ON DOWNEY'S CONJECTURE

MARAT M. ARSLANOV, ISKANDER SH. KALIMULLIN,
AND STEFFEN LEMPP

ABSTRACT. We prove that the degree structures of the d.c.e. and the 3-c.e. Turing degrees are not elementarily equivalent, thus refuting a conjecture of Downey. More specifically, we show that the following statement fails in the former but holds in the latter structure: There are degrees $\mathbf{f} > \mathbf{e} > \mathbf{d} > \mathbf{0}$ such that any degree $\mathbf{u} \leq \mathbf{f}$ is either comparable with both $\mathbf{e}$ and $\mathbf{d}$, or incomparable with both.

1. THE THEOREMS

In 1965, Putnam [Pu65] defined the n-c.e. sets as a generalization of the c.e. (or computably enumerable) sets:

Definition 1. Given an integer $n > 0$, we call a set $A \subseteq \omega$ *n-c.e.* if there is a computable sequence of sets $\{A_s\}_{s\in\omega}$ such that for all $x \in \omega$,

$$A_0(x) = 0,$$
$$A(x) = \lim_s A_s(x), \text{ and}$$
$$\{s \in \omega \mid A_s(x) \neq A_{s+1}(x)\}.$$

(Note that a c.e. set is thus simply a 1-c.e. set; and a 2-c.e. set is a *d.c.e.* set, i.e., a difference of two c.e. sets.)

These sets were first extensively studied (and extended to the α-c.e. sets for computable ordinals α) by Ershov [Er68a, Er68b, Er70] and are nowadays often said to form the *Ershov hierarchy*, which stratifies the Δ^0_2-sets.

The *n-c.e. degrees*, i.e., the Turing degrees of the n-c.e. sets, were first investigated by Lachlan (late 1960's, unpublished), who showed that for any n-c.e. degree $\mathbf{d} > \mathbf{0}$, there is a c.e. degree $\mathbf{a}$ with $\mathbf{d} > \mathbf{a} > \mathbf{0}$, and

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by Cooper [Co71], who showed that there is a *properly d.c.e. degree*, i.e., a Turing degree containing a d.c.e. set, but no c.e. set.

The d.c.e. and, more generally, the n-c.e. Turing degrees form an intermediate degree structure between the computably enumerable and the Δ^0_2-degrees. It is interesting to compare the n-c.e. degrees with both of them, since they share some of the properties of either structure.

By the above-mentioned result of Lachlan and the existence of a minimal Δ^0_2-degree (Sacks [Sa61]), the n-c.e. degrees and the Δ^0_2-degrees do not form elementarily equivalent degree structures. The first elementary difference between the c.e. and the d.c.e. degrees was found by Arslanov [Ar88] (see also Cooper, Lempp, Watson [CLW89]), who showed that every nonzero d.c.e. degree $\mathbf{d}$ *cups to* $\mathbf{0}'$, i.e., that there is a d.c.e. degree $\mathbf{e} < \mathbf{0}'$ with $\mathbf{d} \cup \mathbf{e} = \mathbf{0}'$, whereas Yates and Cooper (1973, unpublished, cf. D. Miller [Mi81]) showed this to fail for the c.e. degrees. Further elementary differences were found by Downey [Do89], who showed that the Lachlan Nondiamond Theorem [La66] for the c.e. degrees fails for the d.c.e. degrees; and by Cooper, Harrington, Lachlan, Lempp, and Soare [CHLLS91], who showed that for all $n > 1$, the n-c.e. degrees are not dense, in contrast to the Sacks Density Theorem [Sa64] for the c.e. degrees.

Given that no one was able to find, or even conceive of, an elementary difference between the n-c.e. degrees for various $n > 1$, Downey formulated the following

Conjecture 2 (Downey [Do89])**.** *For any $m, n > 1$, the degree structures of the m-c.e. and the n-c.e. degrees are elementarily equivalent.*

In this paper, we will refute this conjecture:

Theorem 3. *The degree structures of the d.c.e. and the 3-c.e. degrees are not elementarily equivalent. (More precisely, there is an $\forall\exists$-sentence in the language of partial orderings only on which they differ.)*

This theorem will immediately follow by Corollary 6 and Theorem 7 below. We conjecture that our proof can be extended as follows:

Conjecture 4. *For any distinct $m, n \geq 1$, the degree structures of the m-c.e. and the n-c.e. degrees are not elementarily equivalent.*

We are now ready to state the technical results establishing Theorem 3:

Theorem 5. *Let E and D be d.c.e. sets and X a c.e. set such that $X \leq_T E$, $E \not\leq_T D$, $D \not\leq_T X$, and both D and E are c.e. in X. Then there exists a d.c.e. set U such that $X \leq_T U \leq_T E$ and $U \mid_T D$.*

This theorem will be proved in section 2. From this theorem, we obtain

Corollary 6. *There are no d.c.e. degrees* $\mathbf{f} > \mathbf{e} > \mathbf{d} > \mathbf{0}$ *such that any d.c.e. degree* $\mathbf{u} \leq \mathbf{e}$ *is comparable with* $\mathbf{d}$*, and any d.c.e. degree* $\mathbf{u}$ *with* $\mathbf{d} \leq \mathbf{u} \leq \mathbf{f}$ *is comparable with* $\mathbf{e}$*.*

Proof. For the sake of a contradiction, assume that such degrees $\mathbf{f} > \mathbf{e} > \mathbf{d} > \mathbf{0}$ exist. Note that by Robinson [Ro71] there is no c.e. degree $\mathbf{x}$ such that $\mathbf{d} < \mathbf{x} \leq \mathbf{e}$ or $\mathbf{e} < \mathbf{x} \leq \mathbf{f}$.

Let $F \in \mathbf{f}$, $E \in \mathbf{e}$ and $D \in \mathbf{d}$ be d.c.e. sets. By Lachlan (unpublished), we can choose a c.e. set $X \leq_T E$ such that D and E are c.e. in X. Then by Theorem 5 we have $D \leq_T X$. Since X is c.e. and $X \leq_T E$, the case $D <_T X$ is not possible. Thus, $X \in \mathbf{d}$ and $\mathbf{d}$ is a c.e. degree.

Let $Y \leq_T F$ be a c.e. set such that E and F are c.e. in Y. Without loss of generality we can assume that $D \leq_T Y$ (otherwise we can replace Y by $X \oplus Y$). Since Y is c.e. and $Y \leq_T F$, we cannot have $E \leq_T Y$. Then by Theorem 5 we must have a d.c.e. set U such that $Y \leq_T U \leq_T F$ and $U \mid_T E$. This contradicts our assumption since $D \leq_T U$. □

On the other hand, we will show that Corollary 6 fails in the 3-c.e. degrees:

Theorem 7. *There are 3-c.e. degrees* $\mathbf{f} > \mathbf{e} > \mathbf{d} > \mathbf{0}$ *such that any 3-c.e. degree* $\mathbf{u} \leq \mathbf{e}$ *is comparable with* $\mathbf{d}$*, and any 3-c.e. degree* $\mathbf{u}$ *with* $\mathbf{d} \leq \mathbf{u} \leq \mathbf{f}$ *is comparable with* $\mathbf{e}$*. (In fact,* $\mathbf{e}$ *and* $\mathbf{d}$ *can be chosen d.c.e. and c.e., respectively; and we can arrange that* $\mathbf{f}$ *is c.e. in* $\mathbf{e}$*, and* $\mathbf{e}$ *is c.e. in* $\mathbf{d}$*.)*

This theorem will be proved in section 3. We obtain immediately

Corollary 8. *There are d.c.e. degrees* $\mathbf{f} > \mathbf{e} > \mathbf{0}$ *such that any d.c.e. degree* $\mathbf{u} \leq \mathbf{f}$ *is comparable with* $\mathbf{e}$*.*

In order to make the proof of Theorem 7 more legible, we will, in section 3.3, sketch the proof of Corollary 8.

Note that from the proof of Corollary 6, it follows that if the degrees $\mathbf{f}$ and $\mathbf{e}$ are as in Corollary 8, then the degree $\mathbf{e}$ must be c.e. Our argument can be extended to give an (infinite) definable class in the d.c.e. degrees consisting only of c.e. degrees, namely, those d.c.e. degrees $\mathbf{e}$ which, for some $\mathbf{f} > \mathbf{e}$, satisfy the statement of Corollary 8. Note that this class cannot be equal to the class of all c.e. degrees by the existence of non-isolating c.e. degrees shown by Arslanov, Lempp, Shore [ALS96].

Note also that all known sentences in the language of partial ordering which are true in the n-c.e. degrees and false in the $(n+1)$-c.e. degrees for some $n \geq 1$ (including the sentence from Theorem 3 for $n = 2$ and the sentence from Corollary 8 for $n = 1$) belong to the level $\forall\exists$ or higher. We conjecture that for all $n \geq 1$, the $\exists\forall$-theory of the n-c.e. degrees is a subtheory of the $\exists\forall$-theory of the $(n+1)$-c.e. degrees.

The rest of this paper will be devoted to the proofs of Theorem 5 and Theorem 7.

2. The proof of Theorem 5

2.1. **The requirements for Theorem 5.** To prove Theorem 5 we can suppose that $\overline{E}$ is not c.e. in D (otherwise we can consider the set $E \oplus X$ instead of E since if $\overline{E} \oplus \overline{X}$ is c.e. in D then $X \leq_T D$ and hence $E \leq_T D$).

Let Y be the "Lachlan" set for $E \oplus D$, i.e., if $E \oplus D = E^1 \oplus D^1 - E^2 \oplus D^2$ for some c.e. sets E^1, D^1, E^2, and D^2, where $E_2 \subseteq E_1$ and $D_2 \subseteq D_1$, then $Y = h^{-1}(E^2 \oplus D^2)$, where h is a 1-1 computable function such that $E^1 \oplus D^1 = \mathrm{rng}(h)$. Then, obviously, $Y \leq_T X$. Hence we can suppose that X is a set of the form $Y \oplus Z$ for some c.e. set Z.

Let $\{E^1_s\}_{s\in\omega}$, $\{E^2_s\}_{s\in\omega}$, $\{D^1_s\}_{s\in\omega}$, and $\{D^2_s\}_{s\in\omega}$ be computable enumerations of the c.e. sets E^1, E^2, D^1 and D^2, respectively, such that $E^2_s \subseteq E^1_s$ and $D^2_s \subseteq D^1_s$. Let $E_s = E^1_s - E^2_s$ and $D_s = D^1_s - D^2_s$. Let $\{X_s\}_{s\in\omega}$ be a computable enumeration of X.

Let f and g be partial computable functions such that $f(y) = 2h^{-1}(2y)$ and $g(y) = 2h^{-1}(2y+1)$. (So $f(\omega \oplus \emptyset) = E \oplus \emptyset$ and $g(\emptyset \oplus \omega) = \emptyset \oplus D$.)

To prove Theorem 5 it is sufficient now to construct a d.c.e. set $U \leq_T E \oplus X \equiv_T E$ meeting for all Turing functionals Γ and Λ the following requirements:

$$\mathcal{P}_\Gamma : U = \Gamma(D) \rightarrow \overline{E} = \mathrm{dom}(\Phi(D));$$
$$\mathcal{N}_\Lambda : D = \Lambda(U \oplus X) \rightarrow D = \Psi(X).$$

Here, the functionals Φ and Ψ will be built during the construction below.

We first consider basic modules for $\mathcal{P}$- and $\mathcal{N}$-requirements in isolation.

2.2. **The Basic Module for the $\mathcal{P}$-requirement in isolation.** To satisfy the $\mathcal{P}$-requirement we proceed by an ω-sequence of cycles. For an arbitrary cycle $k \in \omega$ we

(1) pick a witness $u_k \notin U$ and wait for $\Gamma(D; u_k)[s]\downarrow = 0$ at a stage s,

(2) then open cycle $k+1$, define $\Phi(D;k)=0$ with $\varphi(k)=\gamma(u_k)$, and
(3) wait for k to enter into E or $D \upharpoonright \gamma(u_k)$ to change. In the latter case go to Step 1 with the same witness u_k. In the former case,
(4) enumerate u_k into U, stop all cycles $> k$, and
(5) wait for a stage s_1 when $\Gamma(D;u_k)[s_1]\downarrow = 1$. After that
(6) reopen cycles $> k$, and leave $\Phi(D;k)$ undefined. (Note that $\Phi(D;k)$ must now be undefined even though D is only a d.c.e. set since we never define such a computation unless there is a corresponding computation $\Gamma(D;u_k)$ with use at most that of $\Phi(D;k)$.)
(7) If later k leaves E then we define $\Phi(D;k)=0$ with $\varphi(k)=0$, and extract u_k from U (for the sake of $U \leq_T E$).

There are the following possible outcomes of this strategy.

A. There are only finitely many cycles opened. This means that some cycle k waits at Step 1 or 5 forever. We denote this outcome by ω.
B. For some cycle k there is an infinite loop from Step 3 to Step 1. We denote this outcome by k for the least such $k \in \omega$.
C. Otherwise. In this case we will have $\overline{E} = \mathrm{dom}(\Phi(D))$, which is impossible.

To successfully combine this strategy with strategies for the $\mathcal{N}$-requirements we need modify it allowing many different witnesses u_k for each cycle $k > 0$.

2.3. **The Modified Basic Module for the $\mathcal{P}$-requirement.** Now each cycle k can define a number of witnesses u_k^α, where the upper index α is any binary finite string. For any distinct witnesses u_k^α and u_k^β the string α is not a prefix of β (and vice versa).

Cycle $k \in \omega$:

(1) Wait for $\Gamma(D;u_k^\alpha)\downarrow = 0$ for the witness u_k^α with $\alpha \subseteq D$ if such witness exists.
(2) If there is no witness u_k^α defined with $\alpha \subseteq D$ then pick a new witness $u_k^\alpha \notin U$, where $\alpha \subseteq D$ is the $\subseteq$-least string such that $\beta \subseteq \alpha$ and $\Gamma(\alpha;u_l^\beta)\downarrow$ for each witness u_l^β, where $l < k$ and $\beta \subseteq D$ (for $k=0$ only a witness x_k^λ can be defined, where λ is the empty string), and return to Step 1.
(3) Open cycle $k+1$, and if $k \notin E$ then for the witness u_k^α with $\alpha \subseteq D$ define $\Phi(D;k)=0$, where the use $\varphi(k)$ is equal to the maximum of the length of α and the use $\gamma(u_k^\alpha)$. Then

(4) wait for k to enter into E or $\Phi(D;k) = 0$ to become undefined. In the latter case go to Step 1. In the former case, go to Step 5.
(5) While $k \in E$ enumerate each witness u_k^α such that $\alpha \subseteq D$ into U. (So we delay the enumeration of u_k^α until α becomes an initial segment of D. Since D is d.c.e. and c.e. in X we do not lose the permitting provided by $E \equiv_T E \oplus X$. Also this delay does not injure the $\mathcal{P}$-requirement since if $\alpha \not\subseteq D$ then the corresponding $\Phi^D(k)$-definition is not correct because its use is greater than the length of α). The cycles $> k$ will work only at those stages at which $\Phi(D;k)$ is undefined.
(6) If later k leaves E then we define $\Phi(D;k) = 0$ with $\varphi(k) = 0$, and extract all u_k^α from U.

Note that the infinite loop from Step 2 to Step 1 at the cycle k is possible only if some cycle $l < k$ infinitely loops from Step 4 to Step 1. Thus the possible outcomes are exactly the same as in the original module.

A. Outcome ω. There are only finitely many cycles opened.
B. Outcome $k \in \omega$. k is the least cycle which infinitely loops from Step 4 to Step 1.

2.4. **The Basic Module for the $\mathcal{N}$-requirement in isolation.** Again we proceed by an ω-sequence of cycles. For each cycle l,

(1) wait for $D \upharpoonright (l+1) = \Lambda(U \oplus X) \upharpoonright (l+1))$, and
(2) define $\Psi(X;l) = D(l)$ with $\psi(l) = \lambda(l)$, and restrain $U \upharpoonright \lambda(l)$. Open the cycle $l+1$.
(3) Wait for $D \upharpoonright (l+1)$ or $X \upharpoonright \psi(l)$ to change. In the latter case cancel all restraints imposed by cycles $\geq l$ returning these cycles to Step 1. In the former case we stop the cycles $> l$ and
(4) wait for $D \upharpoonright (l+1) = \Lambda(U \oplus X) \upharpoonright (l+1)$ again.
(5) Redefine $\Psi(X)(l) = D(l)$, reopen the cycles $> l$.

There are the following possible outcomes of this strategy.

A. There are only finitely many cycles opened. This means that some cycle l waits at Step 1 or Step 4 forever. We denote this outcome by ω.
B. For some cycle l there is an infinite loop from Step 3 to Step 1. We denote this outcome by l for the least such $l \in \omega$.
C. Otherwise. In this case we will have $D = \Psi(X)$, which is impossible.

Note that for the sake of the requirement $U \leq_T E$ we may need to extract some u_k^α from U, injuring some $\mathcal{N}$-requirement. This can only be because of some k leaving E and, therefore, $f(k)$ entering

$X = Y \oplus Z$. To avoid this difficulty, it suffices to define $\psi(l)$ in Step 2 to be greater than all $f(k)$ for any $k \in E$ such that there is a witness $u_k^\alpha \in U$ (of some $\mathcal{P}$-strategy) such that $u_k^\alpha < \lambda(l)$.

2.5. **$\mathcal{P}$- and $\mathcal{N}$-strategies together.** In contrast to the extraction of elements from U which has to take place immediately, the restraints imposed by $\mathcal{N}$-strategies can delay or even prevent the enumeration of elements into U by $\mathcal{P}$-strategies of lower priority. The $\mathcal{N}$-restraints drop simultaneously with $X \upharpoonright \psi(l)$-changes, so that the change of X permits the enumeration of a witness u_k^α into U, even though the immediate permission by $E(k)$ is already lost during the delay caused by the restraint.

The restraint of an $\mathcal{N}$-strategy does not protect against a $\mathcal{P}$-strategy of higher priority, i.e., if the $\mathcal{N}$-strategy lies below some outcome of the $\mathcal{P}$-strategy. If the outcome is ω and this is the true outcome, then the $\mathcal{P}$-strategy can injure the restraint only finitely many times. If the outcome is $k_0 \in \omega$, then the infinitely many witnesses of the $\mathcal{P}$-strategy can injure the restraint infinitely often, which can cause a problem for the $\mathcal{N}$-strategy. We omit from our consideration the witnesses u_k^α for $k \le k_0$, since this set is finite (if k_0 is the true outcome). Suppose that when we set a restraint we see a witness $u_k^\alpha \notin U$, $k > k_0$, which can potentially injure our restraint in the future. Note that $\alpha \not\subseteq D$ at the current stage since we believe in the outcome k_0 of the $\mathcal{P}$-strategy and since $\Gamma(\alpha; u_{k_0}^\beta)\downarrow$ for some witness $u_{k_0}^\beta$ with $\beta \subseteq \alpha$. But at some previous stage, we must have had $\alpha \subseteq D$ (when u_k^α was assigned). If later the witness u_k^α enters into U, we must have $\alpha \subseteq D$ at this stage, so that injury to our restraint is possible only if some p leaves D and $\alpha(p) = 0$.

Thus, to protect the $\mathcal{N}$-strategy from the infinitely many injuries of higher priority strategies, we must set the use $\psi(l)$ in Step 2 not only greater than $\lambda(l)$ and all $f(k)$ (for any $k \in E$ such that there is a witness $u_k^\alpha \in U$ of some $\mathcal{P}$-strategy such that $u_k^\alpha < \lambda(l)$)) but also greater than all $g(p)$ such that $p \in D$ and there is a $\mathcal{P}$-strategy whose outcome k_0 leads to our $\mathcal{P}$-strategy and which has a witness u_k^α with $k > k_0$ and $\alpha(p) = 0$.

2.6. **The tree of strategies.** In the formal construction below, we will use the tree of strategies $T = (\omega+1)^{<\omega}$. Let $<_L$ and $<$ be the standard relations on T under the ordered alphabet $\{0 < 1 < 2 < \cdots < \omega\}$.

