TOWARDS CHARACTERIZING THE INITIAL SEGMENTS OF THE DEGREES OF THE CEERS

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Abstract. Motivated by the problem of understanding small fragments of the first-order theory of the degree structure of the computably enumerable equivalence relations (ceers), we study which finite partial orders and lattices occur as initial segments of the ceer degrees above the finite degrees. We prove that the following structures are isomorphic to initial segments of $\mathbf{Ceers} \setminus \mathbf{Fin}$ (and, in fact, can be realized inside the light ceers above deg(Id)): (1) every finite distributive lattice, (2) the five-element nondistributive lattices N_5 and M_3, and (3) the bounded six-element partial order P_6, which is neither an upper nor a lower semilattice.

1. Introduction

The classification of mathematical objects up to natural notions of equivalence is a central theme in logic and in effective mathematics. In effective settings, a particularly natural way to compare classification problems is via *computable reducibility* between equivalence relations on ω: given equivalence relations E and F on ω, we write $E \leq F$ if there exists a total computable function $f\colon \omega \to \omega$ such that, for all $x, y \in \omega$,

$$x \, E \, y \iff f(x) \, F \, f(y).$$

This notion is the natural effective analogue of Borel reducibility, which plays a central role in the modern descriptive set-theoretic analysis of classification problems (see, for instance, [10, 11]).

In this paper we work with *computably enumerable equivalence relations*, abbreviated as *ceers*—that is, equivalence relations E on ω whose graphs $\{(x, y) \in \omega^2 : x \, E \, y\}$ are computably enumerable. Ceers arise naturally in several areas of logic and algebra: for example, they occur as relations of provable equivalence in effective formal theories [16, 15, 6], and they also arise from algorithmic problems such as word problems and isomorphism problems for computably presented algebraic structures [14, 7, 8].

The study of ceers is by now a well-established line of research, with roots going back to Ershov's seminal work in the early 1970s on numberings and positive

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equivalence relations [9]. A central viewpoint in the subject is the structural one: to study ceers up to computable reducibility and thus investigate the degree structure naturally induced by this reducibility. In its modern form, this perspective was decisively promoted by Gao and Gerdes [12], partly in analogy with the role of Borel reducibility in the classification theory of equivalence relations, but also as a development firmly rooted in classical computability-theoretic questions. It is this degree-theoretic viewpoint that underlies the present paper.

More precisely, two ceers E and F are said to be *equivalent*, written $E \equiv F$, if $E \leq F$ and $F \leq E$. The *degree* of a ceer E, denoted $\deg(E)$, is its $\equiv$-equivalence class. The partially ordered set of all ceer-degrees under $\leq$ will be denoted by **Ceers**.

A substantial amount is now known about the global structure of **Ceers**. The *finite ceers*, that is, the computable ceers with only finitely many equivalence classes, form an initial segment of order type ω, denoted by **Fin**. At the opposite end, **Ceers** has a top element, namely, the degree of any *universal ceer*, that is, a ceer to which every other ceer computably reduces. Universal ceers have been extensively studied and admit natural realizations (see, e.g., [2, 1]). A natural upper bound for **Fin** is provided by the identity ceer Id on ω; however, $\deg(\mathrm{Id})$ is not the least degree among ceers with infinitely many classes. This leads to the fundamental light/dark dichotomy introduced and studied in detail in [4]:

Definition 1.1 (Andrews, Sorbi [4])**.** A ceer E is *light* if $\mathrm{Id} \leq E$, equivalently, if E admits a computable transversal, while E is *dark* if it is neither finite nor light.

The dark degrees already exhibit striking structural behavior. There are infinitely many dark degrees that are minimal over **Fin**, but there is no greatest dark degree [4]. More generally, **Ceers** is neither an upper semilattice nor a lower semilattice, and its automorphism group is highly nontrivial: in particular, there are continuum many automorphisms related to the light/dark decomposition (see [4] for a systematic study of these phenomena). These structural properties also feed into the logical complexity of the degree structure. Andrews, Schweber, and Sorbi [3] showed that the first-order theory of **Ceers**, viewed as a poset, is computably isomorphic to true first-order arithmetic; the same holds for the induced substructures of the light degrees and of the dark degrees.

1.1. **Initial segments.** A classical route to understanding a degree structure is to analyze which finite posets or lattices occur as initial segments. Such results provide a concrete measure of the internal richness of the structure and often yield useful information about low-quantifier fragments of its first-order theory, in particular its $\forall\exists$-theory.

The first systematic investigation of large initial segments of $\mathbf{Ceers} \setminus \mathbf{Fin}$ is due to Andrews and Sorbi [5]. They show that $(\omega^{<\omega}, \sqsubseteq)$ embeds as an initial segment of $\mathbf{Ceers} \setminus \mathbf{Fin}$, and refine the construction so as to embed the free distributive lattice generated by the lower semilattice $\omega^{<\omega}$. A crucial point—made completely explicit in [5]—is that, in their construction, even if the embedded initial segment is internally a lattice, joins in the embedded copy need not be joins in the ambient structure **Ceers** (see [5, Observation 34]).

The present paper continues this line of research, but with a different emphasis. We focus on *actual* lattice-theoretic embeddings and show that, even beyond lattices, initial segments above **Fin** can realize patterns substantially more varied

than previously known. Our main results are summarized as follows (see Figure 1 for the nondistributive lattices and the nonlattice considered in the paper). The following structures embed as an initial segment of $\mathbf{Ceers} \setminus \mathbf{Fin}$:

- every finite distributive lattice (Theorem 5.1);
- the five-element nondistributive lattices M_3 and N_5 (Theorems 4.1 and 3.1);
- the six-element poset P_6, which is not a lattice (Theorem 6.1).

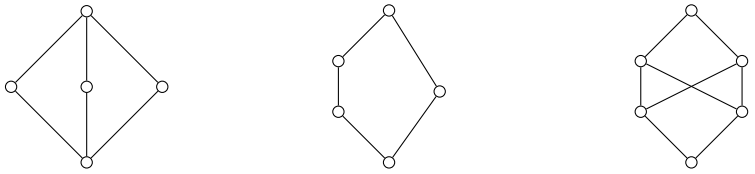


FIGURE 1. From left to right: M_3, N_5, and P_6.

These results highlight the richness of the initial segments of $\mathbf{Ceers} \setminus \mathbf{Fin}$, showing that this degree structure supports not only distributive-lattice behavior but also genuinely nondistributive, and even nonlattice, order types as initial segments. The proofs are presented in order of increasing technical difficulty: we first treat N_5 and M_3, then finite distributive lattices, and finally P_6.

2. Preliminaries

We use standard computability-theoretic notation and conventions; for more background, we refer to any textbook in the area. In particular, $\{\varphi_e\}_{e\in\omega}$ denotes a standard effective listing of the partial computable functions and $\{W_e\}_{e\in\omega}$ a standard effective listing of the c.e. subsets of ω.

2.1. **Ceers.** A *ceer* is a c.e. equivalence relation on ω. For a ceer E and $x \in \omega$, we write $[x]_E$ for the E-class of x. We freely identify ceers with their degrees when no confusion may arise.

Given ceers E and F, their *uniform join* (disjoint sum) $E \oplus F$ is defined by

$$x\,(E \oplus F)\,y \iff \big(x = 2u \;\&\; y = 2v \;\&\; u\,E\,v\big) \vee \big(x = 2u+1 \;\&\; y = 2v+1 \;\&\; u\,F\,v\big).$$

We will frequently appeal to the following

Lemma 2.1. *Let E be a ceer and W an infinite c.e. set such that whenever an element x enters W at a stage s, say, then $[x]_E$ does not change after stage s. Then $E \oplus \mathrm{Id} \equiv E$.*

Proof. Clearly, $E \leq E \oplus \mathrm{Id}$. To show that $E \oplus \mathrm{Id} \leq E$, note that by our assumption on E and W, we can speed up the enumeration of W in order to effectively enumerate an infinite computable subset $V \subseteq W$ such that whenever an element x enters V at a stage s, say, then no $y < s$ is currently E-equivalent to x (and thus, by our assumption on W, $[x]_E \cap [0, s) = \emptyset$). This set V now allows us to give the desired reduction: Reduce E off $[V]_E$ via the identity, and split V into two infinite subsets to reduce $(E \upharpoonright [V]_E) \oplus \mathrm{Id}$ to $E \upharpoonright [V]_E$. □

Following [5], we code lower cones using restrictions to c.e. sets.

Definition 2.2. Let E be a ceer and let $W \subseteq \omega$ be a nonempty c.e. set. Fix a total computable surjection $\pi\colon \omega \to W$. The *restriction* of E to W is the ceer $E \restriction W$ on ω given by

$$x\,(E \restriction W)\,y \iff \pi(x)\,E\,\pi(y).$$

Up to $\equiv$, the resulting ceer does not depend on the choice of π. Whenever no confusion arises, we will also speak informally of "the ceer induced by E on W".

Remark 2.3. Most of the sets to which we restrict in the construction, such as U_i, $U_i \cup U_j$, or $U_\ell^{\mathcal{L}}$, are computable domains. For the sake of readability, we work with them as literal domains. Formally, one fixes computable presentations of all these restrictions. Whenever the text states that an input or a value of a reduction "lies in" such a computable domain, the formal meaning is that its image under the fixed presentation map lies there. Likewise, if $X \subseteq Y$ are c.e. sets and we use the inclusion reduction $E \restriction X \leq E \restriction Y$, the formal map sends an index u for X to the least index v for Y with $\pi_Y(v) = \pi_X(u)$. All diagonalization requirements below are to be read with this convention. This convention only changes notation, not the degree of any restriction.

Lemma 2.4. *If W is a nonempty c.e. set, then $E \restriction W \leq E$. Moreover, if $R \leq E$ via f and $W = \mathrm{ran}(f)$, then $R \equiv E \restriction W$.*

Proof. The first claim is immediate from the definition. For the second, let $\pi\colon \omega \to W$ be any total computable surjection. To reduce R to $E \restriction W$, given x, search for the least $h(x)$ with $\pi(h(x)) = f(x)$; such an index exists because $f(x) \in W = \mathrm{ran}(f)$. To reduce $E \restriction W$ to R, given u, compute $\pi(u)$ and search for the least x_u such that $f(x_u) = \pi(u)$. This search also terminates because $\pi(u) \in \mathrm{ran}(f)$. Then

$$u\,(E \restriction W)\,v \iff \pi(u)\,E\,\pi(v) \iff f(x_u)\,E\,f(x_v) \iff x_u\,R\,x_v,$$

and the two reductions are computable. □

Lemma 2.5. *For all ceers E and F, if $E \leq F$, then there exists a nonempty c.e. set W such that $E \equiv F \restriction W$.*

Proof. Fix a reduction f witnessing $E \leq F$ and let $W = \mathrm{ran}(f)$, which is a nonempty c.e. set. By Lemma 2.4, we have $E \equiv F \restriction W$. □

2.2. **Initial segments and finite lattices.** If $(P, \leq_P)$ is a poset, an *initial segment* is a subset $I \subseteq P$ such that $x \leq_P y \in I$ implies $x \in I$. An *order embedding* $\iota\colon P \to Q$ is an injective map with $p \leq_P q \iff \iota(p) \leq_Q \iota(q)$; it embeds P *as an initial segment* of Q if $\iota[P]$ is an initial segment of Q. If I is an initial segment of a poset Q, then every meet computed inside I coincides with the meet in Q (whenever the meet exists in I). By contrast, joins computed inside I need not coincide with joins in Q, since upper bounds may lie outside I.

For finite distributive lattices we will use the following convenient variant of Birkhoff's representation theorem. If $\mathcal{L}$ is a finite distributive lattice, let $\mathcal{J}^*(\mathcal{L})$ be the poset obtained from the set of nonzero join-irreducibles by adjoining $0_{\mathcal{L}}$ as a least element.

Proposition 2.6. *If $\mathcal{L}$ is a finite distributive lattice, then $\mathcal{L}$ is isomorphic to the lattice of nonempty downward closed subsets of $\mathcal{J}^*(\mathcal{L})$ ordered by inclusion.*

Proof. This is the usual Birkhoff representation theorem, with $0_{\mathcal{L}}$ adjoined to every ideal so that the empty ideal is replaced by the singleton $\{0_{\mathcal{L}}\}$. To be precise, the map

$$\ell \longmapsto \{j \in \mathcal{J}^*(\mathcal{L}) : j \leq_{\mathcal{L}} \ell\}$$

is an isomorphism from $\mathcal{L}$ onto the lattice of nonempty downward closed subsets of $\mathcal{J}^*(\mathcal{L})$. □

3. Embedding N_5

We will now prove our results in the order of increasing difficulty and start with the following

Theorem 3.1. *The five-element non-modular lattice N_5 embeds (as a partial order) into* **Ceers** $\setminus$ **Fin** *(and indeed into the light ceers above* $\deg(\mathrm{Id})$*) as an initial segment.*

We hasten to add that this embedding is unlikely to preserve joins in **Ceers** and thus is probably not a lattice embedding; we have not been able to verify either way. (Of course, it does preserve meets.)

Proof. We represent the lattice N_5 as a partial order on the set $\{0, a, b, c, d\}$, with the orderings $0 < a < c < d$ and $0 < b < d$. We partition ω into three infinite computable sets U_0, U_1 and U_2. We will define a ceer D (on ω) and we heavily rely on Lemma 2.5. In particular, let $A = D \upharpoonright U_0$, $B = D \upharpoonright (U_1 \cup U_2)$, and $C = D \upharpoonright (U_0 \cup U_1)$. Our ceer D will only have equivalence classes of size at most 2.

We next discuss how we split up our task into requirements.

3.1. **The requirements for N_5.** The partial order embedding will map $0 \mapsto \mathrm{Id}$, $a \mapsto A$, $b \mapsto B$, $c \mapsto C$, and $d \mapsto D$ (or rather their degrees in **Ceers**). For technical reasons, we will ensure that $D \upharpoonright U_1 = \mathrm{Id} \upharpoonright U_1$ and $D \upharpoonright U_2 = \mathrm{Id} \upharpoonright U_2$. More importantly, we will also ensure that no element $x' \in U_0$ is D-equivalent to any element $x'' \in U_2$. Finally, we ensure a computable reduction from the ceer Id into A (and so $\mathrm{Id} \leq A$); we will simply fix an infinite computable subset $V_0 \subset U_0$ (coinfinite within U_0) for which we promise to never D-collapse any $x \in V_0$ with any other element into a single D-equivalence class. All this will certainly ensure $\mathrm{Id} \leq A \leq C \leq D$ and $\mathrm{Id} \leq B \leq D$. So it remains to ensure the non-reductions, and that the range is onto an initial segment, namely, onto the lower cone of D in **Ceers** $\setminus$ **Fin**.

In addition to these global requirements mentioned above, we will thus also need to ensure the following local requirements for all partial computable functions φ and ψ, which will be arranged in an infinite-injury priority argument on a tree of strategies:

$$N_\varphi : R \leq D \text{ via } \varphi \implies R \equiv D \text{ or } R \equiv C \text{ or } R \equiv B \text{ or } R \equiv A \text{ or } R \leq \mathrm{Id}$$

$$P^a_\psi : A \not\leq B \text{ via } \psi$$

$$P^b_\psi : B \not\leq C \text{ via } \psi$$

$$P^c_\psi : C \not\leq A \text{ via } \psi$$

3.2. **The strategies for N_5.** Let us first discuss the individual strategies for satisfying the local requirements mentioned above, and then how they can be combined.

Satisfying the local P-requirements in isolation is straightforward: For P^a_ψ, pick two fresh elements $x, x' \in U_0 - V_0$ (currently D-inequivalent to any other number) and wait for $\psi(x)$ and $\psi(x')$ to both be defined. If at least one of $\psi(x)$ and $\psi(x')$ is not in $U_1 \cup U_2$, or if they are already D-equivalent, then we are done; else we D-collapse x and x' into one D-class and keep $\psi(x)$ and $\psi(x')$ in two distinct D-classes. We will use similar strategies for P^b_ψ (picking fresh elements $x \in U_1$ and $x' \in U_2$ and expecting their images to be in $U_0 \cup U_1$); and for P^c_ψ (picking fresh elements $x \in U_1$ and $x' \in U_0 - V_0$ and expecting their images to be in U_0).

The basic strategy for N_φ in isolation splits into several cases, depending on where the range of φ lies. First of all, we may assume that φ looks more and more total. (So φ really determines the ceer R as simply the preimage of D under φ. Recall here that all D-equivalence classes are assumed to be finite, in fact of size at most 2.) There are now several cases to consider, with cases named to be mnemonic and to match later notation. One more item deserves mentioning right away: Since elements already in a D-equivalence class of size 2 cannot D-collapse anymore, we will only focus on elements entering $\mathrm{ran}(\varphi)$ while they are still in a D-equivalence class of size 1.

Case fin/1/2: The range of φ is finite, or is infinite but $\mathrm{ran}(\varphi) \subseteq^* V_0 \cup U_1$ or $\mathrm{ran}(\varphi) \subseteq^* V_0 \cup U_2$: Then clearly $R \leq D \upharpoonright (V_0 \cup U_1) \equiv \mathrm{Id}$ or $R \leq D \upharpoonright (V_0 \cup U_2) \equiv \mathrm{Id}$. (We use here, and will use repeatedly from now on, that since each D-equivalence class is finite, $D \upharpoonright (V_0 \cup U_1) \equiv \mathrm{Id}$ and $D \upharpoonright (V_0 \cup U_2) \equiv \mathrm{Id}$. Indeed, $R \equiv \mathrm{Id}$ iff the range of φ is infinite.)

Case 0/02: $\mathrm{ran}(\varphi) \subseteq^* U_0 \cup U_2$ and $\mathrm{ran}(\varphi) \cap (U_0 - V_0)$ is infinite: Then

$$R \leq (D \upharpoonright (U_0 \cup U_2)) \oplus \mathrm{Id} \leq D \upharpoonright (U_0 \cup U_2) \leq D \upharpoonright U_0 = A,$$

where the second inequality follows by Lemma 2.1, as we will see in the verification in detail. So we will need to ensure that $A \leq R$ as well.

