

BUILDING MODELS OF STRONGLY MINIMAL THEORIES

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ABSTRACT. What information does one need to know in order to build the models of a strongly minimal theory? To answer this question, we first formalize it in two ways. Note that if a theory T has a computable model, then $T \cap \exists_n$ is uniformly Σ_n^0 . We call such theories Solovay theories. A degree is *strongly minimal computing* if it computes a copy of every countable model of every strongly minimal Solovay theory. A second notion, introduced by Lempp in the mid-1990's, is that of a strongly minimal relatively computing degree. A degree \mathbf{d} is *strongly minimal relatively computing* if whenever T is a strongly minimal theory with one computable model, \mathbf{d} computes a copy of every countable model of T . We characterize both classes of degrees as exactly the degrees which are *high over $\mathbf{0}''$* , i.e., $\mathbf{d} \geq \mathbf{0}''$ and $\mathbf{d}' \geq \mathbf{0}^{(4)}$.

1. INTRODUCTION

In computable model theory, we try to understand the difficulty of building mathematical structures. A fundamental question in computable model theory is to understand the relationship between computing facts about a theory and being able to compute a copy of its model(s). If T has a computable model, then $T \cap \exists_n$ is uniformly Σ_n^0 , but many theories with this property have no computable models. We say a theory is a *Solovay theory* if $T \cap \exists_n$ is uniformly Σ_n^0 . Knight and Solovay (see Knight [Kn99]) showed that a degree computes a copy of some model of every Solovay theory if and only if it computes a non-standard model of true arithmetic. In particular, no such degree is arithmetical. For model-theoretically nice theories, we can hope that if T is a Solovay theory, then we can get some reasonable, say, at least arithmetical, copies of its models.

Put another way, we ask: Precisely what information, beyond the obvious necessity of the theory being a Solovay theory, is needed to build a model of T . In the case of an \aleph_0 -categorical theory, Lerman-Schmerl [LS79], improved by Knight [Kn94], showed that if T is an \aleph_0 -categorical Solovay

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theory, then the countable model of T has a $0'$ -computable copy. Further, their proof exactly highlighted what information was needed.

Andrews and Knight [AK18] examined this question for strongly minimal theories. They showed that if T is a strongly minimal Solovay theory, then every countable model of T has a copy computable in $0'''$. They ask whether $0''$ suffices. Now by Khoussainov-Laskowski-Lempp-Solomon [KLLS07], we know that at least $0''$ is necessary. We improve the result of Andrews-Knight slightly, and construct a theory which shows that this bound is sharp. The sharpness of this bound constitutes the bulk of this paper. We say that a degree is *strongly minimal computing* if whenever T is a strongly minimal Solovay theory, then \mathbf{d} computes a copy of every countable model of T .

In the 1990's, Lempp asked about the complexity of computing the models of a strongly minimal theory T which has at least one computable model. Since this implies that T is a Solovay theory, this is a closely related question. We say that a degree is *strongly minimal relatively computing* if whenever T is a strongly minimal theory with at least one computable model, then \mathbf{d} computes a copy of every countable model of T .

In general, for a class of theories \mathcal{C} , we say that a degree is \mathcal{C} -computing if it computes a copy of every countable model of every Solovay theory in \mathcal{C} . We say that a degree is \mathcal{C} -relatively computing if it computes a copy of every countable model of a theory $T \in \mathcal{C}$ which has at least one computable model.

We believe that understanding the \mathcal{C} -computing and the \mathcal{C} -relatively computing degrees are a way of capturing the general question: What information do we need to compute models of theories from \mathcal{C} . Examples of natural such further questions include: What are the \aleph_1 -categorical computing degrees? What are the few models computing degrees, i.e., let \mathcal{C} be the class of theories with countably many countable models?

Our main theorem is the following:

Main Theorem. *A degree \mathbf{d} is strongly minimal computing if and only if \mathbf{d} is strongly minimal relatively computing if and only if \mathbf{d} is high over $0''$, i.e., $\mathbf{d} \geq 0''$ and $\mathbf{d}' \geq 0^{(4)}$.*

It follows immediately from the definitions that every strongly minimal computing degree is strongly minimal relatively computing. We will construct a particular class of theories T_f in section 2, each with a computable prime model, and in section 3, we show that we can choose f so that any presentation A of a positive-dimensional model of T has $0^{(4)} \leq A'$. The construction of these theories T_f constitutes the bulk of this paper. By

Khousainov-Laskowski-Lempp-Solomon [KLLS07], every strongly minimal relatively computing degree is also above $0''$, so this suffices to show that every strongly minimal relatively computing degree is high over $0''$. Lastly, in section 4, we show that every degree which is high over $0''$ is strongly minimal computing. This is done by using the methods and machinery (which require very little change) in Andrews-Knight [AK18], where they prove $0'''$ is strongly minimal computing.

2. THE STRONGLY MINIMAL THEORIES T_f

We will first construct a strongly minimal structure G which does not code any computability-theoretic content. Rather, G will give us a template on which we can easily code a function f to create the structure $M := M_f$. The function f can be any function $f : \omega^2 \rightarrow \{0, 1\}$ computable from $0'''$ so that for each k , there is at most one n so that $f(k, n) = 1$ and such an n is of the form $\langle k, n_0 \rangle$. The goal is to code the set of k for which there exists an n with $f(k, n) = 1$ (which can be of degree $0^{(4)}$) into the jump of the structure M . The way G creates the template is by leaving some tuples as being ready to code information. In particular, we will have a configuration called a k -hook below. In G , k -hooks will never have a relation S_0^n hold on them; but in M , we will place relations S_0^n on a k -hook depending on (a computable approximation to) whether $f(k, n) = 1$. The result will be that for a sufficiently generic k -hook, S_0^n holds on the k -hook if and only if $f(k, n) = 1$.

The zero-dimensional models have the benefit of not having generic k -hooks, while the positive-dimensional models will. At first, it appears that this only provides one jump of distinction (finding sufficiently generic elements), but the structure of a k -hook is so that fixing a subtuple on which $R_2(a, b)$ holds makes it easy (only one jump) to find k -hooks containing ab , but R_2 is only $\forall\exists$ -definable, so finding sufficiently generic tuples satisfying R_2 takes already 3 jumps in the prime model. This, in addition to the one jump (which is the same over any model) to de-code $\emptyset^{(4)}$, will force that $\emptyset^{(4)}$ can only be computed by 4 jumps over the prime model (which will be computable), but by only 1 jump over any other model.

2.1. Building the structure G . In this section, we will use the Hrushovski amalgamation method in a novel way to construct our template structure G . For an introduction to the amalgamation method, we still recommend Hrushovski's original paper [Hr93]. For an introduction to the amalgamation method using an infinite language (which introduces some subtleties over the original construction), we recommend [An10], [An11], or [An13]. For

a more modern presentation of a general construction which covers each of the previous constructions, see [AMta]. In order for G to serve as a template, we need to allow room for the computable model M (constructed in the next subsection) to change its mind about occurrences of R_2 . In particular, M might replace some k - n -coincidences with k -hooks, in the terminology introduced below, and we need our bound μ to count these together. This leads to the strange-looking definition of which extensions are counted together, i.e., which extensions are *of the form* of which other extensions, in Definition 2.11. This is the main cause why we must re-do much of the technical details of the amalgamation construction here and cannot simply cite any existing presentation.

For the construction, we will use the language $\mathcal{L} = \{R_0^j, R_1^j \mid j \in \omega\} \cup \{R_2\} \cup \{S_0^k \mid k \in \omega\}$ where R_2 is binary, and R_1^j , R_0^j , and S_0^j are 4-ary for every j . We will also use the language $\mathcal{L}_0 = \{R_0^j \mid j \in \omega\} \cup \{S_0^k \mid k \in \omega\}$. The structure M we construct in section 2.2 will be in the language \mathcal{L}_0 , though it should be appropriately understood in terms of a definitional expansion M' in the language \mathcal{L} . The subscripts for the symbols in \mathcal{L} represent the number of quantifiers that will be needed to define the symbol in M . R_1^j will be \forall -definable in M , and R_2 will be $\forall\exists$ -definable in M . We do this by making R_1^j holding on a tuple cause that tuple to have fewer than the generic number of extensions for an extension involving only R_0^j . In this way, we will make R_1^j definable via a universal formula which uses the symbol R_0^j . We will do the same with R_2 being \forall -definable via a universal formula which uses the symbol R_1^0 , which makes R_2 $\forall\exists$ -definable using R_0^0 .

We will employ the amalgamation construction machinery, which means that we will define a class \mathcal{C} of finite \mathcal{L} -structures and amalgamate members of this class to form the generic structure G . We first begin with several combinatorial definitions which will be used to describe the class \mathcal{C} .

Definition 2.1. A 4-tuple for which $R_2(x, y)$ and $R_1^k(x, y, z, w)$ hold is called a k -hook. A *hook* is a k -hook for some k .

For A a finite \mathcal{L} -structure, we define $\delta(A)$ to be

$$|A| - \sum_{U \in \mathcal{L}} |\{\bar{a} \in A^{\text{arity}(U)} \mid A \models U(\bar{a})\}| - \#(\text{Hooks appearing on } A).$$

Definition 2.2. For $A, B \subseteq N$, we say that A and B are *freely joined over* $A \cap B$ if there are no relations holding in $A \cup B$ other than those that occur in A or in B . In this case, we write $A \oplus_{A \cap B} B$ for this structure with domain $A \cup B$.

Observation 2.3. If $A, B \subseteq N$, then $\delta(A \cup B) \leq \delta(A) + \delta(B) - \delta(A \cap B)$. Further, equality holds if and only if A and B are freely joined over $A \cap B$.