We also fix some effective listing of all $\mathcal{P}$- and $\mathcal{N}$-requirements such that each $\mathcal{P}$-requirement has an even index in this listing and each $\mathcal{N}$-requirement has an odd index. For the tree T, assign the nth requirement in this listing to all nodes $\xi \in T$ of length n. Let $T = \mathrm{EVN} \cup \mathrm{ODN}$

where EVN and ODN are the sets of nodes of T of even and odd length, respectively.

Let $L_\sigma = \{\langle \tau, k\rangle \in \text{ODN} \times \omega : \tau\hat{\ }k \leq \sigma\}$. It is easy to see that if $\tau_1\hat{\ }k_1 < \tau_2\hat{\ }k_2$ and $\langle \tau_2, k_2\rangle \in L_\sigma$, then $\langle \tau_1, k_1\rangle \in L_\sigma$.

2.7. **The construction.** *Stage* $s = 0$. Let $U_0 = \emptyset$ and $\delta_0 = \lambda$ (where λ is the empty string), set $r_0(\tau, k) = 0$ for all k and all $\tau \in \text{ODN}$, and call no k marked by any $\tau \in \text{ODN}$ at stage $s = 0$. Also, no witness is assigned to any $\sigma \in \text{EVN}$. The functionals $\Phi_{\sigma,0}$ and $\Psi_{\tau,0}$ are totally undefined for all $\sigma \in \text{EVN}$ and all $\tau \in \text{ODN}$. (Therefore, their use functions $\varphi_{\sigma,0}$ and $\psi_{\sigma,0}$ are equal to 0 at all arguments.)

Stage $s + 1$ consists of several parts:

Part I) For all $\tau \in \text{ODN}$ and for all k, define $r'_s(\tau, k) = r_s(\tau, k)$ if $X_{s+1} \restriction \psi_{\tau',s}(k') = X_s \restriction \psi_{\tau',s}(k')$ for all $\langle \tau', k'\rangle$ such that $\tau'\hat{\ }k' \leq_L \tau\hat{\ }k$. (Here we define $\sigma \leq_L \tau$ iff $\sigma = \tau$ or $\sigma <_L \tau$.) Otherwise, set $r'_s(\tau, k) = 0$.

Part II) Define $U_{s+1} = (U_s - \{u : u = \langle \sigma, k, \alpha, t\rangle \ \&\ k \notin E_{s+1}\}) \bigcup \{u : u = \langle \sigma, k, \alpha, t\rangle$ is a witness $\&\ k \in E_{s+1} \ \&\ \alpha \subseteq D_{s+1} \ \&\ R_s(\sigma) \leq u)\}$, where $R_s(\sigma) = \max\{\rho_s(\sigma, \tau, k) : \langle \tau, k\rangle \in L_\sigma\}$ and

$$\rho_s(\sigma, \tau, k) = \begin{cases} +\infty & \text{if } \tau\hat{\ }k \subseteq \sigma \text{ and } r'_s(\tau, k) > 0 \\ r'_s(\tau, k) & \text{otherwise} \end{cases}$$

(Here $\rho_s(\sigma, \tau, k)$ is the restraint $r'_s(\tau, k)$ imposed on the positive σ-strategy. If $r'_s(\tau, k) > 0$ for $\tau\hat{\ }k \subseteq \sigma$ we must set $\rho_s(\sigma, \tau, k) = \infty$ since if $r'_s(\tau, k) > 0$ then it is possible that $r'_s(\tau, k') > 0$ for many $k' > k$. This is a formalization of the fact that if $r'_s(\tau, k) > 0$ then any witness below $\tau\hat{\ }k$ cannot enter into U; such witnesses can enter only if we have a window $r'_s(\tau, k) = 0$; of course, such witnesses must be greater than restraints $r'_s(\tau, k')$, $k' < k$).

Part III) In this part we define δ_{s+1} of length $s + 1$ by induction: Let $\delta_{s+1} \restriction n = \sigma$ be already defined and let $n < s + 1$. Consider the following cases:

Case 1: $\sigma \in \text{EVN}$ and σ is assigned to a requirement $\mathcal{P}_\Gamma$.

Subcase 1a: There exists a witness $y = \langle \sigma, k, \alpha, t\rangle$ such that $U_{s+1}(y) \neq \Gamma_{s+1}(D_{s+1}; y){\downarrow}$: Then define $\delta_{s+1} \restriction (n + 1) = (\delta_{s+1} \restriction n)\hat{\ }\omega$.

Subcase 1b: There exist $k \in \omega$ and a witness $y = \langle \sigma, k, \alpha, t\rangle$ such that $\alpha \subseteq D_{s+1}$ and either $\Gamma_{s+1}(D_{s+1}; y)$ is undefined or, for every $t \leq s$ such that $\sigma \subseteq \delta_t$, we have $D_{s+1} \restriction \gamma_{s+1}(y) \neq D_t \restriction \gamma_t(y)$: In this subcase, we let k_0 be least such k and define $\delta_{s+1} \restriction (n + 1) = (\delta_{s+1} \restriction n)\hat{\ }k_0$.

Subcase 1c: Otherwise: Let k_0 be the least k such that there is no witness $\langle \sigma, k, \alpha, t\rangle$ with $\alpha \subseteq D_{s+1}$, and let $\alpha_0 = D_{s+1} \restriction m$, where m

is the least integer greater than $|\alpha|$ and $\gamma_{s+1}(D_{s+1}, \langle \sigma, k, \alpha, t\rangle)$ for all witnesses $\langle \sigma, k, \alpha, t\rangle$ such that $k < k_0$ and $\alpha \subseteq D_{s+1}$.

Assign $\langle \sigma, k_0, \alpha_0, s+1\rangle$ as a new witness, and set $\delta_{s+1} \restriction (n+1) = (\delta_{s+1} \restriction n)^\frown k_0$. Note that since Subcase 1c is the unique place in the construction where new witnesses are assigned and since we assume that the use-function $\gamma_t(D_t, y)$ is non-decreasing in t (unless an old definition applies again since D has changed back), we have that, for each set Z and for all k and τ, there is at most one witness of the form $\langle \tau, k, \alpha, t\rangle$ where $\alpha \subseteq Z$ and $t \in \omega$.

For any $k < k_0$ such that $k \notin E_{s+1}$ and $\Phi_{\sigma,s}(D_{s+1}; k)$ is undefined, we define $\Phi_{\sigma,s+1}(D_{s+1}; k) = 0$ and, for the witness $\langle \sigma, k, \alpha, t\rangle$ such that $\alpha \subseteq D_{s+1}$, set the use $\varphi_{\sigma,s+1}(k) = \max(|\alpha|, \gamma_{s+1}(\langle \sigma, k, \alpha, t\rangle))$.

Case 2: $\sigma \in$ ODN and σ is assigned to a requirement $\mathcal{N}_\Lambda$: This case is divided into two subcases.

Subcase 2a: There exists k marked by the node σ such that $D_{s+1}(k) \neq \Lambda_{s+1}(U_{s+1} \oplus X_{s+1}; k)\downarrow$: Then define $\delta_{s+1} \restriction (n+1) = (\delta \restriction n)^\frown \omega$.

Subcase 2b: Otherwise: Let k_0 be the least k such that $r'_{s+1}(\sigma, k) = 0$. Define $\delta_{s+1} \restriction (n+1) = (\delta_{s+1} \restriction n)^\frown k_0$. If k_0 is not yet marked then we now declare k_0 marked by the node σ.

Set $r_{s+1}(\sigma, k_0) = 0$ and $\psi_{\sigma,s+1}(k) = 0$ if either $\Lambda_{s+1}(U_{s+1} \oplus X_{s+1}; k_0)$ is undefined or if $\Lambda_{s+1}(U_{s+1} \oplus X_{s+1}; k_0)\downarrow \neq D_{s+1}(k_0)$. Otherwise, set $r_{s+1}(\sigma, k_0) = 1 + \lambda_{s+1}(k_0)$ and $\Psi_{\sigma,s+1}(X_s; k_0) = D_{s+1}(k_0)$ with use

$$\begin{gathered}\psi_{\sigma,s+1}(k_0) = 1 + \max(\{\lambda_{s+1}(k_0)\} \cup \\ \{f(k) : (\exists u < \lambda_{s+1}(k_0))[u = \langle \tau, k, \alpha, t\rangle \ \&\ u \in U_{s+1})]\} \cup \\ \{g(p) : (\exists u < \lambda_{s+1}(k_0))[u = \langle \tau, k, \alpha, t\rangle \text{ is a witness } \ \& \\ u \notin U_{s+1} \ \ \& \ (\exists k' < k)[\tau^\frown k' \subseteq \sigma] \ \&\ p \in D_{s+1} \ \&\ \alpha(p)\downarrow = 0]\} \cup \\ \{\psi_{s,\tau}(k) : \tau^\frown k <_L \sigma^\frown k_0\}).\end{gathered}$$

Part IV) Define $r_{s+1}(\sigma, k) = r'_s(\sigma, k)$ if $\sigma \in$ ODN and $\sigma^\frown k \not\subseteq \delta_{s+1}$. Also for all σ and k such that $\sigma \in$ ODN and $\sigma^\frown k \not\subseteq \delta_{s+1}$, we define $\psi_{\sigma,s+1}(k) = \psi_{\sigma,s}(k)$ if $r_{s+1}(\sigma, k) > 0$, and $\psi_{\sigma,s+1}(k) = 0$ if $r_{s+1}(\sigma, k) = 0$.

Part V) Initialize all $\sigma \in$ EVN such that $\delta_{s+1} <_L \sigma$. The initialization of σ at stage $s+1$ means that all current witnesses of σ cease to be witnesses.

End of the construction.

2.8. **The verification.** Let $F(\sigma, n, \alpha, s, t)$ be the predicate

$$(\forall v \geq t)[\langle \sigma, n, \alpha, s\rangle < R_v(\sigma)].$$

Lemma 9. *There is an X-c.e. predicate $\widetilde{F}(\sigma, n, \alpha, s, t)$ such that for all σ, n, α, s, and t,*

(1) $\widetilde{F}(\sigma, n, \alpha, s, t)$ *implies* $F(\sigma, n, \alpha, s, t)$, *and*
(2) $F(\sigma, n, \alpha, s, t)$ *implies* $\widetilde{F}(\sigma, n, \alpha, s, t)$ *if* $\delta_v \not<_L \sigma$ *for all* $v \geq t$.

Proof. Let $\delta_v \not<_L \sigma$ for all $v \geq t$. To prove the lemma, it is sufficient to prove that if $F(\sigma, n, \alpha, s, t)$ then there is a stage $v_0 \geq t$ such that

1) $\langle \sigma, n, \alpha, s \rangle < R_v(\sigma)$ for all v with $t \leq v < v_0$, and
2) $\langle \sigma, n, \alpha, s \rangle < \rho_{v_0}(\sigma, \tau_0, k_0)$ for some $\langle \tau_0, k_0 \rangle \in L_\sigma$ such that $X_{v_0} \upharpoonright \psi_{\tau,t}(k) = X \upharpoonright \psi_{\tau,t}(k)$ for every $\langle \tau, k \rangle$ such that $\tau\widehat{\ }k \leq_L \tau_0\widehat{\ }k_0$,

since the converse obviously holds for arbitrary σ, n, α, s, and t. (Recall that we defined $\sigma \leq_L \tau$ iff $\sigma = \tau$ or $\sigma <_L \tau$.)

Suppose that $\delta_v \not<_L \sigma$ for all $v \geq t$ and $F(\sigma, n, \alpha, s, t)$. Since $\delta_v \not<_L \sigma$ for all $v \geq t$ we have $r'_v(\tau, k) \geq r'_{v+1}(\tau, k)$ for all $v \geq t$ and all $\langle \tau, k \rangle \in L_\sigma$ such that $\tau\widehat{\ }k \not\subseteq \sigma$ (since the construction can set a new restraint $r'_v(\tau, k)$ only if $\tau\widehat{\ }k \subseteq \delta_v$ and since $\tau\widehat{\ }k <_L \sigma$).

First we consider the case when there is $\langle \tau_0, k_0 \rangle \in L_\sigma$ such that $\tau_0\widehat{\ }k_0 \not\subseteq \sigma$ and $r'_v(\tau_0, k_0) > \langle \sigma, n, \alpha, s \rangle$ for all $v \geq t$. By the definition of r'_s from the construction, we have $X_t \upharpoonright \psi_{\tau,t}(k) = X \upharpoonright \psi_{\tau,t}(k)$ for all $\langle \tau, k \rangle$ such that $\tau\widehat{\ }k \leq_L \tau_0\widehat{\ }k_0$ so that we have claims 1) and 2) for $v_0 = t$.

Thus we can assume that there is a stage $v_1 \geq t$ such that $r'_v(\tau_0, k_0) \leq \langle \sigma, n, \alpha, s \rangle$ for all $v \geq v_1$ and all $\langle \tau_0, k_0 \rangle \in L_\sigma$ with $\tau_0\widehat{\ }k_0 \not\subseteq \sigma$.

Note also that if there are a stage $w \geq v_1$ and $\langle \tau_0, k_0 \rangle \in L_\sigma$ such that $\tau_0\widehat{\ }k_0 \subseteq \sigma$ and $r'_v(\tau_0, k_0) > 0$ for all $v \geq w$, then we have claims 1) and 2) for $v_0 = w$.

Finally, it remains to show that it is not possible that for every $w \geq v_1$ and every $\langle \tau_0, k_0 \rangle$ with $\tau_0\widehat{\ }k_0 \subseteq \sigma$, there is a stage $v \geq w$ such that $r'_v(\tau_0, k_0) = 0$.

Indeed, in that case, there would be some $\langle \tau_0, k_0 \rangle$ such that $\tau_0\widehat{\ }k_0 \subseteq \sigma$ and $\tau_0\widehat{\ }k_0 \subseteq \delta_v$ for infinitely many stages v. Otherwise, if for any such $\langle \tau_0, k_0 \rangle$ there are only finitely many stages v such that $\tau_0\widehat{\ }k_0 \subseteq \delta_v$ then there is a stage $v' \geq v_1$ such that $r'_v(\tau_0, k_0) = 0$ for any $v \geq v'$ so that $\langle \sigma, n, \alpha, s \rangle \leq R_v(\sigma)$ which contradicts with $F(\sigma, n, \alpha, s, t)$.

Choose the $\subseteq$-greatest node τ_0 such that $\tau_0\widehat{\ }k_0 \subseteq \sigma$ for some $k_0 \in \omega$ and $\tau_0\widehat{\ }k_0 \subseteq \delta_v$ for infinitely many stages v.

By the choice of $\langle \tau_0, k_0 \rangle$, for each $\langle \tau_1, k_1 \rangle$ such that $\tau_0\widehat{\ }k_0 \subset \tau_1\widehat{\ }k_1 \subseteq \sigma$, we have $\tau_1\widehat{\ }k_1 \subseteq \delta_v$ for finitely many v. Therefore, there is a stage $v_2 \geq v_1$ such that $r'_v(\tau_1, k_1) = 0$ for all $v \geq v_2$ and all $\langle \tau_1, k_1 \rangle$ with $\tau_0\widehat{\ }k_0 \subset \tau_1\widehat{\ }k_1 \subseteq \sigma$.

Let $v_3 \geq v_2$ be any stage such that $\tau_0\widehat{\ }k_0 \subseteq \delta_{v_3+1}$. By Subcase 2b of the construction, we have $r'_{v_3}(\tau_1, k_1) = 0$ for each $\langle \tau_1, k_1 \rangle$ with $\tau_1\widehat{\ }k_1 \subseteq \tau_0\widehat{\ }k_0$. Thus, by the choice of the stages v_1 and v_2, we have $R_{v_3}(\sigma) \leq \langle \sigma, n, \alpha, s \rangle$. This contradicts $F(\sigma, n, \alpha, s, t)$.

Note that by the part II of the construction, if $F(\sigma, n, \alpha, s, t)$ holds then $U_t(\langle\sigma, n, \alpha, s\rangle) \geq U_v(\langle\sigma, n, \alpha, s\rangle)$ for all $v \geq t$, i.e., the number $\langle\sigma, n, \alpha, s\rangle$ cannot enter U after stage t.

Lemma 10. *$U \leq_T E \oplus X \equiv_T E$.*

Proof. Let $u = \langle\sigma, k, \alpha, s_0\rangle$. By construction, if at stage s_0, u is not declared a witness then $u \notin U$. Suppose that u is chosen as a witness at stage s_0 (and so we have $\alpha \subseteq D_{s_0}$). Obviously, if $k \notin E$ then $u \notin U$. Suppose that $k \in E$ and let $s_1 \geq s_0$ be the least stage such that $k \in E_{s_1}$. Then we check whether α was initialized at a stage s with $s_0 < s < s_1$. If so, then $u \notin U$ (since u ceases to be a witness at stage s). To compute $U(u)$ in the case when α is not initialized at a stage s with $s_0 < s < s_1$, we wait for a stage $s \geq s_1$ such that one of the following clauses holds:

1) $u \in U_s$,
2) $\delta_s <_L \sigma$,
3) $\widetilde{F}(\sigma, k, \alpha, s_0, s)$,
4) $(\exists p)[\alpha(p)\downarrow = 1 \;\&\; p \notin D_s]$, or
5) $(\exists p)[\alpha(p)\downarrow = 0 \;\&\; p \in D_s \;\&\; g(p) \notin X]$.

If such stage s exists then we have $u \in U$ iff $u \in U_s$. Indeed, if clause 3) holds then by Lemma 9 (1) we have $F(\sigma, k, \alpha, s_0, s)$, so $U(u) = U_s(u)$. If one of the clauses 4) or 5) holds then α will not be compatible with D_v for all $v \geq s$ so that by Part II of the construction, u cannot enter U at any stage $> s$, i.e., $U(u) = U_s(u)$.

It remains to argue that there is such stage s. Indeed, if $u \notin U$ and $\delta_s \not<_L \sigma$ for all $s \geq s_1$ then by Part II of the construction, this is possible only because at every stage $s \geq s_1$, either α is not compatible with D_s, or $u < R_s(\sigma)$. If $\alpha \not\subseteq D$ then at some stage $s \geq s_1$, either clause 4) or clause 5) must hold. If $\alpha \subseteq D$ then from some stage $s \geq s_1$, we always will have $u < R_s(\sigma)$, i.e., $F(\sigma, k, \alpha, s_0, s)$. Thus, by Lemma 9 (2), we have $\widetilde{F}(\sigma, k, \alpha, s_0, s)$, i.e., clause 3).

Lemma 11. *$\delta = \liminf_s \delta_s$ exists.*

Proof. Suppose that $\liminf_s \delta_s \upharpoonright n = \sigma$ exists but $\liminf_s \delta_s \upharpoonright (n+1)$ does not exist.

We will consider two possibilities.

Case 1: $\sigma \in \text{EVN}$ and σ is assigned to a requirement $\mathcal{P}_\Gamma$. Note that $\liminf_s R_s(\sigma) < +\infty$. Let s_0 be the least stage such that $\delta_s \not<_L \sigma$ for all $s \geq s_0$.

We prove by induction that, for all k, there exists a permanent witness $\langle\sigma, k, \alpha, t\rangle$ such that $\alpha \subseteq D$. Suppose that for any $k < k_0$, this is true. To prove this for k_0, we first note that by construction, for any k there can be at most one such witness $\langle \sigma, k, \alpha, t\rangle$. Let us denote it by u_k (if it exists).

It is easy to see that $(\forall k < k_0)[\Gamma(D; u_k)\downarrow = U(u_k)]$. Indeed, otherwise either $\liminf_s \delta_s \upharpoonright (n+1) = \sigma\hat{\ }\omega$, or $\liminf_s \delta_s \upharpoonright (n+1) = \sigma\hat{\ }k$ for some $k < k_0$.

By the construction, we choose, at some stage s, a permanent witness $\langle \sigma, k_0, \alpha, s\rangle$ with $\alpha = D \upharpoonright \max\{\gamma_e(D, u_k) : k < k_0\}$.