To this end, we will "freeze" (i.e., make computable) D on $U_0 - \mathrm{ran}(\varphi)$, or rather, since $\mathrm{ran}(\varphi)$ is usually not computable but only c.e., we will freeze D on $U_0 - F_0$ for an infinite computable subset $F_0 \subseteq \mathrm{ran}(\varphi) \cap (U_0 - V_0)$, and we will thus restrict the action of any lower-priority strategies on U_0 to only affect F_0, which is still an infinite set. (Strictly speaking, we will restrict the action of strategies below this outcome to F_0 and require strategies to the right of this outcome to only work with "large" numbers. Furthermore, since numbers already in a D-equivalence class of size 2 cannot D-collapse with other elements anymore, we will only consider elements in a D-equivalence class of size 1 as eligible to enter F_0; so if all but finitely many numbers entering $\mathrm{ran}(\varphi)$ are already in a D-equivalence class of size 2, we will consider ourselves to be in Case *fin*/1/2.) Then we have that $A = D \upharpoonright U_0 \leq D \upharpoonright F_0 \leq D \upharpoonright \mathrm{ran}(\varphi) \leq R$ as desired.

Case 01: $\mathrm{ran}(\varphi) \subseteq^* U_0 \cup U_1$ and both $\mathrm{ran}(\varphi) \cap (U_0 - V_0)$ and $\mathrm{ran}(\varphi) \cap U_1$ are infinite: Then

$$R \leq (D \upharpoonright (U_0 \cup U_1)) \oplus \mathrm{Id} \leq D \upharpoonright (U_0 \cup U_1) = C,$$

where the second inequality again follows by Lemma 2.1, as we will see in the verification in detail. So we will need to ensure that $C \leq R$ as well.

To this end, we will again use a similar technique here: We will "freeze" D on $(U_0 \cup U_1) - \mathrm{ran}(\varphi)$, or rather, since $\mathrm{ran}(\varphi)$ is usually not computable, on $(U_0 - F_0) \cup (U_1 - F_1)$ for infinite computable subsets $F_0 \subseteq \mathrm{ran}(\varphi) \cap (U_0 - V_0)$ and $F_1 \subseteq \mathrm{ran}(\varphi) \cap U_1$, and thus restrict the action of any lower-priority strategies (below this outcome) on $U_0 \cup U_1$ to only affect F_0 and F_1, which are still infinite subsets inside U_0 and U_1, respectively. Now we have that $C = D \upharpoonright (U_0 \cup U_1) \leq D \upharpoonright (F_0 \cup F_1) \leq D \upharpoonright \mathrm{ran}(\varphi) \leq R$ as desired.

Case 12: $\mathrm{ran}(\varphi) \subseteq^* V_0 \cup U_1 \cup U_2$ and both $\mathrm{ran}(\varphi) \cap U_1$ and $\mathrm{ran}(\varphi) \cap U_2$ are infinite: Then

$$R \leq (D \upharpoonright (V_0 \cup U_1 \cup U_2)) \oplus \mathrm{Id} \leq D \upharpoonright (U_1 \cup U_2) = B,$$

where the second inequality again follows by Lemma 2.1 and our convention on V_0, as we will see in the verification in detail. So we will need to ensure that $B \leq R$ as well.

To this end, we will again use a similar technique here: We will "freeze" D on $(U_1 \cup U_2) - \mathrm{ran}(\varphi)$, or rather, since $\mathrm{ran}(\varphi)$ is usually not computable, on $(U_1 - F_1) \cup (U_2 - F_2)$ for infinite computable subsets $F_1 \subseteq \mathrm{ran}(\varphi) \cap U_1$ and $F_2 \subseteq \mathrm{ran}(\varphi) \cap U_2$, and thus restrict the action of any lower-priority strategies (below this outcome) on $U_1 \cup U_2$ to only affect F_1 and F_2, which are still infinite subsets inside U_1 and U_2, respectively. Now we have that $B = D \upharpoonright (U_1 \cup U_2) \leq D \upharpoonright (F_1 \cup F_2) \leq D \upharpoonright \mathrm{ran}(\varphi) \leq R$ as desired.

Case 012: Otherwise, all of $\mathrm{ran}(\varphi) \cap (U_0 - V_0)$, $\mathrm{ran}(\varphi) \cap U_1$ and $\mathrm{ran}(\varphi) \cap U_2$ are infinite: Then $R \leq D$, and we will want to show that $D \leq R$ as well. We will again use a similar technique here: We will "freeze" D on $(U_0 \cup U_1 \cup U_2) - \mathrm{ran}(\varphi)$, or rather, since $\mathrm{ran}(\varphi)$ is usually not computable, on $(U_0 - F_0) \cup (U_1 - F_1) \cup (U_2 - F_2)$ for infinite computable subsets $F_0 \subseteq \mathrm{ran}(\varphi) \cap (U_0 - V_0)$, $F_1 \subseteq \mathrm{ran}(\varphi) \cap U_1$ and $F_2 \subseteq \mathrm{ran}(\varphi) \cap U_2$, and thus restrict the action of any lower-priority strategies (below this outcome) on $U_0 \cup U_1 \cup U_2$ to only affect F_0, F_1 and F_2, which are still infinite subsets inside all of U_0, U_1 and U_2, respectively. Now we have that $D \leq D \upharpoonright (F_0 \cup F_1 \cup F_2) \leq D \upharpoonright \mathrm{ran}(\varphi) \leq R$ as desired.

So we can think of Case *fin*/1/2 as the finitary case, and Cases 0/02 through 012 as the infinitary cases as measured on a tree of strategies, with four different Π^0_2-outcomes.

Let us now analyze some typical conflicts between strategies. First, note that the P-strategies are finitary; so we can restart a lower-priority strategy after a P-strategy acts. Next, if a lower-priority N'-strategy works within F_0, F_1 and F_2 (provided by a higher-priority N-strategy with an infinitary outcome) in place of U_0, U_1 and U_2, then the N'-strategy can act as if in isolation. (Here, of course, if the N-strategy provides only one or two of F_0, F_1 and F_2, then the N'-strategy will use all of U_0, U_1 and U_2, respectively, instead for the remaining sets.) So we can focus on the conflicts of a P-strategy with one or several higher-priority N-strategies.

So let us analyze the possible kinds of actions that could cause conflicts. First of all, notice that unlike in many other constructions of ceers, here the only D-collapses of elements into one D-equivalence class are for the sake of diagonalization; and so we can ensure the properties we promised to uphold: D never collapses elements of V_0 with any other elements, and is the identity on U_1 and on U_2 separately; when witnesses are chosen, they are in a D-equivalence class of size 1, and so D contains

only equivalence classes of size at most 2 by initialization; finally, no elements in U_0 and U_2 can ever become D-equivalent.

To sum things up, collapses are only within the single set $U_0 - V_0$, or between $U_0 - V_0$ and U_1, or between U_1 and U_2. Also, as we go through more and more infinite outcomes of N-strategies, we thin out the sets $U_0 - V_0$, U_1 and U_2 into infinite computable "active" subsets F_0, F_1 and F_2 each time, freezing D outside the active sets to make it computable there.

3.3. **The tree of strategies for N_5.** We next define the tree T of strategies for our construction and assign requirements to the nodes ("strategies") on the tree.

The set of possible *outcomes* of our strategies is

$$\Lambda = \{\infty_{012} < \infty_{12} < \infty_{01} < \infty_0 < sat < fin\},$$

where $\{\infty_{012}, \infty_{12}, \infty_{01}, \infty_0, fin\}$ is the set of possible outcomes of N-strategies (as in the cases outlined above), and $\{sat, fin\}$ is the set of possible outcomes of P-strategies (denoting that the strategy is satisfied by active D-collapse, or by waiting forever to perform diagonalization, respectively).

The *tree of strategies* $T \subset \Lambda^{<\omega}$ and the *assignment of requirements* to nodes ("strategies") on T are now defined by recursion on the length of a strategy. We effectively order all N- and P-requirements in order type ω and assign the ith requirement in this list to all strategies $\sigma \in T$ of length i; we then declare the immediate successors of σ to be all nodes $\sigma^\frown\langle o\rangle$, where o ranges over the possible outcomes of the requirement assigned to σ.

Our construction will now be a standard infinite-injury construction performed on T.

3.4. **The environments of strategies for N_5.** Each strategy $\sigma \in T - \{\langle\rangle\}$ will be provided an *environment*, i.e., a triple of computable sets (F_0, F_1, F_2), by the strategy above it; this triple will indicate the sets $F_i \subseteq U_i$ with which σ can work to satisfy its requirement and will be infinite computable sets if σ is along the true path. The strategy $\langle\rangle$ of length 0 will start with the triple $(U_0 - V_0, U_1, U_2)$. If σ is provided at a stage s the environment (F_0, F_1, F_2), then $\sigma^\frown\langle o\rangle$ is provided at stage s an environment (F_0', F_1', F_2') dynamically by σ so as to ensure that each strategy has a sufficient set of numbers to work with.

In particular, under an infinite outcome of an N-strategy, at least one element of each (finite) D-equivalence class intersecting F_i will be in the range of the N-strategy's reduction φ.

In a nutshell, the environment of $\sigma^\frown\langle o\rangle$ thins out the given sets F_i to infinite subsets; if σ is an N-strategy having to drastically thin out a subset of U_i to establish a reduction to its R, as outlined in section 3.2, then σ thins out enough to establish this reduction in the corresponding component(s) of its environment. The fact that the infinitary outcomes of the N-strategies have to be very selective in defining the sets F_i will require us to use a lot of slow-down features to ensure that there is a sufficient collection of numbers currently not D-equivalent to any other number in each set.

3.5. **The full construction for N_5.** We now specify the full construction. At *substage* $t \leq s$ of a stage $s > 0$, a strategy $\sigma \in T$ of length t will act given its environment (F_0, F_1, F_2) and then, unless $t = s$ or a strategy ends a stage, determine its outcome o so that the strategy $\tau = \sigma^\frown\langle o\rangle$ can act at substage $t + 1$.

At stage 0, we simply *initialize* all strategies on T, i.e., we make all their parameters undefined or empty (except for the environment of the strategy $\langle\rangle$). The environment of $\langle\rangle$ at any stage s will always be $(U_0 - V_0, U_1, U_2)$.

The action of a strategy σ at a stage $s > 0$ is now determined by its environment and the requirement it is assigned to. Here, s_0 is the most recent stage $\leq s$ at which σ was initialized.

σ is a P^a_ψ-strategy: If σ currently has no witnesses, then σ chooses two witnesses $x, x' \in F_0$ currently D-inequivalent to any other element (if such numbers exist; else σ ends the stage). Then σ ends the substage with outcome *fin*.

Otherwise, σ currently has two witnesses x and x'. If currently xDx', then σ ends the substage with outcome *sat*.

Otherwise, check if $\psi(x)$ and $\psi(x')$ are both defined, both lie in $U_1 \cup U_2$, and they are not currently D-equivalent. If so, then σ makes x and x' D-equivalent and ends the substage with outcome *sat*; otherwise, it just ends the substage with outcome *fin*.

In all cases (unless σ ends the stage), let s'_0 be the least stage $\geq s_0$ at which σ first had its current outcome since its most recent initialization. Then σ provides the next strategy to act an environment (F'_0, F'_1, F'_2) such that each F'_i contains all elements $> \max\{s'_0, x, x'\}$ of F_i currently D-inequivalent to any other element.

σ is a P^b_ψ-strategy: If σ currently has no witnesses, then σ chooses two witnesses $x \in F_1$ and $x' \in F_2$ currently D-inequivalent to any other element (if such numbers exist; else σ ends the stage). Then σ ends the substage with outcome *fin*.

Otherwise, σ currently has two witnesses x and x'. If currently xDx', then σ ends the substage with outcome *sat*.

Otherwise, check if $\psi(x)$ and $\psi(x')$ are both defined, both lie in $U_0 \cup U_1$, and they are not currently D-equivalent. If so, then σ makes x and x' D-equivalent and ends the substage with outcome *sat*; otherwise, it just ends the substage with outcome *fin*.

In all cases (unless σ ends the stage), let s'_0 be the least stage $\geq s_0$ at which σ first had its current outcome since its most recent initialization. Then σ provides the next strategy to act an environment (F'_0, F'_1, F'_2) such that each F'_i contains all elements $> \max\{s'_0, x, x'\}$ of F_i currently D-inequivalent to any other element.

σ is a P^c_ψ-strategy: If σ currently has no witnesses, then σ chooses two witnesses $x \in F_1$ and $x' \in F_0$ currently D-inequivalent to any other element (if such numbers exist; else σ ends the stage). Then σ ends the substage with outcome *fin*.

Otherwise, σ has two witnesses x and x'. If currently xDx', then σ ends the substage with outcome *sat*.

Otherwise, check if $\psi(x)$ and $\psi(x')$ are both defined, both lie in U_0, and they are not currently D-equivalent. If so, then σ makes x and x' D-equivalent and ends the substage with outcome *sat*; otherwise, it just ends the substage with outcome *fin*.

In all cases (unless σ ends the stage), let s'_0 be the least stage $\geq s_0$ at which σ first had its current outcome since its most recent initialization. Then σ provides the next strategy to act an environment (F'_0, F'_1, F'_2) such that each F'_i contains all elements $> \max\{s'_0, x, x'\}$ of F_i currently D-inequivalent to any other element.

σ is an N_φ-strategy: We will first determine the outcome of σ since it will determine its action. If s is the first stage at which σ acts since its most recent initialization,

then σ ends the substage with outcome *fin*. Otherwise, for each possible outcome o of σ, let $s_o < s$ be the most recent stage at which σ has taken outcome o since its most recent initialization (or set $s_o = s$ if there is no such stage), and let $s'_o \leq s_o$ be the least stage at which σ acted and since which σ has not been initialized or taken an outcome left of outcome o (or set $s'_o = s$ if there is no such stage).

We distinguish several cases and let σ act according to the first applicable case:
Case 012: A new number, in each of F_0, F_1 and F_2 in the interval $(s_{\infty_{012}}, s)$, has entered $\mathrm{ran}(\varphi)$, which is currently D-inequivalent to any other number: We then choose the least such number from each set and enumerate it into the environment (F'_0, F'_1, F'_2) of $\sigma^\frown\langle\infty_{012}\rangle$. Then σ ends the substage with outcome ∞_{012}.
Case 12: A new number, in each of F_1 and F_2 in the interval $(s_{\infty_{12}}, s)$, has entered $\mathrm{ran}(\varphi)$, which is currently D-inequivalent to any other number: We then choose the least such number from each set and enumerate it into the last two coordinates of the environment (F'_0, F'_1, F'_2) of $\sigma^\frown\langle\infty_{12}\rangle$; σ also enumerates into F'_0 all elements $> s'_{\infty_{12}}$ of F_0 currently D-inequivalent to any other element. Then σ ends the substage with outcome ∞_{12}.
Case 01: A new number, in each of F_0 and F_1 in the interval $(s_{\infty_{01}}, s)$, has entered $\mathrm{ran}(\varphi)$, which is currently D-inequivalent to any other number: We then choose the least such number from each set and enumerate it into the first two coordinates of the environment (F'_0, F'_1, F'_2) of $\sigma^\frown\langle\infty_{01}\rangle$; σ also enumerates into F'_2 all elements $> s'_{\infty_{01}}$ of F_2 currently D-inequivalent to any other element. Then σ ends the substage with outcome ∞_{01}.
Case 0/02: A new number, in F_0 in the interval (s_{∞_0}, s), has entered $\mathrm{ran}(\varphi)$, which is currently D-inequivalent to any other number: We then choose the least such number from this set and enumerate it into the first coordinate of the environment (F'_0, F'_1, F'_2) of $\sigma^\frown\langle\infty_0\rangle$; σ also enumerates into F'_1 and F'_2, respectively, all elements $> s'_{\infty_0}$ of F_1 and F_2 currently D-inequivalent to any other element. Then σ ends the substage with outcome ∞_0.
Case fin/1/2: Otherwise, σ ends the substage with outcome *fin*; σ also provides $\sigma^\frown\langle\mathit{fin}\rangle$ an environment (F'_0, F'_1, F'_2) by enumerating into each F'_i all elements $> s'_{\mathit{fin}}$ of F_i currently D-inequivalent to any other element.

Note in particular that the only direct action of an N-strategy is to thin out the environment for lower-priority strategies.

At the end of stage s, we initialize all strategies $>_L \sigma$ where σ is the longest strategy active during stage s.

3.6. **The verification for N_5.** We now need to verify that our ceers A, B, C and D defined above are as required.

By definition of these ceers, we have $A \leq C \leq D$ and $B \leq D$ (since the domains on which the former are defined are subsets of the domains of the latter in each case). Also, $\mathrm{Id} \leq B$ since $\mathrm{Id} \leq D \upharpoonright U_1 \leq B$; and $\mathrm{Id} \leq A$ since $\mathrm{Id} \leq D \upharpoonright V_0 \leq A$.

The remainder of the verification will consist of a sequence of lemmas in which we prove some technical facts and then the satisfaction of all requirements.

Lemma 3.2. *(1) Each D-equivalence class has size at most 2.*
(2) No P-strategy σ waits at infinitely many stages before it can choose witnesses.

(3) *There is a* true path $f \in [T]$ *such that any strategy* $\sigma \subset f$ *acts infinitely often but no strategy* $\sigma <_L f$ *acts infinitely often. Each* $\sigma \subset f$ *is initialized at most finitely often.*
(4) *For each* $\sigma \subset f$*, the environment* (F_0, F_1, F_2) *of* σ *consists of three infinite computable sets* $F_0 \subseteq U_0 \smallsetminus V_0$ *and* $F_i \subseteq U_i$ *for* $i \in \{1, 2\}$*.*

Proof. Item (1) holds since D-collapses are only caused by P-strategies, and when a P-strategy σ D-collapses x and x', say, then x and x' were initially chosen to be D-inequivalent by σ, and since σ is not initialized until the D-collapse of x and x', no other strategy can have made x or x' D-equivalent to any number.

For the remaining items, we proceed by induction on the length n of σ. The lemma is clear for $\sigma = \langle\rangle$. Suppose it holds for all strategies of length $< n$.

