}$. As $v^{r-1}$ is $\Lambda^{r-1}$-free, $\delta^{r-1}$ is such a $\gamma^{r-1}$. Hence for all $i \geqslant \mathrm{wt}(\delta^{r-1})$, $\{i\} \times \mathbf{N}^{r-1} \times \{e\}$ is controlled along $\Lambda^{r-1}$ and if $\{i\} \times \mathbf{N}^{r-1} \times \{e\}$ is controlled at any $\gamma^{r-1} \subset \Lambda^{r-1}$ with initiator $\delta_i^{r-1}$, then $\delta_i^{r-1} \supseteq \delta^{r-1}$. Fix such an $i$ and let $v_i^{r-1}$ be the controller corresponding to the initiator $\delta_i^{r-1}$ for $\{i\} \times \mathbf{N}^{r-1} \times \{e\}$ at $\gamma^{r-1}$. As $v^{r-1}$ is $\Lambda^{r-1}$-free, it follows from Case 3 of Definition 6.3 that $v_i^{r-1} \supseteq v^{r-1}$.

If $\mathrm{up}(v^{r-1}) = \mathrm{up}(v_i^{r-1})$, then as $v^{r-1}$ is the principal derivative of $\kappa^r$ along $\Lambda^{r-1}$ and $v_i^{r-1} \supseteq v^{r-1}$, it follows from (2.8) and (2.4) that $v_i^{r-1}$ has finite outcome along $\Lambda^{r-1}$ iff $v^{r-1}$ has finite outcome along $\Lambda^{r-1}$. And if $\mathrm{up}(v^{r-1}) \neq \mathrm{up}(v_i^{r-1})$, then we note that as $v^{r-1}$ is $\Lambda^{r-1}$-free, there is no primary $\Lambda^{r-1}$-link restraining $v^{r-1}$; hence by Lemma 6.13 (2-Similarity, with $\sigma^{r-1} = v^{r-1}$ and $\hat{\sigma}^{r-1} = v_i^{r-1}$), $v_i^{r-1}$ has finite outcome along $\Lambda^{r-1}$ iff $v^{r-1}$ has finite outcome along $\Lambda^{r-1}$. Thus for all $i \geqslant \mathrm{wt}(\delta^{r-1})$, $v_i^{r-1}$ is validated along $\Lambda^{r-1}$ iff $v^{r-1}$ is validated along $\Lambda^{r-1}$. By Lemma 7.3(ii) (Convergence and Correctness), $v_i^{r-1}$ is validated along $\Lambda^{r-1}$ iff $L(i, e) = 1$. But as $v^{r-1}$ is the principal derivative of $\kappa^r$ along $\Lambda^{r-1}$, it follows from (2.4) that $v^{r-1}$ is validated along $\Lambda^{r-1}$ iff $\kappa^r$ is validated along $\Lambda^r$. By Lemma 7.4 (Accuracy), $\kappa^r$ is validated along $\Lambda^r$ iff $M_{\kappa^r}$ is true. Hence if $M_{\kappa^r}$ is true then $L(i, e) = 1$ for cofinitely many $i$, and if $M_{\kappa^r}$ is not true then $L(i, e) = 0$ for cofinitely many $i$. Thus $R$ is satisfied. ∎

Our main theorem is now immediate from the definition of the functionals $\Delta_{b,c}^{j,k}$, Lemmas 1.1 and 2.1, Lemma 7.2 (Well-Definedness and Totality), and Lemmas 7.5–7.7 ($j$-Satisfaction for $j \leqslant 2$).