As $\Gamma(D; u_k)\downarrow = U(u_k) = 0$ for all $k \notin E$, we have $\overline{E} \subseteq \text{dom}(\Phi_\sigma(D))$. Suppose now that $k \in E$ and $\Phi_\sigma(D; k)$ is defined. So there exists a witness $\langle \sigma, k, \alpha, t\rangle$ with $\alpha \subseteq D$ such that $\Gamma_e(D; \langle \sigma, k, \alpha, t\rangle)\downarrow = 0$. But this means that either $\langle \sigma, k, \alpha, t\rangle < \liminf_s R_s(\sigma)$ or $\langle \sigma, k, \alpha, t\rangle$ was initialized so that the witness $\langle \sigma, k, \alpha, t\rangle$ was assigned before stage s_0. Therefore, $\overline{E} =^* \text{dom}(\Phi_\sigma(D))$, which is impossible.

Case 2: $\sigma \in$ ODN and σ is assigned to a requirement $\mathcal{N}_\Lambda$. Fix a stage s_0 such that $\delta_s \not<_L \sigma$ and $U_s(u) = U_{s_0}(u)$ for all $s \geq s_0$ and all $u \in S$ where

$$S = \{\langle \tau, k, \alpha, t\rangle : \tau \in \text{EVN} \ \& \\ \langle \tau, k, \alpha, t\rangle \text{ is a witness at stage } t \ \& \ [\tau\hat{\ }k \leq \sigma]\}.$$

Note that the set S is finite. Indeed, it is clear that the set of all witnesses $\langle \tau, k, \alpha, t\rangle$ with $\tau\hat{\ }k <_L \sigma$ is finite. Furthermore, the set of all witnesses $\langle \tau, k', \alpha, t\rangle$ with $\tau\hat{\ }k' \subseteq \sigma$ is also finite, otherwise, by Subcase 1c) of the construction, $\Gamma(D; k)$ is undefined for some $k < k'$ where Γ is such that τ is assigned to $\mathcal{P}_\Lambda$, and this is not possible since $\liminf_s \delta_s \upharpoonright n = \sigma$.

To obtain a contradiction, we will show that $D(k) = \Psi_\sigma(X; k)$ for all numbers k which are not marked by the node σ at stage s_0.

Indeed, for all k, $\Psi_\sigma(X; k)$ is defined (otherwise $\liminf_s \delta_s \upharpoonright (n+1)$ exists). Therefore, for any k, the construction, at some stage s_k, imposes a restraint $r_{s_k}(\sigma, k)$ such that $r_{s_k}(\sigma, k) = r_s(\sigma, k) = r'_s(\sigma, k) > 0$ for all $s \geq s_k$ and which supports the equation $D(k) = \Lambda(U \oplus X; k) = \Psi_\sigma(X; k)$.

Suppose that k is not marked by node σ at stage s_0. Then the restraint $U \upharpoonright r_{s_k}(\sigma, k)$ cannot be injured by witnesses from S. Also it cannot be injured by witnesses entering U via nodes $\tau > \sigma$. Therefore, only the following possibilities to injure the restraint remain:

1) u leaves $U \upharpoonright r_{s_k}(\sigma, k)$ at a stage $s' > s_k$, or

2) u enters into $U \upharpoonright r_{s_k}(\sigma, k)$ at a stage $s' > s_k$ and $\tau\hat{\ }k' \subseteq \sigma$ and $u = \langle \tau, k'', \alpha, t\rangle$ for some $k' < k''$ and $\alpha \subseteq D_{s'}$.

From 1) and the definition of Ψ_σ at stage s_k, it easily follows that $X_{s'} \upharpoonright \psi_{\sigma,s'}(k) \neq X_{s_k} \upharpoonright \psi_{\sigma,s_k}(k)$, which is impossible.

It follows from 2) that $t \leq s_k$ and $\alpha \subseteq D_t$. Therefore, there exists a witness $u' = \langle \tau, k', \alpha', t' \rangle$ with $\alpha' \subseteq \alpha$, $t' \leq t$, and $U_t(u') = \Gamma_t(\alpha; u')\downarrow$. But then $\alpha \not\subseteq D_{s_k}$ since otherwise $\tau\widehat{\ }k' \not\subseteq \delta_{s_k}$. Then $\alpha \subseteq D_{s'}$ implies that for some p we have $\alpha(p) = 0$ and $p \in D_{s_k} - D_{s'}$.

Now it follows from the definition of $\Psi_\sigma(X; k)$ at stage s_k that $X_{s'} \upharpoonright \psi_{\sigma,s'}(k) \neq X_{s_k} \upharpoonright \psi_{\sigma,s_k}(k)$, which is impossible. Therefore, $D =^* \Psi_\sigma(X)$.

Lemma 12. *For all Turing functionals* Λ, $D \neq \Lambda(U \oplus X)$.

Proof. Suppose that $D = \Lambda(U \oplus X)$. Let $\sigma \subseteq \delta$ be assigned to the requirement $\mathcal{N}_\Lambda$. If $\sigma\widehat{\ }\omega \subseteq \delta$ then, obviously, $D \neq \Lambda(U)$. Therefore, $\sigma\widehat{\ }k_0 \subseteq \delta$ for some k_0. Let s_0 be a stage such that for all $s \geq s_0$ the following conditions holds:

(1) $r_{s_0}(\tau, k) = r_s(\tau, k) = r'_s(\tau, k) > 0$ for all $\langle \tau, k \rangle$ such that $\tau\widehat{\ }k <_L \sigma\widehat{\ }k_0$,
(2) $D_{s_0}(k_0) = D_s(k_0) = \Lambda_{e,s_0}(U_{s_0} \oplus X_{s_0}; k_0)$,
(3) $U_{s_0} \upharpoonright \lambda_{s_0}(k_0) = U_s \upharpoonright \lambda_{s_0}(k_0)$, and
(4) $X \upharpoonright a = X_{s_0} \upharpoonright a$,

where

$$a = 1 + \max(\{\lambda_{s_0}(k_0)\} \cup$$
$$\{f(k) : (\exists u < \lambda_{s_0}(k_0))[u = \langle \tau, k, \alpha, t \rangle \text{ is a witness at stage } t \;\&\; u \in U_{s_0}]\}$$
$$\cup \{g(p) : (\exists u < \lambda_{s_0}(k_0))[u = \langle \tau, k, \alpha, t \rangle \text{ is a witness at stage } t \;\&\; u \notin U_{s_0}$$
$$\&\; (\exists k' > k)[\tau\widehat{\ }k' \subseteq \sigma] \;\&\; \alpha(p)\downarrow = 0 \;\&\; p \in D_{s_0}]\} \cup$$
$$\{\psi_{\tau,s_0}(k) : \tau\widehat{\ }k <_L \sigma\widehat{\ }k_0\}).$$

Now let $s_1 = \mu s \geq s_0(\sigma\widehat{\ }k_0 \subseteq \delta_{s_1})$. By the construction, at stage s_1, we set the restraint $r_{s_1}(\sigma, k_0) = r_s(\sigma, k_0) = r'_s(\sigma, k_0) > 0$ for all $s \geq s_1$. This contradicts $\sigma\widehat{\ }k_0 \subseteq \delta$.

Lemma 13. *For all Turing functionals* Γ, $U \neq \Gamma(D)$.

Proof. Let $\sigma \subseteq \delta$ be assigned to the requirement $\mathcal{P}_\Gamma$. Suppose that $U = \Gamma(D)$. It follows that we cannot have $\sigma\widehat{\ }\omega \subseteq \delta$. Therefore, $\sigma\widehat{\ }k_0 \subseteq \delta$ for some k_0. This means that $\Gamma(D; u)$ is undefined for the permanent witness $u = \langle \sigma, k_0, \alpha, t \rangle$ such that $\alpha \subseteq D$, a contradiction.

3. The proof of Theorem 7

3.1. **The requirements for Theorem 7.** Our notation for this proof will generally follow Soare [So87].

We will construct a 3-c.e. set F, a d.c.e. set E, and a c.e. set D such that the Turing degrees $\mathbf{f} = \deg(F \oplus E \oplus D)$, $\mathbf{e} = \deg(E \oplus D)$, and

$\mathbf{d} = \deg(D)$ satisfy Theorem 7. (As mentioned in the statement of Theorem 7, F will be c.e. in E, and E c.e. in D.)

We need to meet, for all Turing functionals Φ, Ψ, Π, Σ, and Ω, the following requirements:

$$\begin{aligned}
\mathcal{R}_{\Phi,U} &: U = \Phi(F \oplus E \oplus D) \rightarrow \\
&\qquad\qquad \exists\Gamma\,(U = \Gamma(E \oplus D)) \text{ or } \exists\Delta\,(E = \Delta(U \oplus D)), \\
\mathcal{S}_{\Psi,V} &: V = \Psi(E \oplus D) \rightarrow \exists\Theta\,(V = \Theta(D)) \text{ or } \exists\Lambda\,(D = \Lambda(V)), \\
\mathcal{P}^F_\Pi &: F \neq \Pi(E \oplus D), \\
\mathcal{P}^E_\Sigma &: E \neq \Sigma(D), \\
\mathcal{P}^D_\Omega &: D \neq \Omega.
\end{aligned}$$

Here, the functionals Γ, Δ, Θ, and Λ will be built by us, i.e., by strategies on our tree of strategies as explained below.

For simplicity, we will assume throughout this section that the use functions on the functionals Φ, Ψ, Γ, Δ, and Π are computed *separately* for each set in the join of sets of the oracle.

3.2. **The intuition for the strategies for Theorem 7, Part I.** In this section and in section 3.4, we give the intuition for the strategies for Theorem 7.

3.2.1. *Strategies for $\mathcal{R}$ and $\mathcal{S}$ in isolation.* In the absence of other strategies, an $\mathcal{R}$-strategy will simply build its functional Γ as it sees $\Phi(F \oplus E \oplus D)$ compute U; it will ensure that $\Gamma(E \oplus D)$ correctly computes U by changing $E \oplus D$ whenever U changes (after an F-change). We proceed similarly for an $\mathcal{S}$-strategy in isolation.

3.2.2. *Strategies for $\mathcal{P}$ in isolation.* In the absence of other strategies, a $\mathcal{P}$-strategy is simply a Friedberg-Muchnik strategy: E.g., a $\mathcal{P}^F$-strategy picks a witness x targeted for F, waits for a computation $\Pi(E \oplus D; x) = 0$, and then enumerates x into F and protects the computation $\Pi(E \oplus D; x) = 0$ from now on by restraining $E \oplus D$.

Since the full construction is very delicate, we will now present in fair detail a large number of examples to illustrate all the various components of the construction.

3.2.3. *$\mathcal{P}^F$-strategies below an $\mathcal{R}$-strategy.* There is an obvious conflict between the above-described $\mathcal{R}$- and $\mathcal{P}^F$-strategies: The former strategy may need to change $E \oplus D$ each time F changes (since the F-change allows a U-change) whereas the latter strategy is trying to change F while restraining $E \oplus D$. In the fashion of an infinite-injury priority argument, the solution is to cancel Γ whenever some $\mathcal{P}^F$-strategy has enumerated another witness x into F and found that it resulted in a change of U via some number u_x entering U, say. The $\mathcal{R}$-strategy will then start building its Turing functional Δ and generate an infinite "stream" of numbers y_x such that strategies below the Δ-outcome of the $\mathcal{R}$-strategy will be restricted to enumerating numbers from that stream into E. For numbers in the stream, we then have that $y_x \in E$ iff $x \notin F$ iff $u_x \notin U$, where the latter equivalence can be ensured by restraining F, E, and D on all other numbers $\leq \varphi(u_x)$; in particular, we must ensure $y_x > \varphi(u_x)$ for this to work.

There is one further complication we need to address now, and in much more detail later on: In order to allow the correction of our functionals Γ to work, the $\mathcal{P}^E$-strategies below the Δ-outcome of the $\mathcal{R}$-strategy have to pick their witness y_x *before* x is enumerated into F; this ensures that there will be no definitions of our functionals Γ while $x \in F$ but with use $< y_x$. On the other hand, of course, y_x must be chosen *after* a computation $\Pi(E \oplus D; x) = 0$ as well as all computations $\Phi(F \oplus E \oplus D; u_x)$ for all "potential" u_x (i.e., for all u with $\gamma(u) \leq \pi(x)$) have been found. (One final note: Later on, y_x will also serve as a correction marker for other Γ-functionals, to be enumerated into E when x is extracted from F for any reason whatsoever.)

More precisely, a $\mathcal{P}^F$-strategy (below the Γ-outcome of an $\mathcal{R}$-strategy) proceeds as follows:

(1) Pick a witness x targeted for F.
(2) Wait for a computation $\Pi(E \oplus D; x) = 0$ at a stage s_x, say.
(3) Check whether there is an "x-request" from a $\mathcal{P}^E$-strategy to the $\mathcal{R}$-strategy. If so, then let that $\mathcal{P}^E$-strategy pick its witness y_x now. (Otherwise, pick y_x to be any big number.)
(4) Enumerate x into F and protect the computation $\Pi(E \oplus D; x) = 0$ by restraining $E \oplus D$.
Now the $\mathcal{R}$-strategy takes over and proceeds as follows:
(5) Wait for $\Phi(F \oplus E \oplus D; u)$ to be defined again (at a stage s'_x, say) for all u for which $\Gamma(E \oplus D; u)$ was defined at stage s_x.
(6) If $\Phi(F \oplus E \oplus D; u) = \Gamma(E \oplus D; u)$ for all these u, then the $\mathcal{P}^F$-strategy's diagonalization succeeded in a finitary way, and the $\mathcal{P}^F$-strategy stops.
(7) Otherwise, cancel Γ and let u_x be the least u with $\Phi(F \oplus E \oplus D; u) \neq \Gamma(E \oplus D; u)\downarrow$. Set $\Delta(U \oplus D; y_x) = E(y_x)$ with use $\delta(y_x) = u_x$. For all $y < y_x$ for which $\Delta(D \oplus U; y)$ is currently undefined, define $\Delta(D \oplus U; y) = E(y)$ with use 0 (since $E(y)$ will not change any more for those y) unless $\Delta(D \oplus U; y)$ was

defined before, in which case we define the use as before. Create a *triple* $\langle x, u_x, y_x\rangle$.

(8) Allow the strategies below the Δ-outcome to be eligible to act once and go back to Step 1.

Note that the $\mathcal{R}$-strategy has three possible outcomes:

finite: There are only finitely many expansionary stages: Then $\Phi(F \oplus E \oplus D) \neq U$.

Γ: There are infinitely many expansionary stages, but Γ is canceled only finitely often by a $\mathcal{P}^F$-strategy below the Γ-outcome of the $\mathcal{R}$-strategy, and so all these $\mathcal{P}^F$-strategies eventually wait at Step 2 forever or stop at Step 6: Then $\Gamma(E \oplus D)$ can compute U correctly, and all $\mathcal{P}^F$-requirements can be satisfied.

Δ: There are infinitely many expansionary stages, and Γ is canceled infinitely often: Then Δ-definitions are made infinitely often, and, by the way the uses are chosen, $\Delta(U \oplus D)$ is a total function. Every time the parameter s'_x is set, the set $E \upharpoonright (s'_x+1)$ can change only at a number of the form y_x from now on. For such numbers, we will ensure for all future expansionary stages s:

$$\begin{aligned} x \in F_s &\implies \Phi(F \oplus E \oplus D; u_x)[s] = \Phi(F \oplus E \oplus D; u_x)[s'_x] \\ &\implies \Delta(U \oplus D; y_x)[s] = E_s(y_x); \text{ and} \\ x \notin F_s &\implies \Phi(F \oplus E \oplus D; u_x)[s] = \Phi(F \oplus E \oplus D; u_x)[s_x] \\ &\implies \Delta(U \oplus D; y_x)[s] = E_s(y_x). \end{aligned}$$

Thus $\Delta(U \oplus D)$ correctly computes E.

A $\mathcal{P}^F$-strategy below the Δ-outcome can act as if in isolation since it does not have to worry about Γ-correction injuring its $E \oplus D$-restraint.

3.2.4. *$\mathcal{P}^E$-strategies below an $\mathcal{R}$-strategy.* A $\mathcal{P}^E$-strategy below the Γ-outcome of an $\mathcal{R}$-strategy can act as if in isolation since it does not have to worry about Γ-correction injuring its D-restraint; and, by the remarks in section 3.2.3, any E-change allowing some set U to change also allows a corresponding Γ-correction via that E-change.

A $\mathcal{P}^E$-strategy below the Δ-outcome of an $\mathcal{R}$-strategy has to use witnesses of the form y_x from triples supplied by the $\mathcal{R}$-strategy. When the $\mathcal{P}^E$-strategy is ready to diagonalize, it will

(1) make an "x-request" for some pair $\langle x, u_x\rangle$,
(2) choose its own witness y_x larger than any number mentioned so far,
(3) wait for a computation $\Sigma(D; y_x)\downarrow = 0$ (if ever),

(4) enumerate y_x into E and extract x from F to ensure Δ-correction as explained in section 3.2.3, namely, we can reset the computation $\Delta(U \oplus D; y_x)$ (having use $\delta(y_x) \geq u_x$) from 0 to 1.

3.2.5. *$\mathcal{P}^E$-strategies below the Δ-outcomes of an $\mathcal{R}_1$- and an $\mathcal{R}_0$-strategy.* A $\mathcal{P}^E$-strategy below the Δ-outcome of an $\mathcal{R}_1$-strategy, which itself is below the Δ-outcome of an $\mathcal{R}_0$-strategy, has to deal with two problems simultaneously when enumerating a number y into E: It has to ensure that each $\mathcal{R}_i$-strategy's $\Delta_i(U_i \oplus D)$ can compute this (for both $i = 0$ and $i = 1$), in spite of the D-restraint imposed by the $\mathcal{P}^E$-strategy.

The extra difficulty in this situation arises from the need not to let the two $\mathcal{R}$-strategies' numbers x_0 and x_1 (supplied by $\mathcal{P}_0^F$-strategies below the Γ-outcome of the $\mathcal{R}_0$-strategy, and $\mathcal{P}_1^F$-strategies below the Γ-outcome of the $\mathcal{R}_1$-strategy, respectively, working as described in section 3.2.3) interfere with each other and destroy the F-uses controlling U_i at $u_i = u_{x_i}$ (for $i = 0, 1$). We overcome this problem by picking the witnesses x_i in reverse order, and then enumerating the witnesses x_i into F and simultaneously picking their numbers y_i in "forward" order. Of course, when we pick each y_i, we do not know yet whether the enumeration of x_i will result in a $U_i(u_i)$-change; if it does not, then we have to choose a new witness x_i (and so also a new number y_i). We proceed this way to ensure that enumerating x_0 into F does not destroy the computation $\Phi_1(F \oplus E \oplus D; u_1)$ for any potential u_1 found later. (Note here that when we choose x_0, we already know all the potential u_1 as well as their uses $\varphi_1(u_1)$!) Furthermore, we can choose y_i when we enumerate the corresponding x_i into F since at that time, we already know all the potential u_i as well as their uses $\varphi_i(u_i)$. We will illustrate in the next subsection why this will be important later on.

Finally, note that we need a separate y_i for each x_i for reasons which will become clearer later: Roughly speaking, when x_1 is enumerated into F, this allows $U_0(u_0)$ to change for at least the second time. We can now "kill" the $\mathcal{R}_0$-requirement by first extracting x_1, and then x_0, from F to force a third, and then an (impossible) fourth $U_0(u_0)$-change. Each extraction of x_i from F may, however, interfere with the definition of other Γ-functionals in the full construction; to prevent this, we accompany the extraction of each x_i from F by the enumeration of the corresponding y_i into E. These y_i are large enough not to interfere with our forcing the $U_0(u_0)$-changes, but they will allow other Γ-functionals to correct since E is part of the Γ-oracle. However, once all x_i have been enumerated into F and corresponding u_i have been found, we don't need the separate y_i any more and can simply use $y = y_0$ as the witness for the $\mathcal{P}^E$-strategy. (See the end of subsection 3.2.6 for more explanation.)