Then item (2) holds since while a P-strategy σ is waiting to choose its witnesses, its environment sets F_i keep growing. The restriction that witnesses cannot be D-equivalent to any other element is now vacuous if the element is to be chosen from a set F_i defined in the infinitary way since all elements in F_i are such; if F_i is determined under a finitary outcome of σ^-, then no other strategy can use any element in F_i; so the only case we need consider more carefully is when F_i is defined in a finitary way under an infinitary outcome of σ^- (namely, F_0 under outcome ∞_{12}; F_2 under outcome ∞_{01}; and F_1 and F_2 under outcome ∞_0). But if all but finitely many elements that could otherwise enter F_i are used as witnesses by P-strategies, then infinitely many of these are never D-collapsed since they are chosen by P-strategies that never see their ψ-images converge.

As for (3), since the tree of strategies is finitely branching, there clearly is a leftmost σ of length n which acts infinitely often unless σ ends the stage at cofinitely many stages at which it acts; however, then σ would have to be a P-strategy unable to find witnesses in its environment contrary to item (2).

It remains to establish (4) for σ. Assume first that σ^- is a P-strategy, or an N-strategy such that $\sigma = \sigma^- {}^\frown \langle \mathit{fin} \rangle$. Then σ^-, given its environment (F_0, F_1, F_2), hands to σ for its environment (F_0', F_1', F_2') all but finitely many numbers from the environment of σ^-.

On the other hand, suppose that σ^- is an N-strategy and $\sigma = \sigma^- {}^\frown \langle o \rangle$ for an infinitary outcome $o \neq \mathit{fin}$: Then each F_i' is either a cofinite subset of F_i (and we can argue as in the previous paragraph), or F_i' is an infinite set of elements by the choice of the outcome. □

Lemma 3.3. *Each* P*-requirement is satisfied by the strategy* $\sigma \subset f$ *to which it is assigned.*

Proof. For a given P-requirement, fix the P-strategy $\sigma \subset f$ and assume that σ first acts at stage s_0 after its last initialization. Then at or after stage s_0, by Lemma 3.2(4), σ will pick two witnesses x and x' which are currently D-inequivalent. Since x and x' are explicitly removed from the environment of any $\tau \supset \sigma$ and cannot be added to the environment of any $\tau >_L \sigma$, and since no strategy $\tau < \sigma$ can D-collapse any elements without σ being initialized, we see that only σ can D-collapse x and x' into one D-equivalence class, and will do so, at a stage s_0', say, iff $\psi(x)$ and $\psi(x')$ are both defined and in the right set (namely, $U_1 \cup U_2$, $U_0 \cup U_1$ or U_0 for a P^a-, P^b- or P^c-requirement, respectively) and not D-equivalent at stage s_0'. (This uses the easy fact that by assumption, for each P-requirement, $\{x, x'\} \neq \{\psi(x), \psi(x')\}$ if ψ maps into the right set, so σ itself will not make $\psi(x)$

and $\psi(x')$ D-equivalent.) But then $\psi(x), \psi(x') < s'_0$, and so neither can be in the environment of any $\tau > \sigma$ acting at a stage $\geq s'_0$, and as before no $\tau < \sigma$ D-collapses any elements $< s'_0$ after stage s'_0, so we have xDx' iff $\neg(\psi(x)D\psi(x'))$ as desired. □

Lemma 3.4. *Suppose we have $\sigma^\frown\langle o\rangle \subset f$ for a strategy σ with outcome o along the true path and with an environment (F_0, F_1, F_2) created for $\sigma^\frown\langle o\rangle$ after the last stage s_0, say, after which $\sigma^\frown\langle o\rangle \subset f$ was initialized. Then, if x is enumerated into F_i for some $i \leq 2$ at a stage $s^x \geq s_0$, say, then no $x' < x$ which is in $U_i - F_i$ at stage s^x can be made D-equivalent to any other element after stage s^x, and x itself can be made D-equivalent to another number only by the action of a P-strategy $\tau \supseteq \sigma^\frown\langle o\rangle$.*

Proof. Each F_i is an infinite computable set: Whenever an element x first enters F_i at a stage $s^x \geq s_0$, any element $x' < x$ in U_i has already entered F_i by stage s^x or never will by initialization. Also, x itself can only be used by a P-strategy $\tau \supseteq \sigma^\frown\langle o\rangle$ for diagonalization after stage s^x, as claimed. □

Lemma 3.5. *Each N-requirement is satisfied by the strategy $\sigma \subset f$ to which it is assigned.*

Proof. For a given N-requirement, fix the N-strategy $\sigma \subset f$ and assume that σ first acts at stage s_0 after its last initialization. We now distinguish cases by the five possible outcomes o of σ, letting (F_0, F_1, F_2) be the environment of $\sigma^\frown\langle o\rangle$:

Case $o = \infty_{012}$: In this case, we will argue that $R \equiv D$. Clearly, $R \leq D$ via the given φ. Conversely, we need to show that $D \leq D \upharpoonright (F_0 \cup F_1 \cup F_2)$, since $\mathrm{Id} \leq D$ and, by the outcome, we have $F_0 \cup F_1 \cup F_2 \subseteq \mathrm{ran}(\varphi)$ and thus $D \upharpoonright (F_0 \cup F_1 \cup F_2) \leq D \upharpoonright \mathrm{ran}(\varphi) \leq R$. But this is immediate by Lemma 3.4.

Case $o = \infty_{12}$: In this case, we will argue that $R \equiv B$. First of all, $R \leq B$ via the given φ: This uses that Lemma 2.1 implies $B \oplus \mathrm{Id} \equiv B$ and that $\mathrm{ran}(\varphi) \cap F_0$ is finite, and so by Lemma 3.4, we have $R \leq D \upharpoonright (U_1 \cup U_2) = B$. Conversely, we need to show that $B \leq D \upharpoonright (F_1 \cup F_2)$, since by the outcome, we have $F_1 \cup F_2 \subseteq \mathrm{ran}(\varphi)$ and thus $D \upharpoonright (F_1 \cup F_2) \leq D \upharpoonright \mathrm{ran}(\varphi) \leq R$. But now Lemma 3.4 implies $B \leq D \upharpoonright (F_1 \cup F_2)$ as desired.

Case $o = \infty_{01}$: In this case, we will argue that $R \equiv C$. First of all, $R \leq C$ via the given φ: This uses that Lemma 2.1 implies $C \oplus \mathrm{Id} \equiv C$ and that $\mathrm{ran}(\varphi) \cap F_2$ is finite, and so by Lemma 3.4, we have $R \leq D \upharpoonright (U_0 \cup U_1) = C$. Conversely, we need to show that $C \leq D \upharpoonright (F_0 \cup F_1)$, since by the outcome, we have $F_0 \cup F_1 \subseteq \mathrm{ran}(\varphi)$ and thus $D \upharpoonright (F_0 \cup F_1) \leq D \upharpoonright \mathrm{ran}(\varphi) \leq R$. But now Lemma 3.4 implies that $C \leq D \upharpoonright (F_0 \cup F_1)$ as desired.

Case $o = \infty_0$: In this case, we will argue that $R \equiv A$. First of all, $R \leq A$ via the given φ: This uses that Lemma 2.1 implies $A \oplus \mathrm{Id} \equiv A$ and that $\mathrm{ran}(\varphi) \cap (F_1 \cup F_2)$ is finite, and so by Lemma 3.4, we have $R \leq D \upharpoonright U_0 = A$. Conversely, we need to show that $A \leq D \upharpoonright F_0$, since by the outcome, we have $F_0 \subseteq \mathrm{ran}(\varphi)$ and thus $D \upharpoonright F_0 \leq D \upharpoonright \mathrm{ran}(\varphi) \leq R$. But now Lemma 3.4 implies that $A \leq D \upharpoonright F_0$ as desired.

Case $o = fin$: In this case, we will argue that $R \leq \mathrm{Id}$. For this, we note, by Lemma 3.4, that we have $R \leq (D \upharpoonright U_1) \oplus \mathrm{Id}$ or $R \leq (D \upharpoonright U_2) \oplus \mathrm{Id}$. But by construction, both $D \upharpoonright U_1 \leq \mathrm{Id}$ and $D \upharpoonright U_2 \leq \mathrm{Id}$. □

This concludes the proof of Theorem 3.1. □

4. Embedding M_3

The next theorem proves the counterpart of Theorem 3.1 for M_3. We mention that the proof follows the same pattern; in fact, the situation is somewhat simplified by the fact that the structure of M_3 is, in some sense, more "symmetric" than the one in N_5. In any case, we will highlight the differences between the cases of M_3 and N_5, and we will state explicitly when some parts of the proof can just be derived from the proof of Theorem 3.1.

Theorem 4.1. *The nondistributive lattice M_3 embeds (as a partial order) into* $\mathbf{Ceers} \setminus \mathbf{Fin}$ *(and indeed into the light ceers above* $\deg(\mathrm{Id})$*) as an initial segment.*

Proof. We represent the lattice M_3 as a partial order on the set $\{0, a, b, c, d\}$, with the orderings $0 < a, b, c < d$ and with a, b and c pairwise incomparable. We partition ω into three infinite computable sets U_0, U_1 and U_2.

As in the proof of Theorem 3.1, relying on the fact that every ceer reducible to D can be thought of as having the form $D \upharpoonright W_i$ for some W_i, we will define a ceer D (on ω) and we will let $A = D \upharpoonright (U_0 \cup U_1)$, $B = D \upharpoonright (U_0 \cup U_2)$, and $C = D \upharpoonright (U_1 \cup U_2)$.

As before, our ceer D will only have equivalence classes of size at most 2. We next discuss how we split up our task into requirements. In contrast with Theorem 3.1, many requirements and strategies are symmetric, and hence it will often be convenient to refer to A, B, and C by just mentioning the restriction of D to the corresponding computable sets U_0, U_1 or U_2.

4.1. **The requirements for M_3.** The partial order embedding will map $0 \mapsto \mathrm{Id}$, $a \mapsto A$, $b \mapsto B$, $c \mapsto C$, and $d \mapsto D$ (or rather their degrees in **Ceers**). Similarly to what was done in Theorem 3.1, we will ensure that $D \upharpoonright U_i = \mathrm{Id} \upharpoonright U_i$ for $i \leq 2$: this ensures that $\mathrm{Id} \leq A, B, C \leq D$. (An important difference to Theorem 3.1 is therefore that we do not need to specify a computable set V_0 within which we promise never to make D-collapses.) It remains to ensure the non-reductions, and that the range is onto an initial segment, namely, onto the lower cone of D in $\mathbf{Ceers} \setminus \mathbf{Fin}$.

In addition to these global requirements mentioned above, we will thus also need to ensure the following local requirements. For all partial computable functions φ and ψ, we require, for $i < j \leq 2$ and $k < \ell \leq 2$ with $(i, j) \neq (k, \ell)$, that

$$N_\varphi : R \leq D \text{ via } \varphi \implies R \equiv D \text{ or } R \equiv C \text{ or } R \equiv B \text{ or } R \equiv A \text{ or } R \leq \mathrm{Id},$$

$$P_\psi^{(ij,k\ell)} : D \upharpoonright (U_i \cup U_j) \not\leq D \upharpoonright (U_k \cup U_\ell) \text{ via } \psi.$$

The satisfaction of the P-requirements ensures that A, B and C are pairwise incomparable. For example, satisfying $P_\psi^{(01,02)}$ for all ψ ensures that $A \not\leq B$. Moreover, the combination of the reductions together with the P-requirements suffices to ensure that Id and D are strictly below and strictly above A, B and C, respectively.

4.2. **The strategies for M_3.** We now discuss how to handle the P- and the N-requirements in isolation.

The P-requirements are straightforward: For instance, to satisfy $P_\psi^{(01,02)}$, we pick two fresh elements $x \in U_0$ and $x' \in U_1$ (currently D-inequivalent to any other number) and wait for $\psi(x)$ and $\psi(x')$ to both be defined. If at least one of

$\psi(x)$ and $\psi(x')$ is not in $U_0 \cup U_2$ or if they are already D-equivalent, then we are done. Otherwise, we D-collapse x and x', and we restrain $\psi(x)$ and $\psi(x')$ from D-collapsing. The other P-requirements are symmetric.

The basic strategy for N_φ in isolation splits into several cases, depending on the range of φ.

Case fin/0/1/2: The range of φ is finite, or it is infinite but almost entirely contained in one of the three sets that partition ω; that is, $\mathrm{ran}(\varphi) \subseteq^* U_i$ for some $i \leq 2$. Then clearly $R \leq D \upharpoonright U_i \equiv \mathrm{Id}$. Hence, R is either in **Fin** or equivalent to Id.

The next cases will use the "freezing" technique already employed in Theorem 3.1.

We first have three cases that are symmetric; so we compress them into one.

Case ij, for $i < j \leq 2$: $\mathrm{ran}(\varphi) \subseteq^* U_i \cup U_j$ and both $\mathrm{ran}(\varphi) \cap U_i$ and $\mathrm{ran}(\varphi) \cap U_j$ are infinite: Then

$$R \leq (D \upharpoonright (U_i \cup U_j)) \oplus \mathrm{Id} \leq D \upharpoonright (U_i \cup U_j),$$

where the second inequality follows by Lemma 2.1, as we will see in the verification in detail. So we will need to ensure that $D \upharpoonright (U_i \cup U_j) \leq R$ as well.

We will use the technique of "freezing" D discussed in the N_5-case. Indeed, we aim at making D computable on $(U_i \cup U_j)^2 - (\mathrm{ran}(\varphi))^2$, or rather, on $(U_i \cup U_j)^2 - (F_i \cup F_j)^2$ for infinite computable subsets $F_i \subseteq \mathrm{ran}(\varphi) \cap U_i$ and $F_j \subseteq \mathrm{ran}(\varphi) \cap U_j$, and thus we will restrict the action of any lower-priority strategies on $U_i \cup U_j$ to only affect $F_i \cup F_j$, which is still an infinite subset inside both U_i and U_j. (Strictly speaking, we will restrict the action of strategies below this outcome to F_i and to F_j and require strategies to the right of this outcome to only work with "large" numbers. Furthermore, since numbers already in a D-equivalence class of size 2 cannot D-collapse with other elements anymore, we will only consider elements in a D-equivalence class of size 1 as eligible to enter F_i or F_j; so if all but finitely many numbers entering $\mathrm{ran}(\varphi)$ are already in a D-equivalence class of size 2, we will consider ourselves to be in Case *fin*/0/1/2.) Now we have that $D \upharpoonright ((U_i \cup U_j)^2 - (F_i \cup F_j)^2) \leq \mathrm{Id}$; so, e.g., for $(i, j) = (0, 1)$, we obtain $A = D \upharpoonright (U_0 \cup U_1) \leq D \upharpoonright (F_i \cup F_j) \leq D \upharpoonright \mathrm{ran}(\varphi) \leq R$ as desired, and similarly in the other two cases.

Case 012: Otherwise, all of $\mathrm{ran}(\varphi) \cap U_0$, $\mathrm{ran}(\varphi) \cap U_1$ and $\mathrm{ran}(\varphi) \cap U_2$ are infinite: Then $R \leq D$, and we will want to show that $D \leq R$ as well. We will "freeze" D on $(U_0 \cup U_1 \cup U_2) - \mathrm{ran}(\varphi)$, or rather, since $\mathrm{ran}(\varphi)$ is usually not computable, on $(U_0 - F_0) \cup (U_1 - F_1) \cup (U_2 - F_2)$ for infinite computable subsets $F_0 \subseteq \mathrm{ran}(\varphi) \cap U_0$, $F_1 \subseteq \mathrm{ran}(\varphi) \cap U_1$ and $F_2 \subseteq \mathrm{ran}(\varphi) \cap U_2$, and we will thus restrict the action of any lower-priority strategies on $U_0 \cup U_1 \cup U_2$ to only affect $F_0 \cup F_1 \cup F_2$, which is still an infinite subset inside all of U_0, U_1 and U_2. Now we have that $D \upharpoonright ((U_0 \cup U_1 \cup U_2)^2 - (F_0 \cup F_1 \cup F_2)^2) \leq \mathrm{Id}$ and so $D \leq D \upharpoonright (F_0 \cup F_1 \cup F_2) \leq D \upharpoonright \mathrm{ran}(\varphi) \leq R$ as desired.

So we can think of Case *fin*/0/1/2 as the finitary case, and Cases ij (for $0 \leq i < j \leq 2$) and 012 as the infinitary cases as measured on a tree of strategies, with four different Π^0_2-outcomes.

The interactions between strategies are similar to the ones of Theorem 3.1. We briefly summarize them:

- All P-strategies are finitary, and hence we can restart a lower-priority strategy after a P-strategy acts.
- The conflicts we need to focus on are the ones involving a P-strategy and a higher-priority N-strategy.
- The only D-collapses of elements into one D-equivalence class are performed to satisfy a diagonalization requirement. This allows us to ensure that
  (1) D is the identity on U_0, U_1 and U_2 individually; and
  (2) when witnesses are chosen, they are D-singletons, and so D contains only equivalence classes of size at most 2 by initialization.

  In other words, we never allow D-collapses within a single set U_i. On the other hand, D-collapses between U_i and U_j, for $i \neq j$, are allowed.
- As we go through more and more infinite outcomes of N-strategies, we thin out the sets U_0, U_1 and U_2 into infinite computable "active" subsets F_0, F_1 and F_2 each time, freezing D outside the active sets to make it computable there.

4.3. **The tree of strategies and the environmentsfor M_3.** The *tree of strategies* $T \subset \Lambda^{<\omega}$ for our construction is similar to the one defined for N_5. The set of possible *outcomes* of our strategies is

$$\Lambda = \{\infty_{012} < \infty_{01} < \infty_{02} < \infty_{12} < \mathit{sat} < \mathit{fin}\},$$

where $\{\infty_{012}, \infty_{01}, \infty_{02}, \infty_{12}, \mathit{fin}\}$ is the set of possible outcomes of N-strategies, and $\{\mathit{sat}, \mathit{fin}\}$ is the set of possible outcomes of P-strategies. Recall that each strategy $\sigma \in T$ will be provided an *environment*, i.e., a triple of computable sets (F_0, F_1, F_2) by the strategy above it; this triple will indicate the sets $F_i \subseteq U_i$ within which σ can work to satisfy its requirement. The strategy $\langle\rangle$ of length 0 will start with the triple (U_0, U_1, U_2). If σ is provided at a stage s the environment (F_0, F_1, F_2), then $\sigma^\frown\langle o\rangle$ is provided at stage s an environment (F_0', F_1', F_2') dynamically by σ so as to ensure that each strategy has a sufficient set of numbers to work with.