Proof. The inequality is because of the usual counting: In $A \cup B$, we subtract one for every relation or hook in A or in B . On the right side of the inequality, we subtract one for every relation in A or in B , but then we have double counted those in $A \cap B$, so subtracting $\delta(A \cap B)$ corrects for this.

Equality holds if and only if A and B are freely joined: If equality holds, then there can be no relation on $A \cup B$ aside from those entirely in A or in B , so A and B are freely joined over $A \cap B$. If A and B are freely joined over $A \cap B$, then there are no relations aside from those in A or in B . Similarly, there can be no hooks aside from those in A or in B . This is because whenever $abcd$ forms a k -hook, then $R_1^k(a, b, c, d)$ holds. Since A and B are freely joined over $A \cap B$, R^k holding on $abcd$ implies that $abcd \subseteq A$ or $abcd \subseteq B$. \square

Definition 2.4. • For finite subsets B and A of a structure M , we set $\delta(B/A) = \delta(B \cup A) - \delta(A)$.

- If X is infinite, set $\delta(B/X) = \min\{\delta(B/A) \mid A \subseteq X, A \text{ finite}\}$.
- If $A \subseteq B$, $\delta(A, B) = \min\{\delta(C) \mid A \subseteq C \subseteq B, C \text{ finite}\}$.
- We say that A is a *strong substructure* of B and write $A \leq B$ to mean that $A \subseteq B$ and $\delta(A, B) = \delta(A)$.
- If $A \subseteq B$ are infinite, we say $A \leq B$ if and only if $\delta(X, A) = \delta(X, B)$ for every finite subset $X \subseteq A$.
- B is *simply algebraic over A* if $A \cap B = \emptyset$, $A \leq A \cup B$, $\delta(B/A) = 0$, and there is no proper subset B' of B so that $\delta(B'/A) = 0$.
- B is *minimally simply algebraic over A* if B is simply algebraic over A and there is no proper subset A' of A so that B is simply algebraic over A' . In this case, we write B/A is a *minimally simply algebraic extension*.

The following lemmas, which are repeated here without proof from Andrews [An11, Lemmas 5 and 6], describe the basic combinatorics which allow us to build amalgamation constructions.

Lemma 2.5 (\leq is transitive). *Let A be a finite \mathcal{L} -substructure of N so that $A \leq N$.*

- (1) $\delta(X \cap A) \leq \delta(X)$ for any finite set $X \subseteq N$.
- (2) $\delta(A', A) = \delta(A', N)$ for any set $A' \subseteq A$.
- (3) If $A' \leq A$ and $A \leq N$, then $A' \leq N$. \square

Lemma 2.6 (Modding out by more can only decrease dimension). *If X , A , and B are finite \mathcal{L} -structures so that $A \subseteq B$, then $\delta(X/A \cup (X \cap B)) \geq \delta(X/B)$. In particular, if $X \cap B = \emptyset$, then $\delta(X/A) \geq \delta(X/B)$. \square*

Fix two minimally simply algebraic extensions Λ and Ω over sets of size 2 and 4, respectively, where no relations hold on the base and each involves a single 4-ary relation symbol R . Let Ω_i be the extension obtained by replacing R by the relation symbol R_0^i . And let Λ be the extension obtained by replacing R by the relation symbol R_1^0 . The lack of symmetry here that we use R_1^0 as the relation for a special extension and no other R_1^k is simply because we only need a single relation R_2 , which will become definable using this extension. Fix K to be the maximum of the size of the extension Ω and the size of the extension Λ .

Definition 2.7. Let B/A be a minimally simply algebraic extension. We say B/A is a Λ -extension if $|A| = 2$ and every relation that holds in Λ holds in B/A .

Similarly, we say B/A is a Ω_i -extension if $|A| = 4$ and every relation which holds in Ω_i holds in B/A .

We now define a function μ that will prescribe the number of allowed copies of a minimally simply algebraic extension over a base inside a member of \mathcal{C} .

Definition 2.8. Let Y/X be a minimally simply algebraic extension. We define

$$\mu(X, Y) = \begin{cases} 8 + 2^K & \text{if } Y/X \text{ is an } \Omega_i\text{-extension} \\ & \text{and } R_1^i(X) \text{ holds,} \\ 4 + 2^K & \text{if } Y/X \text{ is an } \Lambda\text{-extension} \\ & \text{and } R_2(X) \text{ holds,} \\ 2 \cdot |X| + 1 + 2^K & \text{otherwise.} \end{cases}$$

Definition 2.9. A 4-tuple on which both $R_1^k(x, y, z, w)$ and $S_0^n(x, y, z, w)$ hold is called a k - n -coincidence.

Definition 2.10. Fix B/A to be a minimally simply algebraic extension. We say that X is a B/A -base if $|X| = |A|$ and one of the following holds:

- B/A is not an Ω_i - or a Λ -extension.
- B/A is an Ω_i -extension, and $R_1^i(X)$ iff $R_1^i(A)$.
- B/A is a Λ -extension, and $R_2(X)$ iff $R_2(A)$.

Definition 2.11. Fix B/A to be a minimally simply algebraic extension. We say that an extension Y/X is *of the form* B/A if X is a B/A -base and after replacing some k - n -coincidences in $(A \cup B)^4 \setminus A^4$ with k -hooks we get an extension B'/A' so that Y/X realizes every (positive) relation which holds in B'/A' (i.e., holds of a tuple in $(A' \cup B')^m \setminus (A')^m$ where m is

the arity of the relation). Note that for finite \mathcal{L} -structures B/A , there is a first-order formula $\Theta_{B/A}(X, Y)$ which says that Y/X is of the form B/A .

We now have all the components necessary to describe our class of finite structures \mathcal{C} .

Definition 2.12. Let \mathcal{C} be the collection of finite \mathcal{L} -structures A so that the following hold:

- (1) If $X \subseteq A$, then $\delta(X) \geq 0$.
- (2) If F, C^1, \dots, C^r are disjoint subsets of A and each C^i/F is of the form B/A for a minimally simply algebraic extension B/A , then $r \leq \mu(A, B)$.
- (3) If $abcd \in A$ is a k -hook, then there is no $n \in \omega$ so that S_0^n holds on $abcd$.
- (4) If $R_1^k(a, b, c, d)$ then there is at most one n so that $S_0^n(a, b, c, d)$.

Our next major goal is to show that \mathcal{C} has certain amalgamation properties, which will allow us to form the amalgamation generic G for the class \mathcal{C} .

The following shows that among the minimally simply algebraic extensions in \mathcal{C} , if one is of the form of another, then they only differ on the base. Furthermore, since they have the same μ , it doesn't matter which one we work with.

Lemma 2.13. *If $\delta(B/A) = 0$, B'/A' is a minimally simply algebraic extension, and B/A is of the form B'/A' , then they realize exactly the same relations on the extensions. That is, there is a map $f : (A \cup B) \rightarrow (A' \cup B')$ satisfying that for all $\bar{x} \in (A \cup B)^n \setminus A^n$ and all $R \in \mathcal{L}$ that $R(\bar{x})$ iff $R(f(\bar{x}))$. Furthermore, B/A is a minimally simply algebraic extension and $\mu(A, B) = \mu(A', B')$. Lastly, for any other extension E/D , E/D is of the form B/A if and only if it is of the form B'/A' .*

Proof. Note that B/A cannot realize all the relations of any form B'/A' in which we replaced any k - n -coincidences with k -hooks. This is because k -hooks are counted as -3 in δ : -1 for R_2 , -1 for R_1^k , and -1 for being a k -hook, whereas a k - n -coincidence is counted as -2 in δ : -1 for R_1^k and -1 for S_0^n . Thus, since $\delta(B/A) = \delta(B'/A') = 0$, B/A cannot realize all the relations in any form B'/A' in which we replaced any k - n -coincidences with k -hooks. So, B/A realizes all relations in B'/A' . But since $\delta(B/A) = \delta(B'/A') = 0$, we also have B'/A' realizes all relations in B/A .

Since these two extensions satisfy precisely the same relations except on the base, it follows that B/A is also a minimally simply algebraic extension. We now show that $\mu(A, B) = \mu(A', B')$. If B/A is not an Ω_i -extension or a

Λ -extension, then $\mu(A', B') = \mu(A, B)$ since $|A| = |A'|$. Otherwise, we demand in the definition of being “of the form” that the bases look sufficiently similar that $\mu(A', B') = \mu(A, B)$. \square

The following is the main combinatorial tool which we will use below to show \mathcal{C} has nice amalgamation properties.

Lemma 2.14. *Let A , B_1 , and B_2 be \mathcal{L} -structures so that any substructure has non-negative δ , $A = B_1 \cap B_2$, and $A \leq B_1$. Let $E = B_1 \oplus_A B_2$. Let Y/X be a minimally simply algebraic extension and suppose C^1, \dots, C^r , F are disjoint substructures of E so that each C^j/F is of the form Y/X . Then one of the following conditions holds:*

- (1) *One of the C^j is contained in $B_1 \setminus A$ and $F \subseteq A$.*
- (2) *$F \cup \bigcup_j C^j$ is contained entirely in B_1 or entirely in B_2 .*
- (3) *For one of the C^j , $C^j \subseteq B_2$, and setting $Z = (F \cap A) \cup (C^j \cap B_2)$, we have $\delta(Z/Z \cap A) < 0$. Further, at least $r - \delta(F)$ of the C^k are entirely contained in $B_1 \setminus A$. (Note that this cannot happen if $A \leq B_2$.)*

Proof. This proof is an adaptation of Hrushovski [Hr93, Lemma 3], as in Andrews’s thesis [An10, Lemma 42]. The notion of being of the form Y/X was different there, but the proof does not rely on that; rather, it works whenever each C^i/F is either minimally simply algebraic or $\delta(C^i/F) < 0$. \square

We must describe the possible cases where μ allows us to have more realizations of an extension, yet adding such an extension is not possible inside \mathcal{C} .