THEOREM 7.8. *Fix $m \in \mathbf{N}$, and let $\mathscr{P} = \langle P_0, \leqslant_0, P_1, \leqslant_1, f_1, \ldots, P_m, \leqslant_m, f_m \rangle$ be a finite m-jump poset such that $P_0$ has least element* 0 *and greatest element* 1. *Then there is a finite set $\mathbf{G}_0$ of r.e. degrees, and there are finite sets $\mathbf{G}_k = \{\mathbf{d} \colon \exists \mathbf{a} \in \mathbf{G}_0 (\mathbf{a}^{(k)} = \mathbf{d})\}$ for each $k \in [1, m]$ such that the diagram in Fig.* 2 *commutes.*

*Furthermore, the embedding maps $0 \in P_0$ to $\mathbf{0}$ and $1 \in P_0$ to $\mathbf{0}'$.*

We have the following corollary, as proved in the introduction.

$$
\begin{tikzcd}
\langle P_0, \leq_0 \rangle \arrow[r, "f_1"] \arrow[d, "\cong"'] & \langle P_1, \leq_1 \rangle \arrow[r, "f_2"] \arrow[d, "\cong"'] & \cdots \arrow[r, "f_m"] & \langle P_m, \leq_m \rangle \arrow[d, "\cong"'] \\
\langle G_0, \leq \rangle \arrow[r] \arrow[d, "\subseteq"'] & \langle G_1, \leq \rangle \arrow[r] \arrow[d, "\subseteq"'] & \cdots \arrow[r] & \langle G_m, \leq \rangle \arrow[d, "\subseteq"'] \\
\mathcal{R} \arrow[r] & \mathcal{R}(\mathbf{0}') \arrow[r] & \cdots \arrow[r] & \mathcal{R}(\mathbf{0}^{(m)})
\end{tikzcd}
$$

FIGURE 2.

COROLLARY 7.9. *The existential theory of* $\mathscr{R}^{(<\omega)} = \langle \mathbf{R}, \mathbf{0}, \mathbf{0}', \leqslant, \leqslant_1, ..., \leqslant_n, ... \rangle$ *is decidable.*

If $\mathscr{J}$ is any recursively presented $<\omega$-jump-poset, then we can modify our construction to embed $\mathscr{I}$ into $\mathscr{R}^{(<\omega)}$. Requirements are listed as before, and form a recursive list. Each requirement has a well-defined dimension. We assign a given requirement to a tree of the correct dimension. As only finitely many trees will have been defined at any stage of the construction, and when a new tree $T^{k+1}$ is needed, we assign the finitely many requirements already assigned to $T^k$ and which need to be assigned to $T^{k+1}$ in the same order that the requirements were assigned to $T^k$. All lemmas now can be proved as before. It is also not difficult to show that there is a countable universal recursively-presented $<\omega$-jump poset. Hence:

THEOREM 7.10. *Let* $\mathscr{P} = \langle P_0, \leqslant_0, P_1, \leqslant_1, f_1, ..., P_m, \leqslant_m, f_m, \ldots \rangle$ *be a countable* $<\omega$*-jump poset such that* $P_0$ *has least element* 0 *and greatest element* 1. *Then for all m,* $\langle P_m, \leqslant_m \rangle$ *can be embedded isomorphically into* $\mathscr{R}[\mathbf{0}^{(m)}, \mathbf{0}^{(m+1)}]$ *so that Fig.* 2 *commutes for all* $m \in \mathbf{N}$. *Furthermore, the embedding maps* $0 \in P_0$ *to* $\mathbf{0}$ *and* $1 \in P_0$ *to* $\mathbf{0}'$.

Slaman and Sui have noted that the methods of proof of Theorem 7.8 should work for $<\omega$-jump usls in place of posets, and that we can add joins at all levels to our language and decide the corresponding $\exists$-theory if 1 is removed from the language. The construction need not be modified. The fact that the target sets are complements of prime ideals suffices to show that joins are preserved.

The methods presented in this paper will carry over to other priority arguments, if certain basic properties are satisfied. One can weaken the requirement assignment process to simultaneously assign requirements, and their derivatives, to the trees at all levels. Each requirement will have a basic module on each tree, which will be a segment of the tree of finite height. This assignment should provide the sentences generating action at each node of each tree. To study the interaction between requirements, an injury analysis similar to that provided by Lemma 2.2 (Interaction) is needed. A notion of control, different for each requirement, will be needed to determine how axioms are to be declared and elements placed into sets, and implication chains will be needed whenever a requirement needs to act off the true path. One can isolate a *guiding principle* for the definition of implication chains. Thus implication chains are to be built (and control relinquished) when there is a primary link which, if later switched, corrects any action for the requirement.

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