4.4. **The full construction for M_3.** We now specify the full construction; we recall the key ingredients here.

At *substage* $t \leq s$ of a stage $s > 0$, a strategy $\sigma \in T$ of length t will act given its environment (F_0, F_1, F_2) and then, unless $t = s$ or a strategy ends a stage early, determine the strategy to act at substage $t + 1$ by determining its outcome.

At stage 0, we *initialize* all strategies on T, i.e., we make all their parameters undefined or empty (except for the environment of the strategy $\langle\rangle$). The environment of $\langle\rangle$ at any stage s will always be (U_0, U_1, U_2).

The action of a strategy σ at a stage $s > 0$ is now determined by its environment and the requirement it is assigned to. Here, s_0 is the most recent stage $\leq s$ at which σ was initialized.

σ is a $P_\psi^{(ij,k\ell)}$-strategy: If σ currently has no witnesses, then σ chooses two witnesses $x \in F_i$ and $x' \in F_j$ currently D-inequivalent to any other number (if such numbers exist; else σ ends the stage). Then, σ ends the substage with outcome fin.

Otherwise, σ currently has two witnesses x and x'. If currently xDx', then σ ends the substage with outcome sat.

Otherwise, check if $\psi(x)$ and $\psi(x')$ are both defined, both lie in $U_k \cup U_\ell$, and they are not currently D-equivalent. If so, then σ makes x and x' D-equivalent and ends the substage with outcome *sat*; otherwise, it just ends the substage with outcome *fin*.

In all cases (unless σ ends the stage), let s_0' be the least stage $\geq s_0$ at which σ first had its current outcome since its most recent initialization. Then σ provides the next strategy to act an environment (F_0', F_1', F_2') such that each F_m' contains all elements $> \max\{s_0', x, x'\}$ of F_m currently D-inequivalent to any other element.

σ is an N_φ-strategy: This case is almost the same as the one in Theorem 3.1.
Case 012: We proceed exactly as in Case 012 of Theorem 3.1.
Case ij for $i < j \leq 2$: A new number $> s_{\infty_{ij}}$ in each of F_i and F_j has entered $\mathrm{ran}(\varphi)$ such that both of them are currently not D-equivalent to any other number (and they are by convention $< s$): If this holds for more than one pair (i, j), choose the one with $i + j$ least; we then choose the least such number from each set and enumerate them into F_i' and F_j' of the environment (F_0', F_1', F_2') of $\sigma^\frown\langle\infty_{ij}\rangle$; σ also enumerates into F_k' (where $k \leq 2$ and different from i and j) all elements $> s'_{\infty_{ij}}$ of F_k currently D-inequivalent to any other element. Then σ ends the substage with outcome ∞_{ij}.
Case fin/0/1/2: Otherwise, σ ends the substage with outcome *fin*; σ also provides $\sigma^\frown\langle\mathit{fin}\rangle$ an environment (F_0', F_1', F_2') by enumerating into each F_i' all elements $> s'_{\mathit{fin}}$ of F_i currently D-inequivalent to any other element.

As in Theorem 3.1, the only direct action of an N-strategy is to thin out the environment for lower-priority strategies.

At the end of stage s, we initialize all strategies $>_L \sigma$, where σ is the longest strategy active during stage s.

4.5. **The verification for M_3.** We now need to verify that our ceers A, B, C and D defined above are as required.

By definition of these ceers, we have $A, B, C \leq D$ (since the domains on which A, B and C are defined are subsets of the domain of D). Also, $\mathrm{Id} \leq A, B$ since $\mathrm{Id} \leq D \upharpoonright U_0 \leq A, B$, and $\mathrm{Id} \leq C$ since $\mathrm{Id} \leq D \upharpoonright U_1 \leq C$.

The rest of the verification is divided into a series of lemmas as for Theorem 3.1. We will not state such lemmas with the corresponding proofs here as they are essentially the same, and we refer the reader to the verification part of Theorem 3.1: the only exception is Lemma 3.5. We just make a few comments:

- The analogue of Lemma 3.2 has a simplified proof. Indeed we do not have to consider the case $F_0 = U_0 - V_0$ as in Lemma 3.2, but just $F_i = U_i$ for $i \leq 2$.
- The analogue of Lemma 3.4 remains essentially the same.

Lemma 4.2. *Each P-requirement is satisfied by the strategy $\sigma \subset f$ to which it is assigned.*

Proof. In the proof of the analogue of Lemma 3.3, use the set $U_k \cup U_\ell$ for a $P^{(ij,k\ell)}$-requirement. □

Lemma 4.3. *Each N-requirement is satisfied by the strategy $\sigma \subset f$ to which it is assigned.*

Proof. For a given N-requirement, fix the N-strategy $\sigma \subset f$ and assume that σ first acts at stage s_0 after its last initialization. We now distinguish cases by the five possible outcomes o of σ, letting (F_0, F_1, F_2) be the environment of $\sigma^\frown\langle o\rangle$:

Case $o = \infty_{012}$: The proof is the same as for Lemma 3.5.

Case $o = \infty_{ij}$ for $i < j \leq 2$: In this case, we will argue that $R \equiv D \upharpoonright (U_i \cup U_j)$. First of all, $R \leq D \upharpoonright (U_i \cup U_j)$ via the given φ. This uses that Lemma 2.1 implies $D \upharpoonright (U_i \cup U_j) \oplus \mathrm{Id} \equiv D \upharpoonright (U_i \cup U_j)$ and that $\mathrm{ran}(\varphi) \cap F_k$ (for $k \leq 2$ different from i and j) is finite, and so by the analogue of Lemma 3.4, we have $R \leq D \upharpoonright (U_i \cup U_j)$, which equals A, B or C, respectively. Conversely, we need to show that $D \upharpoonright (U_i \cup U_j) \leq D \upharpoonright (F_i \cup F_j)$, since by the outcome, we have $F_i \cup F_j \subseteq \mathrm{ran}(\varphi)$ and thus $D \upharpoonright (F_i \cup F_j) \leq D \upharpoonright \mathrm{ran}(\varphi) \leq R$. But now the analogue of Lemma 3.4 implies that A or B or C reduces to $D \upharpoonright (F_i \cup F_j)$, respectively, as desired.

Case $o =$ fin/0/1/2: In this case, we will argue that $R \leq \mathrm{Id}$. For this, we note, by the analogue of Lemma 3.4, that for some $k \leq 2$ we have $R \leq (D \upharpoonright U_k) \oplus \mathrm{Id}$. But by construction, we have $(D \upharpoonright U_k) \equiv \mathrm{Id}$. □

This concludes the proof of Theorem 4.1. □

5. Embedding finite distributive lattices

Lachlan [13, Theorem 2] proved that all finite distributive lattices embed as an initial segment in the 1-degrees of the non-simple c.e. sets. Also, it is known that the interval $[\mathbf{0}_1, \mathbf{0}'_1]$ of the non-simple c.e. 1-degrees is isomorphic to the interval $[\deg(\mathrm{Id}), \deg(R_K)]$ of the ceers. Yet Lachlan's result does not yield information about initial segments of **Ceers** $\setminus$ **Fin** since it only applies to an interval (indeed, an initial segment) of the light degrees: In particular, it is easy to observe that R_K bounds many dark degrees (e.g., $R_S < R_K$ for any simple set, and clearly $\deg(R_S)$ is dark). Thus, the main result of this section provides some new information:

Theorem 5.1. *Every finite distributive lattice embeds (as a partial order) into* **Ceers** $\setminus$ **Fin** *(and indeed into the light ceers above* $\deg(\mathrm{Id})$*) as an initial segment.*

Proof. We will build a ceer D such that all its equivalence classes are finite (but not necessarily of size at most 2 as in the previous sections).

The proof will be given in the remainder of this section; we start with some lattice-theoretic definitions.

5.1. **Lattice-theoretic definitions for finite distributive lattices.** Given a finite distributive lattice $\mathcal{L}$, we denote by $\mathcal{J}(\mathcal{L})$ the collection of its join-irreducible elements (including $0_{\mathcal{L}}$). Birkhoff's representation theorem tells us that $\mathcal{L}$ is isomorphic to the lattice $\mathcal{DJ}(\mathcal{L})$ of the nonempty $<_{\mathcal{L}}$-downward closed subsets of $\mathcal{J}(\mathcal{L})$ under inclusion. This result, combined with Lemma 2.5 above, provides a strategy for embedding $\mathcal{L}$ as an initial segment of **Ceers** $\setminus$ **Fin**: We will encode the sets in $\mathcal{DJ}(\mathcal{L})$ (and thus $\mathcal{L}$) by suitable restrictions of a ceer D. More precisely, let us fix an enumeration $\{j_0, \ldots, j_n\}$ of $\mathcal{J}(\mathcal{L})$ such that $j_i <_{\mathcal{L}} j_k$ implies $i < k$. (So, in particular, $j_0 = 0_{\mathcal{L}}$.) For each $j \in \mathcal{J}(\mathcal{L})$, let U_j be an infinite computable set so

that the collection of the U_j so defined gives a partition of ω into $n+1$ many sets. Next, we use the sets U_j to encode the elements of $\mathcal{L}$ by setting

$$U_\ell^{\mathcal{L}} := \bigcup_{\substack{j \leq_{\mathcal{L}} \ell \\ j \in \mathcal{J}(\mathcal{L})}} U_j.$$

And now, for all $\ell \in \mathcal{L}$, we denote by A_ℓ the ceer $D \upharpoonright U_\ell^{\mathcal{L}}$.

Finally, looking ahead, we let $\mathcal{D}(\mathcal{I})$ be the $<_{\mathcal{L}}$-downward closure of a nonempty subset $\mathcal{I} \subseteq \mathcal{J}(\mathcal{L})$. Then we fix an ordering $<$ between all nonempty subsets of $\mathcal{J}(\mathcal{L}) - \{j_0\}$ such that for all nonempty $\mathcal{I}, \mathcal{I}' \subseteq \mathcal{J}(\mathcal{L}) - \{j_0\}$:

- $\mathcal{D}(\mathcal{I}) \supset \mathcal{D}(\mathcal{I}')$ implies $\mathcal{I} < \mathcal{I}'$, and
- $\mathcal{D}(\mathcal{I}) = \mathcal{D}(\mathcal{I}')$ and $\mathcal{I} \supset \mathcal{I}'$ implies $\mathcal{I} < \mathcal{I}'$.

We will use this ordering in subsections 5.3 and 5.4 to define the ordering of outcomes and our tree of strategies, respectively.

5.2. **The requirements for finite distributive lattices.** In order to see that all the ceers A_ℓ are above Id, it suffices to ensure $A_{j_0} = D \upharpoonright U_{j_0} = \mathrm{Id} \upharpoonright U_{j_0} \equiv \mathrm{Id}$.

The surjectivity of our embedding onto an initial segment of $\mathbf{Ceers} \setminus \mathbf{Fin}$ is established by meeting the following local requirements for all partial computable functions φ:

$$N_\varphi : R \leq D \text{ via } \varphi \implies R \equiv A_\ell \text{ for some } \ell \in \mathcal{L} \text{ or } R < \mathrm{Id}.$$

The injectivity of our embedding is ensured by meeting the following local requirements for all partial computable functions ψ and for any $j \in \mathcal{J}(\mathcal{L})$ and $\ell \in \mathcal{L}$ such that $j \not\leq_{\mathcal{L}} \ell$ (where we may assume ℓ to be maximal such, in fact, by the distributivity of $\mathcal{L}$, greatest such):

$$P_\psi^{j,\ell} : A_j \not\leq A_\ell \text{ via } \psi.$$

Lemma 5.2. *Our requirements ensure that the map $\ell \mapsto \deg(A_\ell)$ embeds $\mathcal{L}$ onto an initial segment of* $\mathbf{Ceers} \setminus \mathbf{Fin}$.

Proof. The surjectivity follows directly by the N-requirements, so to prove the lemma, it suffices to prove two items:

(1) For all $\ell, \ell' \in \mathcal{L}$, $\ell' \leq_{\mathcal{L}} \ell$ iff $A_{\ell'} \leq A_\ell$; and
(2) for all $\ell \in \mathcal{L}$, $A_\ell \geq \mathrm{Id}$.

The left-to-right direction of (1) is trivial by the definition of A_ℓ. For the converse, assume $\ell' \not\leq_{\mathcal{L}} \ell$. By the distributivity of $\mathcal{L}$, we can fix $j \in \mathcal{J}(\mathcal{L})$ minimal with $j \leq_{\mathcal{L}} \ell'$ and $j \not\leq_{\mathcal{L}} \ell$, and then assume ℓ greatest such. Our requirements will now ensure explicitly that $D \upharpoonright U_j \not\leq A_\ell$, which will imply $A_{\ell'} \not\leq A_\ell$.

As for (2), note that $\mathrm{Id} \equiv D \upharpoonright U_{j_0} \leq A_\ell$ by construction. □

5.3. **The strategies for finite distributive lattices.** First of all, similarly to the last two sections, we work with an "active" part F_j of each U_j, and as soon as an element enters $U_j - F_j$, we will no longer allow it to D-collapse.

Since there is never a need to D-collapse any element in U_{j_0}, we will start by setting $F_{j_0} = \emptyset$. (The precise definitions of the other F_j will follow in section 5.5. Each F_j will be infinite computable for $j \neq j_0$.)

Satisfying the local P-requirements in isolation is straightforward: For $P_\psi^{j,\ell}$, pick two fresh elements x, x' in F_j currently D-inequivalent to any other element, keep them D-inequivalent for now, and wait for $\psi(x)$ and $\psi(x')$ to both be defined. If at

least one of $\psi(x)$ and $\psi(x')$ is not in $U^{\mathcal{L}}_\ell$ or they are already D-equivalent, then we are done. Else, we D-collapse x and x' into one D-class and keep $\psi(x)$ and $\psi(x')$ D-inequivalent.

The basic strategy for N_φ in isolation splits into two main cases (the second of which splits into several subcases). First of all, we may assume that φ looks more and more total. So, by Lemma 2.5, φ really determines the ceer R as simply the preimage of D under φ. For an infinitary outcome for N_φ, we will need to monitor the elements in $\mathcal{J}(\mathcal{L})$ such that infinitely many elements in the range of φ are D-equivalent to some element in U_j for at least one $j \neq j_0$; actually, we will have to be more selective: We will search for infinitely many new elements in the D-closure $[F_j]_D$ of F_j (which may include elements from $U_{j'}$ for some $j' >_{\mathcal{L}} j$; however, these will "count" as if appearing in U_j).

We stress here that our construction will ensure that for each D-equivalence class, there will be a $<_{\mathcal{L}}$-least $j \in \mathcal{J}(\mathcal{L})$ such that the D-equivalence class contains an element of U_j, i.e., there will not be two minimal incomparable $j, j' \in \mathcal{J}(\mathcal{L})$ with U_j and $U_{j'}$ intersecting the same D-equivalence class. For each x, denote by j^x the $<_{\mathcal{L}}$-least $j \in \mathcal{J}(\mathcal{L})$ such that $U_j \cap [x]_D \neq \emptyset$. (More precisely, although we will usually suppress this and use it only implicitly, we will denote by j^x the $<_{\mathcal{L}}$-least $j \in \mathcal{J}(\mathcal{L})$ such that $U_j \cap [x]_D \neq \emptyset$ as measured at the stage s at which x enters $\mathrm{ran}(\varphi)$, or rather the first stage after x enters at which the strategy measuring is eligible to act.)

Case fin: The range of φ is finite, or infinite but

$$\mathrm{ran}(\varphi) \cap \bigcup_{j \in \mathcal{J}(\mathcal{L}) - \{j_0\}} [F_j]_D$$

is finite. Then clearly

$$R \leq D \upharpoonright \Big(\omega^2 - \Big(\bigcup_{j \in \mathcal{J}(\mathcal{L}) - \{j_0\}} [F_j]_D\Big)^2\Big) \leq \mathrm{Id}.$$

Cases $\infty_{\mathcal{I}}$: Otherwise, there is a ($<$-least) nonempty set $\mathcal{I}$ consisting of all $j \in \mathcal{J}(\mathcal{L}) - \{j_0\}$ such that there are infinitely many $x \in \mathrm{ran}(\varphi)$ with $j^x = j$ (and so $\mathrm{ran}(\varphi) \cap [F_j]_D$ is infinite). (See subsection 5.1 for the definition of $<$ used here.)

Let $\ell_{\mathcal{I}} = \bigvee_{j \in \mathcal{I}} j$. Our goal will be to show that $R \equiv A_{\ell_{\mathcal{I}}}$. It is easy to check that

$$R \leq D \upharpoonright \Big(\omega^2 - \Big(\bigcup_{j \not\leq \ell_{\mathcal{I}}} [F_j]_D\Big)^2\Big) \leq A_{\ell_{\mathcal{I}}}.$$

The reverse inequality $A_{\ell_{\mathcal{I}}} \leq R$, i.e., ensuring that

$$D \upharpoonright \bigcup_{j \leq \ell_{\mathcal{I}}} [F_j]_D \leq R,$$

will take some care: There are two obstacles to overcome, one of which we have already seen in the previous sections on embedding N_5 and M_3. This first obstacle is that for $j \in \mathcal{I}$, $\mathrm{ran}(\varphi) \cap [F_j]_D$ is an infinite (c.e.) but likely also a coinfinite subset of $[F_j]_D$, which makes it hard to ensure $D \upharpoonright \bigcup_{j \in \mathcal{I}} [F_j]_D \leq R$; the technique we use is the same as before, namely, *freezing*, i.e., going to an infinite computable subset

$F'_j \subseteq F_j$ with $[F'_j]_D \subseteq [\mathrm{ran}(\varphi)]_D \cap [F_j]_D$ and freezing D on $[U_j]_D - [F'_j]_D$ by not allowing further D-collapses; this will ensure that

$$D \upharpoonright \bigcup_{j\in\mathcal{I}} [U_j]_D \leq D \upharpoonright \bigcup_{j\in\mathcal{I}} [F'_j]_D \leq R.$$

The second obstacle is new: The set $\mathcal{I}$ need not be downward closed in $\mathcal{J}(\mathcal{L}) - \{j_0\}$, so we need to ensure that

$$D \upharpoonright \big(\big(\bigcup_{j\leq_{\mathcal{L}} \ell_{\mathcal{I}}} [F_j]_D\big)^2 - \big(\bigcup_{j\in\mathcal{I}} [F_j]_D\big)^2\big) \leq R. \tag{5.1}$$

For this, we use a new technique we call *covering*: For $j \notin \mathcal{I}$ with $j \leq_{\mathcal{L}} \ell_{\mathcal{I}}$, fix some $j' \in \mathcal{I}$ with $j' >_{\mathcal{L}} j$ and make each element of F_j D-equivalent to some element in $F_{j'}$ in such a way that the set of elements used in $F_{j'}$ is only a coinfinite subset $G_j^{j'}$ of $F'_{j'} \subseteq F_{j'} \cap \mathrm{ran}(\varphi)$. So each element of F_j will be D-equivalent to an element in $\mathrm{ran}(\varphi)$, which allows us to conclude that $[F_j]_D = [G_j^{j'}]_D$ for such j, yielding the desired claim (5.1).