Definition 2.15. Let B/A be a minimally simply algebraic extension and let Y/X be minimally simply algebraic and of the form B/A with $Y \in \mathcal{C}$. Let $Z \supseteq X$ be so that $Z \in \mathcal{C}$. We say that Z is a *B/A -obstruction over X* if Z does not contain $\mu(A, B)$ many disjoint extensions over X of the form B/A , yet $Y \oplus_X Z \notin \mathcal{C}$.

Lemma 2.16. *Let B/A be a minimally simply algebraic extension. Then there is a finite list of extensions $Z_1 \supseteq X_1, \dots, Z_n \supseteq X_n$ so that a set Z is a B/A -obstruction over X if and only if Z contains a set $W \supseteq X$ so that W realizes all the relations in Z_i for some $i \leq n$. Furthermore, $X_i \not\leq Z_i$ for each $i \leq n$.*

Proof. Let Z be a B/A -obstruction over X . Let Y/X be the minimally simply algebraic extension of X of the form B/A (unique by Lemma 2.13).

We consider the possible reasons why $Y \oplus_X Z \notin \mathcal{C}$. Since $Y \oplus_X Z$ is freely joined, we have that for any $V \subseteq Y \oplus_X Z$, $\delta(V) = \delta(V \cap Y) + \delta(V \cap Z) - \delta(V \cap X) = \delta(V \cap Y/V \cap X) + \delta(V \cap Z) \geq 0$ since $X \leq Y$ and $Z \in \mathcal{C}$. Thus, $Y \oplus_X Z$ satisfies the first condition of \mathcal{C} . It also satisfies the third and fourth conditions since it is freely joined, any 4-tuple on which $R_1^k(a, b, c, d)$ holds must be in Y or in Z , and $Y, Z \in \mathcal{C}$. Thus, we must have disjoint subsets F, C^1, \dots, C^r of $Y \oplus_X Z$, each of the form B'/A' for some minimally simply algebraic extension B'/A' , and $r > \mu(A', B')$. Let $Z' = X \cup (F \cap Z) \cup \bigcup_j (C^j \cap Z)$. From Z' , remove all relations aside from those on tuples in $(C^j \cup F)^k \setminus F^k$ or relations R_1^i or R_2 holding on all of X needed to make Y/X be of the form B/A , let X' be this Z' restricted to the set X and let Z'/X' be one of the Z_i/X_i . We will argue below that there are only finitely many extensions Z_i/X_i , even as we range this construction over all possible B/A -obstructions Z over X .

Suppose Z is a B/A -obstruction over X witnessed by F, C^1, \dots, C^r . Then Lemma 2.14 gives four possibilities to consider, and we begin by verifying that Z must be in the fourth case, by deriving contradictions from the first three cases:

- (1) One of the C^j is contained in $Y \setminus X$ and $F \subseteq X$: In this case, since Y/X is minimally simply algebraic, we get that $C^j = Y$ and $F = X$. Thus, the remaining C^j 's are all contained in $Z \setminus X$. Also, B/A is of the form B'/A' , so $\mu(A, B) = \mu(A', B')$. Thus, Z already contains $\mu(A, B)$ extensions of the form B'/A' , and thus, also of the form B/A , contradicting Z being a B/A -obstruction.
- (2) Either $F \cup \bigcup_j C^j$ is entirely contained in Y or is entirely contained in Z : This cannot happen since $Y, Z \in \mathcal{C}$.
- (3) $r \leq \delta(F)$: This is impossible since $\delta(F) \leq |F| = |A'| < \mu(A', B')$.

So we must be in the fourth case, that for one of the $C^j \subseteq Z$, setting $U = (F \cap X) \cup (C^j \cap Z)$, we have $\delta(U/U \cap A) < 0$. Further, $r - \delta(F)$ of the C^j are entirely contained in $Y \setminus X$. Let J be the collection of j so that C^j is entirely contained in $Y \setminus X$. Thus, $F \subseteq Y$, though $F \not\subseteq X$. Since the C^j are disjoint and $|J| \geq r - \delta(F) \geq \mu(A', B') - |A'| \geq |A'| \geq \delta(F)$, it follows that at least one of the C^j over F which is entirely contained in $Y \setminus X$ must have $\delta(C^j/F) = 0$ (otherwise, we would have $\delta(\bigcup_{j \in J} C^j/F) \leq -|J| \leq -\delta(F)$, so $\delta(\bigcup_{j \in J} C^j \cup F) < 0$). Then, by Lemma 2.13, this extension C^j/F must satisfy exactly the relations in the extension B'/A' . In particular, no k - n -coincidence has been replaced by a k -hook. Thus, we can identify from the set $F \cup C^j$ the exact extension B'/A' . Since $F \cup C^j$ is contained in Y , we can identify a particular minimally simply algebraic extension B'/A' (namely C^j/F) that causes Z to be a B/A -obstruction over X . Thus, it

follows that there is a finite list of minimally simply algebraic extensions so that any Z that is a B/A -obstruction is a B/A -obstruction due to one of these finitely many minimally simply algebraic extensions.

Thus, to see that our list has only finitely many Z_i/X_i , it suffices to show that for each fixed minimally simply algebraic extension B'/A' , there are only finitely many Z_i created on behalf of an obstruction with this form as the witness.

For each of these forms B'/A' , we can specify the fragment of $F \cup \bigcup_j C^j$ that is contained in Y , and which elements of Y they are; again this is a finite amount of information. Then, since the C^j are disjoint and $F \subseteq Y$, we have exactly one Z_i/X_i from this configuration. That is,

$$Z_i = (X \cup F \cup \bigcup_{j \leq r} C^j) \setminus (Y \setminus X)$$

with only relations holding on tuples in $(C^j \cup F)^k \setminus F^k$ and also possibly a relation holding on all of X to ensure that X is a B/A -base. Thus, there are only finitely many Z_i/X_i . Furthermore, since Z_i has this form, we see that $X_i \not\leq Z_i$: First, $Z_i \neq X$, since otherwise this would be an obstruction in the second case above, which we ruled out. Thus, there is some C^j so that $C^j \cap (Z \setminus X) \neq \emptyset$. But if $C^j \subseteq Z \setminus X$, then we would have $F \subseteq Z$, and we would be in the first case above, which we already ruled out. Recall that $C^j \subseteq Z$, so we must have $C^j \cap X \neq \emptyset$ and $C^j \cap (Z \setminus X) \neq \emptyset$. Thus, by simple algebraicity, $\delta(C^j \cap (Z \setminus X)/Y) < 0$. But since Z and Y are freely joined over X , this yields $\delta(C^j \cap (Z \setminus X)/X) < 0$, so $X \not\leq Z_i$.

We next check that each Z_i is a B/A -obstruction over X_i . Consider the B/A -obstruction Z over X which caused Z_i/X_i to enter our list. There are F and C^j for $j \leq r$ in $Z \oplus_X Y$ which violate the μ -bound for some extension B'/A' . By the above, we saw that $F \subseteq Y$ and $F \not\subseteq X$. Let Y_i/X_i be the minimally simply algebraic extension of the form B/A . Consider the subset of $Y_i \oplus_{X_i} Z_i$ comprised of the elements in $F \cup \bigcup_j C^j$ in $Z \oplus_X Y$. We have preserved all of the relations between C^j 's and F , and since F is contained in Y but not in X , if R_2 or R_1^i hold on F in Y , then the same relation holds on Y_i . It follows that each C^j/F is still of the form B'/A' showing that $Y_i \oplus_{X_i} Z_i$ also violates the μ -bound and thus Z_i is a B/A -obstruction over X_i . Similarly, we see that any extension of any X that contains a subset W so that W/X realizes all the relations in Z_i/X_i is also a B/A -extension.

By construction of our list of Z_i/X_i 's, it is immediate that any B/A -obstruction Z over X contains a subset W so that W/X realizes all of the

relations in Z_i/X_i for some i . Namely, the Z_i/X_i added to the list for this particular B/A -obstruction Z over X . \square

Corollary 2.17. *It is first-order to say that $X \subseteq M$ is contained in some $Z \subseteq M$ that is a B/A -obstruction over X .*

Proof. This is immediate from the previous lemma. \square

Corollary 2.18. *If $X \subseteq W \leq M$ and M contains a B/A -obstruction over X , then W contains a B/A -obstruction over X .*

Proof. M contains an extension Z'_i so that Z'_i/X realizes all the relations in Z_i/X . We show that each of the C^j in the construction of Z_i must be contained in W . Suppose otherwise, then there is a C^j so that $C^j \cap X \neq \emptyset$ and $C^j \setminus W \neq \emptyset$. Since C^j is of the form B'/A' over F , which is a minimally simply algebraic extension, we see that $\delta(C^j \setminus W/F \cup W) < 0$. Since F is freely joined with C^j over X , we get that $\delta(C^j \setminus W/W) < 0$. But this contradicts $W \leq M$. Thus, each C^j is contained in W and $Z'_i \subseteq W$. \square

Lemma 2.19 (Algebraic Amalgamation Lemma). *If $A, B, C \in \mathcal{C}$ and $A \leq B$ and $A \leq C$ and C is simply algebraic over A , say, minimally simply algebraic over $F \subseteq A$, then either $B \oplus_A C \in \mathcal{C}$, or B contains $\mu(F, C \setminus A)$ many disjoint extensions over F of the form $(C \setminus A)/F$, or A contains a $(C \setminus A)/F$ -obstruction over F .*

Note that if $C \in \mathcal{C}$, then the last possibility cannot hold.