More precisely, we proceed as follows:

(1) Wait for a $\mathcal{P}_1^F$-strategy below the Γ-outcome of the $\mathcal{R}_1$-strategy to have picked a witness x_1 targeted for F for which there is now a computation $\Pi_1(E \oplus D; x_1) = 0$ at a stage s_{x_1}, say. Do not let this $\mathcal{P}_1^F$-strategy enumerate x_1 into F yet. (We say the $\mathcal{P}_1^F$-strategy has "started the x-request" of the $\mathcal{P}^E$-strategy.)
(2) Wait for a $\mathcal{P}_0^F$-strategy below the Γ-outcome of the $\mathcal{R}_0$-strategy to have picked a witness x_0 targeted for F which is larger than any number mentioned by stage s_{x_1} and for which there is now a computation $\Pi_0(E \oplus D; x_0) = 0$ at a stage s_{x_0}, say. (We say the $\mathcal{P}_0^F$-strategy has "started the x-request" of the $\mathcal{P}^E$-strategy.)
(3) Let this $\mathcal{P}_0^F$-strategy enumerate x_0 into F and pick a number $y = y_0 = y_{x_0}$ targeted for E larger than any number mentioned so far.
(4) Check whether there is some (least) number $u_0 = u_{x_0}$ with $\Gamma_0(E \oplus D; u_0) \neq \Phi_0(F \oplus E \oplus D; u_0)$. If not, then cancel x_0 and y_0 and return to the beginning of Step 2.
(5) Otherwise, we say that the above $\mathcal{P}_0^F$-strategy has "fulfilled the x-request" of the $\mathcal{P}^E$-strategy and define $\Delta_0(U_0 \oplus D; y) = 0$ with use $\delta_0(y) = u_0$.
(6) Let the $\mathcal{P}_1^F$-strategy enumerate x_1 into F and pick a number $y_1 = y_{x_1}$ targeted for E larger than any number mentioned so far.
(7) Now first check whether this results in U_0 changing at the above number u_0 again. If so, then we force a permanent diagonalization for $\mathcal{R}_0$ as follows: Extract x_1 from F and enumerate y_1 into E to force a third U_0-change at u_0. (We say that the $\mathcal{R}_0$-strategy has "started diagonalization via the pair $\langle x_0, u_0 \rangle$".) When (and if) this occurs, then extract x_0 from F and enumerate y_0 into E to force an (impossible) fourth U_0-change at u_0. Once we have this permanent diagonalization, we stop.
(8) Otherwise, check whether there is some least number $u_1 = u_{x_1}$ with $\Gamma_1(E \oplus D; u_1) \neq \Phi_1(F \oplus E \oplus D; u_1)$. If not, then cancel x_1, y_1, x_0, and y_0, and return to the beginning of Step 1.
(9) Otherwise, we say that the above $\mathcal{P}_1^F$-strategy has "fulfilled the x-request" of the $\mathcal{P}^E$-strategy. Associate y with the tuple $\langle x_0, u_0, x_1, u_1, y \rangle$, and define $\Delta_1(U_1 \oplus D; y) = 0$ with use $\delta_1(y) = u_1$.
(10) Wait for a computation $\Sigma(D; y) = 0$ at a stage s_y, say.

(11) Enumerate y into E, extract x_i from F (for both $i \leq 1$), protect the computation $\Sigma(D; y) = 0$ by restraining D, and stop.

Note here that the enumeration of y into E in Step 11 will not affect the setup protecting the computations $\Phi_0(F \oplus E \oplus D; u_0)$ and $\Phi_1(F \oplus E \oplus D; u_1)$ since both these computations are found *before* y is chosen.

3.2.6. *$\mathcal{P}^E$-strategies below the Γ-outcome of an $\mathcal{R}_1$-strategy and the Δ-outcome of an $\mathcal{R}_0$-strategy.* We briefly consider this case to point out the role of Γ-correction and the order in which witnesses are chosen. We assume here that the $\mathcal{R}_0$-strategy has higher priority than the $\mathcal{R}_1$-strategy; the other case is very similar. For this analysis, it will be crucial that the witness y is chosen after the x-request has been started but before it has been fulfilled:

(1) Wait for a $\mathcal{P}^F$-strategy below the Γ-outcome of the $\mathcal{R}_0$-strategy to have picked a witness x targeted for F for which there is now a computation $\Pi(E \oplus D; x) = 0$ at a stage s_x, say. (We say the $\mathcal{P}^F$-strategy has "started the x-request" of the $\mathcal{P}^E$-strategy.)
(2) Let this $\mathcal{P}^F$-strategy enumerate x into F and pick a number $y = y_x$ targeted for E larger than any number mentioned so far.
(3) Check whether there is some least number $u = u_x$ with $\Gamma_0(E \oplus D; u) \neq \Phi_0(F \oplus E \oplus D; u)$. If not, then cancel x and y, and return to the beginning of Step 1.
(4) Otherwise, we say the above $\mathcal{P}^F$-strategy has "fulfilled the x-request" of the $\mathcal{P}^E$-strategy.
(5) Associate y with the triple $\langle x, u, y\rangle$, and define $\Delta_0(U_0 \oplus D; y) = 0$ with use $\delta_0(y) = u$.
(6) Wait for a computation $\Sigma(D; y) = 0$ at a stage s_y, say.
(7) Enumerate y into E, extract x from F, protect the computation $\Sigma(D; y) = 0$ by restraining D, and stop.

The point of this case is that the $\mathcal{R}_1$-strategy can make additional $\Gamma_1(E \oplus D; u')$-definitions (for new u') after x is enumerated into F but before y is enumerated into E (and x is extracted from F); but since $y < \gamma_1(u')$ for these u' and so the enumeration of y into E destroys all these $\Gamma_1(E \oplus D; u')$-computations, $\Gamma_1(E \oplus D; u')$ can be corrected for these u'. This is crucial since for these u', we have no Φ_1-computations with $x \notin F$ and thus no control.

3.2.7. *$\mathcal{P}^F$-strategy below the Γ-outcomes of an $\mathcal{R}_1$- and an $\mathcal{R}_0$-strategy.* A $\mathcal{P}^F$-strategy below the Γ-outcome of an $\mathcal{R}_1$-strategy, which itself is below the Γ-outcome of an $\mathcal{R}_0$-strategy, may, when ready to enumerate its witness x upon finding a computation $\Pi(E \oplus D; x)$, be confronted simultaneously with x-requests to both the $\mathcal{R}_0$- and the $\mathcal{R}_1$-strategy by a $\mathcal{P}^E_0$-strategy and a $\mathcal{P}^E_1$-strategy below the Δ-outcomes of the $\mathcal{R}_0$- and the $\mathcal{R}_1$-strategy, respectively. (However, there can be at most one such $\mathcal{P}^E$-strategy for each higher-priority $\mathcal{R}$-strategy.) In this case, we will consider both x-requests "started", enumerate x into F and let both the $\mathcal{P}^E_0$-strategy and the $\mathcal{P}^E_1$-strategy choose its witness y_0 and y_1, respectively. Then the $\mathcal{R}_0$-strategy first checks for a possible number u_0 at which U_0 changes. If there is such u_0, then the x-request of the $\mathcal{P}^E_0$-strategy is "fulfilled" and the $\mathcal{P}^E_1$-strategy is initialized. Otherwise, the $\mathcal{P}^E_0$-strategy has to start a new x-request (and later choose a new witness y_0), while now the $\mathcal{R}_1$-strategy checks for a possible number u_1 at which U_1 changes. If there is such u_1, then the x-request of the $\mathcal{P}^E_1$-strategy is "fulfilled"; else, the $\mathcal{P}^F$-strategy succeeds in diagonalizing without interfering with the $\mathcal{R}_0$- or the $\mathcal{R}_1$-strategy.

3.2.8. *$\mathcal{P}^F$-strategy between a $\mathcal{P}^E$- and an $\mathcal{R}$-strategy.* We next need to analyze the interaction between an $\mathcal{R}$-strategy and a $\mathcal{P}^F$-strategy below its Δ-outcome in the case that some $\mathcal{P}^E$-strategy below the Δ-outcome but of lower priority than the $\mathcal{P}^F$-strategy has already enumerated a number. The problem is that in that case, the late enumeration of a witness x, say, by the $\mathcal{P}^F$-strategy may destroy the setup for the definition of $\Delta(U \oplus D; y)$ for the $\mathcal{P}^E$-strategy's witness y (where y was picked after x, and so x may be less than the number u_y in the triple $\langle x_y, u_y, y\rangle$ used by the $\mathcal{P}^E$-strategy) so that we lose control over the computation $\Delta(U \oplus D; y)$. However, in this case, we can easily diagonalize for $\mathcal{R}$.

More specifically, the $\mathcal{P}^F$-strategy proceeds as follows:

(1) Pick a witness x.
(2) Wait for a computation $\Pi(E \oplus D; x) = 0$.
(3) Enumerate x into F at a stage s'_x, say.
Now the $\mathcal{R}$-strategy takes over and proceeds as follows:
(4) Wait for $\Phi(F \oplus E \oplus D; u)$ to be defined again for all $u \leq$ any number mentioned by stage s'_x.
(5) Check if $\Delta(U \oplus D; y)\downarrow \neq E(y)$ for some y. If not, then the $\mathcal{P}^F$-strategy's diagonalization succeeded in a finitary way, and the $\mathcal{P}^F$-strategy stops.
(6) Otherwise, let $i = \Delta(U \oplus D; y)$ for the least such y. Then there are earlier stages $s_0 < s_1$, say, at which we set $\Delta(U \oplus D; y) = i$ and $\Delta(U \oplus D; y) = 1 - i$, respectively. So, by stage s_1, we must have changed $E(y)$ from value i to value $1 - i$, and, in order to correct $\Delta(U \oplus D; y)$, for some numbers x_y and u_y, we must have changed $F(x_y)$ to force $U(u_y)$ to change for at least the second

time, while, since now again $\Delta(U \oplus D; y) = i$, $U(u_y)$ has now changed for at least the third time. We now force a permanent diagonalization for $\mathcal{R}$ by extracting x from F (and enumerating the corresponding y_x into E) and stop.

A $\mathcal{P}^F$-strategy below the Δ-outcome of an $\mathcal{R}$-strategy is thus still finitary in any case, but it has the potential to force a finite outcome for the $\mathcal{R}$-strategy.

3.3. **The proof of Corollary 8: A preliminary construction.** We interrupt the discussion of the intuition for the strategies for Theorem 7 in order to sketch the simpler proof of Corollary 8 and thereby give the reader some guidance into the combinatorially more difficult full proof of Theorem 7. Note that the strategies described so far, namely, the strategies for the $\mathcal{R}$-, $\mathcal{P}^F$-, and $\mathcal{P}^E$-requirements (setting $D = \emptyset$), suffice to prove Corollary 8, and that nothing in the description so far makes F non-d.c.e., or E non-c.e., as long as all sets U involved are assumed to be only d.c.e.

3.3.1. *The preliminary tree of strategies.* We define a *preliminary tree of strategies* as follows: We fix an arbitrary computable priority ordering of all $\mathcal{R}$-, $\mathcal{P}^F$-, and $\mathcal{P}^E$-requirements of order type ω. Let

$$O = \{\Delta <_O \Gamma <_O \mathit{fin}\}$$

be the *preliminary set of outcomes* of our strategies, with the priority ordering $<_O$ as stated.

For the preliminary tree of strategies $T \subseteq O^{<\omega}$ to be defined, assign the nth requirement from our above listing of all requirements to all nodes $\xi \in T$ of length n. Then, given a node (or "strategy") ξ already on T, we let $\xi^\frown\langle \mathit{fin}\rangle$ always be an immediate successor of ξ, and if ξ is an $\mathcal{R}$-strategy, then we also let $\xi^\frown\langle\Delta\rangle$ and $\xi^\frown\langle\Gamma\rangle$ be immediate successors of ξ.

We also fix some notation for our preliminary tree: Given any strategy $\eta \in T$, we let

$$X(\eta) = \{\xi \mid \xi \text{ is an } \mathcal{R}\text{-strategy and } \xi^\frown\langle\Delta\rangle \subseteq \eta\}$$

be the set of $\mathcal{R}$-strategies with Δ-outcome above η, respectively. (These will be the strategies to which η may have to make *x-requests*, as described in the next subsections.)

For the sake of clarity, we present the $\mathcal{P}^F$- and $\mathcal{P}^E$-strategies first, since they both involve recursive calls to the (far more complicated) $\mathcal{R}$-strategies presented afterwards.

3.3.2. *The preliminary full $\mathcal{P}^F$-strategy.* A $\mathcal{P}^F_\Pi$-strategy $\xi \in T$ acts as follows at substage t of stage s at which it is eligible to act:

(1) If this is the first stage at which ξ is eligible to act since ξ was last initialized (and so ξ does not have a witness), then ξ chooses a witness x larger than any number mentioned so far and *ends the stage*.
(2) Otherwise, let x be ξ's current witness. Next check whether there is a computation $\Pi(E;x)\downarrow = 0$. If not, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next*.
(3) Otherwise, check whether ξ has already enumerated x into F. If so, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next*.
(4) Otherwise, check whether there is an x-request from a $\mathcal{P}^E$-strategy η to an $\mathcal{R}$-strategy $\zeta \in X(\eta)$ with $\zeta^\frown\langle\Gamma\rangle \subseteq \xi$ which has not yet been started (as defined below). If not, then
   - enumerate x into F;
   - choose a corresponding y_x larger than any number mentioned so far;
   - initialize all strategies $> \xi$; and
   - *end the stage*.
(5) Otherwise, for each such $\mathcal{R}$-strategy ζ, fix the highest-priority such $\mathcal{P}^E$-strategy $\eta_\zeta \supseteq \zeta^\frown\langle\Delta\rangle$. We say ξ *is delayed in the enumeration of* x, and we *end the stage*. [In this case, each such ζ wants to reserve ξ's witness x for forming a "pair $\langle x, u\rangle$" for the sake of the $\mathcal{P}^E$-strategy η_ζ. So the highest-priority such ζ (call it ζ_0) takes over for ξ, and at a future ζ_0-expansionary stage, ζ_0 will enumerate x into F unless ζ_0, and thus also ξ, has been initialized before then. From then on, each such ζ will check, in decreasing order of priority, whether a "pair $\langle x, u\rangle$" can be formed for it.]

3.3.3. *The preliminary full $\mathcal{P}^E$-strategy.* A $\mathcal{P}^E_\Sigma$-strategy $\xi \in T$ acts as follows at substage t of stage s at which it is eligible to act:

(1) If this is the first stage at which ξ is eligible to act since ξ was last initialized, then check whether $X(\xi) \neq \emptyset$. If so, then *make x-requests* to all $\mathcal{R}$-strategies $\zeta \in X(\xi)$ and *end the stage*. [In this case, each $\mathcal{R}$-strategy $\zeta \in X(\xi)$ first needs to "start" an x-request via some $\mathcal{P}^F$-strategy $\eta_\zeta \supseteq \zeta^\frown\langle\Gamma\rangle$; then each such ζ will enumerate η_ζ's witness x_ζ into F and check whether it can form a "pair $\langle x_\zeta, u_\zeta\rangle$", thus "fulfilling" each such x-request. When x_ζ enters F, ξ will choose the corresponding number y_ζ, i.e., at a stage at which ξ itself is not eligible to act.]

(2) Otherwise, check whether ξ has a witness y. If not, then ξ chooses a witness y larger than any number mentioned so far and *ends the stage*.
(3) Otherwise, for ξ's current witness y, check whether there is a computation $\Sigma(y)\downarrow = 0$. If not, then end the substage and let $\xi^\frown\langle \mathit{fin}\rangle$ be *eligible to act next*.
(4) Otherwise, check whether ξ has already enumerated its current witness y into E. If so, then end the substage and let $\xi^\frown\langle \mathit{fin}\rangle$ be *eligible to act next*.
(5) Otherwise,
   - enumerate ξ's current witness y into E;
   - extract each x from F for all $\eta \in X(\xi)$ such that η fulfilled ξ's x-request via a pair $\langle x, u\rangle$;
   - initialize all strategies $> \xi$; and
   - *end the stage.*

3.3.4. *The preliminary full $\mathcal{R}$-strategy.* An $\mathcal{R}_{\Phi,U}$-strategy $\xi \in T$ acts as follows at substage t of stage s at which it is eligible to act:

(1) Check whether the *length of agreement* $\ell(\Phi(F \oplus E), U)$ between $\Phi(F \oplus E)$ and U now exceeds any number mentioned by the end of the previous ξ-expansionary stage, which we will call s'. (Here, 0 is always a *ξ-expansionary stage.*) If not, then end the substage and let $\xi^\frown\langle \mathit{fin}\rangle$ be *eligible to act next.*
(2) Otherwise, call s a *ξ-expansionary stage.* Now check whether ξ has previously, at a stage s'', say, fulfilled an x-request of some $\mathcal{P}^E$-strategy η (where $\xi \in X(\eta)$) with a pair $\langle x, u\rangle$ such that
   (a) η has not been initialized since stage s'';
   (b) η had also made an x-request to some $\mathcal{R}$-strategy $\hat{\xi} \in X(\eta)$ with $\hat{\xi} \supset \xi$ at the same time at which the current x-request to ξ was made;
   (c) $\hat{\xi}$ enumerated a number $\hat{x}$ into F at stage s'; and
   (d) $U(u)$ has changed since the last ξ-expansionary stage s'.

   In this case, ξ can permanently diagonalize $\Phi(F \oplus E)$ against U:
   - extract $\hat{x}$ from F;
   - enumerate into E the number $\hat{y}$ which was chosen when $\hat{x}$ was enumerated into F;
   - cancel η's x-requests to all $\tilde{\xi} \in X(\eta)$;
   - initialize all strategies $> \xi$; and
   - *end the stage.*

   (In this case, we say ξ has *diagonalized.* Unless ξ is initialized later, this will be the last ξ-expansionary stage.)

(3) Otherwise, check whether there is some (highest-priority) $\mathcal{P}^E$-strategy η such that
  (a) $\xi \in X(\eta)$;
  (b) all of η's x-requests to $\mathcal{R}$-strategies $\hat{\xi} \in X(\eta)$ with $\xi \subset \hat{\xi} \subset \eta$ have been started (as defined below); and either
  (c) η has made an x-request to ξ, and that request has not yet been started by ξ (as defined below); or
  (d) η has made an x-request to ξ, and that request has been started by ξ but not yet fulfilled by ξ (as defined below).

(4) If there is (a highest-priority) such η satisfying Case 3c but none satisfying Case 3d, then:
  - define $\Gamma(E; u') = U(u')$ for all $u' \leq \ell(\Phi(F \oplus E), U)$ for which $\Gamma(E; u')$ is currently undefined, with old use $\gamma(u')$ (if defined previously), or new use $\gamma(u')$ larger than any number mentioned so far (otherwise);
  - end the substage; and
  - let $\xi^\frown\langle\Gamma\rangle$ be *eligible to act next.*

  If later during this stage, some $\mathcal{P}^F$-strategy $\zeta \supseteq \xi^\frown\langle\Gamma\rangle$ wants to enumerate a number x into F but is delayed (and so ends the stage), then for the highest-priority such η, we say that ξ has *started η's x-request.* [Note that in this case, η is not eligible to act at this stage.]

(5) If there is (a highest-priority) such η satisfying Case 3d, then check whether ξ has enumerated a number x into F for this x-request at the previous ξ-expansionary stage s'.
  (a) If ξ has not enumerated such x into F at stage s', then there is a $\mathcal{P}^F$-strategy $\zeta \supseteq \xi^\frown\langle\Gamma\rangle$ which found a computation $\Pi(E; x) = 0$ at the last substage of stage s' (but was delayed in the enumeration of x into F). Now ξ
    - enumerates this x into F;
    - chooses a number y larger than any number mentioned so far;
    - initializes all strategies $> \zeta$; and
    - *ends the stage.*
  (b) Otherwise, ξ has enumerated a number x into F and chosen a number y at the last substage of the previous ξ-expansionary stage s'. Check whether there is some u such that $\Gamma(E; u)\downarrow \neq U(u)$.
    (i) If there is no such u, then ξ's attempt at fulfilling η's x-request has failed. So
      - cancel the number y;

- let η again *make* new *x-requests* (which have *not* been started) to all $\mathcal{R}$-strategies $\hat{\xi} \in X(\eta)$ with $\hat{\xi} \subseteq \xi$;
- initialize all strategies $\geq \xi^\frown\langle fin\rangle$; and
- *end the stage.*

(ii) If there is (a least) such u, then we say that ξ has *fulfilled η's x-request* with the pair $\langle x, u\rangle$. Then
- if ξ is the shortest node in $X(\eta)$ then declare y to be the witness of η;
- cancel Γ;
- define $\Delta(U; y') = E(y')$ for all $y' \leq y$ for which $\Delta(U; y')$ is currently undefined, with old use $\delta(y')$ (if defined previously), or new use $\delta(y')$ larger than any number mentioned so far (otherwise);
- end the substage; and
- let $\xi^\frown\langle\Delta\rangle$ be *eligible to act next.*

(6) Otherwise, i.e., if there is no η as in Case 3c or Case 3d then proceed as in Case 4 or Case 5(b)ii, depending on whether or not there is some u such that $\Gamma(E; u)\downarrow \neq U(u)$ (except that in Case 4, we do not start any x-request, and in Case 5(b)ii, we do not fulfill any x-request or declare any witness).