Note, however, that this will introduce a new difficulty for lower-priority strategies: A strategy trying to show that $A_{j'} \not\leq A_\ell$ cannot use witnesses $x, x' \in G_j^{j'}$ since D-collapsing them will cause D-collapses in F_j, possibly D-collapsing the ψ-images of x and x' if $j \leq_{\mathcal{L}} \ell$. At the moment, of course, this is no problem since there are infinitely many numbers in $F_{j'} - G_j^{j'}$ we could use as witnesses instead. However, recall that each time we have an infinite outcome of a further N'-strategy, say, the set $F'_{j'}$ will shrink further to a subset $F''_{j'}$, possibly becoming a subset of $G_j^{j'}$, which conflicts with further P-strategies. The solution is to not allow elements of $G_j^{j'}$ to count as "true" elements of $F'_{j'} \cap \mathrm{ran}(\varphi')$; so when the N'-strategy measures whether j' is in its set $\mathcal{I}'$, it will only consider elements in $(F'_{j'} - \bigcup_{j<_{\mathcal{L}} j'} G_j^{j'}) \cap \mathrm{ran}(\varphi')$, i.e., elements x with $j^x = j'$.

5.4. **The tree of strategies for finite distributive lattices.** We next define a tree T of strategies for our construction and assign requirements to the nodes ("strategies") on the tree.

First recall the ordering $<$ between all nonempty subsets $\mathcal{I} \subseteq \mathcal{J}(\mathcal{L}) - \{j_0\}$ from subsection 5.1. The set of possible *outcomes* of our strategies is

$$\Lambda = \{\infty_{\mathcal{J}(\mathcal{L})-\{j_0\}} < \cdots < \infty_{\{j_1\}} < \mathit{sat} < \mathit{fin}\},$$

where $\{\infty_{\mathcal{J}(\mathcal{L})-\{j_0\}}, \ldots, \infty_{\{j_1\}}, \mathit{fin}\}$ is the set of possible outcomes of N-strategies (as in the cases outlined above, using the ordering $<$ on the nonempty subsets of $\mathcal{J}(\mathcal{L}) - \{j_0\}$ and assuming $\{j_1\}$ is $<$-greatest), and $\{\mathit{sat}, \mathit{fin}\}$ is the set of possible outcomes of P-strategies (denoting that the strategy is satisfied by active D-collapse, or by waiting forever to perform active diagonalization, respectively).

The *tree of strategies* $T \subset \Lambda^{<\omega}$ and the *assignment of requirements* to nodes ("strategies") on T are now defined by recursion on the length of a strategy. We effectively order all N- and P-requirements in order type ω and assign the ith requirement in this list to all strategies $\sigma \in T$ of length i; we then declare the immediate successors of σ to be all nodes $\sigma^\frown\langle o\rangle$ where o ranges over the possible outcomes of the requirement assigned to σ.

Our construction will now be a standard infinite-injury construction performed on T.

5.5. **The environments of strategies for finite distributive lattices.** The definition of the environments is similar to the one for Theorem 3.1 but with two twists. First of all, the strategy $\langle\rangle$ of length 0 will start with $F_{j_0} = \emptyset$ (and we will from now on suppress the set F_{j_0}) and with $F_j = U_j$ for $j \neq j_0$. And each strategy $\sigma \in T - \{\langle\rangle\}$ will be provided a tuple of computable sets $(F_{j_1}, \ldots, F_{j_n})$ by the strategy above it. Intuitively, numbers in the environments will be left behind by the infinitary outcomes of N-strategies to let lower-priority P-strategies perform their diagonalization. If σ is provided at a stage s the environment $(F_{j_1}, \ldots, F_{j_n})$, then $\sigma^\frown\langle o\rangle$ is provided at stage s an environment $(F'_{j_1}, \ldots, F'_{j_n})$ dynamically by σ so as to ensure that each strategy has a sufficient set of numbers to work with.

However, as indicated in section 5.3, there is one more change necessitated by the need for covering: Each set F_j (for $j \neq j_0$) is associated with a set

$$G_j = \bigcup_{j' >_{\mathcal{L}} j} G_j^{j'} \quad \text{where} \quad G_j^{j'} = \{x \in U_{j'} \mid [x]_D \cap F_j \neq \emptyset\}. \tag{5.2}$$

So this set G_j, while not a subset of F_j, acts as if it were, even though it is contained in various $F_{j'}$ for $j' >_{\mathcal{L}} j$. Symmetrically, we define the set

$$G^{j'} = \bigcup_{j <_{\mathcal{L}} j'} G_j^{j'}. \tag{5.3}$$

So this set $G^{j'}$, while a subset of $U_{j'}$, consists of all the elements of $U_{j'}$ over which a $P^{j',\ell}$-strategy has no control and must avoid for witnesses.

Our construction will now actively define the environments $(F_{j_1}, \ldots, F_{j_n})$ for each strategy $\sigma \neq \langle\rangle$ at each stage at which σ is eligible to act, whereas the definition of the sets G_j, $G^{j'}$ and $G_j^{j'}$ (which also depend on σ) will be via (5.2) and (5.3).

There is one more new feature we want to highlight before presenting the full construction, which will ensure that the outcomes are measured correctly: The concern is that as an N-strategy waits for various numbers to add to its environment under some outcome, it needs to make sure that no lower-priority outcome snatches away potential candidates for this environment for the sake of that lower-priority outcome's environment. We will say that the N-strategy *reserves* numbers *for an outcome's environment*; it will always reserve a number for the highest-priority outcome for which it does not yet have a sufficient set of numbers reserved before.

5.6. **The full construction for finite distributive lattices.** We now specify the full construction. At *substage* $t \leq s$ of a stage $s > 0$, a strategy $\sigma \in T$ of length t will act given its environment $(F_{j_1}, \ldots, F_{j_n})$ and then, after it performs its own action, unless $t = s$ or the strategy σ ends the stage early, determine its outcome o and the environment for $\tau = \sigma^\frown\langle o\rangle$ so that the strategy τ can act at substage $t + 1$.

At stage 0, we simply *initialize* all strategies on T, i.e., we make all their parameters undefined or empty. Also at stage 0, for the strategy $\langle\rangle$, we permanently set $F_j = U_j$ for all $j \neq j_0$.

The action of a strategy σ at a stage $s > 0$ is now determined by its environment and the requirement it is assigned to. In the following, $s^o < s$ is the least, and $s_o < s$ is the most recent, stage at which $\sigma^\frown\langle o\rangle$ was visited and not initialized since. (If $\sigma^\frown\langle o\rangle$ has not been visited since its most recent initialization, then we set $s^o = s_o = s$.)

σ is a $P_\psi^{j,\ell}$-strategy: If σ currently has no witnesses, then σ chooses two witnesses $x, x' \in F_j$ currently D-inequivalent to any other element in $\bigcup_{j' \leq_{\mathcal{L}} j} U_{j'}$ (if such numbers exist; else σ ends the stage). Then σ ends the substage with outcome *fin*.

Otherwise, σ has two witnesses x and x'. If currently xDx', then σ ends the substage with outcome *sat*.

Otherwise, check if $\psi(x)$ and $\psi(x')$ are both defined, both lie in $U_\ell^{\mathcal{L}}$, and they are not currently D-equivalent. If so, then σ makes x and x' D-equivalent, keeps $\psi(x)$ and $\psi(x')$ D-inequivalent from now on, and ends the substage with outcome *sat*; otherwise, it just ends the substage with outcome *fin*.

In all cases (unless σ ends the stage), σ provides the next strategy eligible to act an environment $(F'_{j_1}, \ldots, F'_{j_n})$ such that each $F'_{j'}$ contains all elements $> \max\{s^o, x, x'\}$ of $F_{j'}$ currently D-inequivalent to any other element in $\bigcup_{j'' \leq_{\mathcal{L}} j'} U_{j''}$.

σ is an N_φ-strategy: If s is the first stage at which σ is eligible to act since its most recent initialization, then σ ends the substage with outcome *fin*.

Otherwise, note that σ has $2^n - 1$ possible distinct infinitary outcomes $\infty_{\mathcal{I}}$ (for all nonempty sets $\mathcal{I} \subseteq \mathcal{J}(\mathcal{L}) - \{j_0\} = \{j_1, \ldots, j_n\}$) and the finitary outcome *fin*. Now σ first checks if there are an outcome $\infty_{\mathcal{I}}$, some $j \in \mathcal{J}(\mathcal{L}) - \{j_0\}$ and a number x such that

(1) x is currently not reserved by σ under an outcome $\leq \infty_{\mathcal{I}}$;
(2) $x \in [\mathrm{ran}(\varphi)]_D \cap F_j \cap (s_{\infty_{\mathcal{I}}}, s)$ if $j \in \mathcal{I}$;
(3) $x \in F_j \cap (s_{\infty_{\mathcal{I}}}, s)$ if $j \notin \mathcal{I}$;
(4) x currently is not D-equivalent to any other number in $\bigcup_{j' \leq_{\mathcal{L}} j} U_{j'}$; and
(5) there are currently not already 2^n many numbers in F_j reserved by σ under outcome $\infty_{\mathcal{I}}$.

If so, then choose the $<$-least such $\mathcal{I}$, and for this $\mathcal{I}$ the least $i \in (0, n]$ with such $j = j_i$, and for these $\mathcal{I}$ and j the least such x; then reserve x for σ under outcome $\infty_{\mathcal{I}}$; and call all numbers currently reserved for σ under an outcome $> \infty_{\mathcal{I}}$ no longer reserved.

Next check if there is an outcome $\infty_{\mathcal{I}}$ such that there are now, for each $j \leq \ell_{\mathcal{I}}$, 2^n many numbers in F_j reserved under outcome $\infty_{\mathcal{I}}$. If so, then for the leftmost (in fact, unique) such outcome $\infty_{\mathcal{I}}$, do the following:

(1) for each $j \leq_{\mathcal{L}} \ell_{\mathcal{I}}$ with $j \notin \mathcal{I}$, make some $x \in F_j$, which is currently reserved for σ under the outcome $\infty_{\mathcal{I}}$ and not yet used, D-equivalent to some $x' \in F_{j'}$, for a $j' \in \mathcal{I}$ with $j' >_{\mathcal{L}} j$, currently reserved for σ under the outcome $\infty_{\mathcal{I}}$ and not yet used; and add x into F'_j;
(2) for each j with $j \in \mathcal{I}$ or $j \not\leq_{\mathcal{L}} \ell_{\mathcal{I}}$, add some $x \in F_j$ currently reserved for σ under the outcome $\infty_{\mathcal{I}}$ and not yet used into F'_j;
(3) let σ take outcome $\infty_{\mathcal{I}}$; and
(4) set the environment for $\sigma^\frown\langle\infty_{\mathcal{I}}\rangle$ to be $(F'_{j_1}, \ldots, F'_{j_n})$ for the F'_j as defined above.

If there is no such outcome $\infty_{\mathcal{I}}$, then σ takes outcome *fin*, and we set the environment for $\sigma^\frown\langle\mathit{fin}\rangle$ to be $(F'_{j_1}, \ldots, F'_{j_n})$, where we let F'_j contain every element of $F_j \cap (s^{\mathit{fin}}, s)$ which is currently not D-equivalent to any other number in $\bigcup_{j' \leq_{\mathcal{L}} j} U_{j'}$.

At the end of stage s, we initialize all strategies $>_L \sigma$ where σ is the longest strategy active during stage s.

5.7. **The verification for finite distributive lattices.** We now verify that our construction has the desired properties in a number of lemmas.

Lemma 5.3. *The ceer D satisfies the following properties:*

1. *Each D-class is finite.*
2. *At every stage of the approximation of D, each D-class $[x]_D$ has a $<_{\mathcal{L}}$-least representative in some U_j, say; so j^x is well-defined for each x at every stage.*

Proof. (1) Let x be the least element of any fixed D-equivalence class. Then strategies ξ can D-collapse another element x' with x only if $x, x' \geq s^o$ for ξ's parameter s^o where o is a potential outcome of ξ, so there can be only finitely many such strategies. Each will only act at most once for $[x]_D$, and each time it can D-collapse at most one element with x, for diagonalization or covering.

(2) By induction, we will show that j^x is well-defined for any x. Any D-collapse can only happen in one of the following two scenarios: A P-strategy D-collapses elements $x, x' \in U_j$ while $j^x = j^{x'} = j$. Or an N-strategy D-collapses $x \in U_j$ with $y \in U_{j'}$ for $j <_{\mathcal{L}} j'$ while $j^x = j$ and $j^y = j'$, thus making now $j^x = j^y = j$ (and so y can never again be used "directly" in any future D-collapse). □

Lemma 5.4. *(1) There is a* true path *$f \in [T]$ such that any strategy $\sigma \subset f$ acts infinitely often but no strategy $\sigma <_L f$ acts infinitely often. Each $\sigma \subset f$ is initialized at most finitely often.*

2. *For each $\sigma \subset f$, the environment $(F_{j_1}, \ldots, F_{j_n})$ of σ consists of n many infinite computable sets $F_{j_i} \subseteq U_{j_i}$, and each of these contains infinitely many x which are not D-equivalent to any $y \in \bigcup_{j \not\leq_{\mathcal{L}} j_i} U_j$.*
3. *For each $\sigma \subset f$,*
$$D \upharpoonright \Big(\omega^2 - \Big(\bigcup_{j \in \mathcal{J}(\mathcal{L})} [F_j]_D\Big)^2\Big)$$
is a computable set.
4. *For each σ and each $\mathcal{I}$ with $\sigma^\frown\langle\infty_{\mathcal{I}}\rangle \subset f$,*
$$D \upharpoonright \Big(\omega^2 - \Big(\bigcup_{j \not\leq_{\mathcal{L}} \ell_{\mathcal{I}}} [F_j]_D\Big)^2\Big)$$
is a computable set.

Proof. By simultaneous induction on $|\sigma|$; the case $\sigma = \langle\rangle$ is trivial, so assume $|\sigma| > 0$:

(1) The argument is standard for infinite-injury arguments, except that a strategy ξ may end the stage early if it cannot find diagonalization witnesses in some F_j; however, since, by induction, F_j is infinite, this delay can never be forever.

(2) By construction, for each $i > 0$, each set F_{j_i} is infinite computable. The remainder of the claim follows by inspection of our construction and a counting argument.

(3) Fix a stage s_0 at which σ is eligible to act and after which σ is no longer initialized. Then after stage s_0, no $\tau <_L \sigma$ can D-collapse any number at all, and no $\tau >_L \sigma$ can D-collapse any number $x \leq s(x)$ where $s(x)$ is the greatest stage $< x$ at which σ is eligible to act (and $\lim_x s(x) = \infty$). For any N-strategy $\tau \subset \sigma$ with $\tau^\frown\langle o\rangle \subseteq \sigma$, τ cannot D-collapse any number $\leq s_o$ (and again s_o converges

to infinity). Any P-strategy $\tau \subset \sigma$ D-collapses at most finitely many numbers. Finally, any $\tau \supseteq \sigma$ only D-collapses numbers in $\bigcup_{j \in \mathcal{J}(\mathcal{L})} F_j$. The claim follows.

(4) The argument is similar but now let stage s_0 be such that also σ no longer reserves any numbers under any outcome $< \infty_{\mathcal{I}}$ after it. □

Lemma 5.5. *Each $P_\psi^{j,\ell}$-requirement is satisfied by the strategy $\sigma \subset f$ to which it is assigned.*

Proof. By Lemma 5.4(1), σ will eventually have stable witnesses x and x', say. Assume that $\psi(x)$ and $\psi(x')$ are both defined and lie in $U_\ell^{\mathcal{L}}$; else the P-requirement is satisfied trivially. Now, if xDx', then σ must have made them D-equivalent and kept $\psi(x)$ and $\psi(x')$ D-inequivalent; otherwise, $\neg(xDx')$, and $\psi(x)$ and $\psi(x')$ are D-equivalent. □

Lemma 5.6. *Each N_φ-requirement is satisfied by the strategy $\sigma \subset f$ to which it is assigned.*

Proof. Assume first that σ has outcome *fin* along the true path and that φ is a total function. Then we need to show that R is computable. By the feature of reserving numbers and since there are only finitely many possible $\mathcal{I}$, we can fix a stage s_0 after which, for any j and any $\mathcal{I}$, σ does not see any new numbers in $F_j \cap (s_{\infty_{\mathcal{I}}}, \infty)$ having entered $[\mathrm{ran}(\varphi)]_D$ which are currently not D-equivalent to any other number in $\bigcup_{j' \leq_{\mathcal{L}} j} U_{j'}$. Now fix $x, y \in \omega$ and compute $\varphi(x) \in U_j$ and $\varphi(y) \in U_{j'}$, say; assume they both converge by a stage $s \geq s_0$ at which σ is eligible to act. If $\varphi(x) \notin F_j$ or $\varphi(y) \notin F_{j'}$ then xRy iff $\varphi(x)$ and $\varphi(y)$ have been D-collapsed by stage s. Otherwise, except for finitely many values of $\varphi(x)$ and $\varphi(y)$, since they are in $[\mathrm{ran}(\varphi)]_D$, both must be D-equivalent to numbers in F_k and $F_{k'}$, respectively, for some $k <_{\mathcal{L}} j$ and $k' <_{\mathcal{L}} j'$, so we can argue by induction on the possible number of such k and k'. This gives that R is indeed computable.