Proof. We consider the structure $B \oplus_A C$. It cannot violate the conditions on being in \mathcal{C} in the first way, since for any $X \subseteq B \oplus_A C$, $\delta(X) = \delta(X \cap B) + \delta(X \cap C) - \delta(X \cap A) = \delta(X \cap C) + \delta(X \cap B/X \cap A) \geq \delta(X \cap C) \geq 0$, since $X \cap B$ and $X \cap C$ are freely joined over $X \cap A$. Similarly, every k -hook or k - n -coincidence in $B \oplus_A C$ is contained in either B or in C , so $B \oplus_A C$ cannot violate the third or fourth condition. Therefore, if $B \oplus_A C$ is not in \mathcal{C} , it is because it violates the μ -bound. Thus, by the definition of obstruction, either $B \oplus_A C \in \mathcal{C}$, or B contains $\mu(F, C \setminus A)$ many disjoint extensions over F of the form $(C \setminus A)/F$, or B contains a $(C \setminus A)/F$ -obstruction. By Corollary 2.18, since $A \leq B$, it follows that A contains a $(C \setminus A)/F$ -obstruction over F . \square

Corollary 2.20. *If $X \subseteq W \in \mathcal{C}$, X is a B/A -base and W does not contain a B/A -obstruction over X , then there is a V so that $W \leq V$ and V contains $\mu(A, B)$ many disjoint B/A -extensions over X .*

Proof. If W does not contain a B/A -obstruction over X or $\mu(A, B)$ many disjoint B/A -extensions over X , then let Y/X be minimally simply algebraic and of the form B/A . Then $V_1 = W \oplus_X Y \in \mathcal{C}$. Since $W \leq V_1$, V_1 also does not contain a B/A -obstruction over X . Thus, either V_1 contains $\mu(A, B)$ many disjoint B/A -extensions over X , or we can pass to $V_2 = V_1 \oplus_X Y$. Continuing in this way, we eventually find a V_k which has $\mu(A, B)$ many disjoint extensions over X of the form B/A . Since $W \leq V_1 \leq V_2 \leq \dots$, we have that $W \leq V_k$. \square

Finally, we can present the Strong Amalgamation Lemma, which shows that we can amalgamate the class \mathcal{C} to build a generic structure G .

Lemma 2.21 (Strong Amalgamation Lemma). *Suppose $A, B, C \in \mathcal{C}$ and $A \leq B$ and $A \leq C$. Then there exists $D \in \mathcal{C}$ so that $B \leq D$ and there is an embedding $f : C \rightarrow D$ with $f(C) \leq D$ and $f \upharpoonright A = \text{id}_A$.*

Proof. We proceed by induction on the size of $C \setminus A$. First suppose that there is a $c \in C$ where there are no relations holding between c and A whatsoever with $A \cup \{c\} \leq C$. Then $A \cup \{c\} \leq B \cup \{c\}$ and $A \cup \{c\} \leq C$. Then, by induction since $|C \setminus (A \cup \{c\})| < |C \setminus A|$, we get our D so that $B \cup \{c\} \leq D$ and $f : C \rightarrow D$ as needed.

Now suppose that there is no such c . Then we can take $X \subseteq C$ minimal so that $A \subset X$ and $A \leq X \leq C$. By inductive hypothesis, we may assume that this $X = C$. Then C is simply algebraic over A , say minimally simply algebraic over $F \subseteq A$. Thus, we can use Lemma 2.19. Either $D = B \oplus_A C$ works, or B has $\mu(F, C \setminus A)$ many disjoint extensions over F of the form $C \setminus A/F$. Not all of these can be contained in A as otherwise C would violate the μ -bound. Thus, one of them must be contained in $B \setminus A$ (since $A \leq B$, it cannot be partly in A and partly in $B \setminus A$). Thus, Lemma 2.13 shows that we can map $C \setminus A$ to this extension in B , and $D = B$ suffices. \square

We will now describe a structure G which is the strong amalgamation limit for the class \mathcal{C} . We will establish the properties of this G over the next several lemmas. In particular, we need to establish that G is saturated, and from there that, the theory of G is strongly minimal.

Corollary 2.22. *There exists G so that*

- (1) G is countable.
- (2) $A \subset G$ and A finite implies $A \in \mathcal{C}$.
- (3) $A \leq G$, $A \leq B$ and $B \in \mathcal{C}$ implies that there exists an embedding f of B into G so that f is the identity on A and $f(B) \leq G$.

Proof. By amalgamating within the class \mathcal{C} repeatedly using Lemma 2.21, and using the fact that \mathcal{C} is countable and closed under substructure, we get G with these three properties. \square

Lemma 2.23. *G satisfies the following conditions:*

- (1) G is countable.
- (2) $A \subset G$ and A finite implies $A \in \mathcal{C}$.
- (3) There is an infinite set I so that $I \leq G$ and no relations hold on I .
- (4) If $A \subseteq G$ is finite and $A \subseteq C \in \mathcal{C}$ is a minimally simply algebraic extension, then either G contains $\mu(A, C)$ many disjoint extensions over A of the form C/A or it contains a C/A -obstruction over A .

Proof. The first two follow from the previous corollary. Since, for any A , $A \leq A \cup \{c\}$ where there is no relation holding involving c , we can apply the third item in the previous corollary infinitely often to attain our I .

Let $A \subseteq G$ be finite and $A \subseteq C \in \mathcal{C}$. Let $B \subseteq G$ be so that $A \leq B$ and so that there is no extension of A of the form C/A which is disjoint from B . Since $B \leq B \oplus_A C$, it must be that $B \oplus_A C \notin \mathcal{C}$. By the Algebraic Amalgamation Lemma, either B contains $\mu(A, C)$ extensions over A of the form C/A , or it contains a C/A -obstruction over A . \square

Lemma 2.24. *Any structure satisfying the four properties in Lemma 2.23 is isomorphic to G .*

Proof. Let G' be a structure satisfying (1)-(4). We use a back-and-forth construction along strong substructures to show that $G \cong G'$. At every stage, we have an isomorphism $f : X \rightarrow X'$ from some $X \leq G$ to an $X' \leq G'$. Without loss of generality, we consider a forth stage. That is, we are given some $Y \subseteq G$ with $X \subset Y \leq G$. We may assume that Y is minimal with $X \subset Y \leq G$, so either Y is a simply algebraic extension of X , or Y is comprised one new element which is unrelated to X . In the latter case, property (3) for G' suffices to give an image for Y extending f , so we consider the former case. Let $F \subseteq X$ be so that $Y \setminus X$ is minimally simply algebraic over F . Suppose G' contains $\mu(F, Y \setminus X)$ many disjoint extensions over F' of the form $(Y \setminus X)/F$. Not all of these extensions can be contained in X' , since then Y would violate the μ -bound, so there is one that is not contained in X' . Since $X' \leq G'$, there cannot be such an extension which is partially in X' and partially not. Thus, there is an extension V/F' of the form $(Y \setminus X)/F$ which is disjoint from X' . Since $X' \leq G'$, we see $F' \leq V$, so V is minimally simply algebraic over F' , and X' is freely joined with V over F' . Thus, by Lemma 2.13, $V/F' \cong (Y \setminus X)/F$, and we can send $Y \setminus X$ to this extension.

Now we consider the other possibility, which is that G' contains a $(Y \setminus X)/F$ -obstruction. Since $X' \leq G$, it follows that X' contains this obstruction. But then X contains the same obstruction, and we have that Y is not in \mathcal{C} , contradicting property (2) of G . Thus, G and G' allow a back-and-forth construction along strong substructures. Since both are countable, $G \cong G'$. \square

Lemma 2.25. *Any countable $H \succeq G$ is isomorphic to G .*

Proof. It suffices to show that H satisfies each of the four conditions in Lemma 2.23. It is countable by assumption. Note that the condition $X \in \mathcal{C}$ is defined by a collection of \forall -sentences, so for each such φ , $G \models \forall X \varphi(X)$. Thus, H satisfies the same collection of formulas, giving (2). The same set I shows that (3) holds. Lastly, let $A \subseteq G$ be finite and $A \subseteq C \in \mathcal{C}$ be a minimally simply algebraic extension. G satisfies that for all X , if X is a base for C/A , then either there exists $\mu(A, C)$ many Z so that Z/A is of the form C/A , or there exists W so that W is a C/A -obstruction over A . This is first-order, since being of the form C/A is first-order and having a C/A -obstruction is first-order (see Lemma 2.17). Thus, H satisfies this as well. \square

Corollary 2.26. *G is saturated.*

Proof. Since every countable elementary superstructure of G is isomorphic to G , we see that there are only countably many types consistent with $\text{Th}(G)$. Thus, $\text{Th}(G)$ has a countable saturated model. But then G elementarily embeds in this and we see that it must be isomorphic to G . So G is saturated. \square

We now move towards showing strong minimality of G . To do so, we need to characterize algebraicity in G . So, we now verify that algebraicity is determined by δ .

Definition 2.27. For any finite $A \subseteq G$, we define $d(A) = \min\{\delta(B) \mid A \subseteq B \subseteq G\}$.

In the following, we use standard model-theoretic notation and write, for example, xA or Ax for the set $A \cup \{x\}$.