We first sketch a proof that there cannot be any number y such that $\Delta(U; y)\downarrow \neq E(y)$. For suppose otherwise, and let y be least such and fix $i = \Delta(U; y)$. There must be stages $s_0 < s_1 < s$ such that at stage s_0, we set $\Delta(U; y) = i$, and at stage s_1, we set $\Delta(U; y) = 1 - i$, whereas at stage s, we again have $U_s \upharpoonright (\delta_{s_0}(y) + 1) = U_{s_0} \upharpoonright (\delta_{s_0}(y) + 1)$. Also, at or just before stage s_1, $E(y)$ must have been changed by a $\mathcal{P}^E$-strategy $\zeta \supseteq \xi^\frown\langle\Delta\rangle$ (otherwise, ξ would have been initialized since stage s_0, or $\Delta(U; y)$ would not have been redefined at stage s_1), and this $E(y)$-change must have been accompanied by an F-change at some number x_y, which forced a U-change at some number $u_y \leq \delta_{s_0}(y)$ (this being the second and thus final $U(u_y)$-change). Now, after stage s_1, there cannot have been a third $U(u_y)$-change (with a corresponding $\Phi(F \oplus E; u_y)$-change), restoring the computation $\Delta(U; y)$ to what it was at stage s_0, which is what we wanted to show. (In the later full construction for Theorem 7, we will not be able to argue this way since U can now be 3-c.e. and so $U(u_y)$ may change three times; instead we have to be ready to actively force a diagonalization for ξ.)

3.3.5. *The construction for Corollary 8.* A strategy $\xi \in T$ is *initialized* by

- making all its parameters and restraints undefined;
- making ξ's functionals totally undefined (if ξ is an $\mathcal{R}$-strategy); and
- canceling all x-requests made by or made to ξ.

The construction now proceeds in stages s. At stage 0, we initialize all strategies $\xi \in T$.

Each stage $s > 0$ consists of substages t. At a substage t, a strategy $\xi \in T$ of length t will be eligible to act. (Thus the unique strategy $\langle\rangle \in T$ of length 0 will always be eligible to act at substage 0.) Each strategy, when eligible to act, will act depending on the requirement assigned to it and as specified in the previous three subsections, and then either end the stage (if the strategy requires this or if $s = t$), or else determine the strategy to be eligible to act at substage $t + 1$. At the end of stage s, we initialize all strategies $>_L f_s$ where f_s is the strategy eligible to act at the last substage of stage s.

3.3.6. *A sketch of the verification for the construction for Corollary 8.* We now briefly sketch the verification that the above construction establishes Corollary 8. First of all, it is not hard to check that F is a d.c.e. set; that E is a c.e. set; that the true path (defined as usual as the leftmost path through T of nodes which are eligible to act infinitely often) exists; and that any strategy along the true path is initialized at most finitely often. Next, observe that any $\mathcal{P}^F$-strategy along the true path will satisfy its requirement via its eventual witness x since either there is no computation $\Pi(E; x)\downarrow = 0$, or the $\mathcal{P}^F$-strategy itself or one of the $\mathcal{R}$-strategies above it will enumerate x into F while the computation $\Pi(E; x)\downarrow = 0$ is no longer destroyed by initialization. Next, any $\mathcal{P}^E$-strategy along the true path will eventually have a permanent witness y for which there is no computation $\Sigma(y)\downarrow = 0$, or it will eventually succeed in having its x-requests to the $\mathcal{R}$-strategies above it fulfilled and so can enumerate its witness y into E. Finally, if for an $\mathcal{R}$-strategy along the true path, we have $\Phi(F \oplus E) = U$, then the strategy has infinitely many expansionary stages, and it will extend the definition of its functional Γ or Δ at infinitely many of these stages. Now either Γ is eventually not canceled, and then $\Gamma(E)$ is total and correctly computes U, or else $\Delta(U)$ is total and correctly computes E as outlined at the end of the description of the preliminary full $\mathcal{R}$-strategy in Section 3.3.4.

3.4. **The intuition for the strategies for Theorem 7, Part II.**

We now resume the discussion of the intuition for the strategies for Theorem 7.

3.4.1. *$\mathcal{P}^D$-strategies below an $\mathcal{R}$-strategy.* There is no conflict between these strategies.

3.4.2. *$\mathcal{P}$-strategies below an $\mathcal{S}$-strategy.* This situation is exactly like the situation for $\mathcal{P}^F$- and $\mathcal{P}^E$-strategies below an $\mathcal{R}$-strategy, with E, D, V, and Ψ in place of F, E, U, and Φ, respectively. (And there is never a conflict between $\mathcal{P}^F$-strategies and an $\mathcal{S}$-strategy.)

We now analyze some typical conflicts between $\mathcal{P}$-strategies below more than one $\mathcal{R}$- or $\mathcal{S}$-strategy.

3.4.3. *$\mathcal{P}^E$-strategies below an $\mathcal{S}$-strategy below the Δ-outcome of an $\mathcal{R}$-strategy.* A $\mathcal{P}^E$-strategy below the Θ-outcome of such an $\mathcal{S}$-strategy has to deal with two problems simultaneously when enumerating a number y into E: It has to ensure that $\Delta(U \oplus D)$ can compute this, and that any V-change allowed by this E-change can be computed by $\Theta(D)$ in spite of the D-restraint imposed by the $\mathcal{P}^E$-strategy. We combine the ideas from sections 3.2.3 and 3.2.4 to let this $\mathcal{R}$-strategy act as follows:

(1) Make an "x-request" to the $\mathcal{R}$-strategy.
(2) Wait for a $\mathcal{P}^F$-strategy below the Γ-outcome of the $\mathcal{R}$-strategy to find a computation $\Pi(E \oplus D; x) = 0$ for its witness x. (Do not yet allow this $\mathcal{P}^F$-strategy to enumerate x into F.)
(3) Let the $\mathcal{P}^F$-strategy enumerate x into F.
(4) Pick a witness $y = y_x$ targeted for E larger than any number mentioned so far.
(5) Let the $\mathcal{R}$-strategy check whether there is some u such that $\Gamma(E \oplus D; u)\downarrow \neq U(u)$. If not, then the $\mathcal{P}^E$-strategy's x-request has failed, so cancel y and return to Step 1.
(6) Otherwise, for the least such $u = u_x$, fix a triple $\langle x, u_x, y_x \rangle$.
(7) Wait for a computation $\Sigma(D; y) = 0$ at a stage s_y, say.
(8) Enumerate y into E, extract x from F, and protect the computation $\Sigma(D; y) = 0$ by restraining D.
(9) Check whether there is a "y-request" from a $\mathcal{P}^D$-strategy to the $\mathcal{S}$-strategy. If so, then let that $\mathcal{P}^D$-strategy pick its witness z_y now. (Otherwise, pick z_y to be any big number.)
Now the $\mathcal{S}$-strategy takes over and proceeds as follows:
(10) Wait for $\Psi(E \oplus D; v)$ to be defined again (at a stage s'_y, say) for all v for which $\Theta(D; v)$ was defined at stage s_y.
(11) If $\Psi(E \oplus D; v) = \Theta(D; v)$ for all these v, then the $\mathcal{P}^E$-strategy's diagonalization succeeded in a finitary way, and the $\mathcal{P}^E$-strategy stops.

(12) Otherwise, cancel Θ and let v_y be the least v with $\Psi(E\oplus D;v) \neq \Theta(D;v)\downarrow$. Set $\Lambda(V;z_y) = D(z_y)$ with use $\lambda(z_y) = v_y$. For all $z < z_y$ for which $\Lambda(V;z)$ is currently undefined, define $\Lambda(V;z) = D(z)$ with use 0 (since $D(z)$ will not change any more for those z). Create a *quintuple* $\langle x, u_x, y_x, v_y, z_y\rangle$.

(13) Allow the strategies below the Λ-outcome of the $\mathcal{S}$-strategy to be eligible to act once and go back to Step 1.

Note that the $\mathcal{S}$-strategy has three possible outcomes:

finite: There are only finitely many expansionary stages: Then $\Psi(E \oplus D) \neq V$.

Θ: There are infinitely many expansionary stages, but Θ is canceled only finitely often by a $\mathcal{P}^E$-strategy below the Θ-outcome of the $\mathcal{S}$-strategy, and so all these $\mathcal{P}^E$-strategies eventually wait at Step 7 forever or stop at Step 11: Then $\Theta(D)$ can compute V correctly, and all $\mathcal{P}^E$-requirements can be satisfied.

Λ: There are infinitely many expansionary stages, and Θ is canceled infinitely often: Then $\Lambda(V)$ is defined infinitely often and, by the way the uses are chosen, is a total function. Every time the parameter s'_y is set, the set $D \upharpoonright (s'_y + 1)$ can change only at a number of the form z_y from now on. For such numbers, we will ensure for all future expansionary stages s:

$$\begin{aligned} y \in E_s &\implies \Psi(E \oplus D; v_y)[s] = \Psi(E \oplus D; v_y)[s'_y] \\ &\implies \Lambda(V; z_y)[s] = D_s(z_y); \text{ and} \\ y \notin E_s &\implies \Psi(E \oplus D; v_y)[s] = \Psi(E \oplus D; v_y)[s_y] \\ &\implies \Lambda(V; z_y)[s] = D_s(z_y). \end{aligned}$$

Thus $\Lambda(V)$ correctly computes D.

A $\mathcal{P}^E$-strategy below the Λ-outcome can act as if being only below an $\mathcal{R}$-strategy since it does not have to worry about Θ-correction injuring its D-restraint.

3.4.4. *$\mathcal{P}^D$-strategies below an $\mathcal{S}$-strategy below the Δ-outcome of an $\mathcal{R}$-strategy.* A $\mathcal{P}^D$-strategy below the Θ-outcome of such an $\mathcal{S}$-strategy can act as if in isolation since it does not impose any restraint.

A $\mathcal{P}^D$-strategy below the Λ-outcome of an $\mathcal{S}$-strategy has to use a witness of the form z_y from a quintuple supplied by the $\mathcal{S}$-strategy. When the $\mathcal{P}^D$-strategy is ready to enumerate z_y (since $\Omega(z_y) = 0$), it will first make a "y-request" for some quintuple $\langle x, u_x, y_x, v_y, z_y\rangle$, then choose its own witness z_y, and then, once it has a computation $\Omega(z_y)\downarrow = 0$, simultaneously enumerate z_y into D, extract y_x from E, and re-enumerate x into F to ensure Λ- and Δ-correction as explained in sections 3.2.4 and 3.4.2.

3.4.5. *$\mathcal{P}^E$-strategies below an $\mathcal{S}$-strategy below the Δ-outcomes of an $\mathcal{R}_1$- and an $\mathcal{R}_0$-strategy.* A $\mathcal{P}^E$-strategy below the Θ-outcome of such an $\mathcal{S}$-strategy has to deal with three problems simultaneously when enumerating a numbers y into E: It has to ensure that the $\mathcal{R}_i$-strategy's $\Delta_i(U_i \oplus D)$ can compute this (for $i = 0, 1$), and that any V-change allowed by this E-change can be computed by $\Theta(D)$ in spite of the D-restraint imposed by the $\mathcal{P}^E$-strategy. The extra difficulty in this situation arises from the need not to let the two $\mathcal{R}$-strategies' numbers x_0 and x_1 (supplied by $\mathcal{P}_0^F$-strategies below the Γ-outcome of the $\mathcal{R}_0$-strategy, and $\mathcal{P}_1^F$-strategies below the Γ-outcome of the $\mathcal{R}_1$-strategy, respectively, working as described in section 3.2.5) interfere with each other and destroy the Φ-uses controlling U_i at $u_i = u_{x_i}$ (for $i = 0, 1$).

We overcome this problem as in section 3.2.5 by picking the witnesses x_i in reverse order, then enumerating the witnesses x_i into F and simultaneously picking their corresponding numbers y_i in "forward" order. (Of course, as in section 3.2.5, when we pick each y_i, we do not know yet whether the enumeration of x_i will result in a $U_i(u_i)$-change; if it does not, then we have to choose a new witness x_i (and so also a new number y_i).) In addition, once the $\mathcal{P}^E$-strategy's witness $y = y_0$ has been enumerated into E, this may allow a $V(v)$-change for some v for which $\Theta(D; v)$ cannot be corrected, so the $\mathcal{P}^E$-strategy will generate a tuple $\langle x_0, u_0, x_1, u_1, y, v, z\rangle$ for some $z > \psi(v)$, which can be used by a $\mathcal{P}^D$-strategy below the Λ-outcome of the $\mathcal{S}$-strategy.

More precisely, we proceed as follows:

(1) Wait for a $\mathcal{P}_1^F$-strategy below the Γ-outcome of the $\mathcal{R}_1$-strategy to have picked a witness x_1 targeted for E for which there is now a computation $\Pi_1(E \oplus D; x_1) = 0$ at a stage s_{x_1}, say. Do not let this $\mathcal{P}_1^F$-strategy enumerate x_1 into F yet. (We say the $\mathcal{P}_1^F$-strategy has "started the x-request" of the $\mathcal{P}^E$-strategy.)
(2) Wait for a $\mathcal{P}_0^F$-strategy below the Γ-outcome of the $\mathcal{R}_0$-strategy to have picked a witness x_0 targeted for E which is larger than any number mentioned by stage s_{x_1} and for which there is now a computation $\Pi_0(E \oplus D; x_0) = 0$ at a stage s_{x_0}, say. (We say the $\mathcal{P}_0^F$-strategy has "started the x-request" of the $\mathcal{P}^E$-strategy.)
(3) Let this $\mathcal{P}_0^F$-strategy enumerate x_0 into F and simultaneously pick a number $y = y_0$ targeted for E larger than any number mentioned so far.

(4) Check whether there is some least number $u_0 = u_{x_0}$ with $\Gamma_0(E \oplus D; u_0)\downarrow \neq \Phi_0(F \oplus E \oplus D; u_0)$. If not, then cancel x_0 and y_0 and return to the beginning of Step 2.
(5) Otherwise, we say that the above $\mathcal{P}_0^F$-strategy has "fulfilled the x-request" of the $\mathcal{P}^E$-strategy and define $\Delta_0(U_0 \oplus D; y) = 0$ with use $\delta_0(y) = u_0$.
(6) Let the above $\mathcal{P}_1^F$-strategy enumerate x_1 into F and simultaneously pick a number y_1 targeted for E larger than any number mentioned so far.
(7) Now first check whether this results in U_0 changing at the above number u_0 again. If so, then we force a permanent diagonalization for $\mathcal{R}_0$ as follows: Extract x_1 from F and enumerate y_1 into F to force a third U_0-change at u_0. (We say that the $\mathcal{R}_0$-strategy has "started diagonalization via the pair $\langle x_0, u_0\rangle$".) When (and if) this occurs, then extract x_0 from F and enumerate y_0 into F to force an (impossible) fourth U_0-change at u_0. Once we have this permanent diagonalization, we stop.
(8) Otherwise, check whether there is some least number $u_1 = u_{x_1}$ with $\Gamma_1(E \oplus D; u_1)\downarrow \neq \Phi_1(F \oplus E \oplus D; u_1)$. If not, then cancel x_1, y_1, x_0, and y_0 and return to the beginning of Step 1.
(9) Otherwise, we say that the above $\mathcal{P}_1^F$-strategy has "fulfilled the x-request" of the $\mathcal{P}^E$-strategy. For both $i \leq 1$, associate y with the triple $\langle x_i, u_i, y\rangle$, and define $\Delta_1(U_1 \oplus D; y) = 0$ with use $\delta_1(y) = u_1$.
(10) Wait for computation $\Sigma(D; y) = 0$ at a stage s_y, say.
(11) Enumerate y into E, extract x_i from F (for both $i \leq 1$), and protect the computation $\Sigma(D; y) = 0$ by restraining D.
(12) Check whether there is a "y-request" from a $\mathcal{P}^D$-strategy to the $\mathcal{S}$-strategy. If so, then let that $\mathcal{P}^D$-strategy pick a witness z larger than any number mentioned so far. (Otherwise, pick z to be any number larger than any number mentioned so far.)
(13) Wait for $\Psi(E \oplus D; v)$ to be defined again (at a stage s'_y, say) for all v for which $\Theta(D; v)$ was defined at stage s_y.
(14) If $\Psi(E \oplus D; v) = \Theta(D; v)$ for all these v, then the $\mathcal{P}^E$-strategy's diagonalization succeeded in a finitary way, and the $\mathcal{P}^E$-strategy stops.
(15) Otherwise, the $\mathcal{S}$-strategy cancels Θ, lets v be least such that $\Psi(E \oplus D; v) \neq \Theta(D; v)$, and sets $\Lambda(V; z) = D(z)$ with use $\lambda(z) = v$. (For all $z' < z$ for which $\Lambda(V; z')$ is currently undefined, the $\mathcal{S}$-strategy can define $\Lambda(V; z') = D(z')$ with use 0

(since $D(z')$ will not change any more for those z')) We also create a *tuple* $\langle x_0, u_0, x_1, u_1, y, v, z\rangle$.

(16) Allow the strategies below the Λ-outcome of the $\mathcal{S}$-strategy to be eligible to act once and go back to Step 1.

3.4.6. *$\mathcal{P}^D$-strategies below an $\mathcal{S}$-strategy below the Δ-outcomes of an $\mathcal{R}_0$- and an $\mathcal{R}_1$-strategy.* A $\mathcal{P}^D$-strategy below the Λ-outcome of an $\mathcal{S}$-strategy has to use a witness z from tuples supplied by the $\mathcal{S}$-strategy. When the $\mathcal{P}^D$-strategy is ready to enumerate z (since $\Omega(z) = 0$, it will first make a "y-request" for some tuple $\langle x_0, u_0, x_1, u_1, y, v\rangle$, letting the $\mathcal{S}$-strategy choose its witness z, and then, once it has a computation $\Omega(z)\downarrow = 0$, simultaneously enumerate z into D, extract y from E, and re-enumerate x_0 and x_1 into F to ensure Λ- as well as Δ_0- and Δ_1-correction as explained in sections 3.2.4 and 3.4.2.

3.4.7. *$\mathcal{P}^D$-strategies below the Λ-outcomes of an $\mathcal{S}_1$- and an $\mathcal{S}_0$-strategy below the Δ-outcome of an $\mathcal{R}$-strategy.* A $\mathcal{P}^D$-strategy below the Λ-outcome of such an $\mathcal{S}$-strategy has to deal with two kinds of problems simultaneously when enumerating a number z into D: It has to ensure that the $\mathcal{S}_j$-strategy's $\Lambda_j(V_j)$ can compute this (for $j = 0, 1$), and that the extraction of y_j from E for the sake of Λ_j-correction is accompanied by the re-enumeration of x_j into F to allow $\Delta(U)$ to compute this E-change. The extra difficulty in this situation arises from the need not to let the two $\mathcal{S}$-strategies' numbers y_0 and y_1 (supplied by a $\mathcal{P}^E_0$-strategy below the Θ-outcome of the $\mathcal{S}_0$-strategy, and a $\mathcal{P}^E_1$-strategy below the Θ-outcome of the $\mathcal{S}_1$-strategy, respectively, working as described in section 3.4.2) as well as the $\mathcal{R}$-strategy's two numbers x_0 and x_1 (supplied by a $\mathcal{P}^F_0$-strategy and a $\mathcal{P}^F_1$-strategy below the Γ-outcome of the $\mathcal{R}$-strategy, respectively, working as described in section 3.2.4) interfere with each other and destroy the Ψ_j-uses controlling V_j at $v_j = v_{y_j}$ (for $j = 0, 1$) or the Φ-uses controlling U at $u_j = u_{x_j}$ (for $j = 0, 1$). We overcome this problem by picking the witnesses x_i in reverse order and then enumerating them into F and E in "forward" order (while choosing the corresponding numbers y_i at the same time) so that the uses for $u_1 = u_{x_1}$ are not affected by x_0, and the uses for $v_1 = v_{y_1}$ are not affected by y_0:

(1) Let the $\mathcal{P}^D$-strategy make y-requests to the $\mathcal{S}_1$- and the $\mathcal{S}_0$-strategy.
(2) Let some $\mathcal{P}^E_1$-strategy, say, below the Θ-outcome of the $\mathcal{S}_1$-strategy make an x-request to the $\mathcal{R}$-strategy.
(3) Let some $\mathcal{P}^E_0$-strategy, say, below the Θ-outcome of the $\mathcal{S}_0$-strategy make an x-request to the $\mathcal{R}$-strategy.