Otherwise, suppose that σ has outcome $\infty_{\mathcal{I}}$ along the true path. Then $\mathcal{I}$ is the $<$-least nonempty subset of $\mathcal{J}(\mathcal{L}) - \{j_0\}$ such that, for all $j \in \mathcal{I}$, σ sees, at infinitely many stages, numbers enter $[\mathrm{ran}(\varphi)]_D$ that are currently in F_j and which are currently not D-equivalent to any other number in $\bigcup_{j' \leq_{\mathcal{L}} j} U_{j'}$. Then, by Lemma 5.4(4), we have

$$R \leq D \upharpoonright \bigcup_{j \leq_{\mathcal{L}} \ell_{\mathcal{I}}} [F_j]_D \leq D \upharpoonright U_{\ell_{\mathcal{I}}}^{\mathcal{L}} = A_{\ell_{\mathcal{I}}};$$

so it remains to show the reverse inequality. Now, by Lemma 5.4(4) applied to the environment $(F'_{j_1}, \ldots, F'_{j_n})$ of $\sigma^\frown\langle\infty_{\mathcal{I}}\rangle$ and by covering, we have

$$A_{\ell_{\mathcal{I}}} = D \upharpoonright U_{\ell_{\mathcal{I}}}^{\mathcal{L}} \leq D \upharpoonright \bigcup_{j \leq_{\mathcal{L}} \ell_{\mathcal{I}}} [F'_j]_D \leq$$
$$D \upharpoonright \Big(\bigcup_{j \in \mathcal{I}} [F'_j]_D \cup \bigcup_{\substack{j \notin \mathcal{I} \\ j \leq_{\mathcal{L}} \ell_{\mathcal{I}}}} [F'_j]_D \Big) \leq D \upharpoonright \mathrm{ran}(\varphi) \leq R.$$

This gives $R \equiv A_{\ell_{\mathcal{I}}}$ as desired. □

This concludes the proof of Theorem 5.1. □

6. Embedding P_6

Theorem 6.1. *The bounded six-element partial order P_6 which does not form an upper or lower semilattice embeds (as a partial order) into* $\mathbf{Ceers} \setminus \mathbf{Fin}$ *(and indeed into the light ceers above* $\deg(\mathrm{Id})$*) as an initial segment.*

Remark 6.2. Since P_6 is neither an upper nor a lower semilattice, this theorem also shows that the whole degree structure **Ceers** is neither an upper nor a lower semilattice. Again, by Lachlan [13, Theorem 2], this was known for the ceers in general since the 1-degrees of the c.e. sets are isomorphic to an interval of the ceers.

Proof of Theorem 6.1. We represent the partial order P_6 as being defined on the set $\{0, a, b, c, d, e\}$, with the ordering given by $0 < a, b$; $a < c, d$; $b < c, d$; $c, d < e$; $a \mid b$; and $c \mid d$. We will build a ceer E and appeal to Lemma 2.5 to define the other ceers below E, setting

$$A = E \upharpoonright (U_0 \cup U_1), \qquad B = E \upharpoonright (U_1 \cup U_3),$$
$$C = E \upharpoonright (U_0 \cup U_1 \cup U_3), \qquad D = E \upharpoonright (U_1 \cup U_2 \cup U_3),$$

where we write $\mathrm{dom}(E) = \{e^i_j \mid i < 4 \text{ and } j \in \omega\}$ for a computable 1–1 enumeration $\{e^i_j \mid i < 4 \text{ and } j \in \omega\}$ of ω, and we let $U_i = \{e^i_j \mid j \in \omega\}$ for $i < 4$. We write $e \sim e'$ to mean that e and e' are E-equivalent. If $x = e^i_j$, then set $\hat{x} = e^{i+2}_j$ if $i < 2$, and $\hat{x} = e^{i-2}_j$ if $i \geq 2$.

We next discuss how we split up our task into requirements.

6.1. **The requirements for P_6.** The following are the global requirements of the construction:

- For every $i, i' < 2$ and $j, j' \in \omega$, $e^i_j \sim e^{i'}_{j'}$ if and only if $e^{i+2}_j \sim e^{i'+2}_{j'}$. This ensures that $A \equiv E \upharpoonright (U_2 \cup U_3)$.
- For any distinct j and j', $e^1_j \not\sim e^1_{j'}$ and $e^3_j \not\sim e^3_{j'}$. This ensures that $E \upharpoonright U_1 \equiv \mathrm{Id} \equiv E \upharpoonright U_3$.

The partial order embedding will map $0 \mapsto \mathrm{Id}$, $a \mapsto A$, $b \mapsto B$, $c \mapsto C$, $d \mapsto D$, and $e \mapsto E$ (or rather to their degrees in **Ceers**). Provided that the global requirements are met, we will obviously have $\mathrm{Id} \leq A, B$ (since $\mathrm{Id} \equiv E \upharpoonright U_1 \leq A, B$); $A \leq C$; $A \leq D$ (since $A \equiv E \upharpoonright (U_2 \cup U_3) \leq D$); $B \leq C, D$; and $C, D \leq E$. Therefore it remains to ensure the non-reductions, and that the range is onto an initial segment, namely, the lower cone of E in $\mathbf{Ceers} \setminus \mathbf{Fin}$. For these, we have to ensure the following (local) requirements, for all e:

$$N_e : R \leq E \text{ via } \varphi_e \implies R \equiv E \text{ or } R \equiv A \text{ or } R \equiv B \text{ or } R \equiv C \text{ or } R \equiv D \text{ or } R \leq \mathrm{Id}$$
$$P^a_e : A \not\leq B \text{ via } \psi_e$$
$$P^b_e : B \not\leq A \text{ via } \psi_e$$
$$P^c_e : C \not\leq D \text{ via } \psi_e$$
$$P^d_e : D \not\leq C \text{ via } \psi_e$$

Here we take $\{\varphi_e\}_{e\in\omega}$ and $\{\psi_e\}_{e\in\omega}$ to be effective lists of all partial computable functions. We assume $\mathrm{dom}(R) = \omega$ and the domains of A, B, C, D, E as labeled sets. That is, we take $\mathrm{dom}(A) = U_0 \cup U_1$, $\mathrm{dom}(B) = U_1 \cup U_3$, $\mathrm{dom}(C) = U_0 \cup U_1 \cup U_3$,

$\mathrm{dom}(D) = U_1 \cup U_2 \cup U_3$ and $\mathrm{dom}(E) = U_0 \cup U_1 \cup U_2 \cup U_3 = \omega$. Without loss of generality, we will assume that φ_e and ψ_e are partial computable maps between the relevant labeled sets, without bothering to renumber them by ω. We will also refer to $\mathrm{dom}(A)$ simply as A, and so on.

6.2. **The strategies for P_6.** The strategies for the P-requirements in isolation are simple: E.g., for P_e^a, we pick fresh inequivalent followers $x, y \in U_0$ and wait for $\psi_e(x)\downarrow$ and $\psi_e(y)\downarrow$. If this ever happens, and if $\psi_e(x), \psi_e(y) \in U_1 \cup U_3$, then collapse x and y iff $\psi_e(x) \not\sim \psi_e(y)$ and restrain these four elements from further collapse from now on. (If we collapse x and y, then we also collapse $\hat{x}$ and $\hat{y}$ to ensure $E \upharpoonright (U_0 \cup U_1) \equiv E \upharpoonright (U_2 \cup U_3)$.)

Similarly, for P_e^b, we pick fresh inequivalent followers $x \in U_1$ and $y \in U_3$ and wait for $\psi_e(x)\downarrow$ and $\psi_e(y)\downarrow$. If this ever happens, and if $\psi_e(x), \psi_e(y) \in U_0 \cup U_1$, then collapse x and y iff $\psi_e(x) \not\sim \psi_e(y)$ and restrain these four elements from further collapse from now on.

Also similarly, for P_e^c, we pick fresh inequivalent followers $x \in U_0$ and $y \in U_3$ and wait for $\psi_e(x)\downarrow$ and $\psi_e(y)\downarrow$. If this ever happens, and if $\psi_e(x), \psi_e(y) \in U_1 \cup U_2 \cup U_3$, then collapse x and y iff $\psi_e(x) \not\sim \psi_e(y)$ and restrain these four elements from further collapse from now on.

Finally, for P_e^d, we pick fresh inequivalent followers $x \in U_1$ and $y \in U_2$ and wait for $\psi_e(x)\downarrow$ and $\psi_e(y)\downarrow$. If this ever happens, and if $\psi_e(x), \psi_e(y) \in U_0 \cup U_1 \cup U_3$, then collapse x and y iff $\psi_e(x) \not\sim \psi_e(y)$ and restrain these four elements from further collapse from now on.

The strategy for N_e is significantly more complicated than the corresponding strategies from the previous sections due to the asymmetric nature of our setup, which appears somewhat unmotivated but unavoidable as far as we can tell. We will describe it in very broad strokes first, distinguishing various cases for the range of the reduction φ_e relative to our four sets U_0, U_1, U_2 and U_3; each case will assume the failure of the previous cases:

Case ∞_{02}: Suppose that both $U_0 \cap \mathrm{ran}(\varphi_e)$ and $U_2 \cap \mathrm{ran}(\varphi_e)$ are infinite: Then we will want to show that $R \equiv E$. Clearly, $R \leq E$ via φ_e, so we only need to show that $E \leq R$. We will ensure this in two different ways: We will collapse each element $u_0 \in (U_0 \cup U_2) - \mathrm{ran}(\varphi_e)$ with an element $x_0 \in (U_0 \cup U_2) \cap \mathrm{ran}(\varphi_e)$ (a variation of "covering" from the previous section); and we will thin out U_1 and U_3 into environment sets F^1 and F^3, respectively, and then collapse each element $u_1 \in F^1 \cup F^3$ with an element $x_1 \in (U_0 \cup U_2) \cap \mathrm{ran}(\varphi_e)$ (a variation of "thinning" and "covering" from the previous sections). This will ensure that $E = E \upharpoonright (U_0 \cup U_1 \cup U_2 \cup U_3) \leq E \upharpoonright (([U_0]_E \cup [U_2]_E) \cap \mathrm{ran}(\varphi_e)) \leq E \upharpoonright \mathrm{ran}(\varphi_e) \leq R$.

Case ∞_{03}: Suppose that both $U_0 \cap \mathrm{ran}(\varphi_e)$ and $U_3 \cap \mathrm{ran}(\varphi_e)$ are infinite: Then we will want to show that $R \equiv C$. Since the failure of Case ∞_{02} implies that $U_2 \cap \mathrm{ran}(\varphi_e)$ is finite, we have that $R \leq E \upharpoonright (U_0 \cup U_1 \cup U_3) = C$ via φ_e, so we need to show that $C \leq R$. We will ensure this in three different ways: We will collapse each element $u_0 \in U_0 - \mathrm{ran}(\varphi_e)$ with an element $x_0 \in U_0 \cap \mathrm{ran}(\varphi_e)$ (a variation of "covering" from the previous section); we will thin out U_1 into an environment set F^1 and then collapse each element $u_1 \in F^1$ with an element $x_1 \in U_0 \cap \mathrm{ran}(\varphi_e)$ (a variation of "thinning" and "covering" from the previous sections); and we will thin out U_3 into an environment set $F^3 \subseteq [\mathrm{ran}(\varphi_e)]_E$ (a variation of "thinning"

from the previous sections). This will ensure that $C = E \upharpoonright (U_0 \cup U_1 \cup U_3) \leq E \upharpoonright (([U_0]_E \cup [U_3]_E) \cap \operatorname{ran}(\varphi_e)) \leq E \upharpoonright \operatorname{ran}(\varphi_e) \leq R$.

Case ∞_{12}: Suppose that both $U_1 \cap \operatorname{ran}(\varphi_e)$ and $U_2 \cap \operatorname{ran}(\varphi_e)$ are infinite: Symmetrically to Case ∞_{03}, we will want to show that $R \equiv D$. Since the failure of Case ∞_{02} implies that $U_0 \cap \operatorname{ran}(\varphi_e)$ is finite, we have that $R \leq E \upharpoonright (U_1 \cup U_2 \cup U_3) = D$ via φ_e, so we need to show that $D \leq R$. We will ensure this in three different ways: We will collapse each element $u_0 \in U_2 - \operatorname{ran}(\varphi_e)$ with an element $x_0 \in U_2 \cap \operatorname{ran}(\varphi_e)$ (a variation of "covering" from the previous section); we will thin out U_3 into an environment set F^3 and then collapse each element $u_1 \in F^3$ with an element $x_1 \in U_2 \cap \operatorname{ran}(\varphi_e)$ (a variation of "thinning" and "covering" from the previous sections); and we will thin out U_1 into an environment set $F^1 \subseteq [\operatorname{ran}(\varphi_e)]_E$ (a variation of "thinning" from the previous sections). This will ensure that $D = E \upharpoonright (U_1 \cup U_2 \cup U_3) \leq E \upharpoonright (([U_1]_E \cup [U_2]_E) \cap \operatorname{ran}(\varphi_e)) \leq E \upharpoonright \operatorname{ran}(\varphi_e) \leq R$.

Case ∞_0: Suppose that $U_0 \cap \operatorname{ran}(\varphi_e)$ is infinite: Then we will want to show that $R \equiv A$. Since the failure of Cases ∞_{02} and ∞_{03} implies that both $U_2 \cap \operatorname{ran}(\varphi_e)$ and $U_3 \cap \operatorname{ran}(\varphi_e)$ are finite, we have that $R \leq E \upharpoonright (U_0 \cup U_1) = A$ via φ_e, so we need to show that $A \leq R$. We will ensure this in two different ways: We will collapse each element $u_0 \in U_0 - \operatorname{ran}(\varphi_e)$ with an element $x_0 \in U_0 \cap \operatorname{ran}(\varphi_e)$ (a variation of "covering" from the previous section); and we will thin out U_1 into an environment set F^1 and then collapse each element $u_1 \in F^1$ with an element $x_1 \in U_0 \cap \operatorname{ran}(\varphi_e)$ (a variation of "thinning" and "covering" from the previous sections). This will ensure that $A = E \upharpoonright (U_0 \cup U_1) \leq E \upharpoonright ([U_0]_E \cap \operatorname{ran}(\varphi_e)) \leq E \upharpoonright \operatorname{ran}(\varphi_e) \leq R$.

Case ∞_2: Suppose that $U_2 \cap \operatorname{ran}(\varphi_e)$ is infinite: Symmetrically to Case ∞_0, we will want to again show that $R \equiv A$. Since the failure of Cases ∞_{02} and ∞_{12} implies that both $U_0 \cap \operatorname{ran}(\varphi_e)$ and $U_1 \cap \operatorname{ran}(\varphi_e)$ are finite, we have that $R \leq E \upharpoonright (U_2 \cup U_3) \equiv A$ via φ_e, so we need to show that $A \leq R$. We will ensure this in two different ways: We will collapse each element $u_0 \in U_2 - \operatorname{ran}(\varphi_e)$ with an element $x_0 \in U_2 \cap \operatorname{ran}(\varphi_e)$ (a variation of "covering" from the previous section); and we will thin out U_3 into an environment set F^3 and then collapse each element $u_1 \in F^3$ with an element $x_1 \in U_2 \cap \operatorname{ran}(\varphi_e)$ (a variation of "thinning" and "covering" from the previous sections). This will ensure that $A \equiv E \upharpoonright (U_2 \cup U_3) \leq E \upharpoonright ([U_2]_E \cap \operatorname{ran}(\varphi_e)) \leq E \upharpoonright \operatorname{ran}(\varphi_e) \leq R$.

Case ∞_{13}: Suppose that both $U_1 \cap \operatorname{ran}(\varphi_e)$ and $U_3 \cap \operatorname{ran}(\varphi_e)$ are infinite: Then we will want to show that $R \equiv B$. Since the failure of Cases ∞_{03} and ∞_{12} implies that both $U_0 \cap \operatorname{ran}(\varphi_e)$ and $U_2 \cap \operatorname{ran}(\varphi_e)$ are finite, we have that $R \leq E \upharpoonright (U_1 \cup U_3) = B$ via φ_e, so we need to show that $B \leq R$. We will ensure this as follows: We will thin out U_1 and U_3 into environment sets $F^1 \subseteq [\operatorname{ran}(\varphi_e)]_E$ and $F^3 \subseteq [\operatorname{ran}(\varphi_e)]_E$, respectively (a variation of "thinning" from the previous sections). This will ensure that $B = E \upharpoonright (U_1 \cup U_3) \leq E \upharpoonright (([U_1]_E \cup [U_3]_E) \cap \operatorname{ran}(\varphi_e)) \leq E \upharpoonright \operatorname{ran}(\varphi_e) \leq R$.

Case fin: Otherwise, by the failure of the above cases, both $U_0 \cap \operatorname{ran}(\varphi_e)$ and $U_2 \cap \operatorname{ran}(\varphi_e)$ are finite, and at most one of $U_1 \cap \operatorname{ran}(\varphi_e)$ or $U_3 \cap \operatorname{ran}(\varphi_e)$ is infinite, say, $U_1 \cap \operatorname{ran}(\varphi_e)$ is the only one that may be infinite: Then we will have that $R \leq E \upharpoonright U_1 \equiv \mathrm{Id}$ since by our construction, we will make $E \upharpoonright U_1 \equiv \mathrm{Id}$.

Note that in all the above cases, whenever we collapse an element $u \in U_i \cup U_{i+1}$ with an element $x \in U_i$ (for $i \in \{0,2\}$), we also need to collapse the corresponding elements $\hat{u}$ and $\hat{x}$ in order to ensure that $E \restriction (U_2 \cup U_3) \equiv E \restriction (U_0 \cup U_1) = A$.

6.3. **The tree of strategies for P_6.** The *tree of strategies* is a finitely branching tree. Nodes of length $5e$ are assigned the requirement N_e, while nodes of length $5e+1$, $5e+2$, $5e+3$ and $5e+4$ are assigned the requirement P_e^a, P_e^b, P_e^c, and P_e^d, respectively. The *outcomes* of a P-node are $sat < fin$. The *outcomes* of an N-node are $\infty_{02} < \infty_{03} < \infty_{12} < \infty_0 < \infty_2 < \infty_{13} < fin$. Our construction will be a standard infinite-injury argument carried out on the tree of strategies.