Lemma 2.28. *If $d(xA) = d(yA) = d(A) + 1$ then $(G, Ax) \cong (G, Ay)$*

Proof. Fix B so that $A \subseteq B \subset G$ and $\delta(B) = d(A)$. Then $B \leq G$ and $d(xB) = d(yB) = d(A) + 1 = d(B) + 1$. Thus, xB and yB are both strong in G and isomorphic. Using the back-and-forth argument along strong substructures of G as in Lemma 2.24, we see that $(G, Bx) \cong (G, By)$. \square

We now know that there is only one type over A of an element x so that $d(xA) > d(A)$. We next see that $d(xA) = d(A)$ implies that $x \in \text{acl}(A)$.

Lemma 2.29. *If $d(xA) = d(A)$ then $x \in \text{acl}(A)$.*

Proof. Let B containing A be minimal so that $\delta(B) = d(A)$. We first show that B is algebraic over A .

Claim 2.30. *If B' is any set containing A so that $\delta(B') = d(A)$, then $B \subseteq B'$.*

Proof. We are assuming that $\delta(B) = \delta(B') = d(A)$, and B is minimal with this property. Then $\delta(B \cup B') \leq \delta(B) + \delta(B') - \delta(B \cap B')$. If $B \cap B' \subsetneq B$, then $\delta(B) + \delta(B') - \delta(B \cap B') < \delta(B') = d(A)$. The strict inequality is due to the minimality of B . But then we have $d(A) < d(A)$, which is a contradiction. Thus, $B \cap B' = B$, so $B \subseteq B'$. \square

To see that B is algebraic over A , suppose that B and B' are automorphic over A . In particular, they both have the property of being minimal containing A with $\delta(B) = \delta(B') = d(A)$. Then the claim shows that $B \subseteq B'$ and $B' \subseteq B$. Therefore, $B' = B$, and B is algebraic over A .

It is direct from the definition of d that $d(A) \leq d(B) \leq d(xB)$. Let C be any set containing xA such that $\delta(C) = d(xA)$. Then $d(xA) \leq \delta(C \cup B) \leq \delta(C) + \delta(B) - \delta(C \cap B) = d(xA) + d(A) - \delta(C \cap B) \leq d(xA) + d(A) - d(A) = d(xA)$. Thus, each inequality is an equality, and we have $\delta(C \cap B) = d(A)$. Thus, by the claim, $B \subseteq C$. Thus, $d(xB) \leq \delta(C) = d(xA) = d(A)$. So $d(xB) = d(B)$, and it suffices to show that $x \in \text{acl}(B)$.

Take a sequence of extensions B_0, B_1, \dots, B_n so that $B_0 = B$ and $x \in B_n$ and so that B_{i+1} is minimal so that $B_i \subsetneq B_{i+1}$ and $\delta(B_{i+1}) = \delta(B)$. Then B_{i+1} is simply algebraic over B_i . By property (2) of G , there are only finitely many copies of this simply algebraic extension over B_i , thus B_{i+1} is algebraic over B_i . Finally, we can conclude that $x \in \text{acl}(B)$. \square

Corollary 2.31. *G is strongly minimal.*

Proof. Over any set there is a unique non-algebraic type realized in G . Since G is saturated, this means that there is a unique non-algebraic type. This is equivalent to strong minimality. \square

Now that we have strong minimality of G , we see that G is a definitional expansion of $G|_{\mathcal{L}_0}$. Below, we write $\exists^K Z \varphi$ to mean that there exists K disjoint sets Z which satisfy φ .

Lemma 2.32. *We have*

- (1) $G \models \forall xy (R_2(x, y) \leftrightarrow \neg(\exists^{5+2^K} Z)(Z/xy \text{ is a } \Lambda\text{-extension}))$, and
(2) $G \models \forall xyzw (R_1^k(x, y, z, w) \leftrightarrow \neg(\exists^{9+2^K} Z)(Z/xyzw \text{ is an } \Omega_k\text{-extension}))$

Proof. The left-to-right direction for both (1) and (2) is due to the μ -bound and property (2) of G .

Conversely, for (1), choose xy such that

$$G \models \neg(\exists^{3+2^K} Z)(Z/xy \text{ is a } \Lambda\text{-extension}).$$

Let $B \leq G$ be such that $xy \subseteq B$. Then $B \oplus_{xy} D \notin \mathcal{C}$ where D/xy is a minimally simply algebraic Λ -extension. This must be because it violates the μ -bound. That is, there are disjoint F, C^1, \dots, C^r in $B \oplus_{xy} D$ so that each C^i/F is of the form Y/X for some minimally simply algebraic extension Y/X and $r > \mu(X, Y)$.

We must consider the four cases from Lemma 2.14. In the first case, since D/xy is a minimally simply algebraic extension, we have $C^j = D \setminus xy$ and $F = xy$. We conclude that each of the other C^j must be contained in B , so B contains already $\mu(xy, D)$ many disjoint extensions over xy of the form D/xy . But then $\mu(xy, D) < 5 + 2^K$. This implies $R_2(x, y)$.

In the second case, we have that either B or D is not in \mathcal{C} , which is impossible. In the third case, $r \leq \delta(F) \leq |F| = |X| < \mu(X, Y)$, which is a contradiction. In the last case, aside from the C^j which are entirely contained in $B_1 \setminus A$, there can be at most $\delta(F)$ many C^j 's. But there can be no more than K many disjoint sets C^j which each intersect $B_1 \setminus A$. So, $r \leq \delta(F) + K < 2 \cdot |X| + 2^K \leq \mu(X, Y)$.

The right-to-left proof of (2) is similar. □

2.2. Creating M from G . Now we turn our attention towards building a theory that codes computability-theoretic content. Fix $f : \omega^2 \rightarrow \{0, 1\}$ to be a $0'''$ -computable total function with the property that for every k , there is at most one n so that $f(k, n) = 1$ and such an n satisfies $n = \langle k, n_0 \rangle$ for some n_0 . We first construct a theory T_f for any such function, and we show that for every such T_f , the prime model has a computable copy. In Theorem 2.36, we construct M coding information about f into the “template” of G . Though M is related to G (as stated precisely in the theorem), our extra coding may (depending on f and its approximation) make M the prime model of its theory. To ensure that M is always the prime model of its theory and make it easier to see dimension in M , we will add constant symbols for all of the elements in M to form M_c , and T_f will be the theory of this M_c .

In Section 3, we will consider T_f for f chosen so that $\{k \mid \exists n f(k, n) = 1\} = \emptyset^{(4)}$, and we consider which degrees compute positive-dimensional models of T_f .

Definition 2.33. Given any \mathcal{L} -structure A , we let \widehat{A} be the result of removing any relations $S_0^n(a, b, c, d)$ that hold on a hook $abcd$.

Definition 2.34. Given M an \mathcal{L}_0 -structure, we define an \mathcal{L} -structure \widetilde{M} to be a definitional expansion as follows: $R_1^i(\bar{x})$ iff

$$\neg(\exists^{9+2^K} Z)(Z/\bar{x} \text{ is a } \Omega_i\text{-extension}).$$

Then we define $R_2(\bar{x})$ iff

$$\neg(\exists^{5+2^K} Z)(Z/\bar{x} \text{ is a } \Lambda\text{-extension}).$$

Definition 2.35. For $\sigma \in 2^{<\omega}$, we define $\sigma' \in 2^\omega$ so that

- For every k , there is exactly one $n = \langle k, n_0 \rangle$ so that $\sigma'(k, n) = 1$.
- If $(\sigma(k, \langle k, m \rangle) = 0$ for every $m < n_0$) and $(\sigma(k, \langle k, n_0 \rangle) = 1$ or is undefined) then $\sigma'(k, \langle k, n_0 \rangle) = 1$.

Theorem 2.36. *There is a computable \mathcal{L}_0 -structure M along with a computable function $g : M^2 \rightarrow 2^{<\omega}$ (written $g : (x, y) \mapsto \sigma_{xy}$) so that \widetilde{M} satisfies the following:*

- (1) *For every k, n , there are at most finitely many pairs $x, y \in M'$ so that $R_2(x, y)$ holds and $\sigma'_{xy}(k, n) \neq f(k, n)$.*
- (2) $\widehat{M} \cong G$
- (3) *If $abcd$ is a k -hook, then $\sigma'_{ab}(k, n) = 1$ iff $M \models S_0^n(a, b, c, d)$.*

Proof. As we give a construction of M and g , we also build an \mathcal{L} -structure N by simultaneously giving a Π_1^0 -approximation to the collection of tuples satisfying the symbols R_1^i and a Π_2^0 -approximation to the collection of tuples satisfying R_2 . As such, we may say that we remove R_1^i from some tuple. This simply means that in the Π_1^0 -approximation that we are building, we determine that R_1^i does not hold on some tuple. We may say that we remove R_2 from some tuple or that we place R_2 on some tuple. It is understood that, for a tuple xy , if we place R_2 on xy infinitely often, then we have $N \models R_2(x, y)$. As M is simply N restricted to \mathcal{L}_0 , we describe the construction of N and then we will show that $N = \widetilde{M}$.

Fix a function $f_0(w, x, y, z) \leq_T 0'$ such that

$$f(w, x) = \lim_y \lim_z f_0(w, x, y, z).$$

For each $j \in \omega$ we say that y_0 is the j -*decider* if y_0 is least so that for every $y \geq y_0$ and $w_0, x_0 \leq j$, $\lim_z f_0(w_0, x_0, j, z) = f_0(w_0, x_0, j, y) =$

$f_0(w_0, x_0, j, y_0)$. Note that the property of being the j -decider is a Π_2^0 -property. Further, since f_0 is Δ_2^0 , it is a Π_2^0 -property for y to be the j -decider and $\sigma : j^2 \rightarrow \{0, 1\}$ to be given by $\sigma(w_0, x_0) = f_0(w_0, x_0, j, y)$. We call this pair (y, σ) the j -decision.