(4) Wait for a $\mathcal{P}_1^F$-strategy below the Γ-outcome of the $\mathcal{R}$-strategy to have picked a witness x_1 targeted for F for which there is a computation $\Pi_1(E \oplus D; x_1) = 0$ at a stage s_{x_1}, say.
(5) Let the $\mathcal{P}_1^F$-strategy enumerate its witness x_1 into F, and pick a witness $y_1 = y_{x_1}$ targeted for E (which is larger than any number mentioned so far) for the sake of the $\mathcal{P}_1^E$-strategy.
(6) Check whether there is some least number $u_1 = u_{x_1}$ with $\Gamma(E \oplus D; u_1)\downarrow \neq \Phi(F \oplus E \oplus D; u_1)$ (at a stage s'_{x_1}, say). If not, then cancel x_1 and y_1 and return to the beginning of Step 4. Otherwise, define $\Delta(U \oplus D; y_1) = E(y_1)$ with use $\delta(y_1) = u_1$, and say that the $\mathcal{P}_1^F$-strategy has "fulfilled the x-request" of the $\mathcal{P}_1^E$-strategy.
(7) Wait for a computation $\Sigma_1(D; y_1) = 0$. (While waiting, we need to let other $\mathcal{P}^E$-strategies below the Θ-outcome of the $\mathcal{S}_1$-strategy start at Step 2, choosing their witnesses x_i and y_i, since we cannot guarantee that our $\mathcal{P}_1^E$-strategy will find its computation. Whichever of these $\mathcal{P}^E$-strategies below the Θ-outcome of the $\mathcal{S}_1$-strategy first finds such a computation here *and* a corresponding number v_1 can then stop the other such $\mathcal{P}^E$-strategies and continue in the role of *the* $\mathcal{P}_1^E$-strategy.)
(8) Wait for a $\mathcal{P}_0^F$-strategy below the Γ-outcome of the $\mathcal{R}$-strategy to have picked a witness x_0 targeted for F which is larger than any number mentioned so far and for which there is a computation $\Pi_0(E \oplus D; x_0) = 0$ at a stage s_{x_0}, say.
(9) Let the $\mathcal{P}_0^F$-strategy enumerate its witness x_0 into F, and pick a witness $y_0 = y_{x_0}$ targeted for E (which is larger than any number mentioned so far) for the sake of the $\mathcal{P}_0^E$-strategy.
(10) Check whether there is some least number $u_0 = u_{x_0}$ with $\Gamma(E \oplus D; u_0)\downarrow \neq \Phi(F \oplus E \oplus D; u_0)$ (at a stage s'_{x_0}, say). If not, then cancel x_0 and y_0 and return to the beginning of Step 8.
(11) Otherwise, define $\Delta(U \oplus D; y_0) = 0$ with use $\delta(y_0) = u_0$ and say that the $\mathcal{P}_0^F$-strategy has "fulfilled the x-request" of the $\mathcal{P}_0^E$-strategy.
(12) Wait for a computation $\Sigma_0(D; y_0) = 0$. (While waiting, we need to let other $\mathcal{P}^E$-strategies below the Θ-outcome of the $\mathcal{S}_1$-strategy start at Step 3, choosing their witnesses x_i and y_i, since we cannot guarantee that our $\mathcal{P}_1^E$-strategy will find its computation. Whichever of these $\mathcal{P}^E$-strategies below the Θ-outcome of the $\mathcal{S}_0$-strategy first finds such a computation here *and* a corresponding number v_0 can then stop the other such $\mathcal{P}^E$-strategies and continue in the role of *the* $\mathcal{P}_0^E$-strategy.)

(13) Let the $\mathcal{P}_0^E$-strategy enumerate y_0 into E and extract x_0 from F, and let the $\mathcal{P}^D$-strategy pick its witness $z = z_0$ larger than any number mentioned so far.
(14) Check whether there is some least $v_0 = v_{y_0}$ with $\Theta_0(D; v_0)\downarrow \neq \Psi_0(E \oplus D; v_0)$ (at a stage s'_{y_0}, say). If not, then cancel x_0, y_0 and z_0, and return to Step 8.
(15) Otherwise, define $\Lambda_0(V_0; z) = 0$ with use $\lambda_0(z) = v_0$ and say that the $\mathcal{P}_0^E$-strategy has "fulfilled the y-request" of the $\mathcal{P}^D$-strategy.
(16) Let the $\mathcal{P}_1^E$-strategy enumerate y_1 into E and extract x_1 from F, and pick a number z_1 larger than any number mentioned so far.
(17) Now, first check whether this results in U changing at the above number u_0 again (i.e., for the third time). If so, then we force a permanent diagonalization for $\mathcal{R}$ as follows: Extract y_1 from E and re-enumerate x_1 into F (to force an (impossible) fourth U-change at u_0) and stop.
(18) Otherwise, check whether this results in V_0 changing at the above number v_0 again. If so, then we force a permanent diagonalization for $\mathcal{S}_0$ as follows: Extract y_1 from E and re-enumerate x_1 into F to force a third V-change at v_0. When (and if) this occurs, then extract y_0 from E, re-enumerate x_0 into F, and enumerate z_0 into D to force an (impossible) fourth V-change at v_0. Once we have this permanent diagonalization, we stop.
(19) Otherwise, check whether there is some least $v_1 = v_{y_1}$ with $\Theta_1(D; v_1)\downarrow \neq \Psi_1(E \oplus D; v_1)$ (at a stage s'_{y_1}, say). If not, then cancel x_1, y_1, z_1, x_0, y_0 and z, and return to Step 4.
(20) Otherwise, set $\Lambda_1(V_1; z) = 0$ with use $\lambda_1(z) = v_1$ and say that the $\mathcal{P}_1^E$-strategy has "fulfilled the y-request" of the $\mathcal{P}^D$-strategy.
(21) Create a *tuple*

$$\langle x_0, u_0, x_1, u_1, y_0, v_0, y_1, v_1, z\rangle.$$

(22) Wait for a computation $\Omega(z) = 0$.
(23) Enumerate z into D, extract y_0 and y_1 from E, re-enumerate x_0 and x_1 into F, and stop.

3.4.8. *$\mathcal{P}^D$-strategies below two $\mathcal{S}$- and four $\mathcal{R}$-strategies.* In our final intuitive example, we will combine the techniques from sections 3.4.5 and 3.4.7 and consider a $\mathcal{P}^D$-strategy below the Λ-outcome of an $\mathcal{S}_1$-strategy which is located below, in increasing order of priority, the Δ-outcomes of an $\mathcal{R}_3$- and an $\mathcal{R}_2$-strategy, the Λ-outcome of an $\mathcal{S}_0$-strategy, and the Δ-outcomes of an $\mathcal{R}_1$- and an $\mathcal{R}_0$-strategy.

A $\mathcal{P}^D$-strategy below the Λ-outcome of such an $\mathcal{S}$-strategy again has to deal with two kinds of problems simultaneously when enumerating a number z into D: It has to ensure that the $\mathcal{S}_j$-strategy's $\Lambda_j(V_j)$ can compute this (for $j = 0, 1$), and that the extraction of y and y' from E due to Λ_j-correction is accompanied by re-enumeration into F to allow $\Delta_i(U_i)$ to compute this E-change. The extra difficulty in this situation arises from the need not to let the two $\mathcal{S}$-strategies' numbers y and y' (supplied by $\mathcal{P}_0^E$-strategies below the Θ-outcome of the $\mathcal{S}_0$-strategy, and $\mathcal{P}_1^E$-strategies below the Θ-outcome of the $\mathcal{S}_1$-strategy, respectively, working as described in section 3.4.2) as well as the $\mathcal{R}_i$-strategy's numbers (supplied by a $\mathcal{P}^F$-strategy below the Γ-outcome of the $\mathcal{R}_i$-strategy, for $i = 0, 1, 2, 3$, working as described in section 3.2.4) interfere with each other and destroy the E-uses controlling V_j at v_j (for $j = 0, 1$) or the F-uses controlling U_i at u_i (for $i = 0, 1, 2, 3$), nor the E-use controlling V_0 at v_0' or the F-uses controlling U_l at u_l' (for $l = 0, 1$). We overcome this problem by picking the witnesses in reverse order and then enumerating them into F and E in “forward” order; we also pick two numbers each, u_i and u_i', targeted for F, to generate two numbers each in U_i (for $i = 0, 1$) for y and y', respectively:

(1) Let the $\mathcal{P}^D$-strategy make y-requests to the $\mathcal{S}_1$- and the $\mathcal{S}_0$-strategy.
(2) Let some $\mathcal{P}_1^E$-strategy, say, below the Θ-outcome of the $\mathcal{S}_1$-strategy make an x-request to each $\mathcal{R}_i$-strategy (for $i = 3$, 2, 1, and 0).
(3) For $i = 3, 2, 1, 0$ (in decreasing order):
    (a) Wait for a $\mathcal{P}_i^F$-strategy below the Γ-outcome of the $\mathcal{R}_i$-strategy to have picked a witness x_i targeted for F which is larger than any number mentioned so far and for which there is a computation $\Pi_i(E \oplus D; x_i) = 0$ at a stage s_{x_i}, say. Do not let this $\mathcal{P}_i^F$-strategy enumerate x_i into F yet.
(4) For $i = 0, 1, 2, 3$ (in increasing order):
    (a) Let this $\mathcal{P}_i^F$-strategy enumerate x_i into F, and choose a corresponding number y_i which is larger than any number mentioned so far.
    (b) For $k = 0, \dots, i-1$ (in increasing order), check whether the enumeration of x_i into F has allowed another U_k-change at the number u_k found before. If so, then we force a permanent diagonalization for $\mathcal{R}_k$ as follows: Extract x_i from F and enumerate y_i into E, to force a third U_k-change at u_k. When (and if) this occurs, then extract x_l from F and enumerate y_l into E (for all $l \in [k, i)$), to force an

(impossible) fourth U_k-change at u_k. Once we have this permanent diagonalization, we stop.

(c) Otherwise, check whether there is some least $u_i = u_{x_i}$ with $\Gamma_i(E \oplus D; u_i)\downarrow \neq \Phi_i(F \oplus E \oplus D; u_i)$ (at a stage s'_{x_i}, say). If not, then cancel x_k and y_k (for $k \leq i$) and return to the beginning of Step 3 with this i.

(5) Set $y = y_0$ and wait for a computation $\Sigma_1(D; y) = 0$. (While waiting, we need to let other $\mathcal{P}^E$-strategies below the Θ-outcome of the $\mathcal{S}_1$-strategy start at Step 2, choosing their numbers x_i, y_i, x'_k and y'_k, since we cannot guarantee that our $\mathcal{P}^E_1$-strategy will find its computation. Whichever of these $\mathcal{P}^E$-strategies below the Θ-outcome of the $\mathcal{S}_1$-strategy first finds such a computation here *and* a corresponding number v_1 can then stop the other such $\mathcal{P}^E$-strategies and continue in the role of *the* $\mathcal{P}^E_1$-strategy.)

(6) Let some $\mathcal{P}^E_0$-strategy, say, below the Θ-outcome of the $\mathcal{S}_0$-strategy make an x-request to the $\mathcal{R}_i$-strategy (for $i = 1, 0$).

(7) For $l = 1, 0$ (in decreasing order):

(a) Wait for a $\mathcal{P}'^F_l$-strategy below the Γ-outcome of the $\mathcal{R}_l$-strategy to have picked a witness x'_l targeted for F larger than any number mentioned so far and for which there is a computation $\Pi'_l(E \oplus D; x'_l) = 0$ at a stage $s_{x'_l}$, say. Do not let this $\mathcal{P}'^F_l$-strategy enumerate x'_l into F yet.

(8) For $l = 0, 1$ (in increasing order):

(a) Let this $\mathcal{P}'^F_l$-strategy enumerate x'_l into F, and choose a corresponding witness y'_l (larger than any number mentioned so far) for the $\mathcal{P}^E_1$-strategy.

(b) If $l = 1$ then check whether the enumeration of x'_1 into F has allowed another U_0-change at the number u'_0 found before. If so, then we force a permanent diagonalization for $\mathcal{R}_0$ as follows: Extract x'_1 from F, and enumerate y'_1 into E, to force a third U_0-change at u'_0. When (and if) this occurs, then extract x'_0 from F, and enumerate y'_0 into E, to force an (impossible) fourth U_0-change at u'_0. Once we have this permanent diagonalization, we stop.

(c) Otherwise, check whether there is some least $u'_l = u'_{x'_l}$ with $\Gamma_l(E \oplus D; u'_l)\downarrow \neq \Phi_l(F \oplus E \oplus D; u'_l)$ (at a stage $s'_{x'_l}$, say). If not, then cancel x'_m and y'_m (for $m \leq l$) and return to the beginning of Step 7 with this l.

(9) Set $y' = y'_0$ and wait for a computation $\Sigma_0(D; y') = 0$. (While waiting, we need to let other $\mathcal{P}^E$-strategies below the Θ-outcome of the $\mathcal{S}_0$-strategy start at Step 6, choosing their numbers x_i, y_i,

x'_k and y'_k, since we cannot guarantee that our $\mathcal{P}_0^E$-strategy will find its computation. Whichever of these $\mathcal{P}^E$-strategies below the Θ-outcome of the $\mathcal{S}_0$-strategy first finds such a computation here *and* a corresponding number v_0 can then stop the other such $\mathcal{P}^E$-strategies and continue in the role of *the* $\mathcal{P}_0^E$-strategy.)

(10) Enumerate y' into E, extract x'_0 and x'_1 from F, and choose a witness z' for the $\mathcal{P}^D$-strategy larger than any number mentioned so far.

(11) Check whether there is some least $v_0 = v_{y'}$ with $\Theta_0(D; v_0)\downarrow \neq \Psi_0(E \oplus D; v_0)$ (at a stage $s'_{y'}$, say). If not, then cancel x'_1, x'_0, y'_0, y'_1 and z' and return to Step 7.

(12) Otherwise, define define $\Lambda_0(V_0; z') = 0$ with use $\lambda_0(z') = v_0$ and say that the $\mathcal{P}_0^E$-strategy has "fulfilled the y-request" of the $\mathcal{P}^D$-strategy.

(13) Set $y = y_0$ and wait for a computation $\Sigma_1(D; y) = 0$.

(14) Enumerate y into E, extract x_i from F (for all $i \leq 3$), and choose a corresponding number z larger than any number mentioned so far.

(15) Now, for $k = 0, 1$ (in increasing order):
   (a) Check whether this results in U_k changing at the above number u'_k again. If so, then we force a permanent diagonalization for $\mathcal{R}_k$ as follows: Extract y from E, re-enumerate x_i (for all $i \leq 3$) into F, and enumerate z, to force an (impossible) fourth U_k-change at u'_k and stop.

(16) Otherwise, check whether this results in V_0 changing at the above number v_0 again. If so, then we force a permanent diagonalization for $\mathcal{S}_0$ as follows: Extract y from E, re-enumerate x_i (for all $i \leq 3$) into F, and enumerate z to force a third V_0-change at v_0. When (and if) this occurs, then extract y' from E, re-enumerate x'_0 and x'_1 into F, and enumerate z' to force an (impossible) fourth V_0-change at v_0. Once we have this permanent diagonalization, we stop.

(17) Otherwise, check whether there is some least $v_1 = v_y$ with $\Theta_1(D; v_1)\downarrow \neq \Psi_1(E \oplus D; v_1)$ (at a stage s'_y, say). If not, then cancel all x_i, y_i, x'_k, and y'_k, and return to Step 3.

(18) Otherwise, set $\Lambda_1(V_1; z') = 0$ with use $\lambda_1(z') = v_0$ and say that the $\mathcal{P}_1^E$-strategy has "fulfilled the y-request" of the $\mathcal{P}^D$-strategy.

(19) Create a *tuple*

$$\langle x_0, u_0, x_1, u_1, x_2, u_2, x_3, u_3, x'_0, u'_0, x'_1, u'_1, y', v_0, y, v_1, z' \rangle.$$

(20) Wait for a computation $\Omega(z') = 0$.

(21) Enumerate z' into D, extract y and y' from E, re-enumerate x_i and x'_l into F (for all $i \leq 3$ and $l \leq 1$), and stop.

3.4.9. *$\mathcal{P}^E$-strategy between a $\mathcal{P}^D$- and an $\mathcal{S}$-strategy.* The situation in the interaction between an $\mathcal{S}$-strategy and a $\mathcal{P}^E$-strategy below its Θ-outcome in the case that some $\mathcal{P}^D$-strategy below the Θ-outcome but of lower priority than the $\mathcal{P}^E$-strategy has already enumerated a number is analogous to the situation in section 3.2.8.

We are now ready to describe the construction formally.

3.5. **The tree of strategies for Theorem 7.** We fix an arbitrary computable priority ordering of all $\mathcal{R}$-, $\mathcal{S}$-, and $\mathcal{P}$-requirements of order type ω. Let

$$O = \{\Lambda <_O \Theta <_O \Delta <_O \Gamma <_O \mathit{fin}\}$$

be the *set of outcomes* of our strategies, with the priority ordering $<_O$ as stated.

For the tree of strategies $T \subseteq O^{<\omega}$ to be defined, assign the nth requirement from our above listing of all requirements to all nodes $\xi \in T$ of length n.

The *tree of strategies* is now defined by recursion as follows: Given a node (or "strategy") ξ already on T, we let the set of immediate successors of ξ be

$$\begin{array}{ll} \{\xi^\frown\langle\Delta\rangle, \xi^\frown\langle\Gamma\rangle, \xi^\frown\langle \mathit{fin}\rangle\}, & \text{if } \xi \text{ is an } \mathcal{R}\text{-strategy;} \\ \{\xi^\frown\langle\Lambda\rangle, \xi^\frown\langle\Theta\rangle, \xi^\frown\langle \mathit{fin}\rangle\}, & \text{if } \xi \text{ is an } \mathcal{S}\text{-strategy;} \\ \{\xi^\frown\langle \mathit{fin}\rangle\}, & \text{if } \xi \text{ is a } \mathcal{P}\text{-strategy.} \end{array}$$

We now fix some notation for our tree: Given any strategy $\eta \in T$, we let

$$X(\eta) = \{\xi \mid \xi \text{ is an } \mathcal{R}\text{-strategy and } \xi^\frown\langle\Delta\rangle \subseteq \eta\}, \text{ and}$$
$$Y(\eta) = \{\xi \mid \xi \text{ is an } \mathcal{S}\text{-strategy and } \xi^\frown\langle\Lambda\rangle \subseteq \eta\}$$

be the sets of $\mathcal{R}$- and $\mathcal{S}$-strategies with Δ- and Θ-outcome above η, respectively. (These will be the strategies to which η may have to make *x-requests* or *y-requests*, respectively, as described in the next sections.)

3.6. **The full strategies for Theorem 7.** In this section, we describe in full detail the action of the individual strategies, while in the next section, we will give the full construction in terms of the description of the strategies. As in section 3.3, we start with the description of the (much easier) $\mathcal{P}$-strategies before presenting the $\mathcal{R}$- and $\mathcal{S}$-strategy.