6.4. **The environments of strategies, notation and parameters for P_6.** Each strategy α on the tree will be provided with an *environment* of the form (F_α^1, F_α^3) determined by the higher-priority strategies. For $i \in \{1,3\}$, we will always have $F_\alpha^i \subseteq U_i$, and if $\alpha \subseteq \beta$ then $F_\beta^i \subseteq F_\alpha^i$. The sets F_α^i will be infinite computable sets if α is along the true path of the construction. (For our current construction, it is easier not to "thin out" the sets U_0 and U_2 but to just use "covering".)

As usual, the definition of these environment sets will be initialized if the construction visits left of α. We will always take $F_{\langle\rangle}^i = U_i$. For $o \in \{sat, fin\}$, we always set $F_{\alpha^\frown\langle o\rangle}^i = \{x \in F_\alpha^i \mid [x]_E \subseteq (s^*, \infty)\}$, where s^* is the first stage at which α was eligible to act since its most recent initialization. (We set $F_{\alpha^\frown\langle o\rangle}^i = \emptyset$ if s^* does not exist.) For an infinitary outcome o, $F_{\alpha^\frown\langle o\rangle}^i$ will be determined dynamically when α plays its outcome o.

To *initialize* a node means to reset all parameters associated with that node. The (requirement of a) P-strategy α is said to be *currently satisfied* at stage s if there are followers x_α and y_α such that $\psi_e(x_\alpha) \downarrow$ and $\psi_e(y_\alpha) \downarrow$ and $x_\alpha \sim y_\alpha$ iff $\psi_e(x_\alpha) \not\sim \psi_e(y_\alpha)$.

6.5. **The full construction for P_6.** At stage $s = 0$, we initialize all requirements and do nothing else.

At stage $s > 0$, we define δ_s of length s. Assume we have defined $\alpha \subset \delta_s$. We will need to establish the value of $\delta_s(|\alpha|)$. When we say that α *takes outcome* o, we mean that we set $\delta_s(|\alpha|) = o$. For a strategy α, we define s^o to be the greatest stage $< s$ at which $\alpha^\frown\langle o\rangle$ was eligible to act since the most recent initialization of $\alpha^\frown\langle o\rangle$. (If such a stage does not exist, we set $s^o = s$.)

Each strategy α has one additional *bounding parameter b at stage s* defined by

$$b = \min\big(\{s\} \cup \{u_0^\beta, \widehat{u_0^\beta} \mid \beta^\frown\langle o\rangle \subseteq \alpha \text{ and } o \in \{\infty_{02}, \infty_{03}, \infty_{12}, \infty_0, \infty_2\}\}\big).$$

For easier notation, we first present the following

Definition 6.3. (1) A number u is 0-*good* if $[u]_E \subseteq U_0$.
(2) A number $u \in F_\alpha^1$ is 1-*good* if $[u]_E \subseteq U_0 \cup F_\alpha^1$.
(3) A number u is 2-*good* if $[u]_E \subseteq U_2$.
(4) A number $u \in F_\alpha^3$ is 3-*good* if $[u]_E \subseteq U_2 \cup F_\alpha^3$.

We now distinguish five cases.

α is a P_e^a-node: There are three subcases:
(i) If α is currently satisfied, then we take outcome *sat* and end the substage.

(ii) If α is currently not satisfied and x_α and y_α are undefined, then check if there are 0-good x and y such that $[x]_E, [y]_E, [\hat{x}]_E, [\hat{y}]_E \subseteq (s^*, b)$; $x \not\curlyeqprec y$; and $\hat{x} \not\curlyeqprec \hat{y}$. If so, then we define $x_\alpha = x$ and $y_\alpha = y$, initialize all nodes extending $\alpha^\frown\langle \mathit{fin}\rangle$, and take outcome *fin*; otherwise end the stage.

(iii) If α is currently not satisfied and x_α and y_α are defined, then check if $\psi_e(x_\alpha) \downarrow$ and $\psi_e(y_\alpha) \downarrow$. If not, then we take outcome *fin*; otherwise, we take outcome *sat* and initialize all nodes extending $\alpha^\frown\langle \mathit{sat}\rangle$. If $\psi_e(x_\alpha) \not\curlyeqprec \psi_e(y_\alpha)$, then we collapse x_α with y_α, and we also collapse $\widehat{x_\alpha}$ with $\widehat{y_\alpha}$.

α **is a** P_e^b**-node:** There are three subcases:

(i) If α is currently satisfied, then we take outcome *sat* and end the substage.

(ii) If α is currently not satisfied and x_α and y_α are undefined, then we check if there are a 1-good x and a 3-good y such that $[x]_E, [y]_E \subseteq (s^*, s)$ and $x \not\curlyeqprec y$. If so, then we define $x_\alpha = x$ and $y_\alpha = y$, initialize all nodes extending $\alpha^\frown\langle \mathit{fin}\rangle$, and take outcome *fin*; otherwise end the stage.

(iii) If α is currently not satisfied and x_α and y_α are defined, then check if $\psi_e(x_\alpha) \downarrow$ and $\psi_e(y_\alpha) \downarrow$. If not, then we take outcome *fin*; otherwise, we take outcome *sat* and initialize all nodes extending $\alpha^\frown\langle \mathit{sat}\rangle$. If $\psi_e(x_\alpha) \not\curlyeqprec \psi_e(y_\alpha)$, then we collapse x_α with y_α.

α **is a** P_e^c**-node:** There are three subcases:

(i) If α is currently satisfied, then we take outcome *sat* and end the substage.

(ii) If α is currently not satisfied and x_α and y_α are undefined, then we check if there are a 0-good x and a 3-good y such that $[x]_E \subseteq (s^*, b)$ and $[y]_E \subseteq (s^*, s)$ and $x \not\curlyeqprec y$. If so, then we define $x_\alpha = x$ and $y_\alpha = y$, initialize all nodes extending $\alpha^\frown\langle \mathit{fin}\rangle$, and take outcome *fin*; otherwise end the stage.

(iii) If α is currently not satisfied and x_α and y_α are defined, then check if $\psi_e(x_\alpha) \downarrow$ and $\psi_e(y_\alpha) \downarrow$. If not, then we take outcome *fin*; otherwise, we take outcome *sat* and initialize all nodes extending $\alpha^\frown\langle \mathit{sat}\rangle$. If $\psi_e(x_\alpha) \not\curlyeqprec \psi_e(y_\alpha)$, then we collapse x_α with y_α.

α **is a** P_e^d**-node:** There are three subcases:

(i) If α is currently satisfied, then we take outcome *sat* and end the substage.

(ii) If α is currently not satisfied and x_α and y_α are undefined, then we check if there are a 1-good x and a 2-good y such that $[x]_E \subseteq (s^*, s)$ and $[y]_E \subseteq (s^*, b)$ and $x \not\curlyeqprec y$. If so, then we define $x_\alpha = x$ and $y_\alpha = y$, initialize all nodes extending $\alpha^\frown\langle \mathit{fin}\rangle$, and take outcome *fin*; otherwise end the stage.

(iii) If α is currently not satisfied and x_α and y_α are defined, then check if $\psi_e(x_\alpha) \downarrow$ and $\psi_e(y_\alpha) \downarrow$. If not, then we take outcome *fin*; otherwise, we take outcome *sat* and initialize all nodes extending $\alpha^\frown\langle \mathit{sat}\rangle$. If $\psi_e(x_\alpha) \not\curlyeqprec \psi_e(y_\alpha)$, then we collapse x_α with y_α.

α **is an** N_e**-node:** The strategy follows the action of the first subcase which applies:

Subcase ∞_{02}*:* If $[0, s^o] \subseteq \mathrm{dom}(\varphi_e)$ and there are pairwise inequivalent elements x_0, x_1, x_2, u_0 and u_1 with the following properties:

- x_0, x_1, x_2, u_0 are 0-good,
- u_1 is 1-good,
- $\widehat{x_0}, \widehat{x_1}, \widehat{x_2}, \widehat{u_0}$ are 2-good,
- $\widehat{u_1}$ is 3-good,
- $[x_0]_E, [x_1]_E, [x_2]_E, [\widehat{x_0}]_E, [\widehat{x_1}]_E, [\widehat{x_2}]_E, [u_1]_E, [\widehat{u_1}]_E \subseteq (s^o, s)$,
- $[u_0]_E, [\widehat{u_0}]_E \subseteq (s^*, s)$,
- $x_0, x_1, x_2, \widehat{x_0}, \widehat{x_1}, \widehat{x_2} \in \mathrm{ran}(\varphi_e)$, and
- u_0 has not been collapsed by α under outcome ∞_{02} before,

then for each $i < 2$ (and for the least such tuple), we collapse x_i with u_i, and collapse $\widehat{x_i}$ with $\widehat{u_i}$. Add u_1 to $F^1_{\alpha^\frown\langle\infty_{02}\rangle}$, and $\widehat{u_1}$ to $F^3_{\alpha^\frown\langle\infty_{02}\rangle}$; then take outcome ∞_{02}.

Subcase ∞_{03}*:* If $[0, s^o] \subseteq \mathrm{dom}(\varphi_e)$ and there are pairwise inequivalent elements x_0, x_1, x_2, u_0, u_1 and y with the following properties:

- x_0, x_1, x_2, u_0 are 0-good,
- $u_1, \hat{y}$ are 1-good,
- $\widehat{x_0}, \widehat{x_1}, \widehat{x_2}, \widehat{u_0}$ are 2-good,
- $\widehat{u_1}, y$ are 3-good,
- $[x_0]_E, [x_1]_E, [x_2]_E, [\widehat{x_0}]_E, [\widehat{x_1}]_E, [\widehat{x_2}]_E, [u_1]_E, [\widehat{u_1}]_E, [y]_E, [\hat{y}]_E \subseteq (s^o, s)$,
- $[u_0]_E, [\widehat{u_0}]_E \subseteq (s^*, s)$,
- $x_0, x_1, x_2, y \in \mathrm{ran}(\varphi_e)$, and
- u_0 has not been collapsed by α under outcome ∞_{03} before,

then for each $i < 2$ (and for the least such tuple), we collapse x_i with u_i, and collapse $\widehat{x_i}$ with $\widehat{u_i}$. Add u_1 and $\hat{y}$ to $F^1_{\alpha^\frown\langle\infty_{03}\rangle}$, and $\widehat{u_1}$ and y to $F^3_{\alpha^\frown\langle\infty_{03}\rangle}$; then take outcome ∞_{03}.

Subcase ∞_{12}*:* If $[0, s^o] \subseteq \mathrm{dom}(\varphi_e)$ and there are pairwise inequivalent elements x_0, x_1, x_2, u_0, u_1 and y with the following properties:

- $\widehat{x_0}, \widehat{x_1}, \widehat{x_2}, \widehat{u_0}$ are 0-good,
- $\widehat{u_1}, y$ are 1-good,
- x_0, x_1, x_2, u_0 are 2-good,
- $u_1, \hat{y}$ are 3-good,
- $[x_0]_E, [x_1]_E, [x_2]_E, [\widehat{x_0}]_E, [\widehat{x_1}]_E, [\widehat{x_2}]_E, [u_1]_E, [\widehat{u_1}]_E, [y]_E, [\hat{y}]_E \subseteq (s^o, s)$,
- $[u_0]_E, [\widehat{u_0}]_E \subseteq (s^*, s)$,
- $x_0, x_1, x_2, y \in \mathrm{ran}(\varphi_e)$, and
- u_0 has not been collapsed by α under outcome ∞_{12} before,

then for each $i < 2$ (and for the least such tuple), we collapse x_i with u_i, and collapse $\widehat{x_i}$ with $\widehat{u_i}$. Add $\widehat{u_1}$ and y to $F^1_{\alpha^\frown\langle\infty_{12}\rangle}$, and u_1 and $\hat{y}$ to $F^3_{\alpha^\frown\langle\infty_{12}\rangle}$; then take outcome ∞_{12}.

Subcase ∞_0*:* If $[0, s^o] \subseteq \mathrm{dom}(\varphi_e)$ and there are pairwise inequivalent elements x_0, x_1, x_2, u_0 and u_1 with the following properties:

- x_0, x_1, x_2, u_0 are 0-good,
- u_1 is 1-good,
- $\widehat{x_0}, \widehat{x_1}, \widehat{x_2}, \widehat{u_0}$ are 2-good,
- $\widehat{u_1}$ is 3-good,
- $[x_0]_E, [x_1]_E, [x_2]_E, [\widehat{x_0}]_E, [\widehat{x_1}]_E, [\widehat{x_2}]_E, [u_1]_E, [\widehat{u_1}]_E \subseteq (s^o, s)$,
- $[u_0]_E, [\widehat{u_0}]_E \subseteq (s^*, s)$,

- $x_0, x_1, x_2 \in \mathrm{ran}(\varphi_e)$, and
- u_0 has not been collapsed by α under outcome ∞_0 before,

then for each $i < 2$ (and for the least such tuple), we collapse x_i with u_i, and collapse $\widehat{x_i}$ with $\widehat{u_i}$. Add u_1 to $F^1_{\alpha^\frown\langle\infty_0\rangle}$, and $\widehat{u_1}$ to $F^3_{\alpha^\frown\langle\infty_0\rangle}$; then take outcome ∞_0.

Subcase ∞_2: If $[0, s^o] \subseteq \mathrm{dom}(\varphi_e)$ and there are pairwise inequivalent elements x_0, x_1, x_2, u_0 and u_1 with the following properties:

- $\widehat{x_0}, \widehat{x_1}, \widehat{x_2}, \widehat{u_0}$ are 0-good,
- $\widehat{u_1}$ is 1-good,
- x_0, x_1, x_2, u_0 are 2-good,
- u_1 is 3-good,
- $[x_0]_E, [x_1]_E, [x_2]_E, [\widehat{x_0}]_E, [\widehat{x_1}]_E, [\widehat{x_2}]_E, [u_1]_E, [\widehat{u_1}]_E \subseteq (s^o, s)$,
- $[u_0]_E, [\widehat{u_0}]_E \subseteq (s^*, s)$,
- $x_0, x_1, x_2 \in \mathrm{ran}(\varphi_e)$, and
- u_0 has not been collapsed by α under outcome ∞_2 before,

then for each $i < 2$ (and for the least such tuple), we collapse x_i with u_i, and collapse $\widehat{x_i}$ with $\widehat{u_i}$. Add $\widehat{u_1}$ to $F^1_{\alpha^\frown\langle\infty_2\rangle}$, and u_1 to $F^3_{\alpha^\frown\langle\infty_2\rangle}$; then take outcome ∞_2.

Subcase ∞_{13}: If $[0, s^o] \subseteq \mathrm{dom}(\varphi_e)$ and there are pairwise inequivalent elements x_0 and y_0 with the following properties:

- $x_0, \widehat{y_0}$ are 1-good,
- $\widehat{x_0}, y_0$ are 3-good,
- $[x_0]_E, [y_0]_E, [\widehat{x_0}]_E, [\widehat{y_0}]_E \subseteq (s^o, s)$, and
- $x_0, y_0 \in \mathrm{ran}(\varphi_e)$,

then add x_0 and $\widehat{y_0}$ to $F^1_{\alpha^\frown\langle\infty_{13}\rangle}$, and $\widehat{x_0}$ and y_0 to $F^3_{\alpha^\frown\langle\infty_{13}\rangle}$; then take outcome ∞_{13}.

Subcase fin: Otherwise, take outcome *fin* and set $F^1_{\alpha^\frown\langle\mathit{fin}\rangle} = F^1_\alpha \cap (s^{\mathit{fin}}, \infty)$ and $F^3_{\alpha^\frown\langle\mathit{fin}\rangle} = F^3_\alpha \cap (s^{\mathit{fin}}, \infty)$.

(So note in particular which collapses are possible: The P-strategies may collapse two elements of U_0 (and then also two elements of U_2); an element of U_1 with an element of U_3; an element of U_0 with an element of U_3; or an element of U_1 with an element of U_2, respectively. The N-strategies may collapse two elements of U_0 (and then also two corresponding elements in U_2), or two elements of U_2 (and then also two corresponding elements in U_0); they may also collapse an element of U_0 with an element of U_1 (and then also the corresponding element in U_2 with the corresponding element in U_3), or an element of U_2 with an element of U_3 (and then also the corresponding element in U_0 with the corresponding element in U_1).)

This ends the description of δ_s. Before concluding stage s, we initialize every node to the right of δ_s.

6.6. **The verification for P_6.** We start by checking the global requirements and some other useful global conditions.

Lemma 6.4. *Our ceer E satisfies the following properties:*

(1) Once a number x is chosen as a follower by a P-strategy α, $[x]_E$ cannot increase in size, and thus remains i-good (where $x \in U_i$), unless α diagonalizes or is initialized.

(2) $E \upharpoonright U_1 = \mathrm{Id} \upharpoonright U_1$ *and* $E \upharpoonright U_3 = \mathrm{Id} \upharpoonright U_3$*, so* $E \upharpoonright U_1 \equiv \mathrm{Id} \equiv E \upharpoonright U_3$*.*
(3) $E \upharpoonright (U_0 \cup U_1) \equiv E \upharpoonright (U_2 \cup U_3)$ *via the permutation* $x \mapsto \hat{x}$*.*

Proof. (1) When x is chosen by a P-strategy α at a stage $s > x$, say, all strategies $\beta \geq \alpha^\frown\langle \mathit{fin} \rangle$ are initialized at stage s, so any number used by such β is greater than s; thus no such β can collapse x. As long as α is not initialized thereafter, only an N-strategy β with $\beta^\frown\langle o \rangle \subseteq \alpha$ for some infinitary outcome o can possibly collapse x. Now, if $i \in \{1, 3\}$, then $x \in F^i_\alpha \subseteq F^i_{\beta^\frown\langle o \rangle}$. Also β can collapse x at a stage $s' > s$ only if we have $x < s \leq s^o < s'$, where s^o is β's stage s^o as measured at stage s', or if $x \in U_0 \cup U_2$ and $x < b_{s'}$, the bounding parameter of α as measured at stage s', both of which lead to a contradiction.