We will attempt to satisfy the following requirements: $\text{Ext}(X, B/A)$ where $X \subseteq \omega$ and B/A is a minimally simply algebraic extension in \mathcal{C} . The requirement says: If it is possible to keep \widehat{N} in \mathcal{C} and add $\mu(A, B)$ many disjoint extensions over X of the form B/A , then do that. We create a tree of strategies. Each $\text{Ext}(X, B/A)$ -strategy has an accompanying guess as to the j -decision (y, σ) where $j = \langle X, A, B \rangle$. On a tree of strategies, this strategy has two outcomes, which we label Π_2^0 and Σ_2^0 , with the Π_2^0 -outcome on the left. Below the Σ_2^0 -outcome (which represents that the strategy failed to have the correct j -decision), we place a new $\text{Ext}(X, B/A)$ -strategy with the next guess (y, σ) at the j -decision, which in turn has two outcomes. We keep placing new $\text{Ext}(X, B/A)$ -strategies below the Σ_2^0 -outcome. We needn't worry that this gives an infinite path on the tree of strategies comprised entirely of $\text{Ext}(X, B/A)$ -strategies. This path cannot be the true path, because there is some correct j -decision. Below the Π_2^0 -outcome we place a $\text{Ext}(X', B'/A')$ strategy so that $\langle X', A', B' \rangle = j + 1$. Whatever the true path is, every requirement $\text{Ext}(X, B/A)$ will have some strategy on the true path that takes the Π_2^0 -outcome infinitely often.

At each stage, we first determine the current true path on the tree (by approximations to the series of Π_2^0 -statements about j -decisions). Then we use this current true path to determine an approximation to N as follows: Any R_2 - or R_1^k -relation placed by a strategy to the right of the true path does not hold on N (since this strategy is injured and re-initialized, this is permanent). For any strategy to the left of the current true path, every instance of an R_1^k continues to hold, but we guess that no R_2 -relation that it placed holds. For strategies on the current true path, if the current true path extends the Σ_2^0 -outcome, then we see the R_1^k -relations as holding, but not the R_2 -relations that it placed; whereas, if the current true path extends the Π_2^0 -outcome, we keep all relations placed by that strategy, including R_2 -relations.

This builds a current approximation to N . We will argue below, once we describe the actions that each strategy takes, that this approximation has the property that $\widehat{N} \in \mathcal{C}$. Now, we take the first strategy on the current true path that has never acted (i.e., has not placed any relations or created any elements since the last time it was re-initialized) and such that the current true path is below the Π_2^0 -outcome of this strategy. Let this be α , where α is an $\text{Ext}(X, B/A)$ -strategy with approximation (y, σ) at the j -decision with

$j = \langle X, A, B \rangle$. Since $\widehat{N} \in \mathcal{C}$, it makes sense to ask if there is some way to extend \widehat{N} to add $\mu(A, B)$ many disjoint extensions over X of the form B/A . This is done via free-amalgamation as in Lemma 2.19. This means checking if X is a B/A -base and if there already is a B/A -obstruction over X . If X is a B/A -base and there is no B/A -obstruction over X , then we build $\mu(A, B)$ many disjoint extensions over X of the form B/A . For each new pair xy , we determine $\sigma_{xy} = \sigma$. Then, for each new k -hook $xyzw$, we decide to place $S_0^n(x, y, z, w)$, where n is so that $\sigma'(k, n) = 1$. Lastly, we add one new element unrelated to anything. This is to ensure that \widehat{N} will satisfy property (3) of Lemma 2.23.

This completes the construction.

Let A_s be the set of elements constructed by stage s . Let $A_s[t]$ be the structure with universe A_s as seen in our approximation to N at stage t . Thus, at stage s , we consider that we have built the structure $A_s[s]$.

Lemma 2.37. *For every $s < t$, $\widehat{A_s[t]} \leq \widehat{A_{s+1}[t]}$.*

Proof. The only relations that can hold between $A_{s+1}[t]$ and $A_s[t]$ are those placed at stage $s+1$. That is, each strategy when it places relations, does so with new elements, so only at this one stage can we place relations between $A_{s+1}[t]$ and $A_s[t]$. At this stage, a strategy is building $A_{s+1}[s+1]$ by freely joining in realizations of a simply algebraic extension over $A_s[s+1]$. As such, we have $\widehat{A_s[s+1]} \leq \widehat{A_{s+1}[s+1]}$. The remaining subtlety is that at a later stage, we may have extra S -relations holding in $\widehat{A_{s+1}[t]}$ between $A_{s+1} \setminus A_s$ and A_s . This can happen because at a stage t , we no longer see a relation $R_2(x, y)$. This might turn a k -hook $xyzw$ into a k - n -coincidence. Regardless, at stage s , on the 4-tuple $xyzw$, we counted -2 in δ : -1 for R_1^k and -1 for being a k -hook (this is in addition to the -1 for $R_2(xy)$). After removing R_2 , we still count -2 for $xyzw$: -1 for R_1^k and -1 for S_0^n . Thus, we have not decreased $\delta(\widehat{X}/\widehat{A_s})$ for any $X \subseteq A_{s+1}$. Thus, we still have $\widehat{A_s[t]} \leq \widehat{A_{s+1}[t]}$. \square

Lemma 2.38. *At every pair of stages $s \leq t$, $\widehat{A_s[t]}$ is in \mathcal{C} .*

Proof. The lemma clearly holds for $s = t = 0$. We proceed to show that it holds for $s \leq t$ assuming that the lemma holds for all pairs $s' \leq t'$ so that (t', s') is lexicographically less than (t, s) . Since $A_0 = \emptyset$, Lemma 2.37 shows that $\emptyset \leq \widehat{A_s[t]}$, so the first condition for being in \mathcal{C} holds. Since we took the hat-operation of $A_s[t]$, the third condition holds by definition. There is only one stage where we can place any relations on a tuple $xyzw$. If we placed $R_1^k(x, y, z, w)$, then we could only have placed one relation

$S_0^n(x, y, z, w)$. As such, at all future stages, there can similarly be only one relation S_0^n on $xyzw$, and the fourth condition holds as well.

We must verify that $\widehat{A_s[t]}$ does not violate the μ -bound. Suppose towards a contradiction that F, C^1, \dots, C^r are disjoint subsets of $\widehat{A_s[t]}$ each of which is of the form B/A and $r > \mu(A, B)$. Let n be least so that $X = \bigcup_{i \leq r} C^i \cup F \subseteq A_n$, and let us consider $A_n[t]$. Since $\widehat{A_n[t]} \in \mathcal{C}$ for any $n < s$ by the inductive hypothesis, we see that $n = s$. By the inductive hypothesis, we can assume $A_{s-1}[s] \in \mathcal{C}$. But then the construction at stage s amalgamates within \mathcal{C} , so $\widehat{A_s[s]} \in \mathcal{C}$. There can be no more R_1^k -relations holding inside X than existed in $A_s[s]$.

The concern is with regard to R_2 -relations and S_0^m -relations, for which $\widehat{A_s[s]}$ and $\widehat{A_s[t]}$ may disagree. Let α be the strategy that acts at stage t , i.e., α represents the true path at stage t , and let β be the strategy that acted at stage s . We handle the four cases that $\beta \wedge \Pi_2^0 <_L \alpha$, $\alpha \wedge \Pi_2^0 <_L \beta$, $\beta \wedge \Pi_2^0 \preceq \alpha$, or $\alpha \wedge \Pi_2^0 \preceq \beta$.

For each $r \in [s, t)$, let β_r be the strategy which acts at stage r . We first suppose $\beta_r \wedge \Pi_2^0 <_L \alpha$ for some $r \in [s, t)$. This includes the case where $\beta \wedge \Pi_2^0 <_L \alpha$. Then $A_s[r]$ may have more R_2 -realizations than $A_s[t]$. In turn, since we are taking the hat operation, $\widehat{A_s[t]}$ may have more S_0^m -relations than $\widehat{A_s[r]}$ does. In the definition of ‘‘of the form’’, if we turn a k -hook into a k - m -coincidence, and we turn Y/X into an extension of the form B/A (for B/A minimally simply algebraic), then Y/X was already of the form B/A . Thus, when we consider the sets F, C^1, \dots, C^r in $\widehat{A_s[r]}$, these sets are of the form B/A already. This is impossible because we know by the inductive hypothesis that $\widehat{A_s[r]} \in \mathcal{C}$.

We next suppose that $\alpha \wedge \Pi_2^0 <_L \beta$. The fear is that we may have relations $R_2(x, y)$ which did not appear in $A_s[s]$ but do appear in $A_s[t]$. Note that such xy must be in A_{s-1} . First we consider the case that B/A is a Λ -extension, in particular the extension uses only R_1^0 -relations. Note that it is impossible to have all of $\bigcup_{i \leq r} C^i$ contained inside A_{s-1} , for if this were the case then $A_{s-1}[t] \not\subseteq A_s[t]$. Thus, we have some element of one C^j which was new at stage s . When we visit $\alpha \wedge \Pi_2^0$ at stage s , we remove every R_1^k -relation that holds on this element, so C^j/F is not of the form B/A after all. Thus, we can assume B/A is not a Λ -extension. So the concern is not appearances of R_2 inside F , but rather holding inside each C^i . Now, since $\alpha \wedge \Pi_2^0 <_L \beta$, any occurrence of R_2 which was placed by β would be removed, so any occurrence of R_2 inside X must be in A_{s-1} . Thus,

each C^j contains at least two elements inside A_{s-1} . Now, we compute $\delta(\widehat{X[t]}/\widehat{A_{s-1}[t]})$. Since $\widehat{A_{s-1}[t]} \leq \widehat{A_s[t]}$, $\delta(\widehat{X[t]}/\widehat{A_{s-1}[t]})$ should be ≥ 0 .