3.6.1. *The general $\mathcal{P}^F$-strategy.* A $\mathcal{P}^F_\Pi$-strategy $\xi \in T$ acts as follows at substage t of stage s at which it is eligible to act:

(1) If this is the first stage at which ξ is eligible to act since ξ was last initialized (and so ξ does not have a witness), then ξ chooses a witness x larger than any number mentioned so far and *ends the stage*.
(2) Otherwise, let x be ξ's current witness. Next check whether there is a computation $\Pi(E \oplus D; x)\downarrow = 0$. If not, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next*.
(3) Otherwise, check whether ξ has already enumerated x into F. If so, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next*.
(4) Otherwise, check whether there is an x-request from a $\mathcal{P}^E$- or $\mathcal{S}$-strategy η to an $\mathcal{R}$-strategy $\zeta \in X(\eta)$ with $\zeta^\frown\langle\Gamma\rangle \subseteq \xi$ which has not yet been started (as defined below). If not, then
   - enumerate x into F;
   - initialize all strategies $> \xi$; and
   - *end the stage*.
(5) Otherwise, for each such $\mathcal{R}$-strategy ζ, fix the highest-priority such $\mathcal{P}^E$-strategy $\eta_\zeta \supseteq \zeta^\frown\langle\Delta\rangle$. We say ξ *is delayed in the enumeration of* x, and we *end the stage*. [In this case, each such ζ wants to reserve ξ's witness x for forming a "pair $\langle x, u\rangle$" for the sake of the $\mathcal{P}^E$-strategy η_ζ. So the highest-priority such ζ (call it ζ_0) takes over for ξ, and at a future ζ_0-expansionary stage, ζ_0 will enumerate x into F unless ζ_0, and thus also ξ, has been initialized before then. From then on, each such ζ will check, in decreasing order of priority, whether a "pair $\langle x, u\rangle$" can be formed for it.]

3.6.2. *The general $\mathcal{P}^E$-strategy.* A $\mathcal{P}^E_\Sigma$-strategy $\xi \in T$ acts as follows at substage t of stage s at which it is eligible to act:

(1) If this is the first stage at which ξ is eligible to act since ξ was last initialized, then check whether $X(\xi) \neq \emptyset$. If so, then *make x-requests* to all $\mathcal{R}$-strategies $\zeta \in X(\xi)$ and *end the stage*. [In this case, each $\mathcal{R}$-strategy $\zeta \in X(\xi)$ first needs to "start" an x-request before ξ can choose its witness y. The highest-priority such ζ will determine the stage at which ξ chooses its witness y, which will be a stage at which ξ itself is not eligible to act. Each such ζ will enumerate its delayed witness x_ζ into F, choose a corresponding number y_ζ, and check whether it can form a "pair $\langle x_\zeta, u_\zeta\rangle$", thus "fulfilling" the x-request.]

(2) Otherwise, check whether ξ has a witness y. If not, then ξ chooses a witness y larger than any number mentioned so far and *ends the stage*.
(3) Otherwise, let y be ξ's current witness. Next check whether there is a computation $\Sigma(D; y)\downarrow = 0$. If not, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next*.
(4) Otherwise, check whether ξ has already enumerated y into E. If so, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next*.
(5) Otherwise, check whether there is a y-request from a $\mathcal{P}^D$-strategy η to an $\mathcal{S}$-strategy $\zeta \in Y(\eta)$ with $\zeta^\frown\langle\Theta\rangle \subseteq \xi$ which has not yet been started (as defined below). If not, then
    - enumerate y into E;
    - extract x from F for all $\mathcal{R}$-strategies $\eta \in X(\xi)$ such that η fulfilled ξ's x-request via a pair $\langle x, u\rangle$;
    - initialize all strategies $> \xi$; and
    - *end the stage*.
(6) Otherwise, for each such $\mathcal{S}$-strategy ζ, fix the highest-priority such $\mathcal{P}^D$-strategy $\eta_\zeta \supseteq \zeta^\frown\langle\Lambda\rangle$. We say ξ *is delayed in the enumeration of* y, and we *end the stage*. [In this case, each such ζ wants to reserve ξ's witness y for forming a "pair $\langle y, v\rangle$" for the sake of the $\mathcal{P}^D$-strategy η_ζ. So the highest-priority such ζ (call it ζ_0) takes over for ξ, and at a future ζ_0-expansionary stage, ζ_0 will enumerate y into E unless ζ_0, and thus also ξ, has been initialized before then. From then on, each such ζ will check, in decreasing order of priority, whether a "pair $\langle y, v\rangle$" can be formed for it.]

3.6.3. *The general $\mathcal{P}^D$-strategy.* A $\mathcal{P}^D_\Omega$-strategy $\xi \in T$ acts as follows at substage t of stage s at which it is eligible to act:

(1) If this is the first stage at which ξ is eligible to act since ξ was last initialized then check whether $Y(\xi) \neq \emptyset$. If so, then *make y-requests* to all $\mathcal{S}$-strategies $\zeta \in Y(\xi)$ and *end the stage*. [In this case, each $\mathcal{S}$-strategy $\zeta \in X(\xi)$ first needs to "start" a y-request before ξ can choose its witness z. Each such ζ now tries to find a $\mathcal{P}^E$-strategy $\nu = \nu_\zeta$, which, in turn, will "start" x-requests to all $\mathcal{P}^F$-strategies $\eta \in X(\nu_\zeta)$. Once each such $\eta \in X(\nu_\zeta)$ has found a $\mathcal{P}^F$-strategy $\mu = \mu_\eta$ below its Γ-outcome with a computation $\Pi_\mu(E \oplus D; x_\mu) = 0$, ν_ζ will pick its own witness y_ζ, and then all x_μ are enumerated in order, each time checking whether a corresponding number u_μ is found and ν's y-request can be "fulfilled". (If no u_μ is found, the process starts

over.) Once all of ν's y-requests have been "fulfilled", ξ picks its own witness z. Each ν waits for a computation $\Sigma_\nu(D; y_\nu) = 0$ and then enumerates y_ν into E (and extracts corresponding numbers x_μ out of F), checking each time whether there is a corresponding v_ν and ξ's y-request can be fulfilled. There are now two potential problems: First of all, ν may never find a computation $\Sigma_\nu(D; y_\nu) = 0$ (but ξ cannot afford to wait for such a computation); thus ξ must start a new y-request to ζ with a new ν_ζ while the current ν_ζ waits for its computation. But even when such a computation is found, there may be no corresponding v_ν, in which case the process has to start over.]

(2) Otherwise, check whether ξ has a witness z. If not, then ξ chooses a witness z larger than any number mentioned so far and *ends the stage*.

(3) Otherwise, let z be ξ's current witness. Next check whether there is a computation $\Omega(z)\downarrow = 0$. If not, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next.*

(4) Otherwise, check whether ξ has already enumerated z into D. If so, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next.*

(5) Otherwise,
- enumerate z into D;
- extract y from E for all $\mathcal{S}$-strategies $\eta \in Y(\xi)$ such that η fulfilled ξ's y-request via a pair $\langle y, v\rangle$;
- re-enumerate x into F for all $\mathcal{R}$-strategies $\zeta \in X(\eta)$ such that $\eta \in Y(\xi)$, and ζ fulfilled η's x-request via a pair $\langle x, u\rangle$; and
- *end the stage.*

3.6.4. *The general $\mathcal{R}$-strategy.* An $\mathcal{R}_{\Phi,U}$-strategy $\xi \in T$ acts as follows at substage t of stage s at which it is eligible to act:

(1) Check whether the *length of agreement* $\ell(\Phi(F \oplus E \oplus D), U)$ between $\Phi(F \oplus E \oplus D)$ and U now exceeds any number mentioned by the end of the previous ξ-expansionary stage, which we will call s'. (Here, 0 is always a *ξ-expansionary stage.*) If not, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next.*

(2) Otherwise, call s a *ξ-expansionary stage* and check whether there is some number y such that $\Delta(U \oplus D; y)\downarrow \neq E(y)$. If so, then ξ can permanently diagonalize $\Phi(F \oplus E \oplus D)$ against U as follows: Let y be least such and fix $i = \Delta(U \oplus D; y)$. There must be stages $s_0 < s_1 < s$ such that at stage s_0, we set $\Delta(U \oplus D; y) = i$, and at stage s_1, we set $\Delta(U \oplus D; y) = 1-i$, but

now $(U \oplus D) \restriction (\delta_{s_0}(y)+1) = (U \oplus D)_{s_0} \restriction (\delta_{s_0}(y)+1)$. Also, at or just before stage s_1, $E(y)$ must have been changed from value i to value $1-i$ by a $\mathcal{P}^E$- or $\mathcal{S}$-strategy $\zeta \supseteq \xi^\frown\langle\Delta\rangle$ (otherwise, ξ would have been initialized since stage s_0, or $\Delta(U \oplus D; y)$ would not have been redefined at stage s_1), and this $E(y)$-change must have been accompanied by an F-change at some number x_y, which forced a U-change at some number u_y (this being at least the second $U(u_y)$-change). Now, since stage s_1, there must have been a third and thus final $U(u_y)$-change (with a corresponding $\Phi(F \oplus E \oplus D; u_y)$-change). This $\Phi(F \oplus E \oplus D; u_y)$-change must be due to the action of some strategy $\zeta' \supseteq \xi^\frown\langle\Delta\rangle$ of higher priority than ζ. Such ζ' must be a $\mathcal{P}^F$-strategy which newly enumerates a number x, say, into F. Now permanently diagonalize for ξ by extracting x from F and *end the stage*. (Unless ξ is initialized later, this will be the last ξ-expansionary stage.)

(3) Otherwise, check whether ξ has previously, at a stage s'', say, fulfilled an x-request of some $\mathcal{P}^E$- or $\mathcal{S}$-strategy η (where $\xi \in X(\eta)$) with a pair $\langle x, u\rangle$ such that
  (a) η has not been initialized since stage s'';
  (b) η had also made an x-request to some $\mathcal{R}$-strategy $\hat{\xi} \in X(\eta)$ with $\hat{\xi} \supset \xi$;
  (c) $U(u)$ has changed since the last ξ-expansionary stage s'; and either
  (d) $\hat{\xi}$ enumerated a number $\hat{x}$ into F at stage s'; or
  (e) ξ started diagonalization via the pair $\langle x, u\rangle$ at stage s' (as defined below), and $\hat{\xi}$ enumerated a number $\hat{x}$ into F at the last ξ-expansionary stage before stage s'.

In either case, ξ can permanently diagonalize $\Phi(F \oplus E \oplus D)$ against U:

If Case 3d holds, then
- extract $\hat{x}$ from F;
- enumerate into E the number $\hat{y}$ which was chosen when $\hat{x}$ was first enumerated into F;
- if η is a $\mathcal{P}^E$-strategy then cancel η's x-requests to all $\tilde{\xi} \in X(\eta)$;
- if η is an $\mathcal{S}$-strategy then
  - let ζ be the $\mathcal{P}^D$-strategy such that $\eta \in Y(\zeta)$ and ζ made the y-request to η, causing η to make the x-request to ξ;
  - cancel ζ's y-requests to all $\mathcal{S}$-strategies $\hat{\eta} \in Y(\zeta)$; and

  – for all $\mathcal{S}$-strategies $\hat{\eta} \in Y(\zeta)$, cancel $\hat{\eta}$'s x-requests to all $\mathcal{R}$-strategies $\hat{\xi} \in X(\hat{\eta})$;
- initialize all strategies $\geq \xi^\frown\langle fin\rangle$; and
- *end the stage.*

(In this case, we say ξ has *started diagonalization via the pair* $\langle x, u\rangle$. There will be at most one more ξ-expansionary stage, corresponding to a third (and thus final) $U(u)$-change, at which time Case 3e will hold and we will force a fourth $\Phi(F \oplus E \oplus D; u)$-change.)

If Case 3e holds, then
- extract from F all numbers $\tilde{x}$ for all $\mathcal{R}$-strategies $\tilde{\xi} \in X(\eta)$ with $\xi \subseteq \tilde{\xi} \subset \hat{\xi}$ (where $\tilde{x}$ is the number last enumerated by $\tilde{\xi}$ into F);
- for each such $\tilde{x}$, enumerate the number $\tilde{y}$ into E which was chosen when $\tilde{x}$ was first enumerated into F;
- initialize all strategies $\geq \xi^\frown\langle fin\rangle$; and
- *end the stage.*

(Unless ξ is initialized later, this will be the last ξ-expansionary stage.)

(4) Otherwise, check whether there is some (highest-priority) $\mathcal{P}^E$- or $\mathcal{S}$-strategy η such that
  (a) $\xi \in X(\eta)$;
  (b) η's x-requests to all $\mathcal{R}$-strategies $\hat{\xi} \in X(\eta)$ with $\xi \subset \hat{\xi} \subset \eta$ have been started (as defined below);
  (c) if η is an $\mathcal{S}$-strategy and so some $\mathcal{P}^D$-strategy $\zeta \supseteq \eta^\frown\langle\Lambda\rangle$ has made a y-request to η, then for all $\mathcal{S}$-strategies $\hat{\eta} \in Y(\zeta)$ with $\hat{\eta} \supset \eta$, all of $\hat{\eta}$'s x-requests have already been fulfilled (as defined below); and either
  (d) η has made an x-request to ξ, and that request has not yet been started by ξ (as defined below); or
  (e) η has made an x-request to ξ, and that request has been started by ξ but not yet fulfilled by ξ (as defined below).

(5) If there is (a highest-priority) such η satisfying Case 4d but none satisfying Case 4e, then
- define $\Gamma(E \oplus D; u') = U(u')$ for all $u' \leq \ell(\Phi(F \oplus E \oplus D), U)$ for which $\Gamma(E \oplus D; u')$ is currently undefined, with old use $\gamma(u')$ (if defined previously), or new use $\gamma(u')$ larger than any number mentioned so far (otherwise);
- end the substage; and
- let $\xi^\frown\langle\Gamma\rangle$ be *eligible to act next.*

If later during this stage, some $\mathcal{P}^F$-strategy $\zeta \supseteq \xi^\frown\langle\Gamma\rangle$ wants to enumerate a number x into F but is delayed, then we say that ξ has *started η's x-request.*

(6) If there is (a highest-priority) such η satisfying Case 4e, then check whether ξ has enumerated a number x into F for this x-request at the previous ξ-expansionary stage s', or whether some $\mathcal{R}$-strategy $\xi' \subset \xi$ has enumerated a number x into F for the sake of an x-request by η to both ξ and ξ' at a previous ξ'-expansionary stage s'.

  (a) If neither ξ nor an $\mathcal{R}$-strategy $\xi' \subset \xi$ has enumerated such x into F, then there is a $\mathcal{P}^F$-strategy $\zeta \supseteq \xi^\frown\langle\Gamma\rangle$ which found a computation $\Pi(E \oplus D; x) = 0$ at the last substage of stage s' (but was delayed in the enumeration of x into F). Now ξ
    - enumerates this x into F;
    - chooses a corresponding number y;
    - initializes all strategies $> \zeta$; and
    - *ends the stage.*

  (b) Otherwise, ξ has enumerated a number x into F at the last substage of the previous ξ-expansionary stage s', or ξ' has enumerated a number x into F at the last substage of a previous ξ'-expansionary stage s'. Check whether there is some u such that $\Gamma(E \oplus D; u){\downarrow} \neq U(u)$.
    (i) If there is no such u, then ξ's attempt at fulfilling η's x-request has failed. So
      - let η again *make x-requests* (which have not been started) to all $\mathcal{R}$-strategies $\hat{\xi} \in X(\eta)$ with $\hat{\xi} \subseteq \xi$; and
      - *end the stage.*
    (ii) If there is (a least) such u, then
      - we say that ξ has *fulfilled η's x-request* with the pair $\langle x, u\rangle$;
      - cancel Γ;
      - define $\Delta(U \oplus D; y) = E(y)$ for all $y \leq s$ for which $\Delta(U \oplus D; y)$ is currently undefined, with use $\delta(y)$ (if defined previously), or new use $\delta(y)$ larger than any number mentioned so far as well as any witness chosen at a later substage of the current stage (otherwise);
      - end the substage; and
      - let $\xi^\frown\langle\Delta\rangle$ be *eligible to act next.*

(7) Otherwise, i.e., if there is no η as in Case 4d or Case 4e, then proceed as in Case 5 or Case 6(b)ii, depending on whether there is some u such that $\Gamma(E \oplus D; u)\downarrow \neq U(u)$ (except that unlike in Case 5, we do not start any x-request, and that unlike in Case 6(b)ii, we do not fulfill any x-request).

3.6.5. *The general $\mathcal{S}$-strategy.* An $\mathcal{S}_{\Psi,V}$-strategy $\xi \in T$ acts as follows at substage t of stage s at which it is eligible to act:

(1) Check whether the *length of agreement* $\ell(\Psi(E \oplus D), V)$ between $\Psi(E \oplus D)$ and V now exceeds any number mentioned by the end of the previous ξ-expansionary stage, which we will call s'. (Here, 0 is always a *ξ-expansionary stage.*) If not, then end the substage and let $\xi^\frown\langle fin\rangle$ be *eligible to act next*.
(2) Otherwise, call s a *ξ-expansionary stage* and check whether there is some number z such that $\Lambda(V; z)\downarrow \neq D(z)$. If so, then ξ can permanently diagonalize $\Psi(E \oplus D)$ against V as follows: Let z be least such. Since D is c.e., there must be stages $s_0 < s_1 < s$ such that at stage s_0, we set $\Lambda(V; z) = 0$, and at stage s_1, we set $\Lambda(V; z) = 1$, but now $V \restriction (\lambda_{s_0}(z) + 1) = V_{s_0} \restriction (\lambda_{s_0}(z) + 1)$. Also, at or just before stage s_1, z must have been enumerated into D by a $\mathcal{P}^D$-strategy $\zeta \supseteq \xi^\frown\langle\Lambda\rangle$ (otherwise, ξ would have been initialized since stage s_0, or $\Lambda(V; z)$ would not have been redefined at stage s_1), and this $D(z)$-change must have been accompanied by an E-change at some number y_z, which forced a V-change at some number v_z (this being at least the second $V(v_z)$-change). Now, since stage s_1, there must have been a third and thus final $V(v_z)$-change (with a corresponding $\Psi(E \oplus D; v_z)$-change). This $\Psi(E \oplus D; v_z)$-change must be due to the action of some strategy $\zeta' \supseteq \xi^\frown\langle\Lambda\rangle$ of higher priority than ζ. We will prove in Lemma 17 that ζ' must be a $\mathcal{P}^E$-strategy which enumerates a number y, say, into E. Now permanently diagonalize for ξ by extracting y from E and re-enumerate into F the x which was first enumerated into F when y was chosen, and *end the stage*. (Unless ξ is initialized later, this will be the last ξ-expansionary stage.)
(3) Otherwise, check whether ξ has previously, at a stage s'', say, fulfilled a y-request of some $\mathcal{P}^D$-strategy η (such that $\xi \in Y(\eta)$) with a pair $\langle y, v\rangle$ such that
   (a) η has not been initialized since stage s'';
   (b) η had also made a y-request to some $\mathcal{S}$-strategy $\hat{\xi} \in Y(\eta)$ with $\hat{\xi} \supset \xi$;

(c) $V(v)$ has changed since the last ξ-expansionary stage s'; and either

(d) $\hat{\xi}$ enumerated a number $\hat{y}$ into E at stage s'; or

(e) ξ started diagonalization via the pair $\langle y, v\rangle$ at stage s' (as defined below), and $\hat{\xi}$ enumerated a number $\hat{y}$ into E at the last ξ-expansionary stage before stage s'.

In either case, ξ can permanently diagonalize $\Psi(E \oplus D)$ against V:

If Case 3d holds, then

- extract $\hat{y}$ from E;
- re-enumerate into F the number $\hat{x}$ which was enumerated into F at the time $\hat{y}$ was chosen;
- enumerate into D the number $\hat{z}$ which was chosen when y was first enumerated into E;
- cancel η's y-requests to all $\mathcal{S}$-strategies $\hat{\xi} \in Y(\eta)$;
- for all $\mathcal{S}$-strategies $\hat{\xi} \in Y(\eta)$, cancel $\hat{\xi}$'s x-requests to all $\mathcal{R}$-strategies $\zeta \in X(\hat{\xi})$;
- initialize all strategies $\geq \xi^\frown\langle fin\rangle$; and
- *end the stage.*

(In this case, we say ξ has *started diagonalization via the pair* $\langle y, v\rangle$. There will be at most one more ξ-expansionary stage, corresponding to a third (and thus final) V-change at a number, at which time Case 3e will hold and we will force a fourth $\Psi(E \oplus D)$-change at that number.)