(2) Collapses of elements in U_1 or U_3 (say, the former, by symmetry) can happen in one of three ways:

(i) By an N-strategy under an infinite outcome o: In that case, we are collapsing an element u with an element x such that $[u]_E \subseteq U_0 \cup U_1$ and $[x]_E \subseteq U_0$, so there can be no collapse between u and another element in U_1.
(ii) By a P^b-strategy during diagonalization: In that case, by (1), we are collapsing elements x and y with $[x]_E \subseteq U_0 \cup U_1$ and $[y]_E \subseteq U_2 \cup U_3$, and by initialization, any further collapse with another element in U_1 or U_3 is not possible.
(iii) By a P^d-strategy during diagonalization: In that case, by (1), we are collapsing elements x and y with $[x]_E \subseteq U_0 \cup U_1$ and $[y]_E \subseteq U_2$, and by initialization, any further collapse with another element in U_1 or U_3 is not possible.

Thus no two numbers in U_1 can be collapsed with each other, nor can any two numbers in U_3 by a symmetric argument.

(3) By symmetry, it suffices to show for $x, y \in U_0 \cup U_1$ that if $x \sim y$ then $\hat{x} \sim \hat{y}$. Now two distinct elements $x, y \in U_0 \cup U_1$ can be collapsed in one of five ways:

(i) By an N-strategy under an infinite outcome o: In that case, we are collapsing an element x with an element y such that $[x]_E \subseteq U_0$ and $[y]_E \subseteq U_0 \cup U_1$, and there is a simultaneous collapse between $\hat{x}$ and $\hat{y}$.
(ii) By a P^a-strategy during diagonalization: In that case, by (1), we are collapsing elements x and y with $[x]_E, [y]_E \subseteq U_0$ and $[\hat{x}]_E, [\hat{y}]_E \subseteq U_2$, and we are also collapsing $\hat{x}$ and $\hat{y}$, and by initialization, any further collapse with another element in $U_0 \cup U_1$ has to be via an N-strategy and would again be mirrored by a corresponding collapse in $U_2 \cup U_3$.
(iii) By a P^b-strategy during diagonalization: In that case, by (1), we are collapsing elements x and y with $[x]_E \subseteq U_0 \cup U_1$ and $[y]_E \subseteq U_2 \cup U_3$, and by initialization, any further collapse with another element in $U_0 \cup U_1$ has to be via an N-strategy and would again be mirrored by a corresponding collapse in $U_2 \cup U_3$.
(iv) By a P^c-strategy during diagonalization: In that case, by (1), we are collapsing elements x and y with $[x]_E \subseteq U_0$ and $[y]_E \subseteq U_2 \cup U_3$, and by initialization, any further collapse with another element in $U_0 \cup U_1$ has to be via an N-strategy and would again be mirrored by a corresponding collapse in $U_2 \cup U_3$.
(v) By a P^d-strategy during diagonalization: In that case, by (1), we are collapsing elements x and y with $[x]_E \subseteq U_0 \cup U_1$ and $[y]_E \subseteq U_2$, and by initialization, any further collapse with another element in $U_0 \cup U_1$ has to be via an N-strategy and would again be mirrored by a corresponding collapse in $U_2 \cup U_3$.

Thus any collapse of x with y must be accompanied by a corresponding collapse of $\hat{x}$ and $\hat{y}$, as claimed. □

Lemma 6.5. *There is a* true path $f \in [T]$ *such that*

1. *any strategy $\sigma \subset f$ acts infinitely often but no strategy $\sigma <_L f$ acts infinitely often;*
2. *each $\sigma \subset f$ is initialized at most finitely often;*
3. *no P-strategy σ waits at infinitely many stages before it can choose witnesses; and*
4. *for each $\sigma \subset f$, the environment (F^1, F^3) of σ consists of two infinite computable sets $F^i \subseteq U_i$ (for $i \in \{1, 3\}$).*

Proof. We proceed by simultaneous induction on $n = |\sigma|$.

For (1), since the tree of strategies is finitely branching, there clearly is a leftmost σ of length n which acts infinitely often unless σ^- ends the stage at cofinitely many stages at which it acts; however, then σ^- would have to be a P-strategy unable to find witnesses in its environment, contrary to items (3) and (4) and since the bounding parameter b which σ uses monotonically increases to infinity.

Item (2) is immediate by construction.

Item (3) holds since while a P-strategy σ is waiting to choose its witnesses, its environment sets F^i keep growing. The restriction that witnesses cannot be E-equivalent to any other element is now vacuous if the element is to be chosen from a set F^i defined in the infinitary way since all elements in F^i are such; and if F^i is determined under a finitary outcome of σ^-, then no other strategy can use any element in F^i until σ chooses its witnesses. Finally, note that the bounding parameter b which σ uses monotonically increases to infinity.

It remains to establish (4) for σ. Assume first that σ^- is a P-strategy, or an N-strategy such that $\sigma = \sigma^- \widehat{\ } \langle \mathit{fin} \rangle$. Then σ^-, given its environment (F^1, F^3), hands to σ for its environment (F'^1, F'^3) all but finitely many numbers from the environment of σ^-.

On the other hand, suppose that σ^- is an N-strategy and $\sigma = \sigma^- \widehat{\ } \langle o \rangle$ for an infinitary outcome $o \neq \mathit{fin}$: Then each F'^i is an infinite set of elements by the construction under this outcome. □

Lemma 6.6. *We note the following properties of the equivalence relation E:*

1. *All E-equivalence classes are finite.*
2. *Each E-equivalence class contains at most one element from U_1, and at most one element from U_3.*
3. *When an element x becomes no longer i-good (for any $i < 4$), then $[x]_E$ will no longer change.*

Proof. For (1), suppose toward a contradiction that an E-equivalence class $[x_0]_E$ is infinite. Then $[x_0]_E \subseteq U_0 \cup U_1$ or $[x_0]_E \subseteq U_2 \cup U_3$ since only i-good numbers from some $i < 4$ can be E-collapsed, so any collapse to $[x_0]_E$ must be due to P^a- or N-strategies. Also note that, by our construction and by Lemma 6.4(1), elements x and y can only be collapsed at stage s if $[x]_E, [y]_E \subseteq [0, s)$.

Now fix a P^a-strategy α, an outcome o and a stage s such that

- $\alpha^\widehat{\ }\langle o \rangle \subset f$,
- α takes outcome o for the first time since its last initialization, and
- $x_0 < b$ for α's bounding parameter.

Then no strategy $\beta <_L \alpha^\frown\langle o\rangle$ is eligible to act after stage s, no strategy $\beta \subseteq \alpha$ can collapse x_0, and no strategy $\beta \geq \alpha^\frown\langle o\rangle$ can collapse any number $< s$; so $[x_0]_E$ has reached its final size.

Item (2) is immediate by Lemma 6.4(2).

Finally, (3) is immediate by the construction and by Lemma 6.4(1) since no element can be collapsed once it is no longer i-good for any $i < 4$. □

Lemma 6.7. *Each P-requirement is satisfied.*

Proof. Fix a P-requirement and the corresponding P-strategy $\sigma \subset f$. Then, by Lemma 6.5(3), σ will eventually choose witnesses x and y, and by Lemma 6.4(1) and Lemma 6.5(1), these witnesses are stable and their equivalence classes will not change except through diagonalization or initialization. By initialization, $x \not\sim y$ unless σ collapses them; and when it does, σ will ensure that $\psi_e(x) \not\sim \psi_e(y)$ from then on, again by initialization. □

Remark 6.8. The following observations will be useful in the next proof for the verification in the case of all the infinitary outcomes:

- The "active" sets only need to cover the "good" numbers. Indeed, since the bounding parameter b tends to infinity along the true path, every element $x \in U_0 \cup U_2$ eventually falls below the current value of b. At the first such stage, either x is "good" and is included in the corresponding set $F^0_{\sigma^\frown\langle o\rangle}$ or $F^2_{\sigma^\frown\langle o\rangle}$ (defined below), or it is not good; in the latter case, Lemma 6.6(3) freezes that class forever.
- The same freezing argument applies to elements $x \in U_1 \cup U_3$ outside the environment sets $F^1_{\sigma^\frown\langle o\rangle}$ and $F^3_{\sigma^\frown\langle o\rangle}$.
- Hence, the part of the equivalence relation omitted from the active sets is computable, up to finitely many exceptions.
- In each infinitary case, the parts omitted from the active sets contain an infinite computable subset of either U_1 or U_3, where E is the identity by Lemma 6.4(2); consequently, the computable set omitted contributes an Id-summand which is absorbed by the active sets. We will not mention this harmless computable summand again.

Lemma 6.9. *Each N-requirement is satisfied.*

Proof. Fix the N-strategy σ and the outcome o such that $\sigma^\frown\langle o\rangle \subset f$ as well as a stage s_0 after which $\sigma^\frown\langle o\rangle$ is no longer initialized.

For $i = 0, 2$, define

$$F^i_{\sigma^\frown\langle o\rangle} = \bigcup\{[u]_E \mid u \text{ is } i\text{-good at the stage } s \text{ when first } u < b\}$$

for σ's bounding parameter b. (These sets will play the role of the "active sets" for U_0 and U_2, which we did not need to introduce formally during the construction; but any collapse "off" these sets is computable.)

We now distinguish cases by the possible outcomes o of σ along the true path.

Outcome ∞_{02}: Eventually, all u_0 with $[u_0]_E \subseteq F^0_{\sigma^\frown\langle o\rangle} \cap (s^*, \infty)$ and $[\widehat{u_0}]_E \subseteq F^2_{\sigma^\frown\langle o\rangle} \cap (s^*, \infty)$ are collapsed with an element $x_0 \in U_0 \cap \mathrm{ran}(\varphi_e)$, and all u_0 with $[u_0]_E \subseteq F^2_{\sigma^\frown\langle o\rangle} \cap (s^*, \infty)$ and $[\widehat{u_0}]_E \subseteq F^0_{\sigma^\frown\langle o\rangle} \cap (s^*, \infty)$ are collapsed with an element $x_0 \in U_2 \cap \mathrm{ran}(\varphi_e)$ if they are found to currently be outside $\mathrm{ran}(\varphi_e)$. In addition, all u_1 with $[u_1]_E \subseteq F^1_{\sigma^\frown\langle o\rangle}$ and $[\widehat{u_1}]_E \subseteq F^3_{\sigma^\frown\langle o\rangle}$ are collapsed with an element $x_1 \in U_0 \cap \mathrm{ran}(\varphi_e)$,

and all u_1 with $[u_1]_E \subseteq F^3_{\sigma^\frown\langle o\rangle}$ and $[\widehat{u_1}]_E \subseteq F^1_{\sigma^\frown\langle o\rangle}$ are collapsed with an element $x_1 \in U_2 \cap \operatorname{ran}(\varphi_e)$ if they are found to currently be outside $\operatorname{ran}(\varphi_e)$.

Thus $R \leq E \leq E \upharpoonright [F^0_{\sigma^\frown\langle o\rangle} \cup F^1_{\sigma^\frown\langle o\rangle} \cup F^2_{\sigma^\frown\langle o\rangle} \cup F^3_{\sigma^\frown\langle o\rangle}]_E \leq E \upharpoonright \operatorname{ran}(\varphi_e) \leq R$ as desired.

Outcome ∞_{03}*:* Eventually, all u_0 with $[u_0]_E \subseteq F^0_{\sigma^\frown\langle o\rangle} \cap (s^*, \infty)$ and $[\widehat{u_0}]_E \subseteq F^2_{\sigma^\frown\langle o\rangle} \cap (s^*, \infty)$ are collapsed with an element $x_0 \in U_0 \cap \operatorname{ran}(\varphi_e)$. In addition, all u_1 with $[u_1]_E \subseteq F^1_{\sigma^\frown\langle o\rangle}$ and $[\widehat{u_1}]_E \subseteq F^3_{\sigma^\frown\langle o\rangle}$ are collapsed with an element $x_1 \in U_0 \cap \operatorname{ran}(\varphi_e)$. And only y with $[y]_E \cap \operatorname{ran}(\varphi_e) \neq \emptyset$ are allowed into $F^3_{\sigma^\frown\langle o\rangle}$. Furthermore, by the failure of outcome ∞_{02}, $F^2_{\sigma^\frown\langle o\rangle} \cap \operatorname{ran}(\varphi_e)$ is finite.

Thus $R \leq E \upharpoonright (U_0 \cup U_1 \cup U_3) = C \leq E \upharpoonright [F^0_{\sigma^\frown\langle o\rangle} \cup F^1_{\sigma^\frown\langle o\rangle} \cup F^3_{\sigma^\frown\langle o\rangle}]_E \leq E \upharpoonright \operatorname{ran}(\varphi_e) \leq R$ as desired.

Outcome ∞_{12}*:* Eventually, all u_0 with $[u_0]_E \subseteq F^2_{\sigma^\frown\langle o\rangle} \cap (s^*, \infty)$ and $[\widehat{u_0}]_E \subseteq F^0_{\sigma^\frown\langle o\rangle} \cap (s^*, \infty)$ are collapsed with an element $x_0 \in U_2 \cap \operatorname{ran}(\varphi_e)$. In addition, all u_1 with $[u_1]_E \subseteq F^3_{\sigma^\frown\langle o\rangle}$ and $[\widehat{u_1}]_E \subseteq F^1_{\sigma^\frown\langle o\rangle}$ are collapsed with an element $x_1 \in U_2 \cap \operatorname{ran}(\varphi_e)$. And only y with $[y]_E \cap \operatorname{ran}(\varphi_e) \neq \emptyset$ are allowed into $F^1_{\sigma^\frown\langle o\rangle}$. Furthermore, by the failure of outcome ∞_{02}, $F^0_{\sigma^\frown\langle o\rangle} \cap \operatorname{ran}(\varphi_e)$ is finite.

Thus $R \leq E \upharpoonright (U_1 \cup U_2 \cup U_3) = D \leq E \upharpoonright [F^1_{\sigma^\frown\langle o\rangle} \cup F^2_{\sigma^\frown\langle o\rangle} \cup F^3_{\sigma^\frown\langle o\rangle}]_E \leq E \upharpoonright \operatorname{ran}(\varphi_e) \leq R$ as desired.

Outcome ∞_0*:* Eventually, all u_0 with $[u_0]_E \subseteq F^0_{\sigma^\frown\langle o\rangle} \cap (s^*, \infty)$ and $[\widehat{u_0}]_E \subseteq F^2_{\sigma^\frown\langle o\rangle} \cap (s^*, \infty)$ are collapsed with an element $x_0 \in U_0 \cap \operatorname{ran}(\varphi_e)$. In addition, all u_1 with $[u_1]_E \subseteq F^1_{\sigma^\frown\langle o\rangle}$ and $[\widehat{u_1}]_E \subseteq F^3_{\sigma^\frown\langle o\rangle}$ are collapsed with an element $x_1 \in U_0 \cap \operatorname{ran}(\varphi_e)$ if they are found to currently be outside $\operatorname{ran}(\varphi_e)$. Furthermore, by the failure of outcomes ∞_{02} and ∞_{03}, $(F^2_{\sigma^\frown\langle o\rangle} \cup F^3_{\sigma^\frown\langle o\rangle}) \cap \operatorname{ran}(\varphi_e)$ is finite.

Thus $R \leq E \upharpoonright (U_0 \cup U_1) = A \leq E \upharpoonright [F^0_{\sigma^\frown\langle o\rangle} \cup F^1_{\sigma^\frown\langle o\rangle}]_E \leq E \upharpoonright \operatorname{ran}(\varphi_e) \leq R$ as desired.

Outcome ∞_2*:* Eventually, all u_0 with $[u_0]_E \subseteq F^2_{\sigma^\frown\langle o\rangle} \cap (s^*, \infty)$ and $[\widehat{u_0}]_E \subseteq F^0_{\sigma^\frown\langle o\rangle} \cap (s^*, \infty)$ are collapsed with an element $x_0 \in U_2 \cap \operatorname{ran}(\varphi_e)$. In addition, all u_1 with $[u_1]_E \subseteq F^3_{\sigma^\frown\langle o\rangle}$ and $[\widehat{u_1}]_E \subseteq F^1_{\sigma^\frown\langle o\rangle}$ are collapsed with an element $x_1 \in U_2 \cap \operatorname{ran}(\varphi_e)$ if they are found to currently be outside $\operatorname{ran}(\varphi_e)$. Furthermore, by the failure of outcomes ∞_{02} and ∞_{12}, $(F^0_{\sigma^\frown\langle o\rangle} \cup F^1_{\sigma^\frown\langle o\rangle}) \cap \operatorname{ran}(\varphi_e)$ is finite.

Thus $R \leq E \upharpoonright (U_2 \cup U_3) \equiv A \leq E \upharpoonright [F^2_{\sigma^\frown\langle o\rangle} \cup F^3_{\sigma^\frown\langle o\rangle}]_E \leq E \upharpoonright \operatorname{ran}(\varphi_e) \leq R$ as desired.

Outcome ∞_{13}*:* Eventually, only y with $[y]_E \subseteq \operatorname{ran}(\varphi_e)$ are allowed into $F^1_{\sigma^\frown\langle o\rangle}$, and only y with $[y]_E \cap \operatorname{ran}(\varphi_e) \neq \emptyset$ are allowed into $F^3_{\sigma^\frown\langle o\rangle}$. Furthermore, by the failure of outcome ∞_{02}, $(F^0_{\sigma^\frown\langle o\rangle} \cup F^2_{\sigma^\frown\langle o\rangle}) \cap \operatorname{ran}(\varphi_e)$ is finite.

Thus $R \leq E \upharpoonright (U_1 \cup U_3) \equiv B \leq E \upharpoonright [F^1_{\sigma^\frown\langle o\rangle} \cup F^3_{\sigma^\frown\langle o\rangle}]_E \leq E \upharpoonright \operatorname{ran}(\varphi_e) \leq R$ as desired.

Outcome fin: By the failure of the previous outcomes, $(F^0_{\sigma^\frown\langle o\rangle} \cup F^2_{\sigma^\frown\langle o\rangle} \cup F^3_{\sigma^\frown\langle o\rangle}) \cap \operatorname{ran}(\varphi_e)$ is finite, or $(F^0_{\sigma^\frown\langle o\rangle} \cup F^1_{\sigma^\frown\langle o\rangle} \cup F^2_{\sigma^\frown\langle o\rangle}) \cap \operatorname{ran}(\varphi_e)$ is finite; by symmetry, assume the former.

Then $R \leq E \upharpoonright U_1 \equiv \mathrm{Id}$ as desired. □

This completes the proof of Theorem 6.1. □

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