Case 1: $F \subseteq A_{s-1}$. Then there is one C^j that intersects $A_s \setminus A_{s-1}$. It also intersects A_{s-1} . This witnesses that $\widehat{A_{s-1}[t]} \not\leq \widehat{A_s[t]}$ contradicting Lemma 2.37. Thus, Case 1 is impossible.

Case 2: $F \not\subseteq A_{s-1}$. We partition the C 's into two groups: $I = \{j \mid C^j \subseteq A_{s-1}\}$ and $J = \{j \mid C^j \not\subseteq A_{s-1}\}$. In the following sequence of inequalities, we omit the $[t]$ after the first instance, but all computations are done at stage t . Then

$$\begin{aligned}
& \delta(\widehat{X[t]}/\widehat{A_{s-1}[t]}) \\
&= \delta(\widehat{\bigcup_J C^j}/\widehat{(F \cup \bigcup_I C^j \cup A_{s-1})}) + \delta(\widehat{(F \cup \bigcup_I C^j)}/\widehat{A_{s-1}}) \\
&\leq \delta(\widehat{\bigcup_J C^j}/\widehat{F \cup A_{s-1}}) + \delta(\widehat{F}/\widehat{A_{s-1}}) \\
&\leq -|J| + \delta(\widehat{F}/\widehat{A_{s-1}}) \\
&\leq -|J| + \delta(\widehat{F}) - |I| \\
&= \delta(\widehat{F}) - r \\
&\leq \delta(\widehat{F}) - \mu(B/A) \\
&\leq \delta(\widehat{F}) - (2 \cdot |F| + 2^K) < 0.
\end{aligned}$$

This witnesses that $\widehat{A_{s-1}[t]} \not\leq \widehat{A_s[t]}$, again contradicting Lemma 2.37.

Thus, neither case is possible.

We next consider the case where $\alpha \wedge \Pi_2^0 \preceq \beta$. Since at stage s , β was the least node on the current true path which hadn't acted, we see that α acted before stage s . Since it acted again at stage t , we see that both α and β have been re-initialized by the current path at some stage $r \in (s, t)$ visiting some $\gamma <_L \alpha$. We thus are in the case where $\beta_r \wedge \Pi_2^0 <_L \alpha$, which we considered above.

Lastly, we suppose that $\beta \wedge \Pi_2^0 \preceq \alpha$. If β has not been re-initialized since stage s , then we have $A_s[s] = A_s[t]$, thus since we know $\widehat{A_s[s]} \in \mathcal{C}$, we have that also $\widehat{A_s[t]} \in \mathcal{C}$. If β has been re-initialized, say by visiting a $\gamma <_L \beta$, then there is an $r \in (s, t)$ so that $\beta_r \wedge \Pi_2^0 <_L \alpha$, which we considered above.

Thus, in any one of the cases, we get a contradiction to $\widehat{A_s[t]}$ violating the μ -bound, and we have shown that $\widehat{A_s[t]} \in \mathcal{C}$. \square

Since, by definition, the true N is the structure defined in this way by the true path, this shows that N satisfies property (2) of Lemma 2.23.

Lemma 2.39. *For every minimally simply algebraic extension B/A with $B \in \mathcal{C}$ and for every B/A -base $X \subseteq \widehat{N}$, either there exist $\mu(A, B)$ many disjoint extensions over X of the form B/A , or there exists a B/A -obstruction over X in \widehat{N} .*

Proof. Consider a requirement of the form $\text{Ext}(X, B/A)$ that is on the true path and has the correct guess at the j -decision for $j = \langle X, A, B \rangle$. Let s be a stage when this requirement has a chance to act, and after which it will never be re-initialized. Since the requirement is on the true path, its beliefs about N are correct. That is, at this stage, $A_s[s]$ actually is the structure N on the set A_s . Either the strategy ensures that there are $\mu(A, B)$ many disjoint extensions over X of the form B/A , and since it is correct about N , this extension exists in N over X , or it finds a B/A -obstruction over X . Since its approximation to N is correct, this obstruction is actually in N . \square

Lastly, since we explicitly build a sequence of elements, one from each A_s , which have no relation holding with any other tuple from A_s in N , we build a sequence I so that no relation holds on a tuple from I , and $\widehat{I} \leq \widehat{N}$.

We have now verified that \widehat{N} satisfies each of the properties from Lemma 2.23. Therefore, it is isomorphic to G .

Lemma 2.40. $N = \widetilde{M}$.

Proof. Since $\widehat{N} \cong G$, we see that the restriction of N to the language involving only the R -relations (no S -relations) is the same as the restriction of G to these relations.

Thus, by Lemma 2.32,

$$N \models \forall xy(R_2(x, y) \leftrightarrow \neg(\exists^{5+2^K} Z)(Z/xy \text{ is a } \Lambda\text{-extension})) \text{ and}$$

$$N \models \forall xyzw(R_1^k(x, y, z, w) \leftrightarrow \neg(\exists^{9+2^K} Z)(Z/xyzw \text{ is a } \Omega_i\text{-extension}))$$

since G satisfies these sentences. But the formulas on the right are precisely the definitions in \widetilde{M} of the relations R_2 and R_1^k . Thus, $N = \widetilde{M}$. \square

The first condition of Theorem 2.36 follows from the equation $f(w, x) = \lim_y \lim_z f(w, x, y, z)$, and the third follows from the construction: If we have $\sigma'_{ab}(k, n) = 1$ and, when we construct the 4-tuple $abcd$, believe that it forms a k -hook, then we place $S_0^n(a, b, c, d)$ and no other S -relations. Since

no other strategy can place any relations on the tuple $abcd$, it either is a k -hook and $S_0^n(a, b, c, d)$, or it is not a k -hook. Either way, we have that if it is a k -hook, then $S_0^n(a, b, c, d)$ for the unique n so that $\sigma'_{ab}(k, n) = 1$. \square

We expand M to M_c in the language $\mathcal{L}_0^c = \mathcal{L} \cup \{c_i \mid i \in \omega\}$ computably for new distinct constant symbols c_i . Since M is a computable structure, it has universe ω , and we let c_i be interpreted as the i th element in M .

Let T_f be the theory of the structure M_c . It is immediate that the prime model of T_f is M_c , which is computable.

Theorem 2.41. *T_f is strongly minimal.*

Proof. It suffices to show that \tilde{M} is strongly minimal. We do this by showing that \tilde{M} is definable in G , or equivalently, \tilde{M} is definable in \widehat{M} .

For the sake of this proof, we refer to the relations S_0^n that appear in \widehat{M} as \widehat{S}_0^n , to distinguish them from the relations S_0^n that hold in M . Note that $\widehat{S}_0^n \subseteq S_0^n$, and if $n = \langle k, n_0 \rangle$, then $S_0^n \setminus \widehat{S}_0^n$ consists only of k -hooks. In particular, a 4-tuple $abcd$ is in $S_0^n \setminus \widehat{S}_0^n$ if and only if it is a k -hook and $\sigma'_{ab}(k, n) = 1$.

If $f(k, n) = 1$, then we have that all but finitely many k -hooks $abcd$ satisfy $\sigma'_{ab}(k, n) = 1$, so $S_0^n \setminus \widehat{S}_0^n$ is definable and thus S_0^n is definable in \widehat{M} (note that we use the fact that each R -relation is definable in G).

If $f(k, n) = 0$, then we have that all but finitely many k -hooks $abcd$ satisfy $\sigma'_{ab}(k, n) = 0$, so $S_0^n \setminus \widehat{S}_0^n$ is finite, and thus definable. Once again, we obtain that S_0^n is definable in \widehat{M} . Thus, since G is strongly minimal, we see that M is strongly minimal, showing that M_c is strongly minimal. \square

3. THE POSITIVE-DIMENSIONAL MODELS OF T_f

Now fix a total $0'''$ -computable function $f : \omega^2 \rightarrow \{0, 1\}$ such that for every k , there is at most one n with $f(k, n) = 1$ and such an n is of the form $\langle k, n_0 \rangle$ for some n_0 , and such that $\{k \mid \exists n f(k, n) = 1\} = \emptyset^{(4)}$.