If Case 3e holds, then

- extract from E all numbers $\tilde{y}$ for all $\mathcal{S}$-strategies $\tilde{\xi} \in Y(\eta)$ with $\xi \subseteq \tilde{\xi} \subset \hat{\xi}$ (where $\tilde{y}$ is the number last enumerated into E by $\tilde{\xi}$);
- for each such $\tilde{y}$, re-enumerate into F the number $\tilde{x}$ which was first enumerated into F when $\tilde{y}$ was chosen;
- for each such $\tilde{y}$, also enumerate into D the number $\tilde{z}$ which was chosen when $\tilde{y}$ was enumerated into E;
- initialize all strategies $\geq \xi^\frown\langle fin\rangle$; and
- *end the stage.*

(Unless ξ is initialized later, this will be the last ξ-expansionary stage.)

(4) Otherwise, check whether there is some (highest-priority) $\mathcal{P}^D$-strategy $\eta \supseteq \xi^\frown\langle\Delta\rangle$ such that

(a) $\xi \in Y(\eta)$;

(b) η's y-requests to all $\mathcal{S}$-strategies $\hat{\xi} \in Y(\eta)$ with $\xi \subset \hat{\xi} \subset \eta$ have been started (as defined below);

(c) η has made a y-request to ξ, and that request has not yet been started by ξ (as defined below); or
(d) η has made a y-request to ξ, and that request has been started by ξ but not yet fulfilled by ξ (as defined below).

(5) If there is (a highest-priority) such η satisfying Case 4c but none satisfying Case 4d, then
- define $\Theta(D; v') = V(v')$ for all $v' \leq \ell(\Psi(E \oplus D), V)$ for which $\Theta(D; v')$ is currently undefined, with old use $\vartheta(v')$ (if defined previously), or new use $\vartheta(v')$ larger than any number mentioned so far (otherwise);
- end the substage; and
- let $\xi^\frown\langle\Theta\rangle$ be *eligible to act next.*

If later during this stage, some $\mathcal{P}^E$-strategy $\zeta \supseteq \xi^\frown\langle\Theta\rangle$ wants to enumerate a number y into E but is delayed, then we say that ξ has *started η's y-request.*

(6) If there is (a highest-priority) such η satisfying Case 4d, then check whether ξ has enumerated a number y into E for this y-request at the previous ξ-expansionary stage s', or whether some $\mathcal{S}$-strategy $\xi' \subset \xi$ has enumerated a number y into E for the sake of a y-request by η to both ξ and ξ' at a previous ξ'-expansionary stage s'.
(a) If neither ξ nor an $\mathcal{S}$-strategy $\xi' \subset \xi$ has enumerated such y into E, then there is a $\mathcal{P}^E$-strategy $\zeta \supseteq \xi^\frown\langle\Theta\rangle$ which found a computation $\Sigma(D; y) = 0$ at the last substage of stage s' (but was delayed in the enumeration of y into E). Now ξ
- enumerates this y into E;
- chooses a corresponding number z;
- extracts from F the number x such that y was chosen when x was enumerated into F;
- initializes all strategies $> \zeta$; and
- *ends the stage.*

(b) Otherwise, ξ has enumerated a number y into E at the last substage of the previous ξ-expansionary stage s', or ξ' has enumerated a number y into E at the last substage of a previous ξ'-expansionary stage s'. Check whether there is some v such that $\Theta(D; v)\downarrow \neq V(v)$.
(i) If there is no such v, then ξ's attempt at fulfilling η's y-request has failed. So
- let η again *make y-requests* (which have not been started) to all $\mathcal{S}$-strategies $\hat{\xi} \in Y(\eta)$ with $\hat{\xi} \subseteq \xi$;

- let each $\hat{\xi} \in Y(\eta)$ with $\hat{\xi} \subseteq \xi$ again *make x-requests* (which have not been started) to all $\mathcal{R}$-strategies $\zeta \in X(\hat{\xi})$; and
- *end the stage.*

(ii) If there is (a least) such v, then

- ξ has *fulfilled η's y-request* with the pair $\langle y, v\rangle$;
- cancel Θ;
- define $\Lambda(V; z) = D(z)$ for all $z \leq s$ for which $\Lambda(V; z)$ is currently undefined, with old use $\vartheta(z)$ (if defined previously), or new use $\vartheta(z)$ larger than any number mentioned so far (otherwise);
- end the substage; and
- let $\xi^\frown\langle\Lambda\rangle$ be *eligible to act next.*

(7) Otherwise, i.e., if there is no η as in Case 4c or Case 4d, then proceed as in Case 5 or Case 6(b)ii, depending on whether there is some v such that $\Theta(D; v)\downarrow \neq V(v)$ (except that unlike in Case 5, we do not start any y-request, and that unlike in Case 6(b)ii, we do not fulfill any y-request).

This concludes the description of the individual strategies.

3.7. **The construction for Theorem 7.** Before describing the full construction, we need to define some more terms.

A strategy $\xi \in T$ is *initialized* by

- making all its witnesses undefined (if ξ is a $\mathcal{P}$-strategy);
- making all its functionals totally undefined (if ξ is an $\mathcal{R}$- or $\mathcal{S}$-strategy); and
- canceling all x- and y-requests made by or made to ξ.

The construction now proceeds in stages s. At stage 0, we initialize all strategies $\xi \in T$.

Each stage $s > 0$ consists of substages t. At a substage t, a strategy $\xi \in T$ of length t will be eligible to act. (Thus the unique strategy $\langle\rangle \in T$ of length 0 will always be eligible to act at substage 0.) Each strategy, when eligible to act, will act depending on the requirement assigned to it and as specified in section 3.6 and then either end the stage (if the strategy requires this or if $s = t$), or else determine the strategy to be eligible to act at substage $t + 1$. At the end of stage s, we initialize all strategies $>_L f_s$ where f_s is the strategy eligible to act at the last substage of stage s.

3.8. **The verification for Theorem 7.** Our first lemma checks some routine facts, the proofs of which we leave to the reader.

Lemma 14. (1) *The* true path *of the construction exists, i.e., there exists an infinite path f through T such that each strategy $\xi \subset f$ is eligible to act infinitely often but each strategy $\xi <_L f$ is eligible to act only finitely often.*
(2) *Each strategy $\xi \subset f$ is initialized only finitely often.*
(3) *F, E, and D are a 3-c.e. set, a d.c.e. set, and a c.e. set, respectively.* □

Next, we prove some technical facts about the definitions of functionals in our construction.

Lemma 15. *For any $\mathcal{R}$-strategy ξ and any ξ-expansionary stage s, if a computation $\Delta(U \oplus D; y)$ as well as computations $U \upharpoonright (\delta(y)+1) = \Phi(F \oplus E \oplus D) \upharpoonright (\delta(y)+1)$ are defined at stage s and (with the same oracles!) at a later ξ-expansionary stage s', then there is no witness w in the interval $(\delta(y), \varphi(\delta(y))]$ such that F, E, or D changes at w at a stage $\geq s'$.*

Proof. We fix the stage s_0 at which the use $\delta(y)$ was first defined in Step 6(b)ii or Step 7 of the $\mathcal{R}$-strategy. Note that $s_0 \leq s$ and that this use does not change after stage s_0. We now proceed by induction on the ξ-expansionary stage $s \geq s_0$: By the way the use $\delta(y)$ is chosen initially, our lemma holds at stage $s' = s_0$. Since any witness w chosen at a stage $s' \geq s_0$ satisfies $w \geq s'$ and by initialization, we only have to consider witnesses chosen by strategies $\eta \supseteq \xi\hat{\ }\langle\Delta\rangle$ for this lemma. Such η can only act at a ξ-expansionary stage s, and since the length of agreement $\ell(\Phi(F \oplus E \oplus D), U)$ at such a stage s must exceed y, any witness chosen by η at stage s must be $\geq \varphi_s(\delta_s(y))$. Now, after w is chosen, the use $\varphi(\delta(y))$ may increase at a stage s', say, but only due to a change of the oracle sets F, E, or D at another witness $w' \leq \varphi(\delta(y))$, say, which was thus chosen by a strategy η', say, before w was chosen, and so $\eta' < \eta$ as well as $w' \leq \delta_{s'}(y)$ by induction. (Recall here that we assume that the use functions of functionals for which the oracle is a "named" join of sets are always computed separately on the sets in the join.) Now either η' itself, or some $\xi' < \eta'$ using the witness w' for some request, changes one of F, E, or D at w' at stage s'; thus η is initialized at stage s'. Therefore, the witness w is used after stage s' for the sake of a witness w'' of a $\mathcal{P}$-strategy η'', say, such that w'' had already been chosen by stage s', and so, again by initialization at stage s', $\eta'' \leq \eta'$. This implies $w'' \leq w'$, again by initialization, so $w'' < w$. But the witness w is later used for a request on behalf of the witness w'', and by the way such requests are fulfilled in Step 6(b)ii of the $\mathcal{R}$-strategy or Step 6(b)ii of the $\mathcal{S}$-strategy before the witness w is chosen, we must have $w'' > w$, a contradiction. □

Lemma 16. *Each $\mathcal{R}$-requirement is satisfied.*

Proof. If the hypothesis $U = \Phi(F \oplus E \oplus D)$ of an $\mathcal{R}_{\Phi,U}$-strategy $\xi \subset f$ holds, then, since ξ is eligible to act infinitely often by Lemma 14, there are infinitely many ξ-expansionary stages. Again by Lemma 14, we now fix a stage s_0 after which ξ is no longer initialized and distinguish two cases:

Case 1: ξ cancels its functional Γ only finitely often: Then Step 5 or Step 7 of the $\mathcal{R}$-strategy will apply infinitely often, and by the way Γ is defined and the uses are chosen, $\Gamma(E \oplus D)$ will be a total function. Finally, after the last stage $\geq s_0$ at which Step 6(b)ii or 7 applies, there cannot be any argument at which $\Gamma(E \oplus D)$ and U disagree since this would cause a disagreement between $\Phi(F \oplus E \oplus D)$ and U.

Case 2: ξ cancels its functional Γ infinitely often: Then Step 6(b)ii, or Step 7 of the $\mathcal{R}$-strategy will apply infinitely often, and by the way Δ is defined and the uses are chosen, $\Delta(U \oplus D)$ will be a total function. We need to verify that $\Delta(U \oplus D)$ computes the set E correctly.

For the sake of a contradiction, let s be the first stage at which ξ defines a Δ-computation $\Delta(U \oplus D; y)$ with true oracle for some y (least for this s) such that $\Delta(U \oplus D; y) \neq E(y)$. Then at stage s, every strategy $>_L \xi\widehat{\ }\langle\Delta\rangle$ is initialized and thus cannot cause or allow $E(y)$, $\Delta(U \oplus D; y)$, or $\Phi(F \oplus E \oplus D) \restriction (\delta(y) + 1)$ to change after stage s. Similarly, no strategy $<_L \xi$ will be eligible to act after stage s and thus cannot cause or allow such a change after stage s. Finally, if a strategy $\zeta \subset \xi$ allows $E(y)$, $\Delta(U \oplus D; y)$, or $\Phi(F \oplus E \oplus D) \restriction (\delta(y) + 1)$ to change, then, since ξ is not initialized after stage s, ζ must be an $\mathcal{R}$- or $\mathcal{S}$-strategy, and $\zeta\widehat{\ }\langle\Delta\rangle \subseteq \xi$ or $\zeta\widehat{\ }\langle\Lambda\rangle \subseteq \xi$, respectively, and the change must be due to a request by a $\mathcal{P}$-strategy $\eta \supseteq \xi\widehat{\ }\langle\Delta\rangle$. Thus, in any case, any change allowing or causing $E(y)$, $\Delta(U \oplus D; y)$, or $\Phi(F \oplus E \oplus D) \restriction (\delta(y) + 1)$ to change must be initiated (possibly via a request) by a $\mathcal{P}$-strategy $\eta \supseteq \xi\widehat{\ }\langle\Delta\rangle$, which we now fix. By Lemma 15, the number being used by η must be $\leq \delta(y)$. We now distinguish cases by the type of $\mathcal{P}$-requirement of η.

If η is a $\mathcal{P}^D$-strategy, then η's action will permanently change D and destroy the computation $\Delta(U \oplus D; y)$.

If η is a $\mathcal{P}^E$-strategy, then η will have made an x-request to ξ, and some $\eta' \supseteq \xi\widehat{\ }\langle\Gamma\rangle$ will have supplied a witness x for the witness y' used by η; and since η' fulfilled the x-request of η via x, there must be some u such that $u \leq y$, and $x \in F$ iff $u \in U$, as explained in section 3.2. Since we ensure $y' \in E$ iff $x \notin F$, $\Delta(U \oplus D; y)$ can be corrected when $E(y)$ changes.

If η is a $\mathcal{P}^F$-strategy with a witness x, then there are three possibilities:

- x is already used together with a number u as in the previous paragraph, and we can argue as there.
- η enumerates x for diagonalization (possibly with a delay). If that allows a U-change, then the mechanism of Step 2 of the $\mathcal{R}$-strategy (as explained in Section 3.2.8) will ensure that ξ diagonalizes permanently (using η in the role of ζ' there). The ζ' mentioned in Step 2 of the $\mathcal{R}$-strategy must be a $\mathcal{P}^F$-strategy (since otherwise one of the previous paragraphs must apply).
- η had already started the process of permanent diagonalization of Step 2 of the $\mathcal{R}$-strategy, in which case there cannot have been a further ξ-expansionary stage.

This establishes our lemma. □

Lemma 17. *Each $\mathcal{S}$-requirement is satisfied.*

Proof. This proof is very similar to that of Lemma 16, the main difference being that Ψ has only oracle $E \oplus D$.

If the hypothesis $V = \Psi(E \oplus D)$ of an $\mathcal{S}_{\Psi,V}$-strategy $\xi \subset f$ holds, then, since ξ is eligible to act infinitely often by Lemma 14, there are infinitely many ξ-expansionary stages. Again by Lemma 14, we now fix a stage s_0 after which ξ is no longer initialized and distinguish two cases:

Case 1: ξ cancels its functional Θ only finitely often: Then Step 5 or Step 7 of the $\mathcal{S}$-strategy will apply infinitely often, and by the way Θ is defined and the uses are chosen, $\Theta(D)$ will be a total function. Finally, after the last stage $\geq s_0$ at which Step 6(b)ii or 7 applies, there cannot be any argument at which $\Theta(D)$ and V disagree since this would cause a disagreement between $\Psi(E \oplus D)$ and V.

Case 2: ξ cancels its functional Θ infinitely often: Then Step 6(b)ii or Step 7 of the $\mathcal{S}$-strategy will apply infinitely often, and by the way Λ is defined and the uses are chosen, $\Lambda(V)$ will be a total function. We need to verify that $\Lambda(V)$ computes the set D correctly.

For the sake of a contradiction, let s be the first stage at which ξ defines a Λ-computation $\Lambda(V;z)$ with true oracle for some z (least for this s) such that $\Lambda(V;z) \neq D(z)$. Then at stage s, every strategy $>_L \xi^\frown\langle\Lambda\rangle$ is initialized and thus cannot cause or allow $D(z)$, $\Lambda(V;z)$, or $\Psi(E \oplus D) \restriction (\lambda(z)+1)$ to change. Similarly, no strategy $<_L \xi$ will be eligible to act after stage s. Finally, if a strategy $\zeta \subset \xi$ allows $D(z)$, $\Lambda(V;z)$, or $\Psi(E \oplus D) \restriction (\lambda(z)+1)$ to change, then, since ξ is not initialized after stage s, ζ must be an $\mathcal{R}$- or $\mathcal{S}$-strategy, and $\zeta^\frown\langle\Delta\rangle \subseteq \xi$

or $\zeta^\frown\langle\Lambda\rangle \subseteq \xi$, respectively, and the change must be due to a request by a $\mathcal{P}$-strategy $\eta \supseteq \xi^\frown\langle\Lambda\rangle$. Thus, in any case, any change allowing $D(z)$, $\Lambda(V;z)$, or $\Psi(E \oplus D) \upharpoonright (\lambda(z)+1)$ to change must be due to a request by a $\mathcal{P}^E$- or $\mathcal{P}^D$-strategy $\eta \supseteq \xi^\frown\langle\Lambda\rangle$, which we now fix. We now distinguish cases by the type of $\mathcal{P}$-requirement of η.

If η is a $\mathcal{P}^D$-strategy, then η will have made a y-request to ξ, and some $\eta' \supseteq \xi^\frown\langle\Theta\rangle$ will have supplied a witness y for the witness z' used by η; and since η' fulfilled the y-request of η via y, there must be some v such that $v \leq z$, and $y \in E$ iff $v \in V$, as explained in section 3.4. Since we ensure $z' \in D$ iff $y \notin E$, $\Lambda(V;z)$ can be corrected when $D(z)$ changes.

If η is a $\mathcal{P}^E$-strategy, then there are three possibilities:

- y is already used together with a number v as in the previous paragraph, and we can argue as there.
- η enumerates x for diagonalization (possibly with a delay). If that allows a V-change, then the mechanism of Step 2 of the $\mathcal{S}$-strategy (as explained in Section 3.4.9) will ensure that ξ diagonalizes permanently (using η in the role of ζ' there). The ζ' mentioned in Step 2 of the $\mathcal{S}$-strategy must be a $\mathcal{P}^E$-strategy (since otherwise the previous paragraph must apply).
- η had already started the process of permanent diagonalization of Step 2 of the $\mathcal{S}$-strategy, in which case there cannot have been a further ξ-expansionary stage.

This establishes our lemma. □

Lemma 18. *Each $\mathcal{P}^F$-requirement is satisfied.*

Proof. Fix a $\mathcal{P}^F_\Pi$-strategy $\xi \subset f$. By the construction, ξ must eventually have a fixed witness x, say, targeted for F such that there are no x-requests to an $\mathcal{R}$-strategy η with $\eta^\frown\langle\Gamma\rangle \subseteq \xi$ delaying the enumeration of x. Then ξ succeeds in meeting its requirement in the usual Friedberg-Muchnik fashion. □

Lemma 19. *Each $\mathcal{P}^E$-requirement is satisfied.*

Proof. Fix a $\mathcal{P}^E_\Sigma$-strategy $\xi \subset f$. By the construction, ξ must eventually have a fixed witness y, say, targeted for E such that there are no y-requests to an $\mathcal{S}$-strategy η with $\eta^\frown\langle\Theta\rangle \subseteq \xi$ delaying the enumeration of y. Furthermore, for each $\mathcal{R}$-strategy η with $\eta^\frown\langle\Delta\rangle \subseteq \xi$, since $\eta^\frown\langle\Delta\rangle \subseteq \xi \subset f$, ξ's x-request must eventually be satisfied (namely, by the next stage at which ξ is eligible to act after ξ has picked its witness y). Now ξ succeeds in meeting its requirement in the usual Friedberg-Muchnik fashion. □

Lemma 20. *Each $\mathcal{P}^D$-requirement is satisfied.*

Proof. Fix a $\mathcal{P}^D_\Omega$-strategy $\xi \subset f$. By the construction, ξ must eventually have a fixed witness z, say, targeted for D. Furthermore, for each $\mathcal{S}$-strategy η with $\eta^\frown\langle\Lambda\rangle \subseteq \xi$, since $\eta^\frown\langle\Lambda\rangle \subseteq \xi \subset f$, ξ's y-request must eventually be satisfied (namely, by the next stage at which ξ is eligible to act after ξ has picked its witness x). Now ξ succeeds in meeting its requirement in the usual Friedberg-Muchnik fashion. □

This completes the proof of Theorem 5.

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(Arslanov/Kalimullin) Department of Mathematics, Kazan State University, ul. Kremlevskaya 18, 420008 Kazan, RUSSIA
E-mail address: Marat.Arslanov@ksu.ru

E-mail address: Iskander.Kalimullin@ksu.ru

(Lempp) Department of Mathematics, University of Wisconsin, Madison, WI 53706-1388, USA
E-mail address: lempp@math.wisc.edu
URL: http://www.math.wisc.edu/~lempp