Lemma 3.1. *If \mathbf{d} computes a positive-dimensional model of T_f , then $\mathbf{0}^{(4)} \leq \mathbf{d}'$.*

Proof. Let N be a positive-dimensional model of T_f . Fix $a, b \in N \setminus M$ so that $N \models R_2(a, b)$. The lemma is now immediate from the following

Claim 3.2. $k \in \emptyset^{(4)}$ if and only if

$$\tilde{N} \models \forall x \forall y (R_1^k(a, b, x, y) \rightarrow \exists n S_0^n(a, b, x, y)),$$

or equivalently

$$\tilde{N} \models \exists x \exists y (R_1^k(a, b, x, y) \wedge \exists n S_0^n(a, b, x, y)).$$

Proof. Suppose $k \in \emptyset^{(4)}$. Let n be so that $f(k, n) = 1$. Then for any pair xy , if we have $R_1^k(a, b, x, y)$, then we must have $S_0^n(a, b, x, y)$, showing that the first condition is true. Now suppose that $\tilde{N} \models \forall x \forall y (R_1^k(a, b, x, y) \rightarrow \exists n S_0^n(a, b, x, y))$; then certainly

$$\tilde{N} \models \exists x \exists y (R_1^k(a, b, x, y) \wedge \exists n S_0^n(a, b, x, y)),$$

since there is a pair xy on which $R_1^k(a, b, x, y)$ holds. Lastly, suppose that $N \models \exists x \exists y (R_1^k(a, b, x, y) \wedge \exists n S_0^n(a, b, x, y))$. In M , either all k -hooks \bar{z} aside from finitely many satisfy $S_0^n(\bar{z})$, or all k -hooks \bar{z} aside from finitely many satisfy $\neg S_0^n(\bar{z})$, depending on whether $f(k, n) = 1$ or $f(k, n) = 0$. Thus, since \tilde{N} models the theory of M and $ab \notin \text{acl}(\emptyset)$, we have $S_0^n(a, b, x, y)$ iff $f(k, n) = 1$. Thus, if $\tilde{N} \models \exists x \exists y (R_1^k(a, b, x, y) \wedge \exists n S_0^n(a, b, x, y))$, then there is an n so that $f(k, n) = 1$ and $k \in \emptyset^{(4)}$. \square

Since R_1^k is \forall_1 in the atomic diagram of N , thus in \mathbf{d} , we see that $\emptyset^{(4)} \in \Delta_2^0(\mathbf{d})$, i.e., $\mathbf{0}^{(4)} \leq \mathbf{d}'$. \square

We can now explicitly answer a question which was posed and left open in Andrews-Knight [AK18]:

Corollary 3.3. *There is no positive-dimensional model of T_f which is computable in $\mathbf{0}''$. Thus, $\mathbf{0}''$ is not strongly minimal relatively computing.*

Proof. $\mathbf{0}^{(4)} \not\leq (\mathbf{0}'')$. \square

4. THE STRONGLY MINIMAL COMPUTING AND THE STRONGLY MINIMAL RELATIVELY COMPUTING DEGREES

In this section, we put together the work done already to show that a degree being strongly minimal computing implies that it is strongly minimal relatively computing, which in turn implies that it is high over $\mathbf{0}''$. Lastly, we will conclude the cycle of implications by showing that every degree high over $\mathbf{0}''$ is strongly minimal computing.

Definition 4.1. We call a degree \mathbf{d} *strongly minimal computing* if whenever T is a strongly minimal Solovay theory, then \mathbf{d} computes a copy of every countable model of T .

We call a degree \mathbf{d} *strongly minimal relatively computing* if whenever T is a strongly minimal theory with a computable model, then \mathbf{d} computes a copy of every countable model of T .

The following observation follows trivially from the definitions since the theory of a computable structure is always Solovay.

Observation 4.2. If \mathbf{d} is strongly minimal computing, then \mathbf{d} is strongly minimal relatively computing.

Lemma 4.3. *If \mathbf{d} is strongly minimal relatively computing, then \mathbf{d} is high over $\mathbf{0}''$, i.e., $\mathbf{d} \geq_T \mathbf{0}''$, and $\mathbf{d}' \geq \mathbf{0}^{(4)}$.*

Proof. By Khoussainov-Laskowski-Lempp-Solomon [KLLS07], we have that every strongly minimal relatively computing degree \mathbf{d} is $\geq \mathbf{0}''$. By Lemma 3.1, $\mathbf{d}' \geq \mathbf{0}^{(4)}$. \square

It remains to show that degrees which are high over $\mathbf{0}''$ are strongly minimal computing. In the proof of the following lemma, we rely heavily on both the results and the proofs in Andrews-Knight [AK18]. In particular, in order to prove that a structure M has a $\mathbf{0}'''$ -computable copy, they prove first that there must be a $\mathbf{0}^{(4)}$ -computable P^1 -labeling of a copy of M . A P^1 -labeling is a function f from finite tuples in ω to pairs $(\theta(\bar{x}), k)$ so that θ is a B_1 -formula (i.e. a Boolean combination of existential formulas), $k = \text{MR}(\theta)$, and there is a unique B_1 -type containing θ of Morley rank k . To be a labeling of a copy of M , there must be a copy of M so that for every tuple, the function $f(\bar{a})$ gives a pair (θ, k) so that the unique B_1 -type containing θ of Morley rank k is the B_1 -type of \bar{a} in M .

Lemma 4.4. *Let \mathbf{d} be high over $\mathbf{0}''$. Then \mathbf{d} is strongly minimal computing.*

Proof. First suppose that M is a 1-saturated model (i.e. for every $\bar{a} \in M$ every B_1 -type $p(\bar{a}x)$ consistent with the type of \bar{a} is realized in M) of a strongly minimal Solovay theory T . By the work of Andrews-Knight [AK18] (see the penultimate step of the proofs of Theorems 4.7, 5.7, and Theorems 6.3 and 7.1 combined with Lemma 4.4), we see that there is a $\mathbf{0}^{(4)}$ -computable P^1 -labeling of a copy of M . The degree $\mathbf{0}''$ suffices to decode any P^1 -code for a B_1 -type by [AK18, Lemma 3.7-2]. Thus, a careful reading of the proof of [AK18, Lemma 4.6 (Second Pull-Down Lemma)] shows that \mathbf{d} suffices to give a computable copy of M .

Now, let us suppose that M is not 1-saturated. Andrews-Knight [AK18, Lemma 5.6] shows that M has a copy A where a P^1 -labeling is $\mathbf{0}'''$ -computable. Then, we use Robinson low guessing over \mathbf{d} to give a \mathbf{d} -computable copy of M . Note that since \mathbf{d} is high over $\mathbf{0}''$, $\mathbf{0}'''$ is low and c.e.

over \mathbf{d} . Thus, we can “certify” computations and after only finitely many errors, we will have correct $\mathbf{0}'''$ -computations. The process of certifying computations from $\mathbf{0}'''$ involves constructing a single set V c.e. in \mathbf{d} , and when we see a computation from our approximation to $\mathbf{0}'''$, which is c.e. in \mathbf{d} , with use σ , we add σ to V . We only consider the computation certified if our approximation to $\mathbf{0}^{(4)}$, which is Δ_2^0 in \mathbf{d} , includes the statement $\exists \sigma \in V(\sigma \preceq \mathbf{0}''')$. This lets us demand that either we see a certified computation or our approximation changes. If we see certified computations infinitely often, then in fact $\exists \sigma \in V(\sigma \preceq \mathbf{0}''')$ is true, and one of our certified computations must be correct.¹

We now describe the pull-down strategy for moving from A with a $\mathbf{0}'''$ -computable P^1 -labeling to a \mathbf{d} -computable copy. As in [AK18, Lemma 4.6 (Second Pull-Down Lemma)], we use \mathbf{d} to build a structure B and \mathbf{d} -computable approximations to a \mathbf{d}' -computable function $f : B \rightarrow A$. Since $\mathbf{0}''$ can compute the B_1 -type associated to a P^1 -index (θ, k) , assigning the quantifier-free formulas to tuples in B can be done and checked to be consistent with the P^1 -index assigned to the image of f . Of course, \mathbf{d} must use its approximation to the $\mathbf{0}'''$ -computable P^1 -labeling and will unassign some values of f when its approximation to the assigned P^1 -index of a tuple in the range of f changes.

Our requirements are that every element $b \in B$ eventually has a stable image $f(b)$ and that every $a \in A$ eventually has a stable preimage $f^{-1}(a)$. These second requirements are not difficult and handled exactly as in [AK18, Lemma 4.6 (Second Pull-Down Lemma)].

The requirements to handle every $b \in B$ need a new idea. In [AK18, Lemma 4.6 (Second Pull-Down Lemma)], the strategy to find an image for b uses the fact that, even if we were to fail to find an image for b , we could ensure that b realizes some consistent B_1 -type over the higher-priority elements. But by 1-saturation, this type is realized. Thus, at a sufficiently late stage, if f chooses to send b to this realization of the type, it will never later be injured. In other words, by 1-saturation, we would “luck into” a good image for b . Here, we do not have 1-saturation, but we have the fact that $\mathbf{0}'''$ is c.e. and low over \mathbf{d} (as opposed to a full jump above \mathbf{d}). This is the key to the solution. We will simply ask $\mathbf{0}'''$ for a certified computation telling us exactly where we should send b . This includes asking for the computation showing the P^1 -indices on all the appropriate subtuples, so

¹A further summary of Robinson’s guessing method can be found in Soare [So87, detailed hint to exercise XI.3.5], summarizing the original argument in Robinson [Rob66]. An alternate approach, via cost functions, can be found, e.g., in Nies [Nie02, proof of Theorem 5.1].

that the approximation will not change and make f undefined on b . Since we will eventually find a certified computation that is correct, we will find our image for b .

In more detail: At any given stage, when we consider this requirement, we have a finite $\bar{b} \in B$ so that $f(\bar{b}) = \bar{a}$ is fixed by higher priority requirements. And we have a tuple $\bar{b}\bar{d}$ on which we have committed to some formula $\theta(\bar{b}\bar{b}\bar{d})$. We now look for a certified computation from $\mathbf{0}'''$ — and do not proceed further until one appears — providing the following:

- an element $c \in A$ so that extending f to add $f(b) = c$ is consistent, and
- the P^1 -index of the type of $\bar{a}c$ and all of its subtuples for the c provided in the previous bullet.

Once we see a certified computation, we continue our construction with $f(b) = c$. This strategy is only injured if the P^1 -index for $\bar{a}c$ or a subtuple changes after the computation is certified. In this case, we repeat this strategy. As only finitely many certified computations can be wrong, eventually we succeed in the requirement. \square

Putting the previous results together, we have proved:

Main Theorem. *A degree is strongly minimal computing if and only if it is strongly minimal relatively computing if and only if it is high over $\mathbf{0}''$.*

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